Winter term 2012/13

Exercise Sheet 5, Theoretical Quantum and Atom Optics

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To be returned on Tuesday, 27/11/2012, in the tutorials

Exercise 9. Atom-field interaction Hamiltonian

Consider the minimal-coupling Hamiltonian for an electron in an external electromagnetic field,

$$H = \frac{1}{2m}[\boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{r},t)]^2 + eU(\boldsymbol{r},t) + V(r),$$

where the potentials $U, \boldsymbol{A}$ produce the external field, while V binds the electron to a nucleus at $\boldsymbol{r}_0$. Show that, in the radiation gauge: $U(\boldsymbol{r},t) = 0, \nabla \cdot \boldsymbol{A}(\boldsymbol{r},t) = 0$, and in the dipole approximation: $\boldsymbol{k} \cdot \boldsymbol{r} \ll 1$, so that $\boldsymbol{A}(\boldsymbol{r}_0 + \boldsymbol{r}, t) \approx \boldsymbol{A}(t)\exp(i\boldsymbol{k}\cdot\boldsymbol{r}_0) \Longrightarrow \boldsymbol{A}(\boldsymbol{r},t) \equiv \boldsymbol{A}(\boldsymbol{r}_0,t)$, H can be written in the following forms, where $H_0 = \frac{p^2}{2m} + V(r)$ is the Hamiltonian in the absence of the external field:

(a) $H_r = H_0 - e\boldsymbol{r}\cdot\boldsymbol{E}(\boldsymbol{r}_0,t)$

Hint: Gauge-transform the electronic wave function ψ into $\phi(\boldsymbol{r},t) = \exp\left[-\frac{ie}{\hbar}\boldsymbol{A}(\boldsymbol{r}_0,t)\cdot\boldsymbol{r}\right]\psi(\boldsymbol{r},t)$.

(b) $H_p = H_0 - \frac{e}{m}\boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{r}_0,t)$

Hint: Use the fact that the commutator $[\boldsymbol{p},\boldsymbol{A}]$ vanishes in the radiation gauge, and neglect quadratic terms $\sim A^2$.

4 Points

Exercise 10. Rabi oscillations

Fill in the gaps in the calculation from the lecture on Rabi oscillations. To do so, first remember that for the coefficients c_i (which are rotated with respect to the original amplitudes b_i), the following system of ordinary differential equations (ODEs) was shown to hold:

$$\dot{c}_1(t) = ic_2(t)\frac{\Omega^*}{2}e^{i\Delta t}, \qquad \dot{c}_2(t) = ic_1(t)\frac{\Omega}{2}e^{-i\Delta t}.$$

(a) Check that the following is a solution of this ODE system:

$$c_1(t) = \left(a_1e^{i\Omega_g t/2} + a_2e^{-i\Omega_g t/2}\right)e^{i\Delta t/2}$$
$$c_2(t) = \left(a_3e^{i\Omega_g t/2} + a_4e^{-i\Omega_g t/2}\right)e^{-i\Delta t/2},$$

if suitable relations between the constants a_i are fulfilled. Here, $\Omega_g = \sqrt{\Delta^2 + |\Omega|^2}$.

(b) Determine how the constants a_i depend on the initial conditions $c_1(0)$ and $c_2(0)$.

(c) Show that for $c_1(0) = 1$, $c_2(0) = 0$ the solution reduces to the one given in the lecture:

$$c_1(t) = e^{i\Delta t/2}\left[\cos(\Omega_g t/2) - i\frac{\Delta}{\Omega_g}\sin(\Omega_g t/2)\right], \qquad c_2(t) = e^{-i\Delta t/2}\left[i\frac{\Omega}{\Omega_g}\sin(\Omega_g t/2)\right].$$

(d) Thus, rotating back, you have found the following solution for the time-evolution of the original amplitudes for a two-level system interacting with a classical light field:

$$b_1(t) = e^{-i(\omega_1-\Delta/2)t}\left[\cos(\Omega_g t/2) - i\frac{\Delta}{\Omega_g}\sin(\Omega_g t/2)\right],$$
$$b_2(t) = e^{-i(\omega_2+\Delta/2)t}\left[i\frac{\Omega}{\Omega_g}\sin(\Omega_g t/2)\right].$$

Check that both $|b_1(t)|^2$ and $|b_2(t)|^2$ perform harmonic oscillations. Explicitely determine the frequency and the amplitude of these oscillations.

2 Points

Please turn the page

Exercise 11. Avoided crossing

Consider a two-level system governed by the Hamiltonian

$$\hat{H} = -\frac{\hbar}{2}\begin{pmatrix} 0 & \Omega^* \\ \Omega & 2\Delta \end{pmatrix}, \qquad \Delta \in \mathbb{R}, \Omega \in \mathbb{C}.$$

(a) Calculate the eigenvalues $E_{\pm}(\Omega, \Delta)$. It is useful to introduce $\Omega_g = \sqrt{\Delta^2 + |\Omega|^2}$.

(b) Sketch $E_{\pm}$ as a function of Δ for $\Omega = 0$ and $\Omega \neq 0$, respectively.

(c) Calculate the corresponding normalized eigenvectors $\psi_{\pm}(\Omega, \Delta)$.

(d) Simplify the results for $E_{\pm}$ and $\psi_{\pm}$ in the parameter regimes $|\Delta| \to 0$ and $|\Delta| \gg |\Omega|$ by carefully performing the appropriate limits.

4 Points