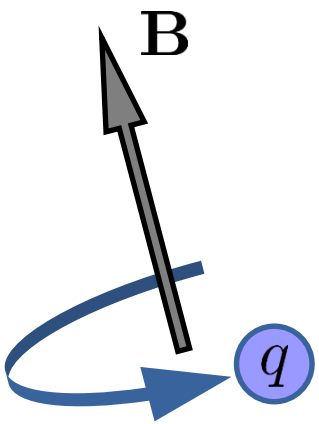
# Ultracold Quantum Gases

## Part 3: Artificial gauge potentials

- Coupling between electromagnetic fields and **charged** particles central for many phenomena:
  - » Integer and fractional Quantum Hall effect
  - » Spin-orbit coupling
  - » Topological insulators
  - » ….




- Quantum simulation with quantum gases
  - » Well controlled systems to study solid-state models
  - » **Neutral** atoms (q = 0)

**Simulating magnetic effects with quantum gases is a challenge:**
**Requires the creation of "substitutes" to real electromagnetic fields:**
**"Artificial gauge potentials"**

# Ultracold Quantum Gases

## Part 3: Artificial gauge potentials

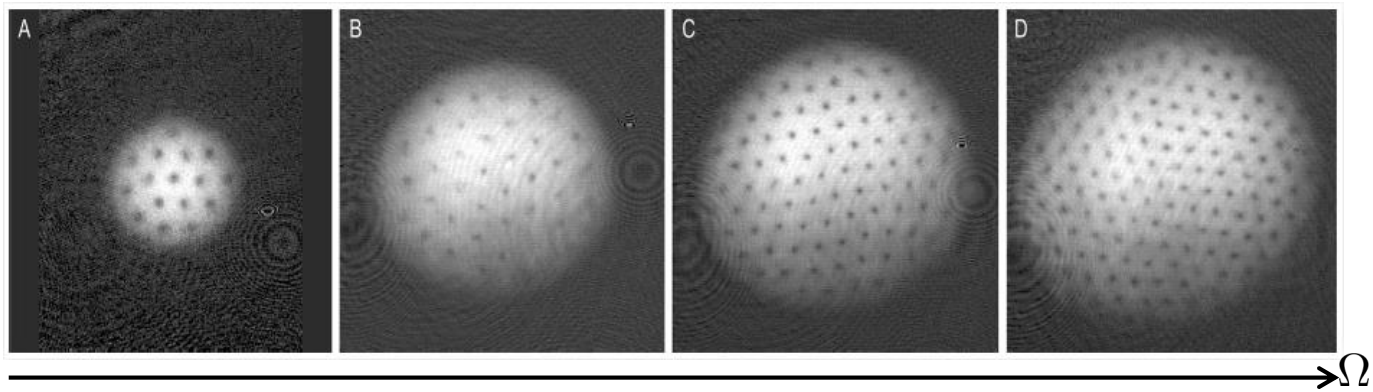
Part 3

# Ultracold Quantum Gases

## 3.1 Lorentz force for neutral particles

- Description of electromagnetic fields in quantum physics
  - » New natural length scale in quantum mechanics
  - » Landau levels

- How to simulate magnetic fields for neutral particle?
  - » Lorentz force for neutral particles




Dalibard, Introduction to the physics of artificial gauge fields, Cours du Collège de France

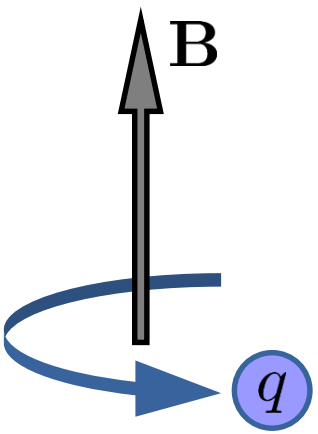
# Motion of a charged particle in a static magnetic field

- Static magnetic field
  - » Divergence free (no magnetic monopole) $\boldsymbol{B} = \nabla \times \boldsymbol{A}$
  - » Gauge freedom: $\boldsymbol{B}$ corresponds to a set of vector potentials

$$\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \boldsymbol{\nabla}\chi(\boldsymbol{r})$$

  - » Gauge transformation

$$\boldsymbol{A}(\boldsymbol{r}) \to \mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \boldsymbol{\nabla}\chi(\boldsymbol{r})$$

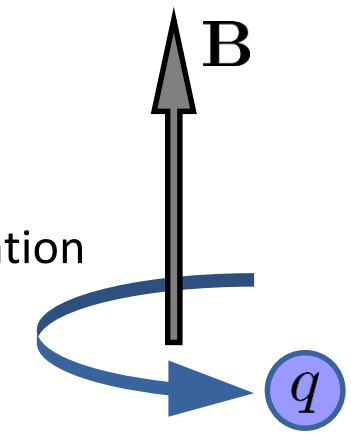



- Classical motion of a **charged** particle
  - » Newtonian equation of motion $m\ddot{\boldsymbol{r}} = q\dot{\boldsymbol{r}} \times \boldsymbol{B}$
  - » Lorentz force $\boldsymbol{F} = q\dot{\boldsymbol{r}} \times \boldsymbol{B}$
  - » Classical cyclotron orbit
    - Cyclotron frequency $\omega_c = \dfrac{|q|B}{M}$
    - Cyclotron orbit $v_0 = \omega_c r_0$
    - Strong magnetic fields

$$\omega_c \to \infty \Rightarrow r_0 \to 0$$

# Motion of a charged particle in a static magnetic field

- Lagrangian mechanics
  - » Lagrange function $L(\boldsymbol{r},\dot{\boldsymbol{r}}) = \frac{1}{2}M\dot{\boldsymbol{r}}^{\boldsymbol{2}} + q\,\dot{\boldsymbol{r}}\cdot\boldsymbol{A}(\boldsymbol{r})$




