Excitation of Resonant Cavities by Magnetic Moments

Wolfgang Hillert

Physics Institute of Bonn University

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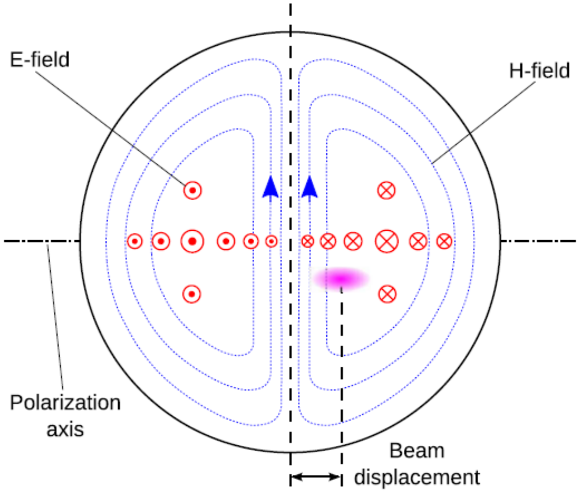
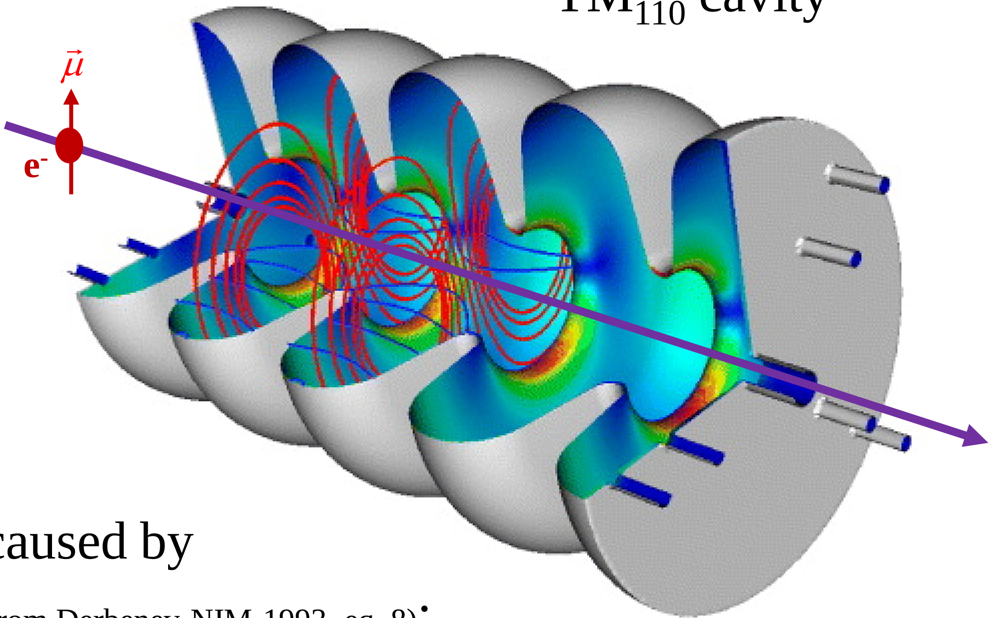
1. Functional Principle
2. Relativistic Stern-Gerlach Force
3. Cavity Modes
4. Energy Transfer per Particle Passage
5. Signal Power
6. Example: Respol with TE_{011}, TE_{111}

Resonant Polarimetry

Principle Idea (Derbenev 1993):



TM$_{110}$ cavity

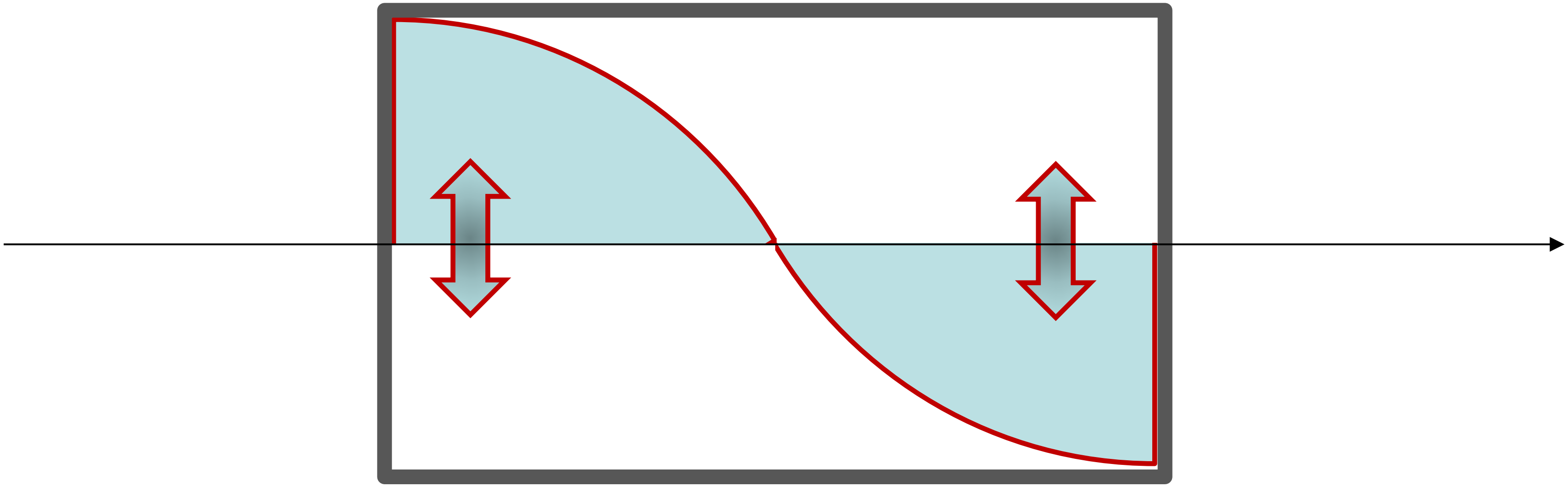


Coupling of the magnetic moment (caused by the spin) to the cavity’s B-field (taken from Derbenev-NIM-1993, eq. 8):

$$W_C = \omega_c |a|^2 = \omega_c N^2 \left| \left\langle \frac{e}{2mc\sqrt{2\omega_c}} \left(\left(G + \frac{1}{\gamma} \right) B_\perp^c + \frac{1+G}{\gamma} B_\parallel^c \right) \vec{e} \cdot e^{ik\theta} \right\rangle \right|^2 \frac{\hbar^2 t^2}{4} P_e \sin^2 \alpha$$

