Winter term 2012/13

Exercise Sheet 7, Theoretical Quantum and Atom Optics

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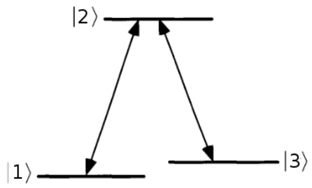
To be returned on Tuesday, 11/12/2012, in the tutorials

Exercise 13. Λ system: Dark state and adiabatic passage



Consider the semiclassical treatment of the three-level Λ system you know from the lectures. Expanding the state vector in the form $|\Psi(t)\rangle = c_1 e^{-i\omega_1 t}|1\rangle + c_2 e^{-i\omega_2 t}|2\rangle + c_3 e^{-i\omega_3 t}|3\rangle$, you have seen that within the rotating wave approximation and assuming zero detuning for both classical LASER fields the time-evolution of the coefficients is governed by the matrix Hamiltonian

$$H = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega_{12} & 0 \\ \Omega_{12}^* & 0 & \Omega_{23} \\ 0 & \Omega_{23}^* & 0 \end{pmatrix}, \tag{1}$$

where Ω_{12} and Ω_{23} denote the complex Rabi frequencies of the two transitions.

(a) Find the eigenvalues and eigenstates of this Hamiltonian. Show that one of these eigenstates does not have any contribution of the excited level $|2\rangle$. This one is called the *dark state* $|\Psi_d\rangle$ of the Λ system.

(b) Check that the dark state can be written as

$$|\Psi_d\rangle = \cos\frac{\theta}{2}|1\rangle + \sin\frac{\theta}{2}e^{-i\psi}|3\rangle, \tag{2}$$

where

$$\tan\frac{\theta}{2} := \frac{|\Omega_{12}|}{|\Omega_{23}|}, \qquad \psi := \arg(\Omega_{12}) + \arg(\Omega_{23}) \pm \pi. \tag{3}$$

Check also that in the special case of $|\Omega_{12}| = |\Omega_{23}|$ this reduces to the stationary solution you have seen in the lectures.

(c) Simplify this expression for the dark state in the limits of $|\Omega_{12}| \gg |\Omega_{23}|$ and $|\Omega_{12}| \ll |\Omega_{23}|$. Show that in both cases the dark state reduces to one of the ground states of the Λ system.

(d) The *Stimulated Raman Adiabatic Passage* (STIRAP) scheme makes use of the dark state's dependence on the two Rabi frequencies to coherently "pump" an ensemble of atoms from one of the ground states of the Λ system to the other. Give a short explanation why a pulse sequence starting from $|\Omega_{12}| \ll |\Omega_{23}|$ and then slowly tuning to $|\Omega_{12}| \gg |\Omega_{23}|$ leads to a transfer of atoms from $|1\rangle$ to $|3\rangle$. What is the role of *adiabaticity* in this context?

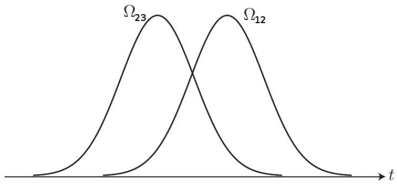
(e) In order to estimate how fast the STIRAP pulse sequence can be performed without losing too many atoms from the dark state, let us consider two subsequent Gaussian pulses of the same shape, with a time separation τ between them. The time-dependent Rabi frequencies read

$$\Omega_{23}(t) = A\exp\left(-\frac{(t+\tau/2)^2}{2d^2}\right), \qquad \Omega_{12}(t) = A\exp\left(-\frac{(t-\tau/2)^2}{2d^2}\right), \tag{4}$$

where we assume $d, A \in \mathbb{R}$.

From the first part of the exercise you know that, for a fixed time t, the absolute value of the energy difference between the dark state and each of the other eigenstates is given by

$$\Delta E(t) = \frac{\hbar}{2}\Omega(t) = \frac{\hbar}{2}\sqrt{\Omega_{12}(t)^2 + \Omega_{23}(t)^2}. \tag{5}$$



Check that this energy gap has an extremum (which for a reasonable pulse sequence with $\tau > \sqrt{2}d$ is a minimum) at $t = 0$, i.e. when the amplitudes of the pulses intersect and $\Omega_{12}(0) = \Omega_{23}(0) =: \Omega_c$.

(f) To apply the standard Landau-Zener formula, expand the Rabi frequencies around this minimum to first order in time and plug the linearized expressions into the above formula for the energy gap. You should find

$$\Delta E(t) \approx \hbar \sqrt{\frac{\Omega_c^2}{2} + \frac{\Omega_c^2 \tau^2}{8d^4} t^2}. \tag{6}$$

In the Landau-Zener problem on sheet 6, the energy difference between the adiabatic eigenstates was given by (recovering $\hbar$ that was set to 1 in the exercise)

$$\Delta E_{LZ} = \hbar \sqrt{4\alpha^2 t^2 + 4f^2}, \tag{7}$$

and the probability of non-adiabatic loss was found to be $P_{LZ} = \exp\left(-2\pi \frac{f^2}{2\alpha}\right)$. Compare eqs. (7) and (6) to find the effective Landau-Zener parameters α and f of the STIRAP sweep and estimate the probabiltity of losing atoms from the dark state.

5 Points

Exercise 14. Mollow triplet

Consider a two-level atom (levels 1 and 2) in a single-mode light field (photon number n), and the full Hamiltonian describing the system:

$$H = H_A + H_F + H_{AF} = \frac{\hbar\omega}{2}\sigma_z + \hbar\omega_L a^\dagger a + \hbar g(\sigma_+ a + a^\dagger \sigma_-),$$

written in terms of the Pauli operators $\sigma_z = |2\rangle\langle 2| - |1\rangle\langle 1|, \sigma_+ = |2\rangle\langle 1|, \sigma_- = |1\rangle\langle 2|$ and field ladder operators $a, a^\dagger$, where $\hbar\omega = E_2 - E_1$ is the atomic level spacing, ω_L the frequency of the light field, and g the atom-field coupling strength, which is assumed to be real without loss of generality.

(a) Let us now introduce the manifolds $\mathcal{M}(n) = \{|1, n+1\rangle, |2, n\rangle\}, n \in \mathbb{N}_0$, consisting of decoupled product states between an atomic state and a Fock state of the light field. Why is it sufficient to look at these manifolds individually when diagonalizing H?
Keep n fixed and find the eigenvalues $E_\pm^n$ of H. Show that the corresponding *dressed state* eigenvectors are given by

$$|+, n\rangle = \sin\theta_n |1, n+1\rangle + \cos\theta_n |2, n\rangle$$
$$|-, n\rangle = \cos\theta_n |1, n+1\rangle - \sin\theta_n |2, n\rangle,$$

where the mixing angle $\theta_n \in [0, \pi/2)$ is defined by $\tan(\theta_n) = -\Omega_R^n/\Delta$, where $\Omega_R^n = 2g\sqrt{n+1}$ is the (photon number dependent) Rabi frequency and $\Delta = \omega_L - \omega$ the detuning.

(b) Determine the transition matrix elements

$$\langle i, m|\sigma_+|j, n\rangle, \quad i, j \in \{+, -\}$$

of the spontaneous transitions between the two-level manifolds $\mathcal{M}(n) = \{|1, n+1\rangle, |2, n\rangle\}$ with $n = m, m \pm 1, m \pm 2, \dots$.
Hint: Verify that only transitions between adjacent manifolds are allowed.

(c) Show, using a sketch of the energy levels, that the central frequencies of these allowed spontaneous transitions form a fluorescence (Mollow) triplet structure around the field frequency ω_L.

5 Points