Lectures on classical optics

Part III

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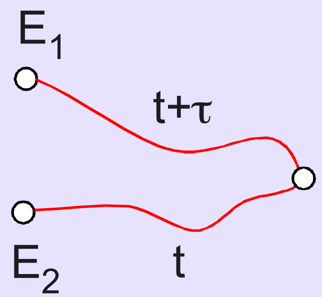
The concept of coherence

consider harmonic fields E_1, E_2 at positions r_1, r_2 at time t=0:



$$E = E_1(t+\tau) + E_2(t)$$

$\langle I_n \rangle \equiv \langle E_n(t) E_n^*(t) \rangle$, $n \in \{1,2\}$

$$\langle I \rangle = \langle EE^* \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2\,\mathrm{Re}[\,\langle E_1(t+\tau) E_2^*(t) \rangle\,] \quad \text{where} \quad \langle f \rangle \equiv \underset{T \to \infty}{\mathcal{Lim}} \langle f \rangle_T\,, \quad \langle f \rangle_T \equiv \frac{1}{T} \int_{-T/2}^{T/2} f(t)\,dt$$

$\underset{T \to \infty}{\mathcal{Lim}}$ means T finite but sufficiently large such that $\langle f \rangle_T$ does not depend on T

Normalized pair correlation function: $$\gamma_{12}(\tau) \equiv \frac{\langle E_1(t+\tau) E_2^*(t) \rangle}{(\langle I_1 \rangle \langle I_2 \rangle)^{1/2}}$$

$$\Rightarrow \quad \langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2\,(\langle I_1 \rangle \langle I_2 \rangle)^{1/2}\,\mathrm{Re}[\,\gamma_{12}(\tau)\,]$$

$$\gamma_{12}(\tau) = |\gamma_{12}(\tau)| \exp(i\phi_{12}(\tau)) \Rightarrow \langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2\,(\langle I_1 \rangle \langle I_2 \rangle)^{1/2}\, |\gamma_{12}(\tau)| \cos(\phi_{12}(\tau))$$

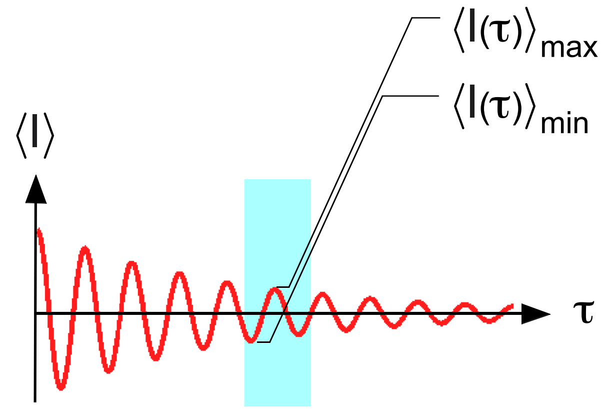
Assume: $|\gamma_{12}(\tau)|$ changes much slower than $\phi_{12}(\tau)$ (weakly coherent light)

$$\Rightarrow \langle I \rangle_{max/min} = \langle I_1 \rangle + \langle I_2 \rangle \pm 2\,(\langle I_1 \rangle \langle I_2 \rangle)^{1/2}\, |\gamma_{12}(\tau)|$$

Interference visibility:

$$\kappa \equiv \left| \frac{\langle I \rangle_{max} - \langle I \rangle_{min}}{\langle I \rangle_{max} + \langle I \rangle_{min}} \right| = \frac{2\,(\langle I_1 \rangle \langle I_2 \rangle)^{1/2}}{(\langle I_1 \rangle + \langle I_2 \rangle)}\, |\gamma_{12}(\tau)|$$

$$\langle I_1 \rangle = \langle I_2 \rangle \Rightarrow \kappa(\tau) = |\gamma_{12}(\tau)|$$



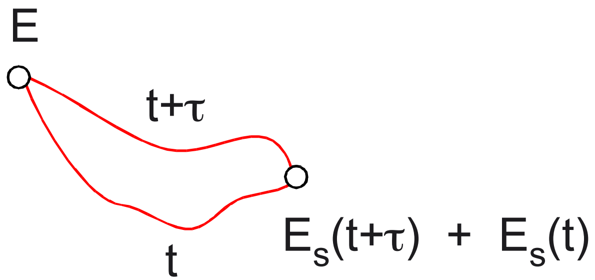
Definition:

$|\gamma_{12}(\tau)| = 1$ for all τ $\Rightarrow$ complete coherence

$0 < |\gamma_{12}(\tau)| < 1$ for some τ $\Rightarrow$ partial coherence

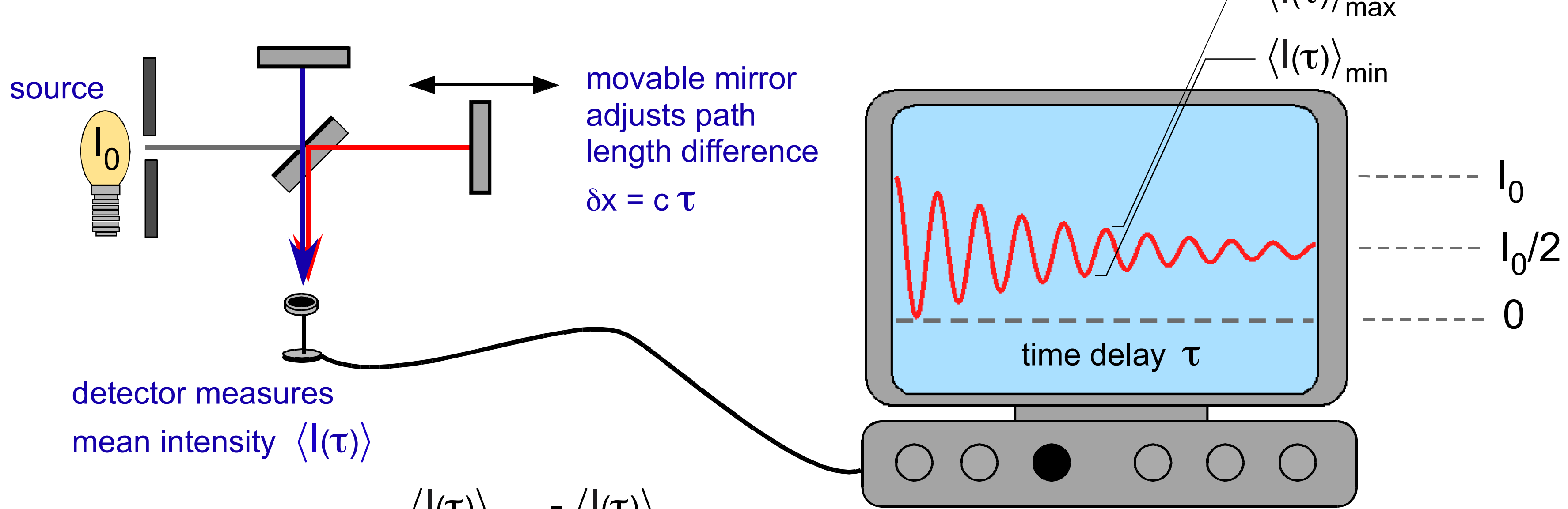
$|\gamma_{12}(\tau)| = 0$ for all τ $\Rightarrow$ no coherence

Normalized autocorrelation function: $g^{(1)}(\tau) \equiv \dfrac{\langle E(t+\tau)E^*(t)\rangle}{\langle I\rangle}$ degree of first order coherence



$g^{(1)}: \mathbb{R} \rightarrow \{c \in \mathbb{C}: |c| \leq 1\}$ with $g^{(1)}(0) = 1$, $g^{(1)}(-\tau) = g^{(1)*}(\tau)$

measurement of $g^{(1)}(\tau)$ in a Michelson Interferometer



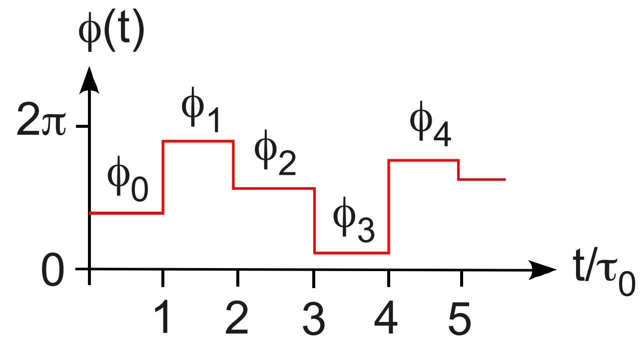
Interference contrast: $\kappa(\tau) \equiv \dfrac{\langle I(\tau)\rangle_{max} - \langle I(\tau)\rangle_{min}}{\langle I(\tau)\rangle_{max} + \langle I(\tau)\rangle_{min}}$

maximal coherence: Interference contrast maximal for all τ

partial coherence: Interference contrast decreases for large τ

Normalized autocorrelation function: $g^{(1)}(\tau) \equiv \dfrac{\langle E(t+\tau)E^*(t)\rangle}{\langle I\rangle}$