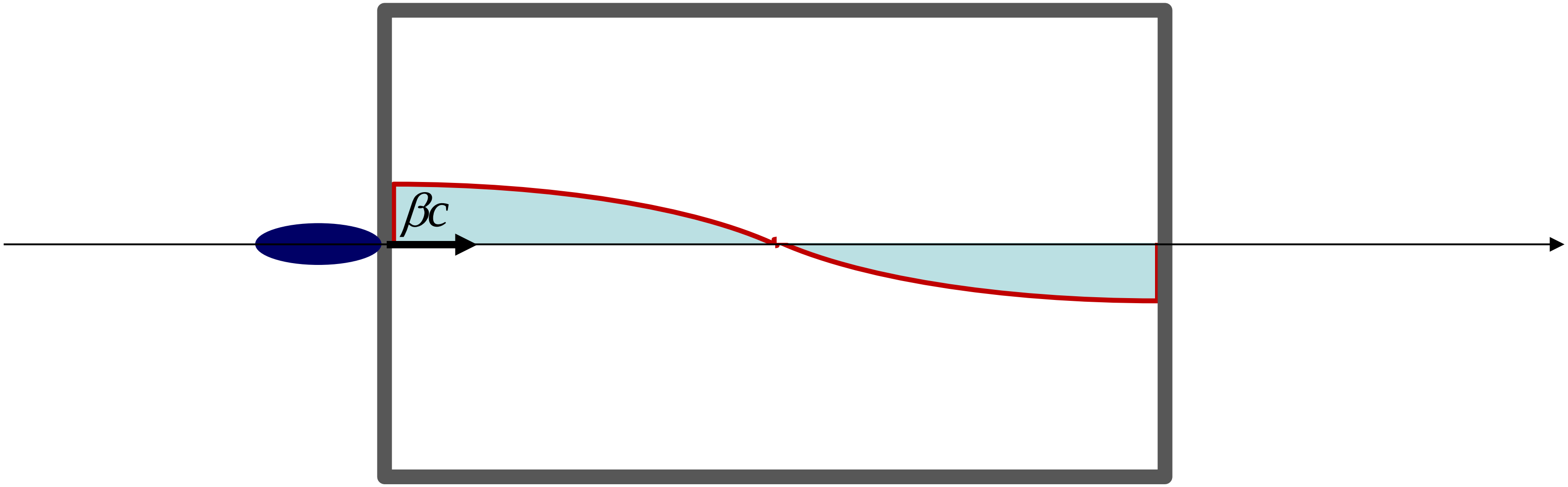
?Physical understanding? **?γ and G scaling?**

Transverse Mode



$$\Delta W = \int \frac{\partial}{\partial z}\left(\vec{\mu}\bullet\vec{B}\right)\cdot dz$$

Transverse Mode



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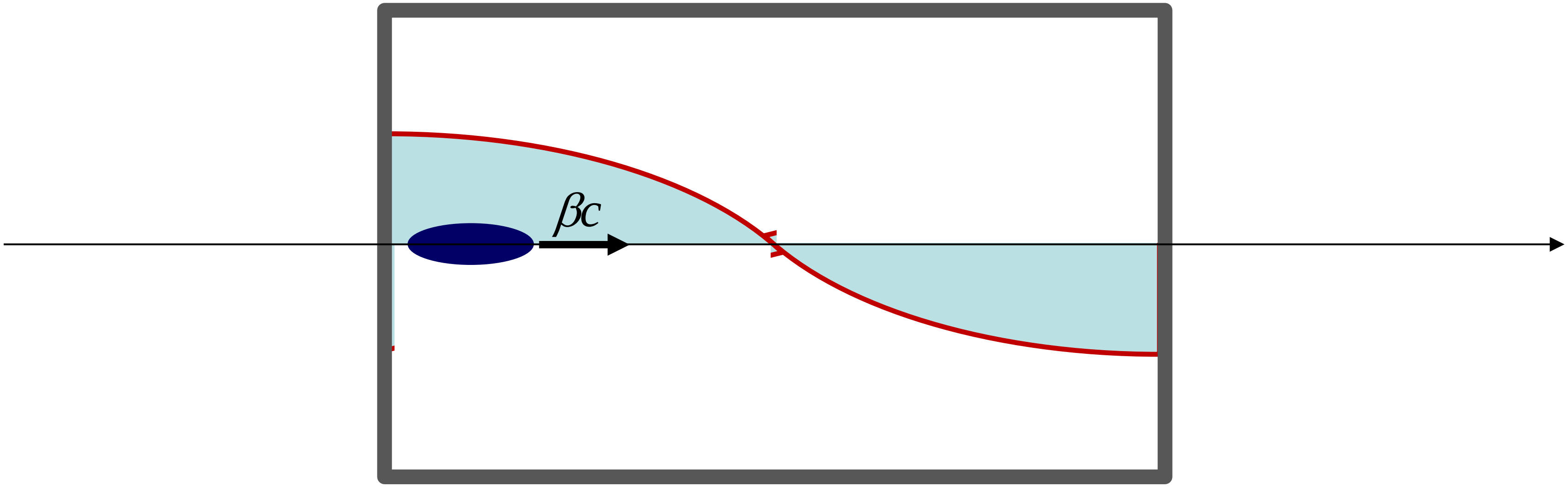

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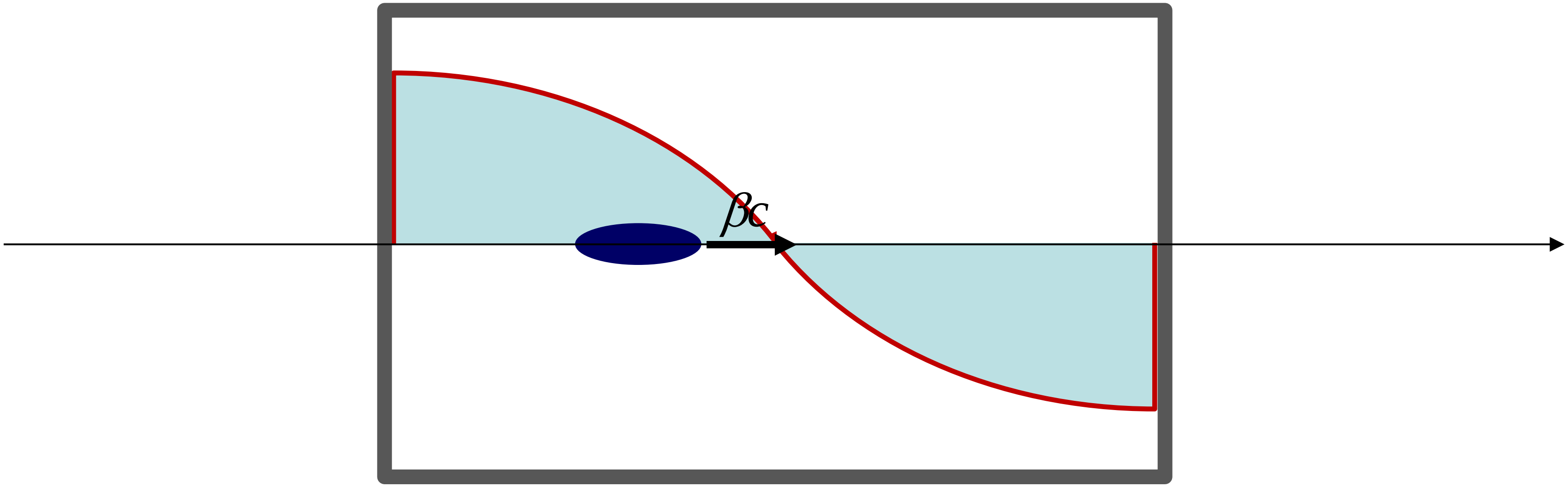