  - » Newtonian equation of motion obtained via Euler Lagrange equation

$$\frac{\partial L}{\partial r_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) \qquad \Rightarrow m\ddot{\boldsymbol{r}} = q\dot{\boldsymbol{r}}\times\boldsymbol{B}$$

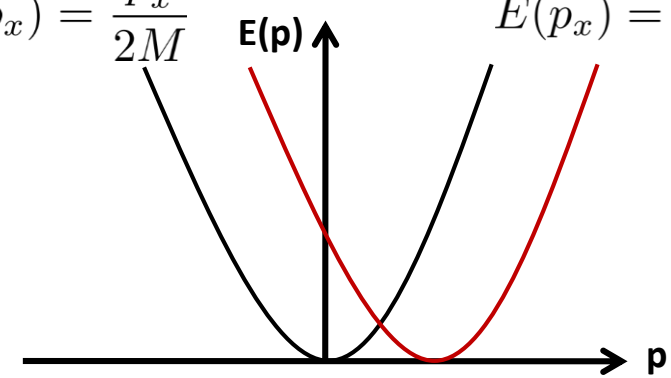
- Hamiltonian
  - » Legendre transform to obtain the Hamiltonian

$$\boldsymbol{p} = \boldsymbol{\nabla}_{\dot{\boldsymbol{r}}} L(\boldsymbol{r},\dot{\boldsymbol{r}})$$

$$H(\boldsymbol{r},\boldsymbol{p}) = \boldsymbol{p}\cdot\dot{\boldsymbol{r}} - L(\boldsymbol{r},\dot{\boldsymbol{r}})$$

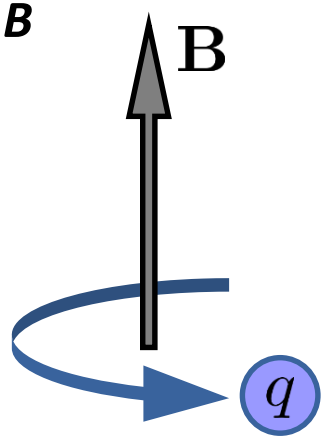
  - » Hamiltonian (minimal coupling): $E(p_x) = \frac{p_x^2}{2M}$ $\qquad E(p_x) = \frac{(p_x - eA)^2}{2M}$

$$\hat{H} = \frac{\left(\hat{\boldsymbol{p}} - q\,\boldsymbol{A}(\hat{\boldsymbol{r}})\right)^2}{2M}$$



# Landau levels

- Energy spectrum of a **charged** particle in a uniform magnetic field ***B***
  - » Hamiltonian $$\hat{H} = \frac{\left(\hat{\boldsymbol{p}} - q\,\boldsymbol{A}(\hat{\boldsymbol{r}})\right)^2}{2M}$$
  - » Eigenvalues**: Landau levels** (analog to an harmonic oscillator)
$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c$$




- Magnetic length
  - » **New natural length scale** $l_{mag} = \sqrt{\frac{\hbar}{eB}}$
  - » $l_{mag}$ is the minimal cyclotron orbit size (Heisenberg inequality)
$$\Delta \mathrm{r} \Delta p \geq \frac{\hbar}{2} \Rightarrow (\Delta \mathrm{r})^2 \geq \frac{\hbar}{2M\omega_c}$$
  - » For B = 1 T: $l_{mag}$ = 25 nm

$$\mathbf{A} = \begin{pmatrix} 0 \\ Bx \\ 0 \end{pmatrix}$$

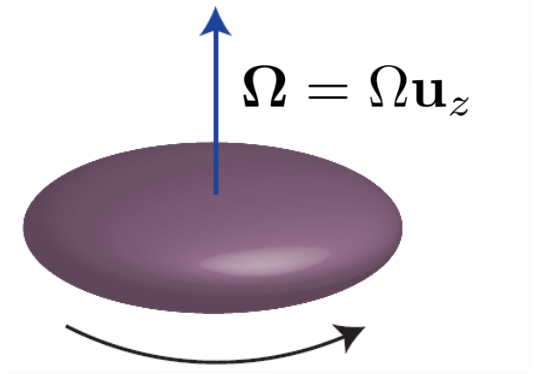
**Quantum Hall effect:**

**Direct consequence of the minimal cyclotron orbit size and quantized energies**

# Lorentz force for neutral particles

- "Artificial" magnetic field:

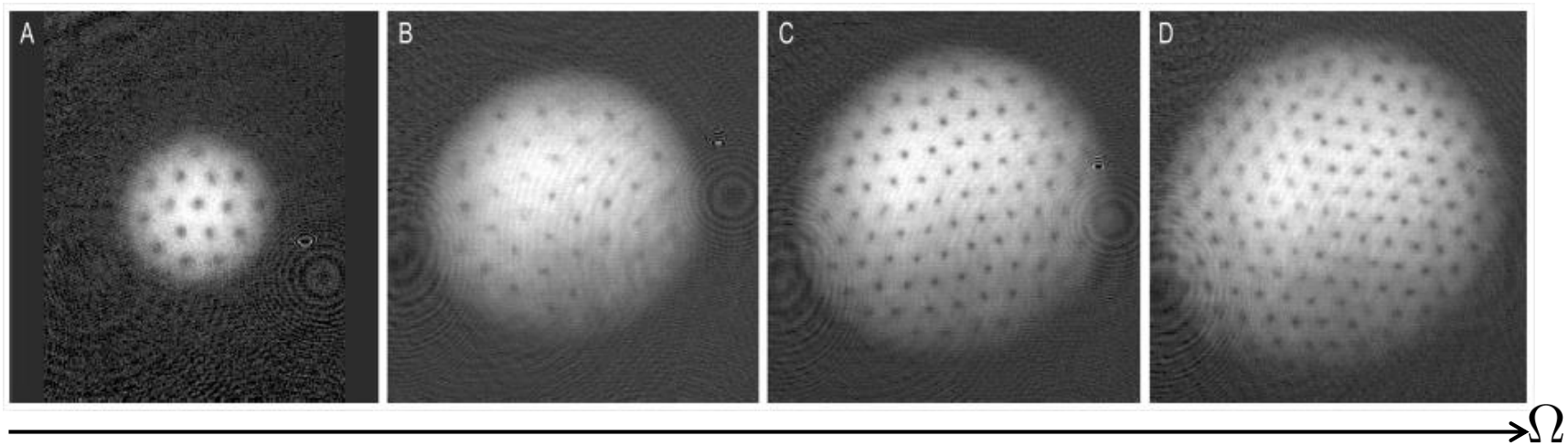
  Lorentz force for neutral particles




- Rotation of the trap around the *z* axis
  - » Coriolis force appearing in the rotating frame

$$\mathbf{F}_{\text{Coriolis}} = 2M\mathbf{v} \times \mathbf{\Omega} \qquad \mathbf{F}_{\text{Lorentz}} = q\mathbf{v} \times \mathbf{B}$$

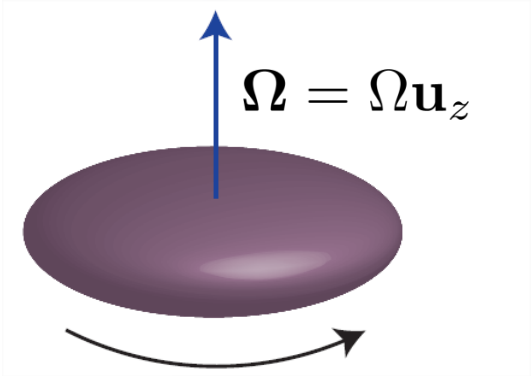
  - » Rotating superfluid: vortex lattice




Ketterle, Science 292, 476 (2001)

# Lorentz force for neutral particles

- Rotation of the trap $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2M} + V(\hat{\mathbf{r}}) - \Omega \hat{L}_z$




- Effective vector potential

$$\hat{H} = \frac{(\hat{\mathbf{p}} - q\mathbf{A})^2}{2M} + V(\hat{\mathbf{r}}) + V_{\text{centrif.}}(\hat{\mathbf{r}})$$

$$q\mathbf{A} = M\Omega(x\mathbf{u}_y - y\mathbf{u}_x)$$

$$V_{\text{centrif.}} = -\frac{1}{2} M\Omega^2 \hat{\mathbf{r}}^2$$

- Effective magnetic field $qB = 2M\Omega$

- Flux density $n_\Phi = \frac{qB}{h} = \frac{2M\Omega}{h}$

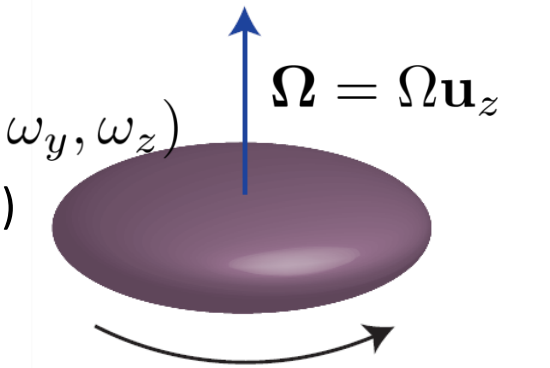
- Advantages
  - » Simple (no special set-up)
  - » Applicable for any atoms / molecules

# Lorentz force for neutral particles

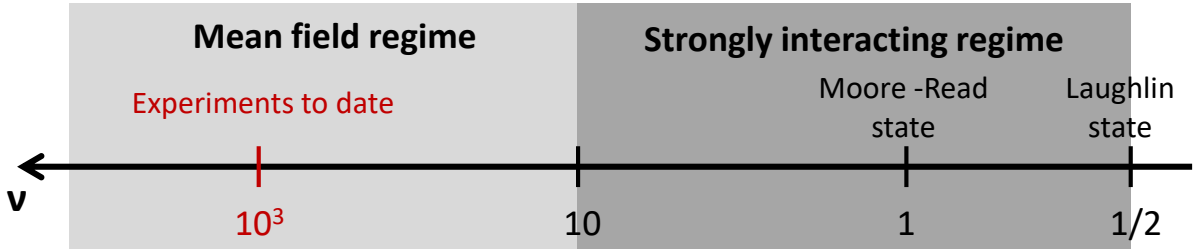
Challenges

» Heating due to the rotation of non-circular potentials $(\omega_x, \omega_y, \omega_z)$
(Equivalent to time dependent potentials in rotating frame)

» The trapping potential has to balance the
centrifugal potential $\Omega \leq \omega$