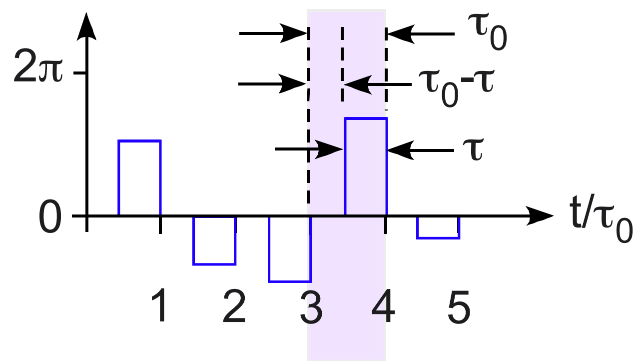
Example: sucessive wave trains of duration τ_0 and length $c\,\tau_0$



$$E(t) = E_0 \exp[i\omega t + i\phi(t)] \quad \text{with} \quad \phi(t):$$

$$\Rightarrow \quad g^{(1)}(\tau) = e^{i\omega\tau} \langle e^{i(\phi(t+\tau) - \phi(t))}\rangle$$

$\phi(t+\tau)-\phi(t)$ for $0 \leq \tau \leq \tau_0$



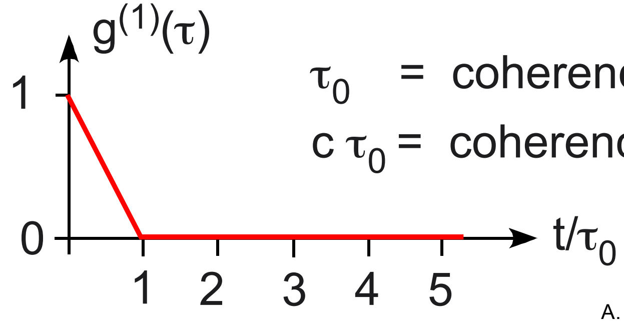
For $\tau_0 < \tau$: $\phi(t+\tau)-\phi(t) \neq 0$ random $\Rightarrow \langle e^{i(\phi(t+\tau) - \phi(t))}\rangle = 0$

For $0 \leq \tau \leq \tau_0$:

$$\langle e^{i(\phi(t+\tau) - \phi(t))}\rangle = \frac{1}{N\tau_0} \sum_{n=0}^{N-1} \int_{n\tau_0}^{(n+1)\tau_0} dt\, e^{i(\phi(t+\tau) - \phi(t))}$$

$$= \frac{1}{N\tau_0} \sum_{n=0}^{N-1} \Big((\tau_0-\tau) + \tau \exp(i(\phi_{n+1} - \phi_n)) \Big) = (\tau_0 - \tau)/\tau_0$$

$$\Rightarrow \quad g^{(1)}(\tau) = e^{i\omega\tau}(\tau_0 - \tau)/\tau_0 \times \begin{cases} 0 & \text{if } \tau_0 < \tau \\ 1 & \text{if } 0 \leq \tau \leq \tau_0 \end{cases}$$

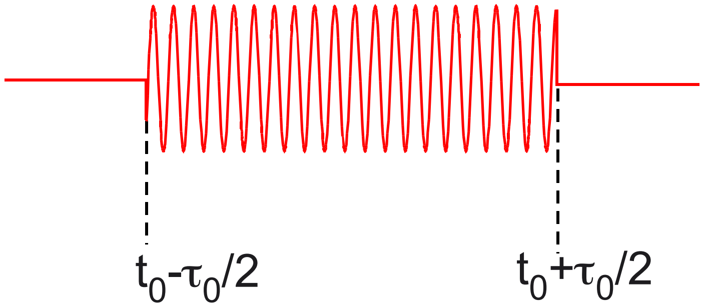


τ_0 = coherence time

$c\,\tau_0$ = coherence length

Coherence and emission spectrum:

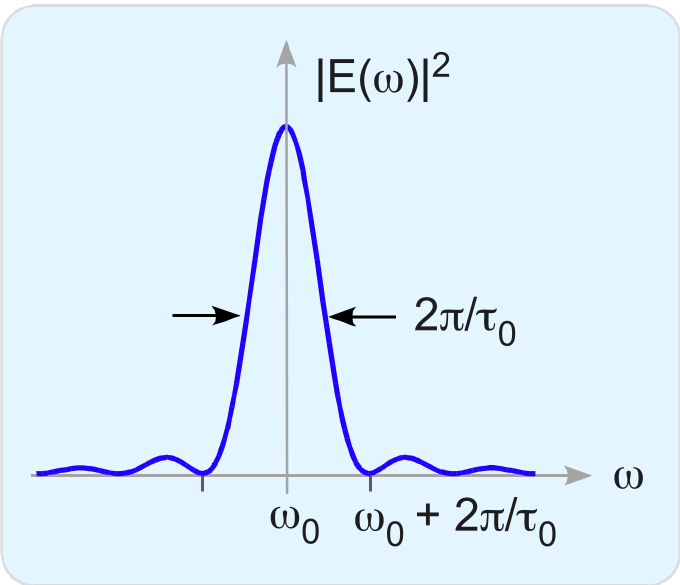
consider single wave train of duration τ_0, phase ϕ_0, frequency ω_0



$$E(t) = \exp[-i\omega_0 t - i\phi_0] \times \begin{cases} 0 & \text{otherwise} \\ 1 & \text{if } t_0-\tau_0/2 \leq t \leq t_0+\tau_0/2 \end{cases}$$

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt\ E(t)\, e^{i\omega t} = \sqrt{\frac{2}{\pi}} \frac{\sin[(\omega-\omega_0)\tau_0/2]}{(\omega-\omega_0)} \exp(-i\phi_0)$$

N wave trains with the same frequency ω_0 but arbitrary phases ϕ_n, durations τ_n, starting times t_n:
(includes previous example of successive wave trains: $\tau_n=\tau_0$, $t_n = n\,\tau_0$)



$$E(\omega) = \sum_{n=1}^{N} \sqrt{\frac{2}{\pi}} \frac{\sin[(\omega-\omega_0)\tau_n/2]}{(\omega-\omega_0)} \exp(i(\omega-\omega_0)t_n - i\phi_n)$$

$$|E(\omega)|^2 \approx \sum_{n=1}^{N} |E_n(\omega)|^2 = \frac{2}{\pi} \sum_{n=1}^{N} \frac{\sin^2[(\omega-\omega_0)\,\tau_n/2]}{(\omega-\omega_0)^2}$$

Emission bandwidth $\Delta\nu \approx 1/\tau$ with $\tau \equiv \frac{1}{N} \sum_{n=1}^{N} \tau_n$

Wiener-Khintchine Theorem:

$$E(\omega) \equiv \mathcal{F}[E(t)] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt\; E(t)\, e^{i\omega t} \,, \qquad F(\omega) \equiv \frac{|E(\omega)|^2}{\int_{-\infty}^{\infty} d\omega\, |E(\omega)|^2}$$

normalized spectral power density

$$\Rightarrow \quad F(\omega) = \frac{1}{\sqrt{2\pi}} \mathcal{F}[g^{(1)}] \,, \quad \mathcal{F} \equiv \text{Fourier-Transform}$$

Proof:
(for physicists)

$$\int_{-\infty}^{\infty} d\omega\, |E(\omega)|^2 = \int_{-\infty}^{\infty} dt\, |E(t)|^2 \approx \int_{-T/2}^{T/2} dt\, |E(t)|^2 \approx T\, \langle E(t)E^*(t)\rangle$$

$$|E(\omega)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt dt'\; E(t')E^*(t)\, e^{i\omega(t'-t)}$$

$t'=t+\tau$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt\, d\tau\; E(t+\tau)E^*(t)\, e^{i\omega\tau} \approx \frac{T}{2\pi} \int_{-\infty}^{\infty} d\tau\, \langle E(t+\tau)E^*(t)\rangle\, e^{i\omega\tau}$$

$$F(\omega) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau\; e^{i\omega\tau}\, \frac{\langle E(t+\tau)E^*(t)\rangle}{\langle E(t)E^*(t)\rangle} = \frac{1}{\sqrt{2\pi}} \mathcal{F}[g^{(1)}]$$