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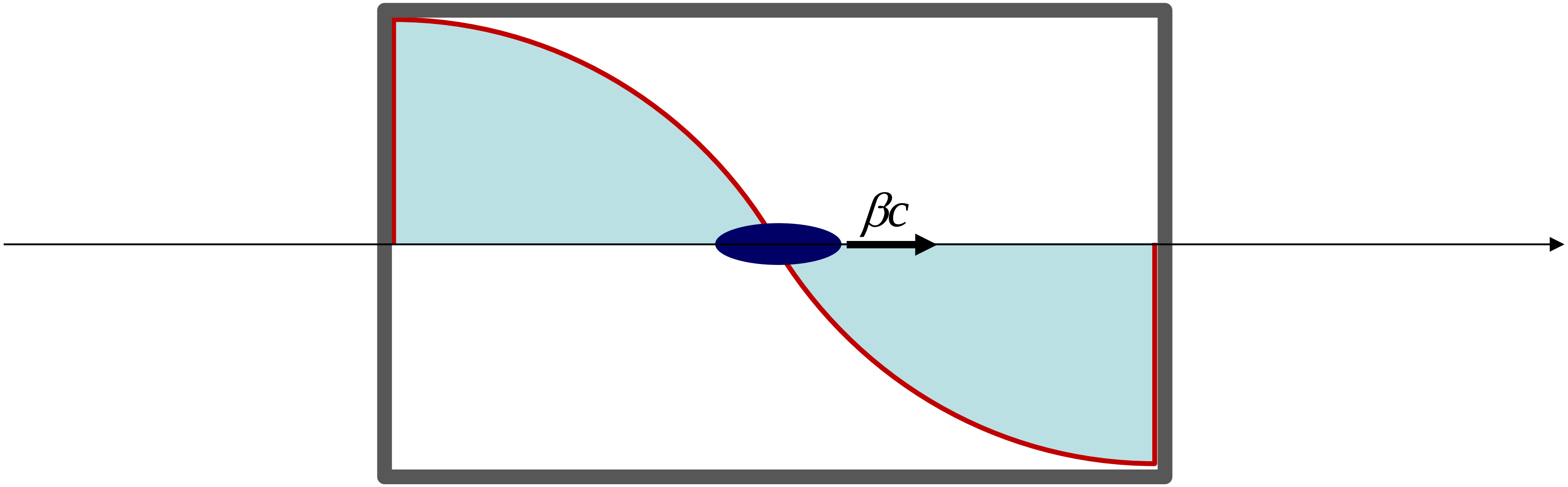

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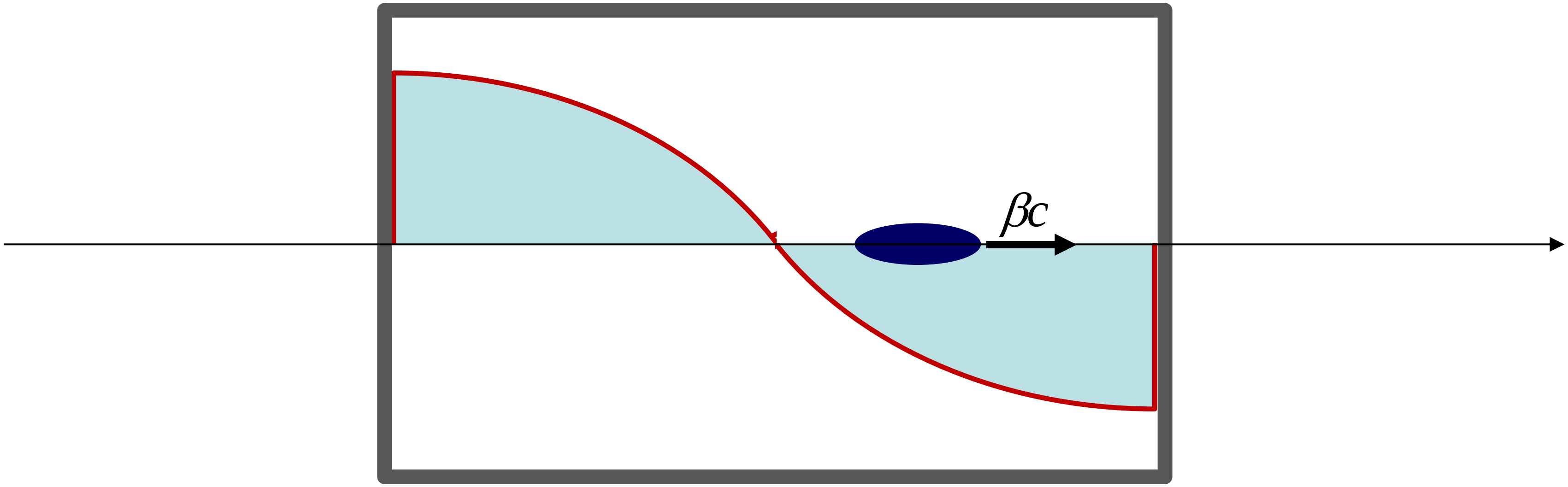

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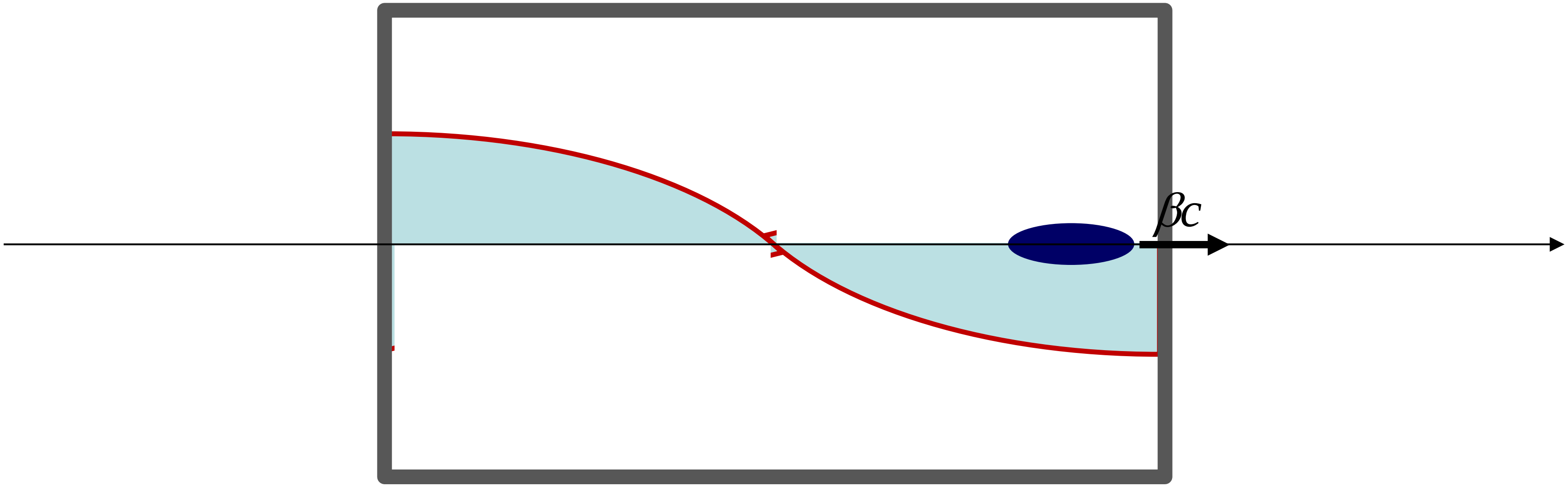