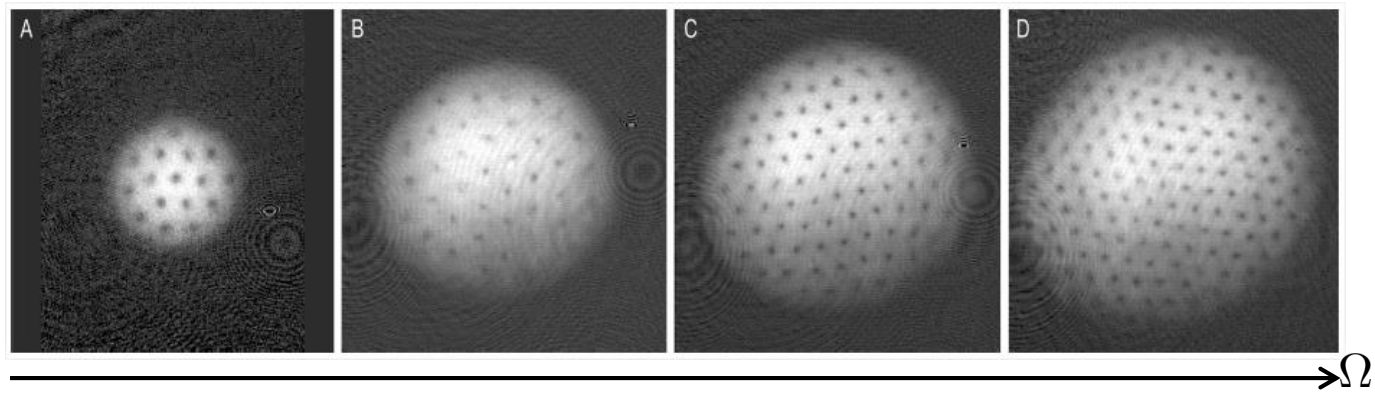



Flux density limited to $n_\Phi \leq \dfrac{2M\omega}{h}$




Filling factor:

$$\nu = \frac{n_{2D}}{n_\Phi}$$



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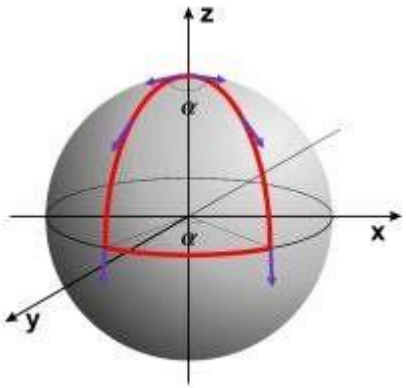
# Ultracold Quantum Gases

## 3.2 Berry curvature and artificial magnetic field




- Gauge transformation in quantum mechanics
  - » Aharonov-Bohm phase (gauge invariant, geometric)
  - » Central concept for the quantum simulation with **neutral** atoms

- Berry phase
  - » Quantum mechanical phase imprinted on the system wave function while modifying external parameters.
  - » Analog to the Aharonov-Bohm phase (gauge invariant, geometric)
  - » Induced by a vector field analog to a magnetic field – **Berry curvature**.

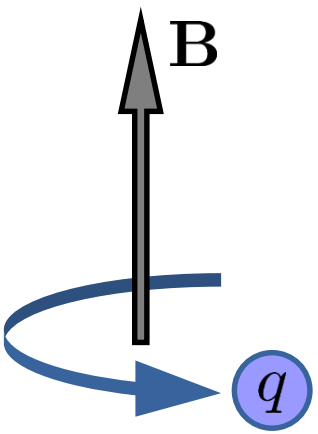
- Realization of gauge fields for free atoms: Optically dressed states

  Well-designed light-matter interaction induces electric or magnetic fields for **neutral** atoms.

# Gauge transformation



- Classical physics
  - » Equation of motion $m\ddot{\boldsymbol{r}} = q\dot{\boldsymbol{r}} \times \boldsymbol{B}$
  - » Gauge transformation $\boldsymbol{A}(\boldsymbol{r}) \to \mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \boldsymbol{\nabla}\chi(\boldsymbol{r})$




- Quantum mechanics
  - » Schrödinger equation

$$\imath\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = \frac{(\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{r}}))^2}{2M}\psi(\mathbf{r},t)$$

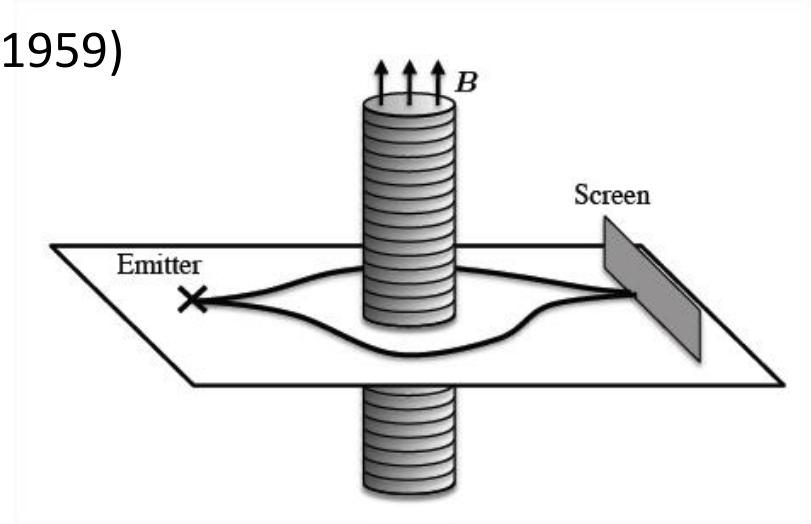
  - » Gauge transformation (imposed by the Schrödinger equation)

$$\mathbf{A} \to \mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla\chi(\mathbf{r})$$
$$\psi(\mathbf{r},t) \to \psi'(\mathbf{r},t) = \exp[\imath q\chi(\hat{\mathbf{r}})/h]\psi(\mathbf{r},t)$$