Example: Collision broadened light source

N molecules of a gas radiate monochromatic light $E_0 \exp(-i(\omega t + \phi_\nu(t)))$, $\nu \in \{1,...,N\}$

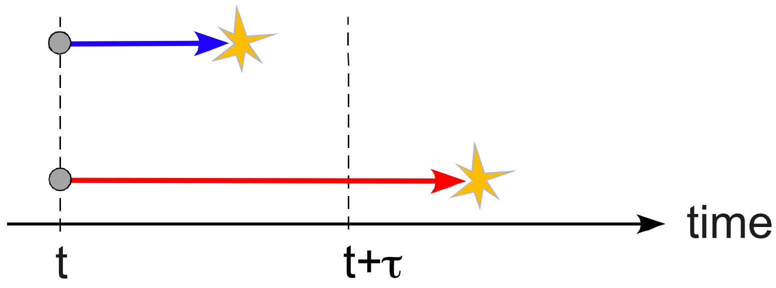
Collisions yield random phase jumps, i.e., phase $\phi_\nu(t) \in [0,2\pi]$ fluctuates.

$$E(t) = E_0 \sum_{\nu=1}^{N} \exp(-i(\omega t + \phi_\nu(t)))$$

$$\langle E(t)\, E(t+\tau)^* \rangle = |E_0|^2 \left\langle \sum_{\nu,\mu=1}^{N} e^{i\omega\tau}\, e^{i(\phi_\nu(t+\tau) - \phi_\mu(t))} \right\rangle = |E_0|^2 e^{i\omega\tau} \left\langle \sum_{\nu=1}^{N} e^{i(\phi_\nu(t+\tau) - \phi_\nu(t))} \right\rangle$$

Summanden mit $\nu \neq \mu$ mitteln sich zu Null

$$e^{i(\phi_\nu(t+\tau) - \phi_\nu(t))} = \begin{cases} e^{i\chi_\nu} & \text{if free flight} < \tau \quad \chi_\nu \in [0,2\pi] \text{ random} \\ 1 & \text{if free flight} > \tau \end{cases}$$



$$\sum_{\nu=1}^{N} e^{i(\phi_\nu(t+\tau) - \phi_\nu(t))} = P(<\tau) \sum_{\nu=1}^{N} e^{i\chi_\nu} + N\, P(>\tau) = N\, P(>\tau)$$

$P(<\tau)$ = Probability for free flight shorter than τ

$P(>\tau)$ = Probability for free flight longer than τ

$$\Rightarrow \langle E(t)\, E(t+\tau)^* \rangle = |E_0|^2 e^{i\omega\tau} \langle N\, P(>\tau) \rangle = N\, |E_0|^2\, e^{i\omega\tau}\, P(>\tau)$$

$$\text{coherence function:} \quad g(\tau) \equiv \frac{\langle E(t)\,E(t+\tau)^* \rangle}{\langle E(t)\,E(t)^* \rangle} = e^{i\omega\tau}\, P(>\tau)$$

$P(>\tau)$ = Probability for free flight longer than τ

Calculation of $P(>\tau)$ (kinetic gas theory):

Probability for a free flight of duration $t \in [\tau, \tau + d\tau]$:

$$p(t)\, d\tau = \frac{1}{\tau_0} \exp(-\tau/\tau_0)\, d\tau$$

τ_0 = mean duration of a free flight

$$P(>\tau) \equiv \int_\tau^\infty p(s)\, ds = \exp(-\tau/\tau_0)$$

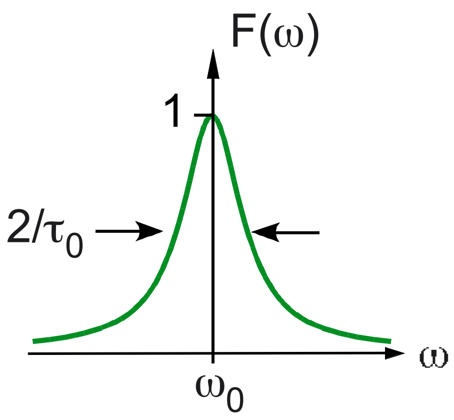
$$\Rightarrow \quad g^{(1)}(\tau) = e^{i\omega_0\tau} \exp(-\tau/\tau_0)$$

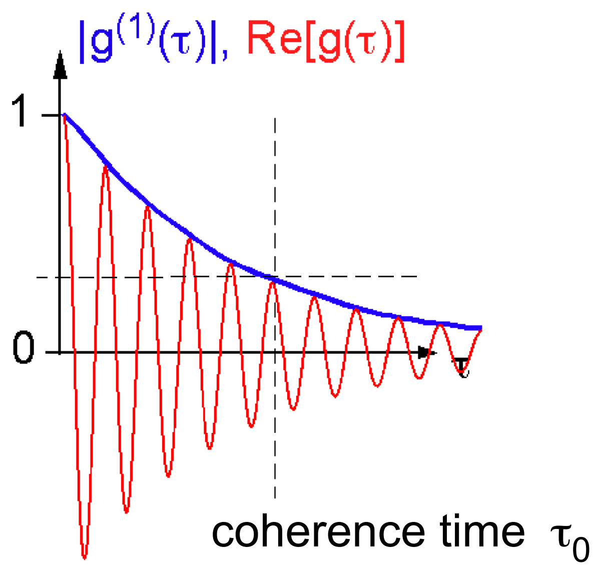
$$\mathrm{Re}[g(\tau)] = \cos(\omega\tau) \exp(-\tau/\tau_0)$$

$$|g(\tau)| = \exp(-\tau/\tau_0)$$

$$\Rightarrow \quad F(\omega) \sim \frac{1}{1 + (\omega-\omega_0)^2 \tau_0^2}$$

W-K-Theorem





Example: Doppler broadened light source

N Molecules of a gas radiate light $E_n(t) = E_0 \exp[i(\omega_0 t + \phi_n)]$ at frequency ω_0 in their restframe.

Motion of the molecules $\Rightarrow$ Doppler-shift:

Probability for molecule with velocity $v \in [v,v+dv]$: $P(v)dv \sim \exp(-v^2/v_0^2)\, dv$ with $v_0 = (m/2k_BT)^{1/2}$

Probability for emission with frequency $\omega_0+\delta$: $P(\delta) \sim \exp(-\delta^2/\delta_0^2)$ with $\delta_0 = (m/2k_BT)^{1/2}\, \omega/c$

Total field: $$E(t) = E_0 \sum_{n=1}^{N} \exp[i(\omega_n t + \phi_n)]\,, \quad \omega_n = \omega_0 + \delta_n$$

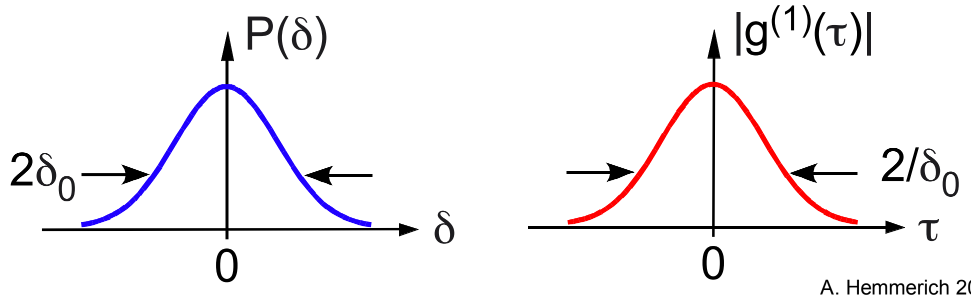
$$\langle E(t+\tau)E^*(t)\rangle = |E_0|^2 \sum_{n,m=1}^{N} \exp(i\omega_n\tau) \left\langle \exp[i(\omega_n-\omega_m)t]\, \exp[i(\phi_n - \phi_m)] \right\rangle$$

$$= |E_0|^2 \sum_{n=1}^{N} \exp(i\omega_n\tau) = N\,|E_0|^2 \exp(i\omega_0\tau)\; N^{-1} \sum_{n=1}^{N} \exp(i\delta_n\tau)$$

$$= N\,|E_0|^2 \exp(i\omega_0\tau) \int_{-\infty}^{\infty} P(\delta) \exp(i\delta\tau)\, d\delta = N\,|E_0|^2 \exp(i\omega_0\tau) \exp(-\delta_0^2\tau^2)$$

$$g^{(1)}(\tau) = \exp(i\omega_0\tau) \exp(-\delta_0^2\tau^2)$$

(cf. W-K-Theorem)