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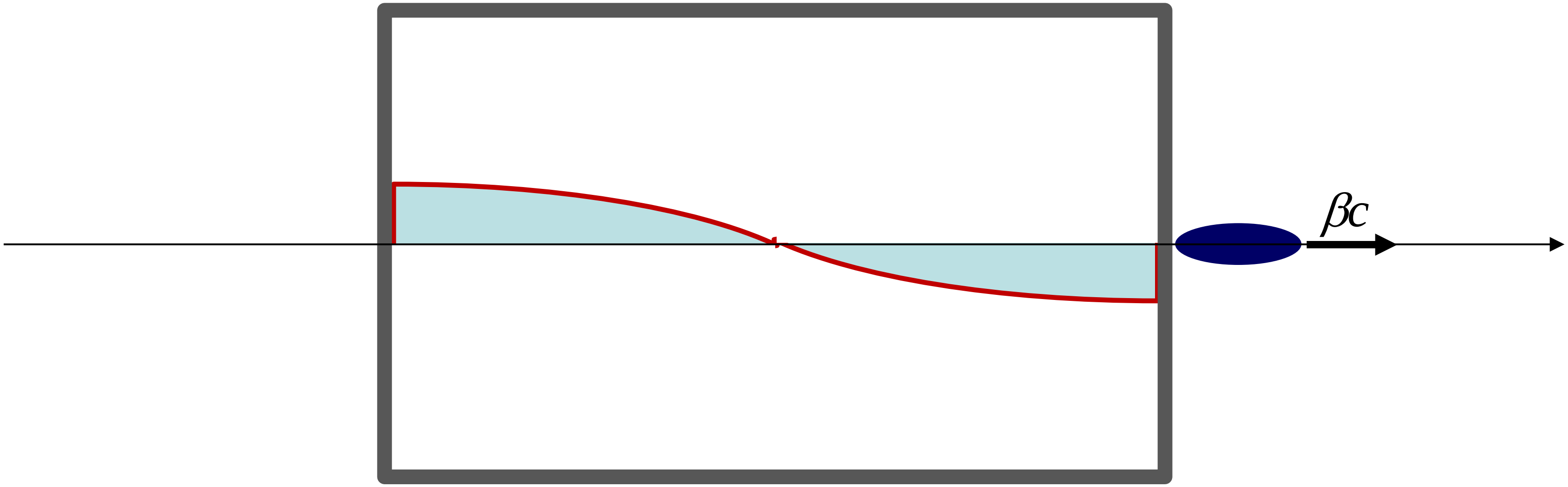

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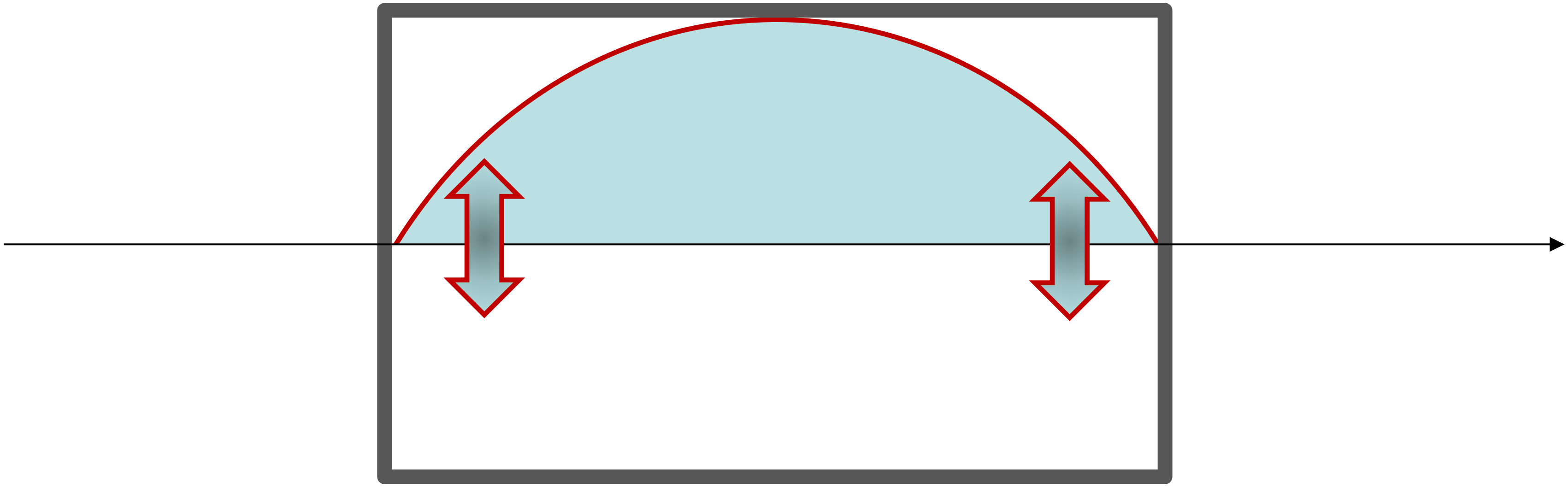
Longitudinal Mode



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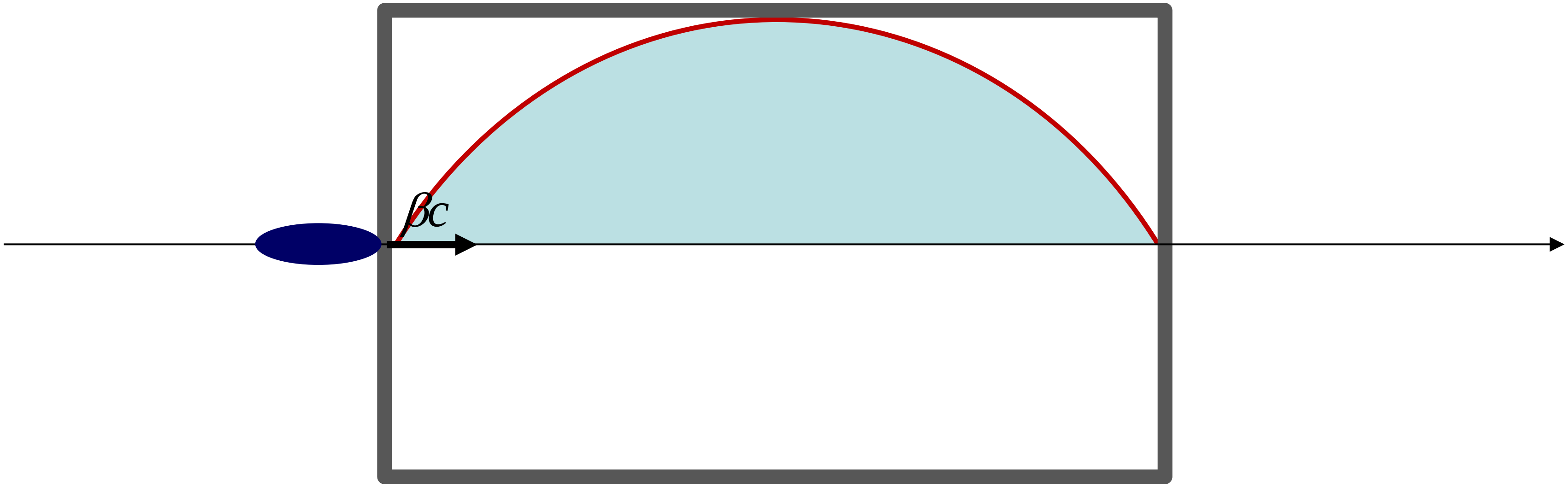

