Winter term 2012/13

Exercise Sheet 8, Theoretical Quantum and Atom Optics

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To be returned on Tuesday, 18/12/2012, in the tutorials

Exercise 15. Expectation value of the ground state energy

To describe the many-body condensate state, one may adopt a Hartree Ansatz assuming a symmetrized product of single-particle wavefunctions ϕ (which are normalized to unity). Taking

$$\Psi = \prod_{i=1}^{N} \phi(\mathbf{r}_i), \tag{1}$$

where N gives the number of condensed bosons and having the total Hamiltonian

$$H = \sum_{i=1}^{N} \Big[\frac{\mathbf{p}_i^2}{2m} + V(\mathbf{r}_i) \Big] + U_0 \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j)$$

show that the energy of the state (1) in the limit of large N is given by

$$E = \int d\mathbf{r} \Big[\frac{\hbar^2}{2m} |\nabla \psi(\mathbf{r})|^2 + V(\mathbf{r}) |\psi(\mathbf{r})|^2 + \frac{1}{2} U_0 |\psi(\mathbf{r})|^4 \Big], \tag{2}$$

where $\psi(\mathbf{r}) = \sqrt{N}\phi(\mathbf{r})$ denotes the so-called "condensate wavefunction".

3 Points

Exercise 16. Derivation of GP equation

Derive the time-independent Gross-Pitaevskii equation by finding the "optimal" ψ, i.e. by extremizing the energy (2) with respect to independent variations of $\psi(\mathbf{r})$ and $\psi(\mathbf{r})^*$, subject to the constraint that the total number of particles $N = \int d\mathbf{r} |\psi(\mathbf{r})|^2$ is conserved.
This can be done by the method of Lagrange multipliers, i.e. by demanding $\delta E - \mu \delta N = 0$, where the Lagrange multiplier μ (the chemical potential) will enter the time-independent GP equation.

3 Points

Exercise 17. Hydrodynamic equations

The velocity field of the condensate is defined as $\mathbf{v} = \frac{\hbar}{2mi} \frac{\psi^* \nabla \psi - \psi \nabla \psi^*}{|\psi|^2}$.
Writing ψ in terms of its amplitude f and phase φ, $\psi = f e^{i\varphi}$, check that density $n = |\psi|^2$ and velocity may be obtained as

$$n = f^2, \quad \mathbf{v} = \frac{\hbar}{m} \nabla \varphi.$$

Show that the time-dependent Gross-Pitaevskii equation can then be cast into the following form:

$$\frac{\partial n}{\partial t} = -\nabla \cdot (n\mathbf{v}),$$

$$-\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} + \frac{1}{2} m \mathbf{v}^2 + V + nU_0.$$

The first of these equations is the familiar continuity equation from hydrodynamics of an ideal fluid. Take the gradient of the second equation to obtain the equation of motion for the velocity field (the analogue to Euler's equation). **4 Points**