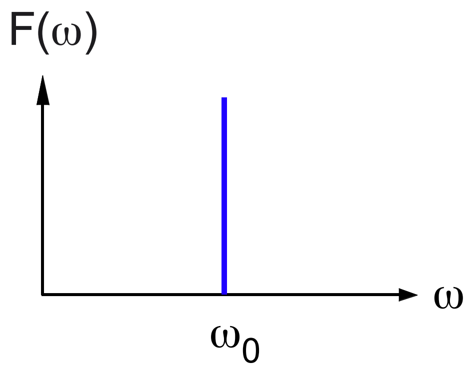


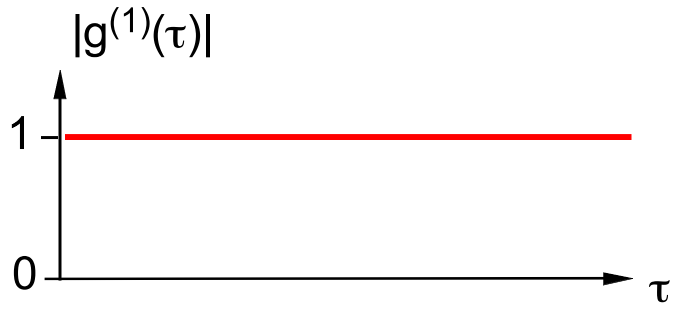
Example: monochromatic light

$E(t) = E_0 \exp(i(\omega_0 t + \phi))$

$g^{(1)}(\tau) = \exp(i\omega_0\tau)$

$|g^{(1)}(\tau)| = 1$





Example: monochromatic light with frequency noise (Laser)

$E(t) = E_0 \exp(i(\omega_0 t + \phi(t)))$, $\phi(t) = \int_0^t ds\, \Omega(s)$, $\Omega(s)$ = frequency fluctuations

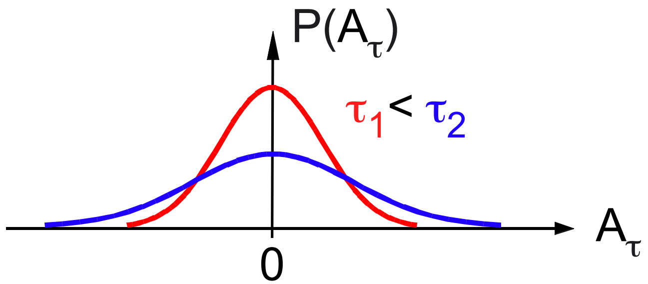
(1) $\langle E(t+\tau)E^*(t)\rangle = |E_0|^2 \exp(i\omega_0\tau) \langle \exp[i(\phi(t+\tau) - \phi(t))] \rangle$

$= |E_0|^2 \exp(i\omega_0\tau) \exp[-\frac{1}{2}\langle[\phi(t+\tau) - \phi(t)]^2\rangle]$

↑ Gaussian moment theorem

Gaussian phase noise: evolution of $\phi(t)$ is an example of a Wiener-Lévy process

$$P(\phi_2, t+\tau \,|\, \phi_1, t) \equiv \frac{1}{\sqrt{2w\,\pi\,\tau}} \exp\left[\frac{-(\phi_2 - \phi_1)^2}{2w\,\tau}\right]$$



$$A_\tau(t) \equiv (\phi(t+\tau) - \phi(t)) \quad \Rightarrow \quad P(A_\tau) \sim \exp(-f(\tau)\,A_\tau^{\,2})$$

moments: $$\langle A^{2n}\rangle \equiv \int_{-\infty}^{\infty} A^{2n}\,P(A)\,dA\,, \quad \text{with } P(A) = \exp(-f\,A^2)$$

Gaussian moment theorem:

A = Gaussian variable $\Rightarrow$

(a) $$\langle A^{2n}\rangle = \frac{(2n)!}{2^n\,n!}\langle A^2\rangle^n$$

(b) $$\langle A^{2n+1}\rangle = 0$$

$$\langle \exp(iA)\rangle = \sum_{n=0}^{\infty} \frac{\langle (iA)^n\rangle}{n!} \underset{(b)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{\langle A^{2n}\rangle}{(2n)!} \underset{(a)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{\langle A^2\rangle^n}{2^n\,n!} = \exp[-\tfrac{1}{2}\langle A^2\rangle]$$

(2) $$\langle[\phi(t+\tau) - \phi(t)]^2\rangle = \left\langle \int_t^{t+\tau}\int_t^{t+\tau} ds\, ds'\, \Omega(s)\, \Omega(s') \right\rangle = \int_0^{\tau}\int_0^{\tau} ds\, ds'\, \langle\Omega(s+t)\, \Omega(s'+t)\rangle$$

$$\langle\Omega(s+t)\, \Omega(s'+t)\rangle = \langle\Omega(t)\, \Omega(s'-s+t)\rangle = \langle\Omega(s-s'+t)\, \Omega(t)\rangle$$

$$\Rightarrow \quad \langle\Omega(s+t)\, \Omega(s'+t)\rangle = f(|s-s'|)$$

(3) coordinate transformation

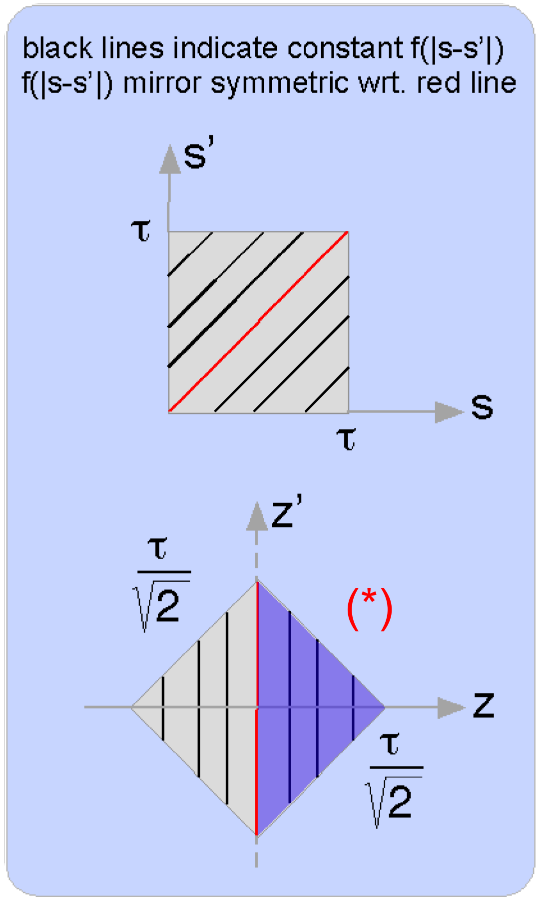
$$\begin{pmatrix} z \\ z' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s \\ s' \end{pmatrix} - \frac{\tau}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad dz\, dz' = ds\, ds'$$

$$x = s-s' = \sqrt{2}\, z$$

$$\langle[\phi(t+\tau) - \phi(t)]^2\rangle =$$

$$\int_0^{\tau}\int_0^{\tau} ds\, ds'\; f(|s-s'|) \underset{(*)}{=} 2\int_0^{\tau/\sqrt{2}} dz \int_{-(\tau/\sqrt{2}-z)}^{\tau/\sqrt{2}-z} dz'\; f(|\sqrt{2}\, z|)$$

$$= \sqrt{2}\int_0^{\tau} dx \int_{-((\tau-x)/\sqrt{2})}^{(\tau-x)/\sqrt{2}} dz'\; f(|x|) = 2\int_0^{\tau} dx\, f(|x|)\, (\tau-x) = 2\int_0^{\tau} ds\, (\tau-s)\, \langle\Omega(s+t)\, \Omega(t)\rangle$$