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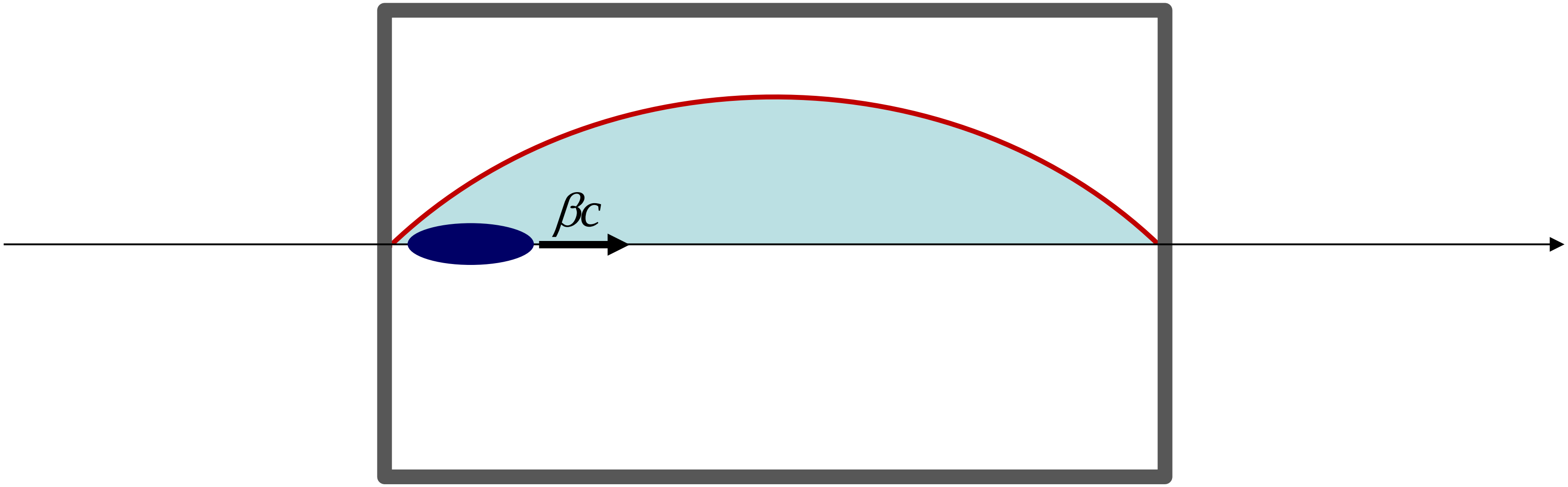

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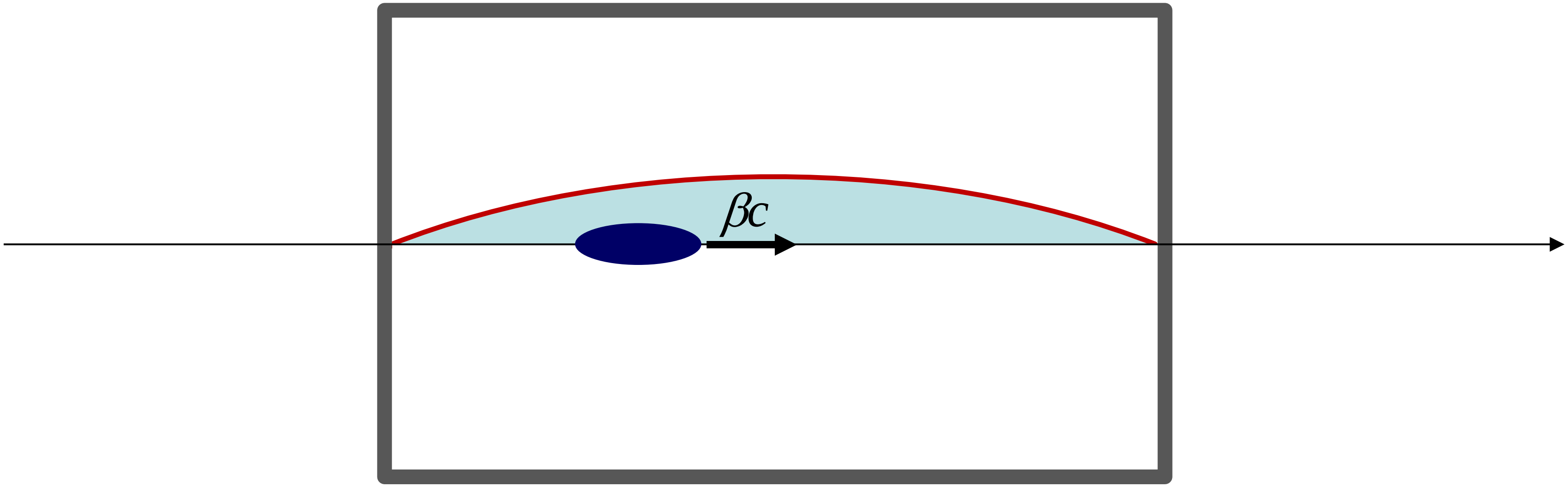

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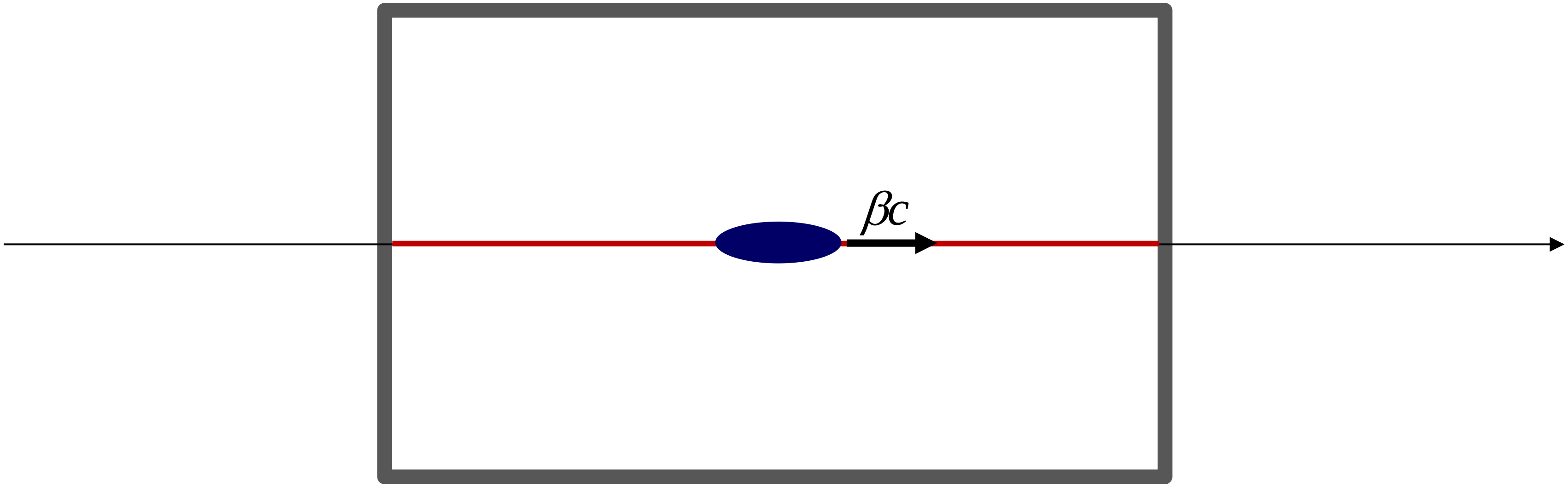