**The wave-function is modified by a gauge transformation: it acquires a phase!**

# The Aharonov-Bohm effect

- Gedanken experiment of Aharonov and Bohm (1959)
  - » Two path interferometer for single electrons
  - » Infinite solenoid: $\mathbf{B}_{in} = \mathbf{B}$ and $\mathbf{B}_{out} = \mathbf{0}$




- Probing a magnetic field without seeing it
  - » Zero Lorentz force outside the solenoid
  - » **BUT:** Shift of the interference pattern

- One of the "seven wonders of the quantum world" [*New Scientist* magazine]
  - » Several experimental demonstrations
  - » Questions the locality of electromagnetic fields
    - Local electromagnetic fields (**B**, **E**) and delocalized particle in the solenoid,
    - Gauge potentials (**A**, V) and particle localized around the solenoid.
  - » Global action versus local forces: Lagrangian formalism (based on energies) is not just a computational aid to the Newtonian formalism (based on forces).

Aharonov and Bohm, Phys. Rev. 115, 485 (1959)

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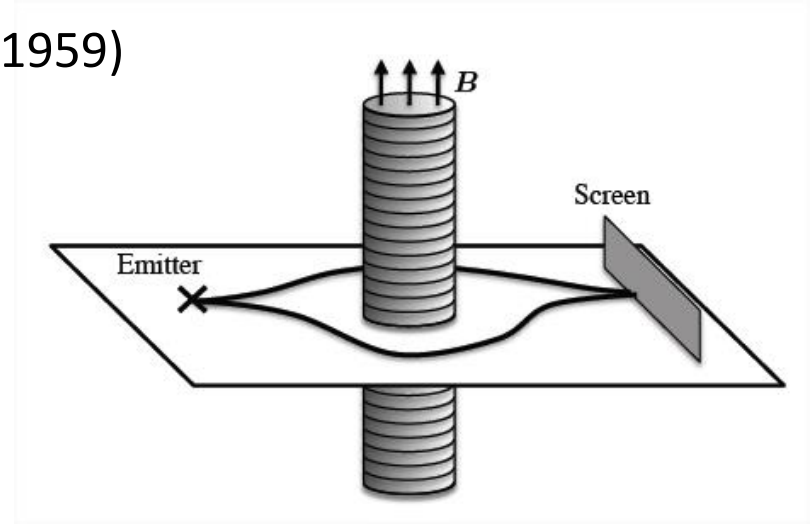


- Probing a magnetic field without seeing it
  - » Zero Lorentz force outside the solenoid
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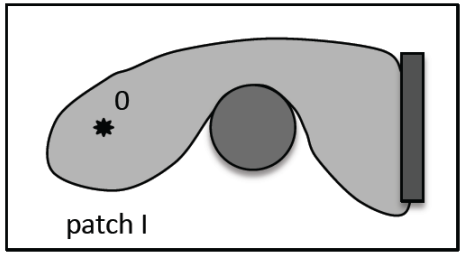
- Aharonov-Bohm phase
  - » Switching the current corresponds to a gauge change:

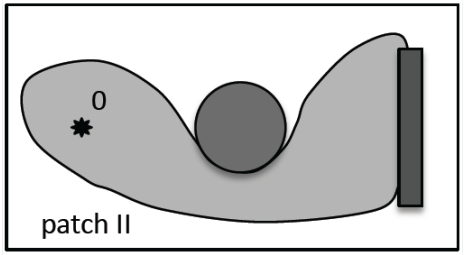
$$\mathbf{A}(\mathbf{r}) = \mathbf{0} \rightarrow \mathbf{A}(\mathbf{r}) = \nabla \chi_{I,II}(\mathbf{r})$$

  - » The matter-wave interference at ***r*** is related to:

$$\psi_l^*(\mathbf{r})\psi_r(\mathbf{r}) = \exp[\imath q(\chi_{II}(\mathbf{r}) - \chi_I(\mathbf{r}))]\psi_l^{(0)*}\psi_r^{(0)}$$

$$\chi_I(\mathbf{r}) - \chi_{II}(\mathbf{r}) = \int_{\mathbf{0},\mathbf{CI}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}')d\mathbf{r}' - \int_{\mathbf{0},\mathbf{CII}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}')d\mathbf{r}'$$




Aharonov and Bohm, Phys. Rev. 115, 485 (1959)

