Condensed-Matter Theory - Special Topics

Problem 7 — Potenzialstreuung II
a) Leiten Sie für den Hamiltonian (Fermionen)

$$H = H_0 + V = \sum_{\alpha\beta} t_{\alpha\beta} c_\alpha^\dagger c_\beta + \sum_{\alpha\beta} V_{\alpha\beta} c_\alpha^\dagger c_\beta$$

diagrammatisch (d.h. mittels Störungstheorie in V) den Zusammenhang zwischen großkanonischem Potenzial und reduzibler Selbstenergie ab!

b) Stellen Sie Ω über eine Kopplungskonstantenintegration dar ($V_{\alpha\beta} \to \lambda V_{\alpha\beta}$)!

c) Verifizieren Sie diese Darstellung auch durch direktes Nachrechnen!

d) Definieren Sie diagrammatisch das Luttinger-Ward-Funktional $\Phi[\boldsymbol{G}]$, und zeigen Sie, dass sich dieses explizit berechnen lässt! Verifizieren Sie, dass

$$\frac{\delta \Phi[\boldsymbol{G}]}{\delta G_{\alpha\beta}(i\omega)} = \frac{1}{\beta} V_{\beta\alpha} \quad !$$

e) Überprüfen Sie, dass

$$\Omega = \Phi + \frac{1}{\beta} \sum_\omega e^{i\omega 0^+} \mathsf{tr} \ln \boldsymbol{G}(i\omega) - \frac{1}{\beta} \sum_\omega e^{i\omega 0^+} \mathsf{tr}(\boldsymbol{V}\boldsymbol{G}(i\omega)) \; !$$

Problem 8 — Self-energy functionals
Consider:

(1) $\Omega_{\boldsymbol{t},\boldsymbol{U}}[\boldsymbol{\Sigma}] = \Phi_{\boldsymbol{U}}[\boldsymbol{G}_{\boldsymbol{U}}[\boldsymbol{\Sigma}]] + \mathsf{Tr} \ln \boldsymbol{G}_{\boldsymbol{U}}[\boldsymbol{\Sigma}] - \mathsf{Tr}(\boldsymbol{\Sigma}\boldsymbol{G}_{\boldsymbol{U}}[\boldsymbol{\Sigma}])$

(2) $\Omega_{\boldsymbol{t},\boldsymbol{U}}[\boldsymbol{\Sigma}] = \Phi_{\boldsymbol{U}}[\boldsymbol{G}_{\boldsymbol{U}}[\boldsymbol{\Sigma}]] + \mathsf{Tr} \ln \dfrac{1}{\boldsymbol{G}_{\boldsymbol{t},0}^{-1} - \boldsymbol{\Sigma}} - \mathsf{Tr}\left(\boldsymbol{\Sigma} \dfrac{1}{\boldsymbol{G}_{\boldsymbol{t},0}^{-1} - \boldsymbol{\Sigma}}\right)$

(3) $\Omega_{\boldsymbol{t},\boldsymbol{U}}[\boldsymbol{\Sigma}] = \Phi_{\boldsymbol{U}}[\boldsymbol{G}_{\boldsymbol{U}}[\boldsymbol{\Sigma}]] + \mathsf{Tr} \ln \boldsymbol{G}_{\boldsymbol{U}}[\boldsymbol{\Sigma}] - \mathsf{Tr}\left(\boldsymbol{\Sigma} \dfrac{1}{\boldsymbol{G}_{\boldsymbol{t},0}^{-1} - \boldsymbol{\Sigma}}\right)$

(4) $\Omega_{\boldsymbol{t},\boldsymbol{U}}[\boldsymbol{\Sigma}] = \Phi_{\boldsymbol{U}}[\boldsymbol{G}_{\boldsymbol{U}}[\boldsymbol{\Sigma}]] + \mathsf{Tr} \ln \dfrac{1}{\boldsymbol{G}_{\boldsymbol{t},0}^{-1} - \boldsymbol{\Sigma}} - \mathsf{Tr}(\boldsymbol{\Sigma}\boldsymbol{G}_{\boldsymbol{U}}[\boldsymbol{\Sigma}])$

(5) $\Omega_{\boldsymbol{t},\boldsymbol{U}}[\boldsymbol{\Sigma}] = \Phi_{\boldsymbol{U}}[(\boldsymbol{G}_{\boldsymbol{t},0}^{-1} - \boldsymbol{\Sigma})^{-1}] + \mathsf{Tr} \ln \boldsymbol{G}_{\boldsymbol{U}}[\boldsymbol{\Sigma}] - \mathsf{Tr}(\boldsymbol{\Sigma}\boldsymbol{G}_{\boldsymbol{U}}[\boldsymbol{\Sigma}])$

(6) $\Omega_{\boldsymbol{t},\boldsymbol{U}}[\boldsymbol{\Sigma}] = \Phi_{\boldsymbol{U}}[(\boldsymbol{G}_{\boldsymbol{t},0}^{-1} - \boldsymbol{\Sigma})^{-1}] + \mathsf{Tr} \ln \dfrac{1}{\boldsymbol{G}_{\boldsymbol{t},0}^{-1} - \boldsymbol{\Sigma}} - \mathsf{Tr}\left(\boldsymbol{\Sigma} \dfrac{1}{\boldsymbol{G}_{\boldsymbol{t},0}^{-1} - \boldsymbol{\Sigma}}\right)$

(7) $\Omega_{\boldsymbol{t},\boldsymbol{U}}[\boldsymbol{\Sigma}] = \Phi_{\boldsymbol{U}}[(\boldsymbol{G}_{\boldsymbol{t},0}^{-1} - \boldsymbol{\Sigma})^{-1}] + \mathsf{Tr} \ln \boldsymbol{G}_{\boldsymbol{U}}[\boldsymbol{\Sigma}] - \mathsf{Tr}\left(\boldsymbol{\Sigma} \dfrac{1}{\boldsymbol{G}_{\boldsymbol{t},0}^{-1} - \boldsymbol{\Sigma}}\right)$

(8) $\Omega_{\boldsymbol{t},\boldsymbol{U}}[\boldsymbol{\Sigma}] = \Phi_{\boldsymbol{U}}[(\boldsymbol{G}_{\boldsymbol{t},0}^{-1} - \boldsymbol{\Sigma})^{-1}] + \mathsf{Tr}\ln\frac{1}{\boldsymbol{G}_{\boldsymbol{t},0}^{-1} - \boldsymbol{\Sigma}} - \mathsf{Tr}(\boldsymbol{\Sigma}\boldsymbol{G}_{\boldsymbol{U}}[\boldsymbol{\Sigma}])$

where $\boldsymbol{G}_{\boldsymbol{t},0}^{-1} = i\omega + \mu - \boldsymbol{t}$, where $\boldsymbol{G}_{\boldsymbol{U}}[\boldsymbol{\Sigma}]$ is the inverse functional of $\boldsymbol{\Sigma}_{\boldsymbol{U}}[\boldsymbol{G}]$, and where the additional parameter dependencies of the various functionals are made explicit by the subscripts $\boldsymbol{t}$ and $\boldsymbol{U}$.

Which functionals yield a valid variational principle $\delta\Omega_{\boldsymbol{t},\boldsymbol{U}}[\boldsymbol{\Sigma}] = 0$?