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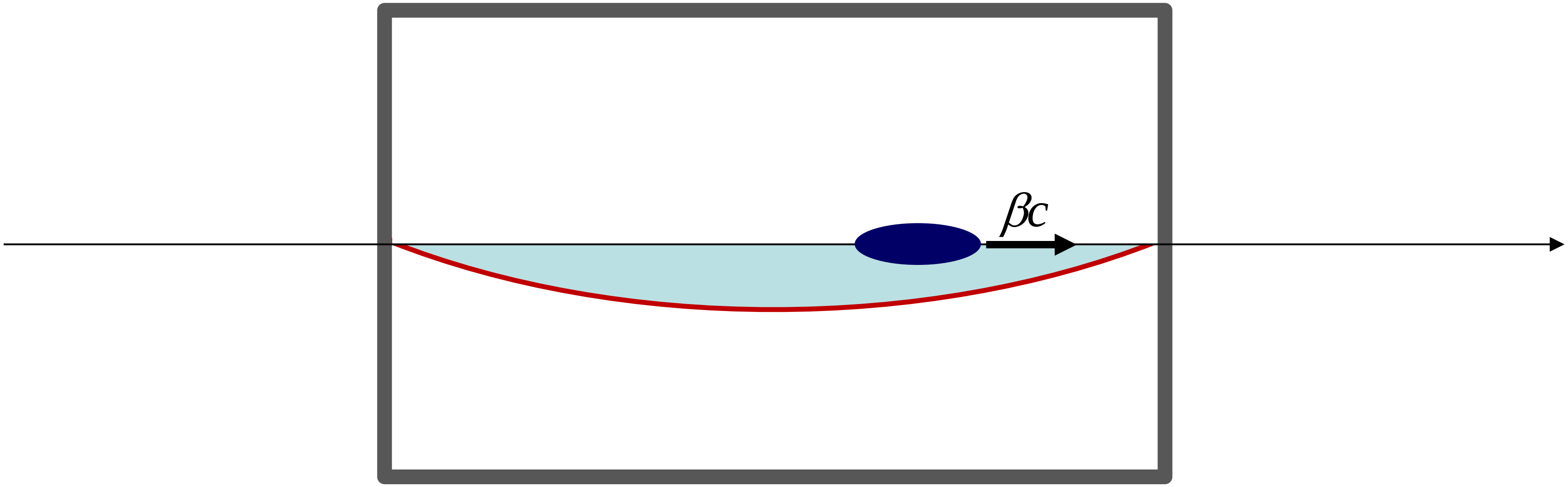

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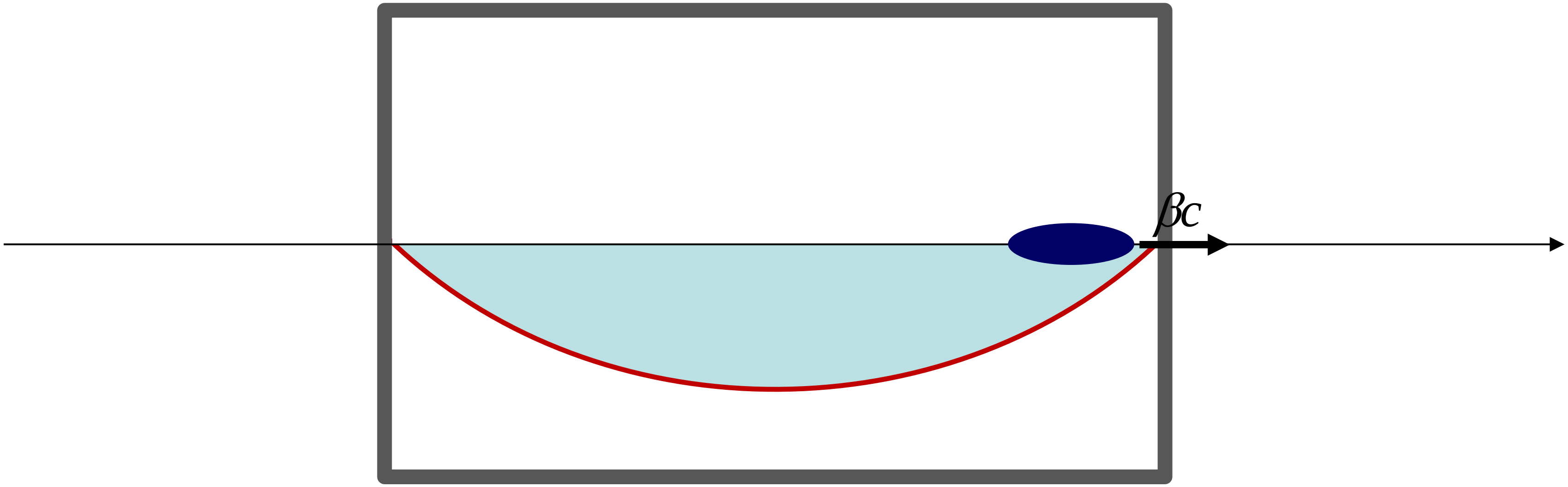