WKT: $$|\mathcal{F}[\Omega](\omega)|^2 \approx \frac{T}{2\pi}\int_{-\infty}^{\infty} ds'\langle\Omega(t+s')\Omega(t)\rangle\, e^{i\omega s'}$$

$$\Rightarrow \int_{-\infty}^{\infty} d\omega\,|\mathcal{F}[\Omega](\omega)|^2\, e^{-i\omega s} = T\int_{-\infty}^{\infty} ds'\,\langle\Omega(t+s')\Omega(t)\rangle\,\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(s'-s)} = T\,\langle\Omega(t+s)\Omega(t)\rangle \qquad (4)$$

$$\langle[\phi(t+\tau)-\phi(t)]^2\rangle = 2\int_0^{\tau} ds\,\langle\Omega(s+t)\,\Omega(t)\rangle\,(\tau-s) \underset{(4)}{\approx} \frac{2}{T}\int_{-\infty}^{\infty} d\omega\,|\mathcal{F}[\Omega](\omega)|^2\int_0^{\tau} ds\,(\tau-s)\,e^{-i\omega s} \qquad (5)$$

$$\int_0^{\tau} ds\,(\tau-s)\,e^{-i\omega s} = \tau\int_0^{\tau} ds\,e^{-i\omega s} + \int_0^{\tau} ds\,s\,e^{-i\omega s} = i\tau\omega^{-1}(e^{-i\omega\tau}-1) + \frac{d}{d\omega}\,\omega^{-1}(e^{-i\omega\tau}-1)$$

use: $i\dfrac{d}{d\omega}e^{-i\omega s} = s\,e^{-i\omega s}$

$$= -i\tau\omega^{-1} - \omega^{-2}\,(e^{-i\omega\tau}-1) = \frac{2\sin^2(\omega\tau/2)}{\omega^2} + \frac{i}{\omega}\left(\frac{\sin(\omega\tau)}{\omega} - \tau\right) \qquad (6)$$

$$\Rightarrow \langle[\phi(t+\tau)-\phi(t)]^2\rangle \underset{(5,6)}{=} \frac{2}{T}\int_{-\infty}^{\infty} d\omega\,|\mathcal{F}[\Omega](\omega)|^2\left(\frac{2\sin^2(\omega\tau/2)}{\omega^2} + \frac{i}{\omega}\left(\frac{\sin(\omega\tau)}{\omega} - \tau\right)\right)$$

$$\langle[\phi(t+\tau) - \phi(t)]^2\rangle = \frac{2}{T} \int_{-\infty}^{\infty} d\omega \, |\mathcal{F}[\Omega](\omega)|^2 \left(\frac{2\sin^2(\omega\tau/2)}{\omega^2} + \frac{i}{\omega} \left(\frac{\sin(\omega\tau)}{\omega} - \tau \right) \right)$$

$$= \frac{4}{T} \int_{-\infty}^{\infty} d\omega \, |\mathcal{F}[\Omega](\omega)|^2 \, \frac{\sin^2(\omega\tau/2)}{\omega^2} = \frac{4\tau}{T} \int_{-\infty}^{\infty} dz \, |\mathcal{F}[\Omega](z/\tau)|^2 \, \frac{\sin^2(z/2)}{z^2}$$

Imaginary part is odd

$$\Rightarrow \quad g^{(1)}(\tau) = \frac{\langle E(t+\tau)E^*(t)\rangle}{\langle E(t)E^*(t)\rangle} = \exp(i\omega_0\tau) \, \exp\left(-\frac{4\tau}{T} \int_{-\infty}^{\infty} dz \, |\mathcal{F}[\Omega](z/\tau)|^2 \, \frac{\sin^2(z/2)}{z^2} \right)$$

(1,2,3)

$$g^{(1)}(\tau) = \exp(i\omega_0\tau)\ \exp\left(-\frac{4\tau}{T}\int_{-\infty}^{\infty} dz\, |\mathcal{F}[\Omega](z/\tau)|^2\ \frac{\sin^2(z/2)}{z^2}\right)$$

white frequency noise: $S(\omega) \equiv \frac{1}{T} |\mathcal{F}[\Omega](\omega)|^2$ constant, $\tau_0^{-1} \equiv 2\pi S$

$$\int_{-\infty}^{\infty} \frac{\sin^2(z/2)}{z^2} dz = \frac{\pi}{2} \quad \Rightarrow \quad \boxed{g^{(1)}(\tau) = \exp(i\omega_0\tau)\exp(-\tau/\tau_0)}$$

W-K-Theorem $\Rightarrow$ Lorenzian emission spectrum with $2/\tau_0$ FWHM

Eample for white frequency noise: Schalow Townes quantum noise

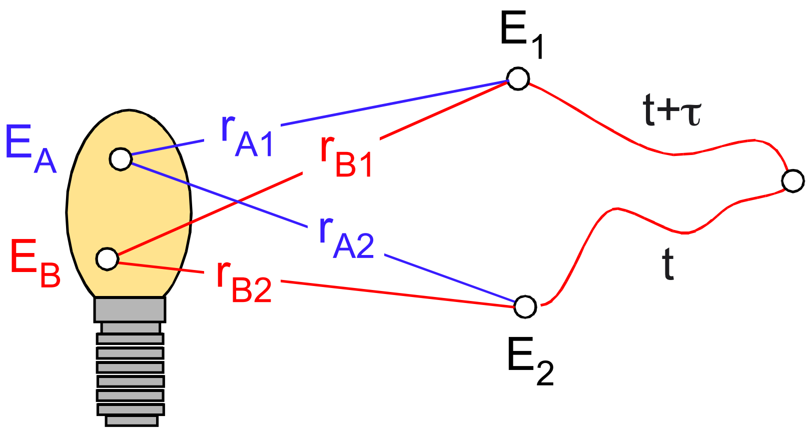
1/f frequency noise: $S \equiv \frac{1}{T}|\mathcal{F}[\Omega]|^2 = \frac{S_0}{|\omega|}$, S_0 constant, $\tau_0 \equiv \left(8S_0\int_{0+}^{\infty} dz\, \frac{\sin^2(z/2)}{z^3}\right)^{-1/2}$

(Note: Integral does not exist, if integration extends to zero!)

$$\Rightarrow \quad \boxed{g^{(1)}(\tau) = \exp(i\omega_0\tau)\exp[-(\tau/\tau_0)^2]}$$

W-K-Theorem $\Rightarrow$ Gaussian emission spectrum with $2/\tau_0$ 1/e-Full Width

Spatial Coherence



$$E_1 = E_{A1} + E_{B1} \qquad E_{An} = E_A \exp(i\, r_{An}\, \omega/c)$$
$$E_2 = E_{A2} + E_{B2} \qquad E_{Bn} = E_B \exp(i\, r_{Bn}\, \omega/c)$$

$$\langle E_1(t+\tau)E_2^*(t)\rangle = \langle E_{A1}(t+\tau)E_{A2}^*(t)\rangle + \langle E_{B1}(t+\tau)E_{B2}^*(t)\rangle$$
$$+ \cancel{\langle E_{A1}(t+\tau)E_{B2}^*(t)\rangle} + \cancel{\langle E_{B1}(t+\tau)E_{A2}^*(t)\rangle}$$

E_{An}, E_{Bn} mutually incoherent

Light Source: consisting of mutually incoherent point sources with equal coherence properties described by $g^{(1)}(\tau) = \exp(i\omega\tau - \tau/\tau_0)$

$$\langle I_n\rangle = \langle E_n(t)E_n^*(t)\rangle = \langle E_{An}(t)E_{An}^*(t)\rangle + \langle E_{Bn}(t)E_{Bn}^*(t)\rangle$$
$$+ \cancel{\langle E_{An}(t)E_{Bn}^*(t)\rangle} + \cancel{\langle E_{Bn}(t)E_{An}^*(t)\rangle}$$

E_{An}, E_{Bn} mutually incoherent

$$\Rightarrow \langle I_1\rangle = \langle I_2\rangle$$

$$E_{An} = E_A \exp(i\, r_{An}\, \omega/c) \;\Rightarrow$$

$$\langle E_{A1}(t+\tau)E_{A2}^*(t)\rangle = \langle E_A(t+\tau)E_A^*(t)\rangle \exp[i(r_{A1} - r_{A2})\omega/c] = \langle E_A(t+\tau_A)E_A^*(t)\rangle \quad \text{with } \tau_A \equiv \tau + (r_{A1} - r_{A2})/c$$

$$E_{Bn} = E_B \exp(i\, r_{Bn}\, \omega/c) \;\Rightarrow$$

$$\langle E_{B1}(t+\tau)E_{B2}^*(t)\rangle = \langle E_B(t+\tau)E_B^*(t)\rangle \exp[i(r_{B1} - r_{B2})\omega/c] = \langle E_B(t+\tau_B)E_B^*(t)\rangle \quad \text{with } \tau_B \equiv \tau + (r_{B1} - r_{B2})/c$$

$$\Rightarrow \quad \langle E_1(t+\tau)E_2^*(t)\rangle = \langle E_{A1}(t+\tau)E_{A2}^*(t)\rangle + \langle E_{B1}(t+\tau)E_{B2}^*(t)\rangle = \langle E_A(t+\tau_A)E_A^*(t)\rangle + \langle E_B(t+\tau_B)E_B^*(t)\rangle$$

$$\langle I_n\rangle = \langle E_n(t)E_n^*(t)\rangle = \langle E_{An}(t)E_{An}^*(t)\rangle + \langle E_{Bn}(t)E_{Bn}^*(t)\rangle$$

assume $\langle E_{An}(t)E_{An}^*(t)\rangle = \langle E_{Bn}(t)E_{Bn}^*(t)\rangle$ and hence $\langle I_n\rangle = 2\langle E_{An}(t)E_{An}^*(t)\rangle = 2\langle E_{Bn}(t)E_{Bn}^*(t)\rangle$

$$\gamma_{12}(\tau) \equiv \frac{\langle E_1(t+\tau)E_2^*(t)\rangle}{(\langle I_1\rangle\langle I_2\rangle)^{1/2}} = \frac{1}{2}\,[\, g^{(1)}(\tau_A) + g^{(1)}(\tau_B)\,] = \frac{1}{2}[\, \exp(i\omega\tau_A - \tau_A/\tau_0) + \exp(i\omega\tau_B - \tau_B/\tau_0)\,]$$