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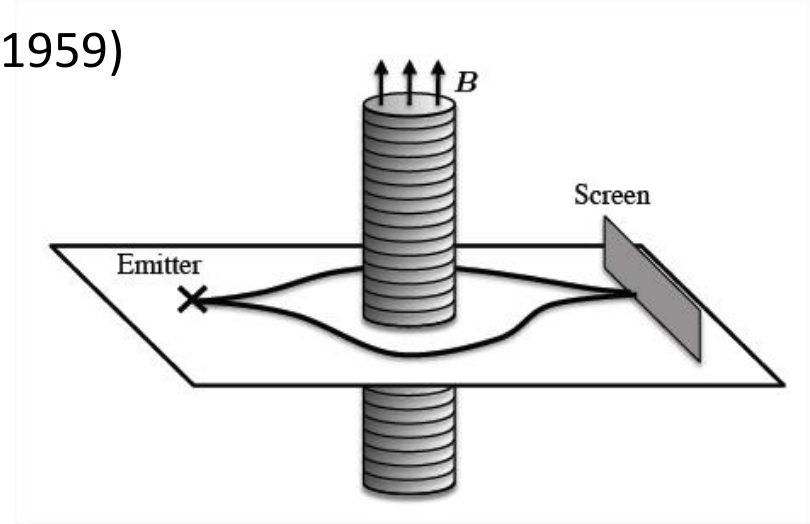


- Probing a magnetic field without seeing it
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  - » **BUT:** Shift of the interference pattern

- Aharonov-Bohm phase

$$\Delta\varphi = \frac{1}{\hbar}\oint q\mathbf{A}(\mathbf{r}).d\mathbf{r} = \frac{2\pi}{\Phi_0}\int\int B_z(x,y)dydx = 2\pi\frac{\Phi}{\Phi_0}$$

Flux quantum $\Phi_0 = \frac{h}{q}$

  - » Gauge invariant
  - » Geometric phase (no dependency on velocity)
  - » Even topological (constant under path deformation)

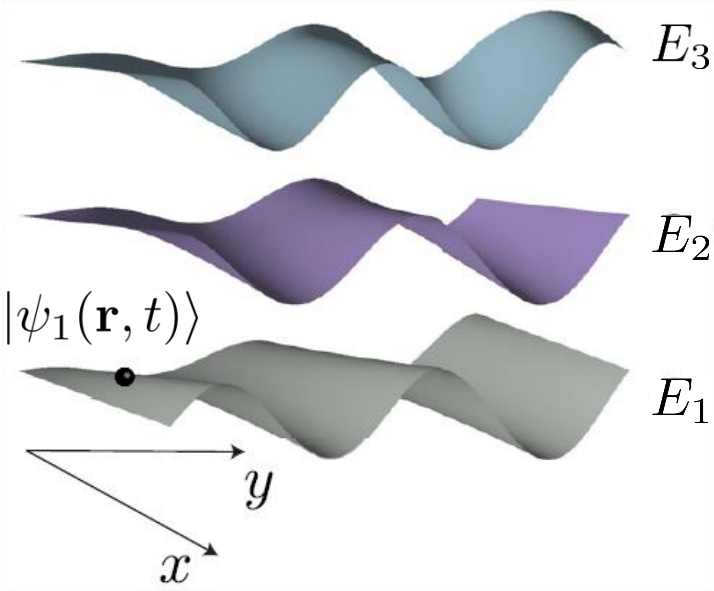
**Is there something analog for neutral particles?**

Aharonov and Bohm, Phys. Rev. 115, 485 (1959)

# Berry phase

- Adiabatic evolution of a quantum system
  - » Quantum system depending on a set of parameters $\lambda$
  - » Quantum system governed by $\hat{H}(\lambda)$

$$\hat{H}(\lambda)\,|\psi_n(\lambda)\rangle = E_n(\lambda)\,|\psi_n(\lambda)\rangle$$




  - » Evolution on a closed trajectory

$$\lambda(0) \to \lambda(t) \to \lambda(T) = \lambda(0)$$

  - » Time evolution of the state vector of the system

$$|\psi(t)\rangle = \sum_n c_n(t)\,|\psi_n(\lambda(t))\rangle$$

  - » Adiabatic approximation for a system initially prepared in $|\psi_l\rangle$:

$$|\psi(t)\rangle = c_l(t)\,|\psi_l(\lambda(t))\rangle$$

# Berry phase

- Berry connection
  - » Schrödinger equation $\imath\hbar\dot{c}_l = \left[E_l(t) - \imath\hbar\dot{\lambda}\,\langle\psi_l|\nabla\psi_l\rangle\right] c_l$
  - » Definition of the Berry connection $\mathbf{A}_l(\lambda) = \imath\hbar\,\langle\psi_l|\nabla\psi_l\rangle$

$$\imath\hbar\dot{c}_l = \left[E_l(t) - \dot{\lambda}\cdot\mathbf{A}_l(\lambda)\right] c_l$$

- Geometric and dynamical phases
  - » For a close contour in parameter space $c_l(T) = e^{\imath\Phi^{\text{dyn.}}(T)} e^{\imath\Phi^{\text{geom.}}(T)} c_l(0)$
  - » Dynamical phase (usual phase for time-dependent problems, gauge invariant)

$$\Phi^{\text{dyn}}(T) = -\frac{1}{\hbar}\int_0^T E_l(t)dt$$

  - » Geometric phase “Berry phase” (gauge invariant, only depends on the trajectory)

$$\Phi^{\text{geom}}(T) = \frac{1}{\hbar}\int_0^T \dot{\lambda}\cdot\mathbf{A}_l(\lambda)dt = \frac{1}{\hbar}\oint \mathbf{A}_l(\lambda)d\lambda$$

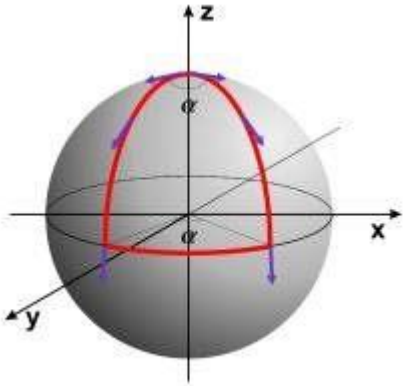
# Berry curvature and artificial magnetic fields