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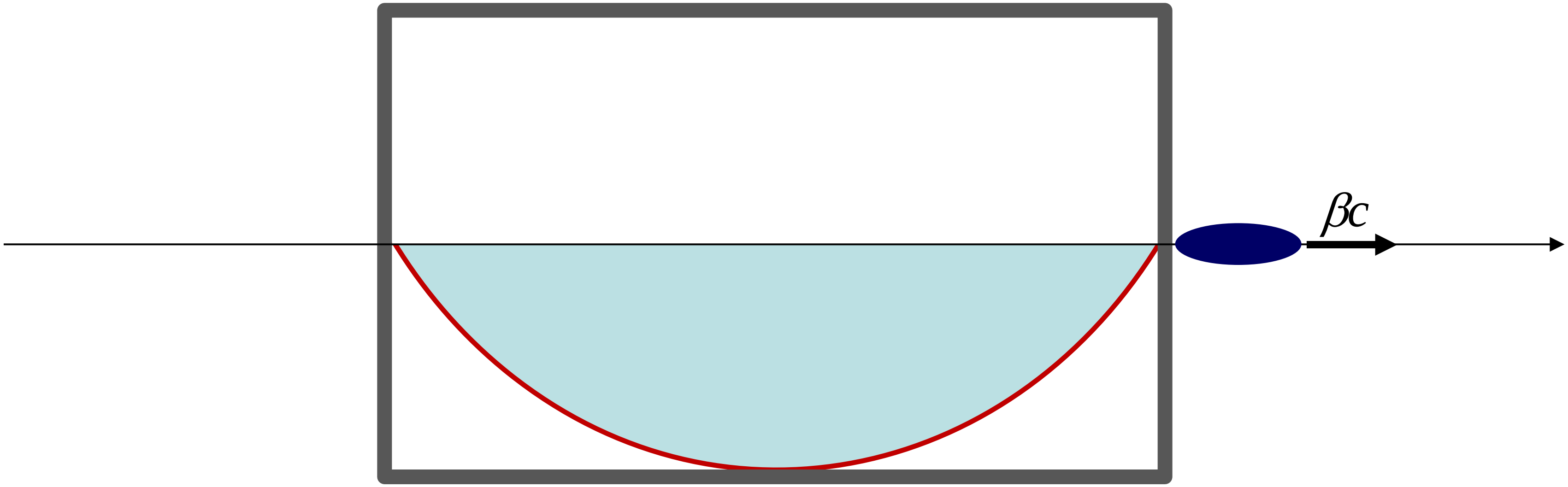

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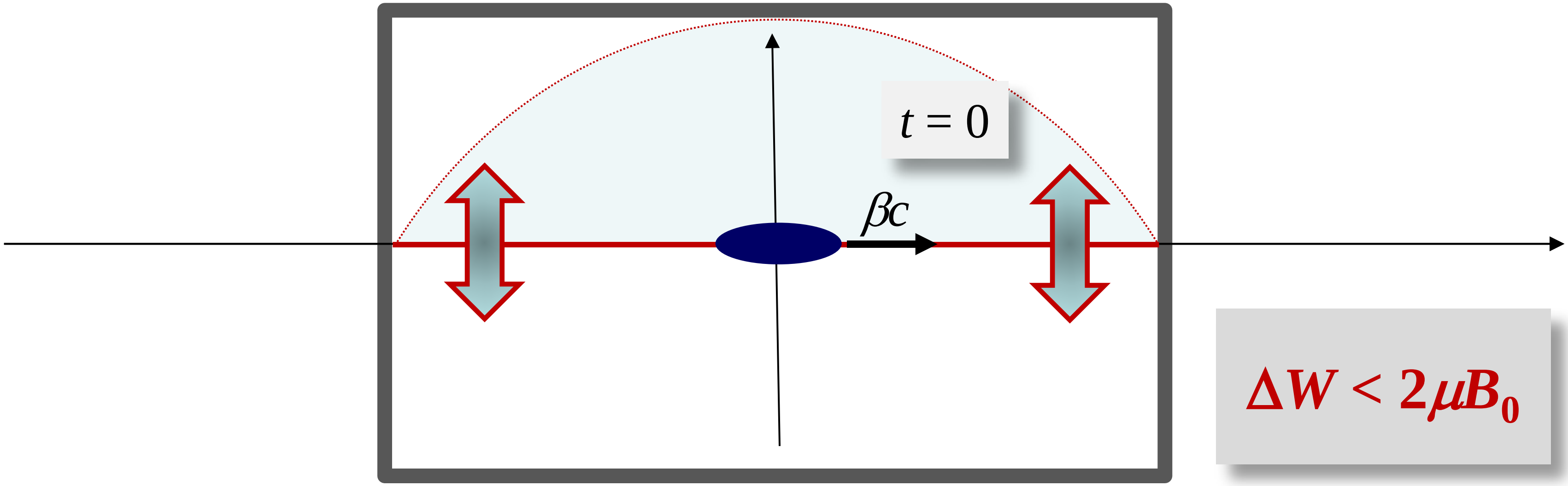


$$\mathbf{\Delta W < 2\mu B_0}$$

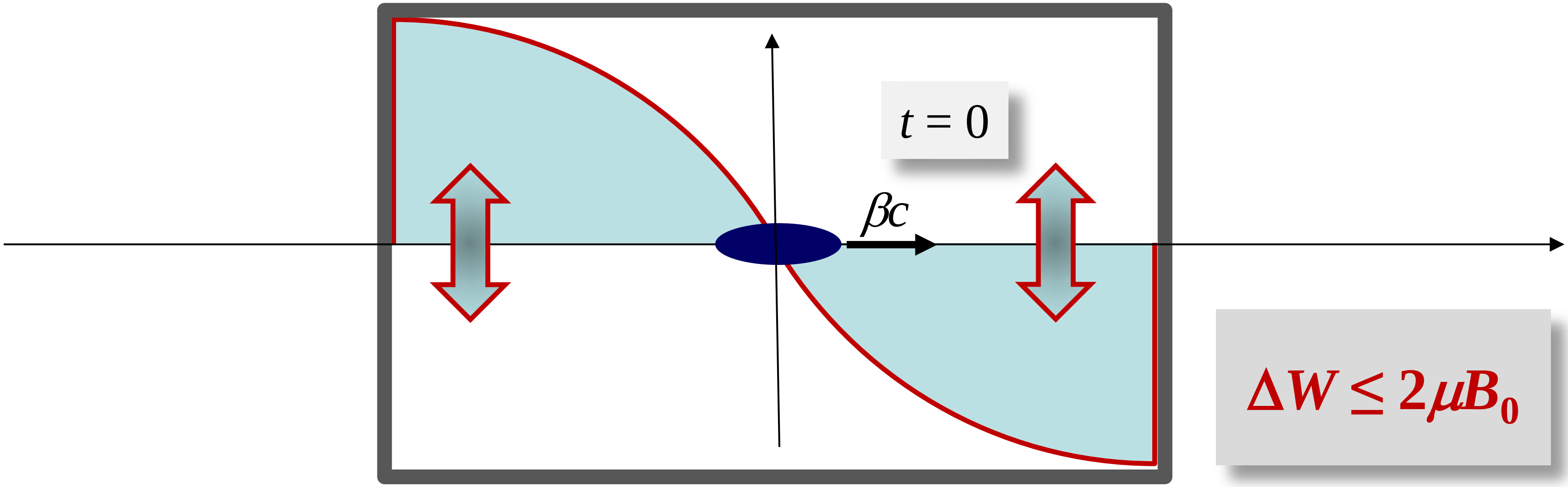
Findings:

$$B_\perp = B_0 \cdot \cos\left(\omega t + \phi\right) \quad \Rightarrow \quad \phi_{opt} = -\frac{\pi}{2}, \quad \beta_{ph} \approx 1$$

?