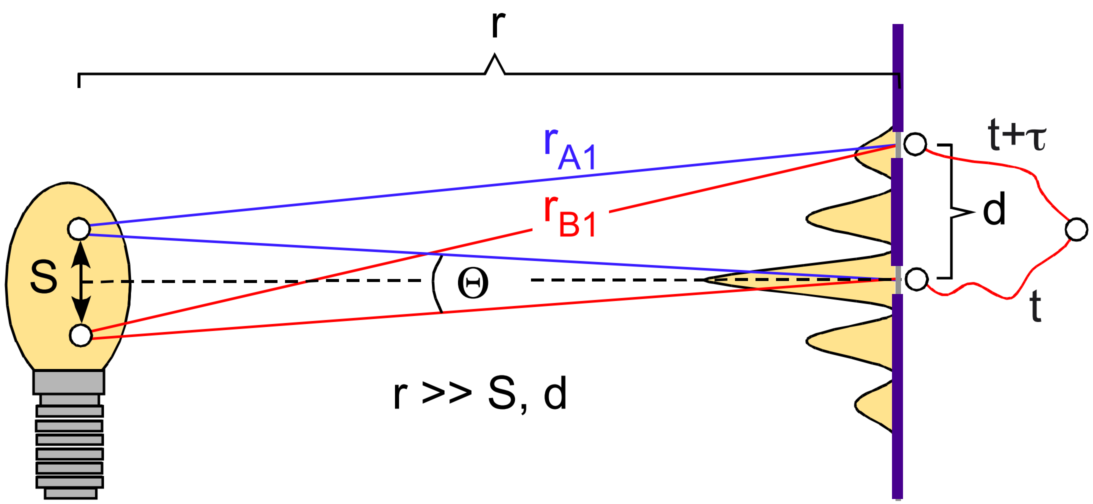
pair correlation is sum of $g^{(1)}$-functions of each point source

$$4\,|\gamma_{12}(\tau)|^2 = |g^{(1)}(\tau_A)|^2 + |g^{(1)}(\tau_B)|^2 + 2\,|g^{(1)}(\tau_A)||g^{(1)}(\tau_B)| \cos(\omega\,(\tau_A - \tau_B))$$

Interference term !

$$4\,|\gamma_{12}(\tau)|^2 = |g^{(1)}(\tau_A)|^2 + |g^{(1)}(\tau_B)|^2 + 2\,|g^{(1)}(\tau_A)||g^{(1)}(\tau_B)|\cos(\omega(\tau_A - \tau_B))$$

Interference term depends on $\tau_A - \tau_B = (r_{A1} - r_{A2})/c - (r_{B1} - r_{B2})/c$



$$r_{A2} - r_{B2} = 0 \quad \Rightarrow \quad \tau_A - \tau_B = (r_{A1} - r_{B1})/c$$

$$r_{A1} \approx r + \frac{(d-S/2)^2}{2r}, \quad r_{B1} \approx r + \frac{(d+S/2)^2}{2r}$$

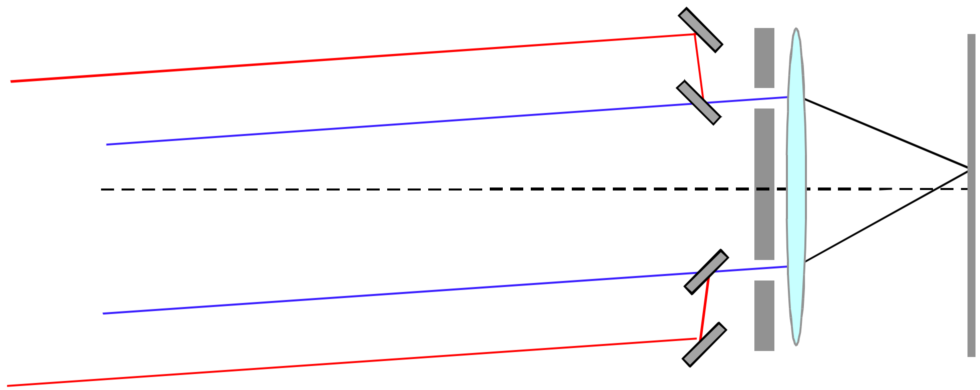
$$\Rightarrow \tau_A - \tau_B \approx -\frac{S\,d}{2r\,c}$$

Light Source: mutually incoherent point sources $g^{(1)}(\tau) = \exp(i\omega\tau - \tau/\tau_0)$

First minimum of $|\gamma_{12}(\tau)|^2$:

$$\omega(\tau_A - \tau_B) = \pi,\ S \approx r\,\Theta \quad \Rightarrow \quad d \approx \lambda/\Theta$$

transverse coherence length



Michelson stellar interferometer: adjustable slits, extension of slit separation by mirrors

Measurement of angular diameter of stars, angular separation of double stars, etc.

Second Order Coherence

Normalized autocorrelation function: $g^{(2)}(\tau) \equiv \dfrac{\langle I(t+\tau)I(t)\rangle}{\langle I(t)\rangle^2}$ degree of second order coherence

(1) $g^{(2)} : \mathbb{R} \rightarrow \{r \in \mathbb{R}: r \geq 0\}$

(2) $g^{(2)}(-\tau) = g^{(2)}(\tau)$

(3) $g^{(2)}(0) \geq 1$

(4) $g^{(2)}(\tau) \leq g^{(2)}(0)$

(5) $g^{(2)}(\tau\rightarrow\infty) = 1$ if correlations vanish

Proof (3):

$$\left(\frac{1}{N}\sum_{n=1}^{N} I_n\right)^2 = \frac{1}{N^2}\left(\sum_n I_n^2 + \sum_{n\neq m} I_n I_m\right) \leq \frac{1}{N^2}\left(\sum_n I_n^2 + \sum_{n\neq m}(I_n^2+I_m^2)/2\right)$$

$$= \frac{1}{N^2}\sum_{n,m}(I_n^2+I_m^2)/2 = \frac{1}{N}\sum_{n=1}^{N} I_n^2 \quad \Rightarrow \quad g^{(2)}(0) = \frac{\langle I(t)^2\rangle}{\langle I(t)\rangle^2} = \frac{\frac{1}{N}\sum_{n=1}^{N} I_n^2}{\left(\frac{1}{N}\sum_{n=1}^{N} I_n\right)^2} \geq 1$$

Proof (4):

$$\left(\langle I(t+\tau)I(t)\rangle\right)^2 = \left(\frac{1}{N}\sum_{n=1}^{N} I(t_n+\tau)I(t_n)\right)^2 \leq \left(\frac{1}{N}\sum_{n=1}^{N} I(t_n+\tau)^2\right)\left(\frac{1}{N}\sum_{n=1}^{N} I(t_n)^2\right) = \left(\langle I(t)^2\rangle\right)^2$$

Cauchy-Schwartz

Proof (5):

$$\tau \to \infty \quad \Rightarrow \quad \langle I(t+\tau)I(t)\rangle = \langle I(t+\tau)\rangle\langle I(t)\rangle = \langle I(t)\rangle^2$$

no correlations

Example: monochromatic light

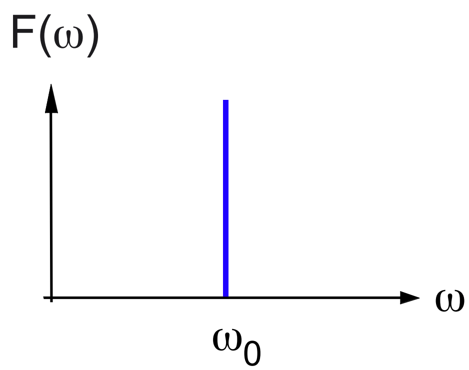
$$E(t) = E_0 \exp(i(\omega_0 t + \phi))$$