- Berry curvature
  - » Real, gauge invariant vector field $$\mathbf{B}_l = \nabla \times \mathbf{A}_l$$
    $$\mathbf{A}_l(\lambda) = \imath\hbar \langle \psi_l | \nabla \psi_l \rangle$$
  - » Analog to a magnetic field (relative to any set of parameters λ, not only position)

- Berry phase and Aharonov-Bohm phase

$$\Phi^{\text{geom}}(T) = \frac{1}{\hbar} \oint \mathbf{A}_l(\lambda) d\lambda = \frac{1}{\hbar} \int \int \mathbf{B}_l \cdot d^2 S$$

The Berry phase accumulated by a particle moving around a closed contour C is fully analog to the Aharanov-Bohm phase.

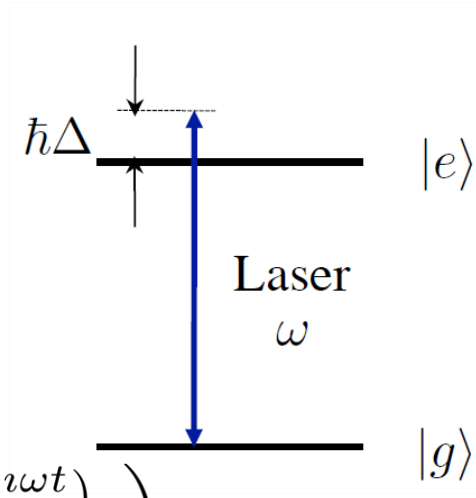



- How to impose such a geometric phase on ultracold gases?
  - » Atom: different internal states
  - » $H_{\text{atom-light}}$ depends on intensity and detuning
  - » These parameters can be easily varied in space

# Toy model: two-level system



- Two level system coupled by a light field
  - » Spontaneous emission neglected (long lived electronic state)
  - » Atom-laser coupling characterized by
    - Rabi frequency $\kappa = |\kappa| e^{\imath\Phi}$
    - Detuning $\Delta$




$$\hat{H}_{\text{int}} = \hbar \begin{pmatrix} 0 & \frac{1}{2}(\kappa e^{\imath\omega t} + \kappa^* e^{-\imath\omega t}) \\ \frac{1}{2}(\kappa^* e^{-\imath\omega t} + \kappa e^{\imath\omega t}) & \omega_0 \end{pmatrix}$$

- Rotating wave approximation
  - » In the co-rotating frame

$$\hat{H}_{\text{int}} = \hbar \begin{pmatrix} \frac{\Delta}{2} & \frac{1}{2}(\kappa + \kappa^* e^{-2\imath\omega t}) \\ \frac{1}{2}(\kappa^* + \kappa e^{2\imath\omega t}) & -\frac{\Delta}{2} \end{pmatrix}$$

  - » Rotating wave approximation $\omega_0 >> \Delta, \kappa$

$$\hat{H}_{\text{int}} = \frac{\hbar}{2} \begin{pmatrix} \Delta & \kappa^* \\ \kappa & -\Delta \end{pmatrix}$$

Dalibard and Gerbier, Artificial gauge potentials for neutral atoms, Reviews of Modern Physics **83**, 1523 (2011)

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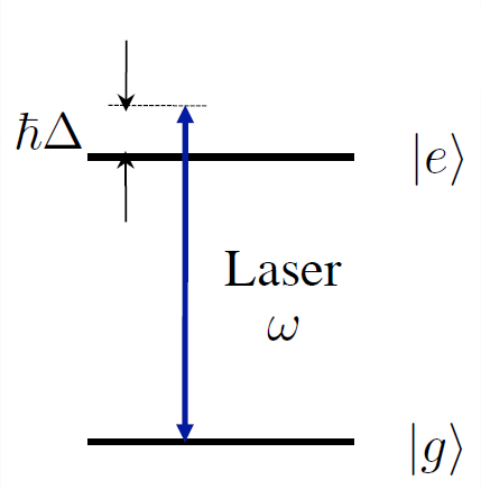



$$\hat{H}_{\text{int}} = \frac{\hbar}{2} \begin{pmatrix} \Delta & \kappa^* \\ \kappa & -\Delta \end{pmatrix}$$

- Eigenstates and eigenvalues
  - » Eigenvalues

$$E_\pm = \pm \frac{\hbar\Omega}{2} \qquad \Omega = \sqrt{\Delta^2 + |\kappa|^2}$$

  - » Eigenstates

$$\hat{H}_{\text{int}} = \frac{\hbar\Omega}{2} \begin{pmatrix} \cos(\theta) & e^{-\imath\Phi}\sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \qquad |\psi_+\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{\imath\Phi}\sin(\theta/2) \end{pmatrix}$$

$$\cos(\theta) = \frac{\Delta}{\Omega} \quad \sin(\theta) = \frac{|\kappa|}{\Omega} \qquad |\psi_-\rangle = \begin{pmatrix} -e^{\imath\Phi}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$$