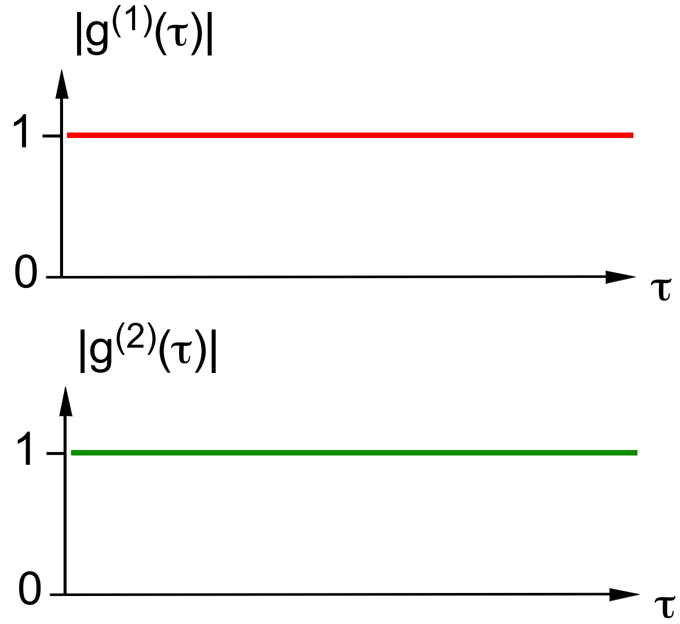
$$I(t) = E_0 E_0^*$$

$$g^{(1)}(\tau) = \exp(i\omega_0\tau)$$

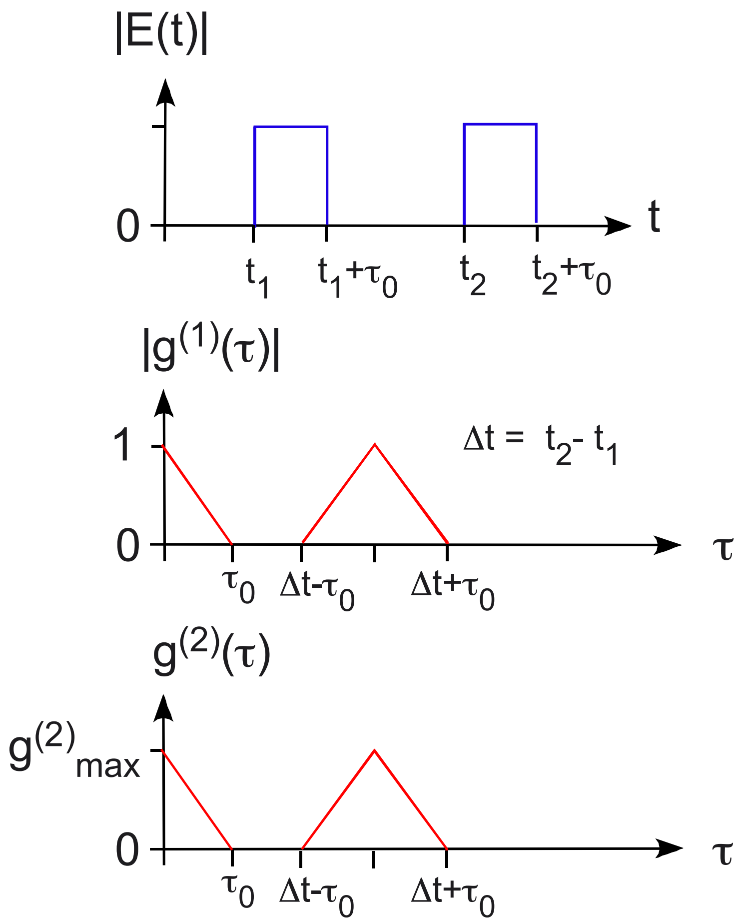
$$|g^{(1)}(\tau)| = 1$$

$$g^{(2)}(\tau) \equiv \frac{\langle I(t+\tau)I(t)\rangle}{\langle I(t)\rangle^2} = 1$$





Example: pulse train



$g^{(2)}_{max} = (N\,\tau_0)^{-1}$

N = mean number of pulses/time

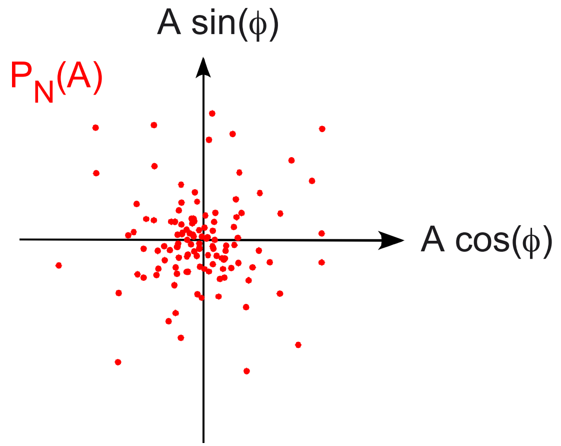
Gaussian chaotic Light:

Consider N emitters with random phases $\phi_n(t)$: $A(t)\exp[i\phi(t)] \equiv \sum_{n=1}^{N} \exp[i\phi_n(t)]$

Probability for $A(t)\exp[i\phi(t)]$ to fall within unit area at the point (A,ϕ) in the complex plane C:

$$P_N(A,\phi) = \frac{1}{\pi N} \exp(-A^2/N)$$

$$\int_0^\infty \int_0^{2\pi} A\, dA\, d\phi\, P_N(A,\phi) = 1$$



Probability for measuring an intensity $\in [\, I, I + dI \,]$:

$$P(I)\, dI = \frac{1}{\langle I \rangle} \exp(-I / \langle I \rangle)\, dI$$

moments: $$\langle I^n \rangle \equiv \int_0^\infty dI\, P(I)\, I^n = n!\, \langle I \rangle^n$$

$$\Delta I \equiv (\langle I^2 \rangle - \langle I \rangle^2)^{1/2} = \langle I \rangle$$

Consider N emitters with random phases $\phi_n(t)$: $A(t)\exp[i\phi(t)] \equiv \sum_{n=1}^{N} \exp[i\phi_n(t)]$

$$\Rightarrow \quad \langle I \rangle = \Big| \sum_{\nu} \exp[i\phi_\nu] \Big|^2 = \sum_{\nu,\mu} \exp[i(\phi_\nu - \phi_\mu)] = \sum_{\nu=\mu} 1 = N$$

$$\Rightarrow \quad \langle I^n \rangle = \Big| \sum_{\nu} \exp[i\phi_\nu] \Big|^{2n} = \Big| \sum_{\nu,\mu} \exp[i(\phi_\nu - \phi_\mu)] \Big|^n$$

$$= \sum_{\nu 1,\mu 1} \cdots \sum_{\nu n,\mu n} \exp[i(\phi_{\nu 1} - \phi_{\mu 1})] \cdots \exp[i(\phi_{\nu n} - \phi_{\mu n})]$$

$$= \sum_{p} \sum_{\nu 1 = p(\mu 1)} \cdots \sum_{\nu n = p(\mu n)} \exp[i(\phi_{\nu 1} - \phi_{\mu 1})] \cdots \exp[i(\phi_{\nu n} - \phi_{\mu n})]$$

$$= \sum_{p} \sum_{\nu 1 = p(\mu 1)} \cdots \sum_{\nu n = p(\mu n)} 1 = \sum_{p} N^n = n!\, N^n = n!\, \langle I \rangle^n$$

$$\Rightarrow \quad P(I)\, dI = \frac{1}{\langle I \rangle} \exp(-I / \langle I \rangle)\, dI$$

Probability distribution is completely determined by its momenta

NOTE: for chaotic light: $g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2$

$$E(t) = \sum_{n=1}^{N} E_n(t)\text{ , with } E_n(t), E_m(t) \text{ uncorrelated for } n \neq m:$$

$$\Rightarrow \text{ for } n \neq m \quad \langle E_n(t+\tau)\, E_m(t)^* \rangle = \langle E_n(t+\tau) \rangle \langle E_m(t)^* \rangle = 0$$

$$\langle E(t+\tau)E(t)\, E(t)^* E(t+\tau)^* \rangle = \sum_{n,m,k,l}^{N} \langle E_n(t+\tau)E_m(t)^*\, E_k(t)E_l(t+\tau)^* \rangle$$

non-zero contribution only for n=m & k=l or n=l & m=k

$$= \sum_{n=1}^{N} \langle E_n(t+\tau)E_n(t)\, E_n(t)^* E_n(t+\tau)^* \rangle$$

N >>1, non-diagonal terms dominate

$$+ \sum_{n \neq m}^{N} \langle E_n(t+\tau)E_n(t)^*\, E_m(t)E_m(t+\tau)^* \rangle + \sum_{n \neq m}^{N} \langle E_n(t+\tau)E_n(t+\tau)^*\, E_m(t)^* E_m(t) \rangle$$

$$\approx \sum_{n \neq m}^{N} \langle E_n(t+\tau)E_n(t)^* \rangle \langle E_m(t)E_m(t+\tau)^* \rangle + \sum_{n \neq m}^{N} \langle E_n(t+\tau)E_n(t+\tau)^* \rangle \langle E_m(t)^* E_m(t) \rangle$$

$$\approx \left| \sum_{n=1}^{N} \langle E_n(t+\tau)E_n(t)^* \rangle \right|^2 + \left| \sum_{n=1}^{N} \langle E_n(t)^* E_n(t) \rangle \right|^2$$

$$\langle E(t+\tau)\, E(t)^* \rangle = \sum_{n,m}^{N} \langle E_n(t+\tau) E_m(t)^* \rangle = \sum_{n}^{N} \langle E_n(t+\tau) E_n(t)^* \rangle$$

$$\langle E(t+\tau)E(t)\, E(t)^* E(t+\tau)^* \rangle = \left| \sum_{n=1}^{N} \langle E_n(t+\tau) E_n(t)^* \rangle \right|^2 + \left| \sum_{n=1}^{N} \langle E_n(t)^* E_n(t) \rangle \right|^2$$

$$= |\langle E(t+\tau)\, E(t)^* \rangle|^2 + |\langle E(t)\, E(t)^* \rangle|^2$$

$$\Rightarrow \quad g^{(2)}(\tau) = \frac{\langle E(t+\tau)E(t)\, E(t)^* E(t+\tau)^* \rangle}{|\langle E(t)\, E(t)^* \rangle|^2} = \frac{|\langle E(t+\tau)\, E(t)^* \rangle|^2}{|\langle E(t)\, E(t)^* \rangle|^2} + 1 = |g^{(1)}(\tau)|^2 + 1$$

An example of chaotic light: collisional broadened source revisited

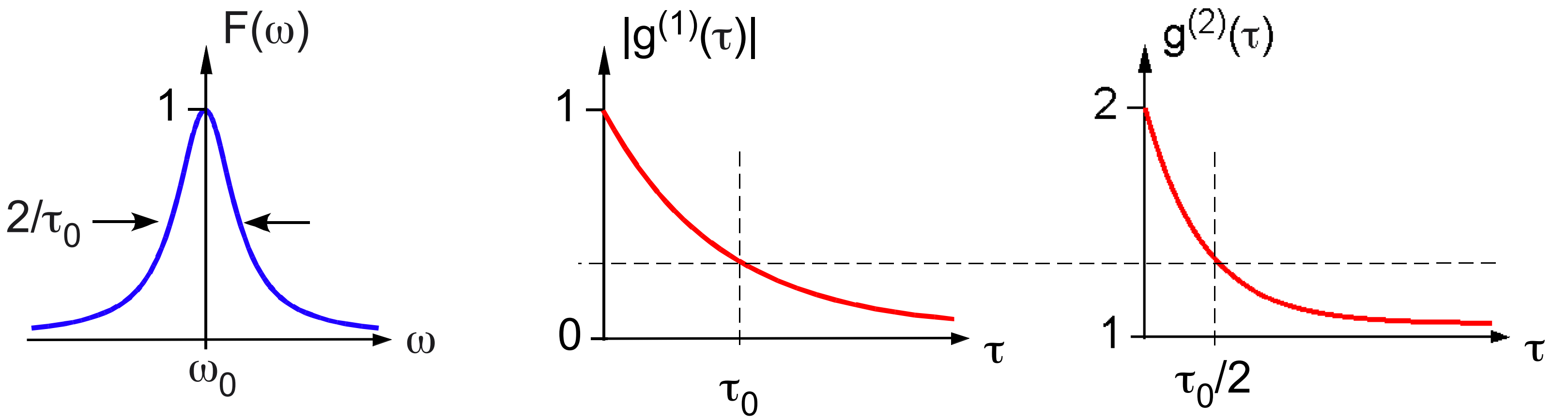
$$E(t) \;=\; E_0 \sum_{n=1}^{N} \exp[i\phi_n(t)]\,, \quad \phi_n(t) \;=\; -\,\omega_n\, t + \xi_n\,, \quad \xi_n = \text{random phase} \quad \Rightarrow$$

$$g^{(1)}(\tau) \;=\; \sum_{n=1}^{N} \left\langle \exp[i(\phi_n(t+\tau) - \phi_n(t))] \right\rangle \;=\; \sum_{n=1}^{N} \left\langle \exp(i\omega_n \tau) \right\rangle \;=\; \int_{-\infty}^{\infty} d\omega \, \exp(i\omega\tau)\, P(\omega)$$

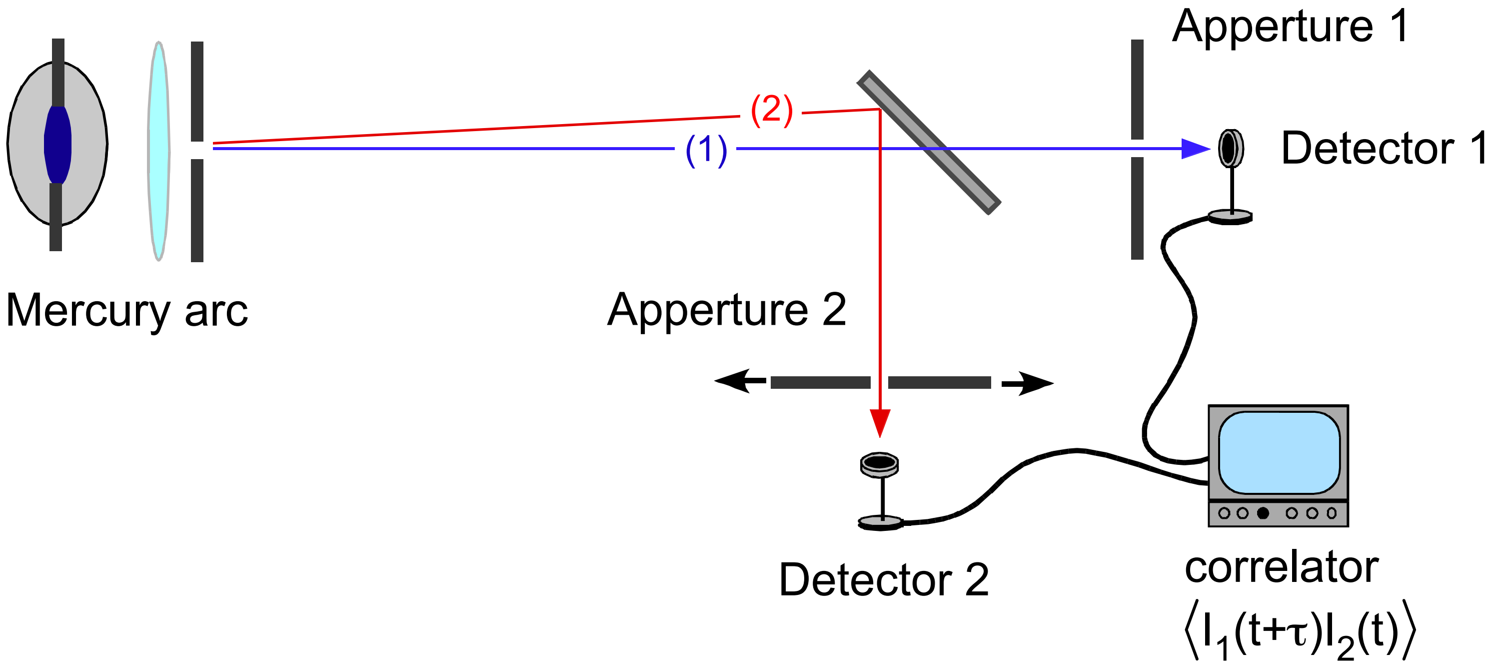
Example: assume Lorentzian spectrum (collision broadened light source)

$$P(\omega) \;=\; \frac{\tau_0}{\pi} \frac{1}{1+\tau_0^{\,2}(\omega_0-\omega)^2} \qquad \Rightarrow \qquad g^{(1)}(\tau) \;=\; \exp(-i\omega_0\tau - |\tau|/\tau_0)$$

$$g^{(2)}(\tau) \;=\; 1 + \exp(-2|\tau|/\tau_0)$$



Measurement of $g^{(2)}(\tau)$: Hanbury Brown & Twiss (1956)



Variation of apperture 2 permits measurement of transverse coherence length
→ determination of opening angle of source

R. Hanbury Brown, R. Q. Twiss, Nature 177,27 (1957)