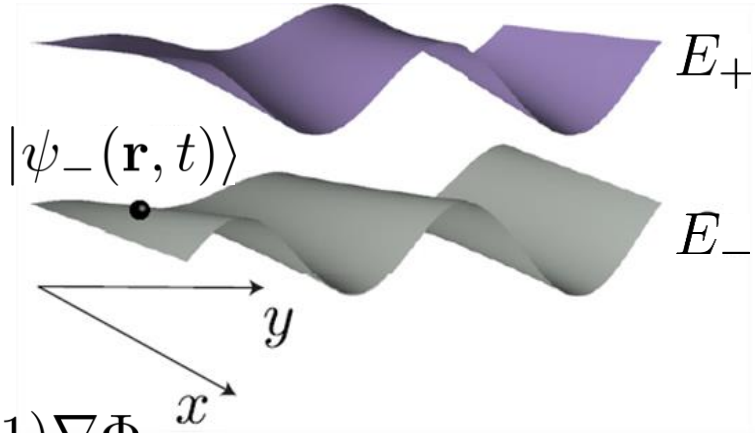
Dalibard and Gerbier, Artificial gauge potentials for neutral atoms, Reviews of Modern Physics **83**, 1523 (2011)

# Toy model: two-level system



- Space dependent atom-light coupling

$$\hat{H}_{\text{int}}(\mathbf{r}) = \frac{\hbar}{2}\begin{pmatrix} \Delta(\mathbf{r}) & \kappa^*(\mathbf{r}) \\ \kappa(\mathbf{r}) & -\Delta(\mathbf{r}) \end{pmatrix}$$




- Resulting gauge potentials
  - » Berry connection

$$\mathbf{A}_\pm(\mathbf{r}) = \imath\hbar\,\langle\psi_\pm|\nabla\psi_\pm\rangle = \pm\frac{\hbar}{2}(\cos(\theta)-1)\nabla\Phi$$

  - » Berry curvature

$$\mathbf{B}_\pm(\mathbf{r}) = \pm\frac{\hbar}{2}\nabla(\cos(\theta))\times\nabla\Phi$$

- Conditions to obtain non-vanishing gauge potentials
  - » Non-zero Berry curvature achieved only for non-zero gradients of the phase and the mixing angle

$$\kappa = |\kappa|e^{\imath\Phi} \qquad \cos(\theta) = \frac{\Delta}{\Omega} \quad \sin(\theta) = \frac{|\kappa|}{\Omega}$$

$$\Omega = \sqrt{\Delta^2 + |\kappa|^2}$$

  - » Non-zero gradients gradient of the mixing angle can be achieved via a gradient of intensity $(\nabla\kappa)$ or a gradient of detuning $(\nabla\Delta)$

Dalibard and Gerbier, Artificial gauge potentials for neutral atoms, Reviews of Modern Physics **83**, 1523 (2011)

# Toy model: two-level system

- Berry connection and curvature
  - » Arise in simple experimental schemes
  - » Give rise to magnetic-like behavior (Lorentz force for neutral atoms)

- Drawbacks of this toy model
  - » Scheme only applicable to alkaline-earth species as Yb
    - The occupation of the excited state must be non-negligible in order to create non-vanishing artificial magnetic fields
    - Spontaneous emission is detrimental
    - Scattering rate must be negligible on the experimental time scale (10 - 100 ms)

  - » Strength of the realized magnetic field
    - One travelling wave: spatial scale for the variation of the mixing angle limited to the beam waist *w*.
    - More travelling waves: interference phenomena introduce a much shorter wave-length $\lambda/2$.

**Part 3.4**

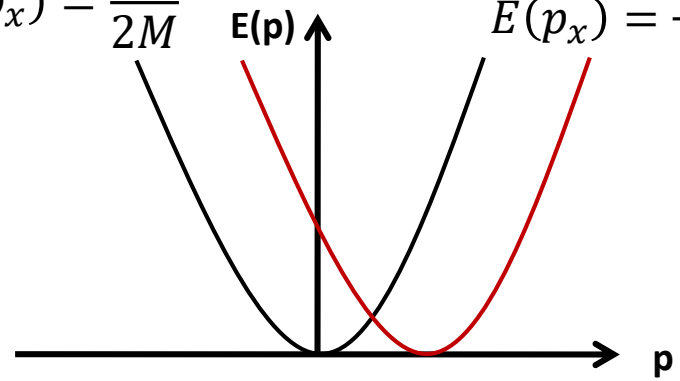
Dalibard and Gerbier, Artificial gauge potentials for neutral atoms, Reviews of Modern Physics **83**, 1523 (2011)

# Summary: Generating *artificial* gauge potentials

## Momentum space

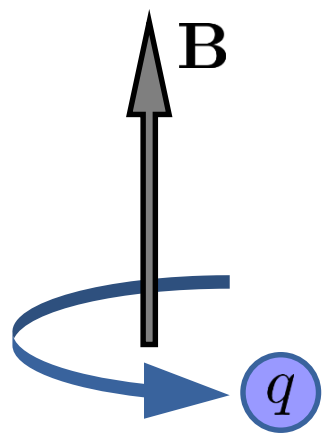
$$E(p_x) = \frac{p_x^2}{2M}$$

$$E(p_x) = \frac{(p_x + qA)^2}{2M}$$




$$\hat{H} = \frac{\left(\hat{\boldsymbol{p}} + q\,\boldsymbol{A}(\hat{\boldsymbol{r}})\right)^2}{2M}$$

## Real space




$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r})$$

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{\partial \boldsymbol{A}(\boldsymbol{r})}{\partial t}$$

$$\Delta\varphi = \frac{1}{\hbar}\oint q\boldsymbol{A}(\mathbf{r}) \cdot d\boldsymbol{r} = 2\pi\frac{\Phi}{\Phi_0}$$