Transverse Mode

$$\Delta W = \int \frac{\partial}{\partial z}\left(\vec{\mu} \bullet \vec{B}\right) \cdot dz$$



$$\Delta W \leq 2\mu B_0$$

Findings:

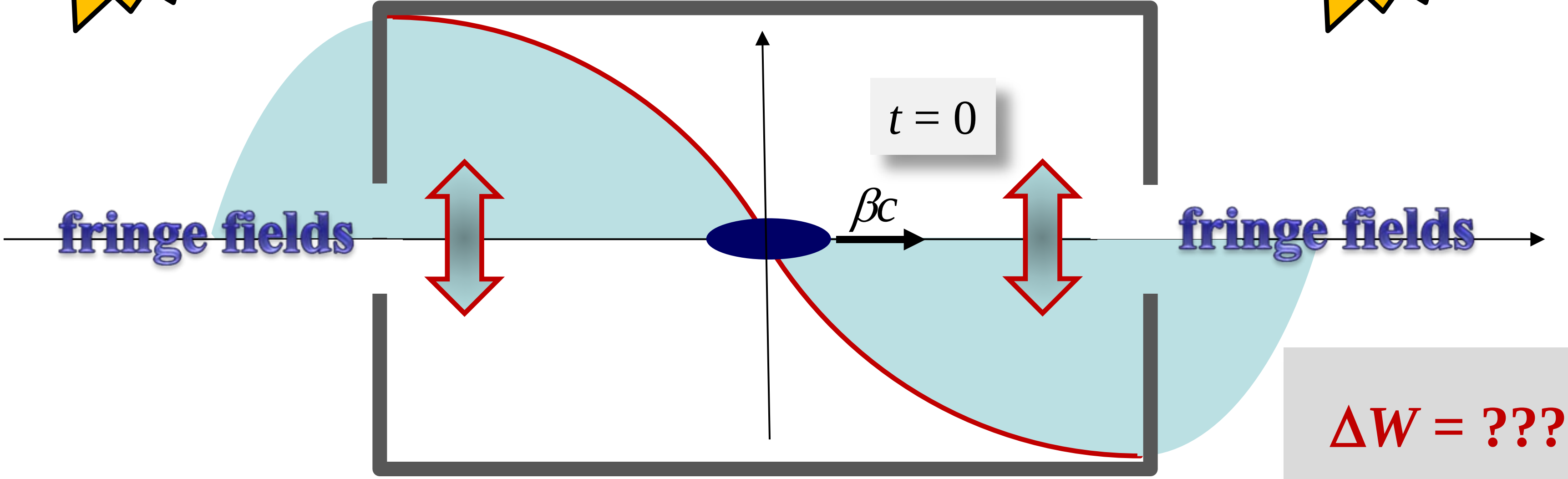
$$B_\perp = B_0 \cdot \cos\left(\omega t + \phi\right) \quad \Rightarrow \quad \phi_{opt} = 0, \quad \beta_{ph} \gg 1$$

Transverse Mode

but:

$$\Delta W = \int \frac{\partial}{\partial z}\left(\vec{\mu}\bullet\vec{B}\right)\cdot dz$$

but:



$\Delta W = ???$

Findings:

$$B_\perp = B_0 \cdot \cos\left(\omega t + \phi\right) \Rightarrow \phi_{opt} = 0, \quad \beta_{ph} = ???$$

Some Approaches

Derbenev (NIM-1993)

Hamiltonian:
$$H = \frac{1}{2}\left(P^2 + \omega_c^2 Q^2\right) + \sum_j \vec{\Omega}_j^{ext} \bullet \vec{S}^j + \sum_j \vec{\Omega}_j^{c} \bullet \vec{S}^j$$

Cavity fields:
$$\vec{E}(\vec{r},t) = -\frac{1}{c} P(t)\vec{E}^c(\vec{r}), \qquad \vec{B}(\vec{r},t) = Q(t)\vec{B}^c(\vec{r})$$

Spin precession:
$$\vec{\Omega} = -\frac{e}{mc}\left[\left(G + \frac{1}{\gamma}\right)\vec{B}_\perp + \frac{1+G}{\gamma}\vec{B}_\parallel + \left(G + \frac{1}{1+\gamma}\right)\vec{E} \times \vec{\beta}\right]$$

Magnetic moment:
$$\vec{\mu} = (1+G)\frac{e}{mc}\vec{S}$$

Equations of motion:

Canonical variables:
$$\dot{P} = \{H,P\} = -\omega_c^2 Q - \sum_j \frac{\partial \vec{\Omega}_j^c}{\partial Q} \bullet \vec{S}^j$$

$$\dot{Q} = \{H,Q\} = P + \sum_j \frac{\partial \vec{\Omega}_j^c}{\partial P} \bullet \vec{S}^j$$

Spin:
$$\dot{\vec{S}} = \{H,\vec{S}\} = \vec{\Omega} \times \vec{S}$$
(not verified ☺)

Some Approaches

Conte (arXiv: 0907.2161v1-2009)

Longitudinal Stern-Gerlach force:

$$F_z^{SG} = \frac{\partial}{\partial z^*}\left(\vec{\mu}^* \bullet \vec{B}^*\right) = \gamma\left(\frac{\partial}{\partial z} + \frac{\beta}{c}\frac{\partial}{\partial t}\right)\left(\vec{\mu}^* \bullet \gamma\left[\left(\vec{B} - \frac{\vec{\beta}}{c} \times \vec{E}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta} \bullet \vec{B}\right)\right]\right)$$

Energy transfer to the cavity:

$$\Delta U = \int_0^L F_z^{SG} \cdot dz = \boxed{\gamma^2} \cdot \int_0^L \left(\frac{\partial}{\partial z} + \frac{\beta}{c}\frac{\partial}{\partial t}\right)\vec{\mu} \bullet \left(\vec{B}_\perp - \frac{\vec{\beta}}{c} \times \vec{E}_\perp + \frac{1}{\gamma}\vec{B}_\parallel\right) \cdot dz$$

Some Approaches

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Energy transfer to the cavity:

$$\Delta U = \int_0^L F_z^{SG}\cdot dz = \boxed{\gamma^2}\cdot\int_0^L\left(\frac{\partial}{\partial z} + \frac{\beta}{c}\frac{\partial}{\partial t}\right)\vec{\mu}\bullet\left(\vec{B}_\perp - \frac{\vec{\beta}}{c}\times\vec{E}_\perp + \frac{1}{\gamma}\vec{B}_\parallel\right)\cdot dz$$

Improper procedures in Conte:

- Treatment of the fringe fields: $F \simeq \gamma^2 \dfrac{\partial B_y}{\partial z}$ neglecting temporal changes
- No relativistic cancellation by taking use of the total derivative
- Neglecting beam deflection and spin precession in the transverse magnetic fields in the cavity using

$$\vec{B}^* = \gamma\left(\vec{B}_\perp - \frac{\vec{\beta}}{c}\times\vec{E}\right) + \vec{B}_\parallel$$

A simple but (hopefully) correct Approach

Transformation of derivatives:

$$\frac{\partial}{\partial z^*} = \gamma\left(\frac{\partial}{\partial z} + \frac{\beta}{c}\frac{\partial}{\partial t}\right) = \gamma\frac{d}{dz} - \frac{1}{\beta\gamma c}\frac{\partial}{\partial t}$$

Transformation of the fields:

$$\vec{\mu}^* \bullet \vec{B}^* = \vec{\mu}\bullet\left[\frac{\gamma}{1+G}\left\{\left(G+\frac{1}{\gamma}\right)\vec{B}_\perp - \left(G+\frac{1}{1+\gamma}\right)\frac{\vec{\beta}}{c}\times\vec{E}\right\} + \vec{B}_\parallel\right]$$

Taking use of the relativistic compensation:

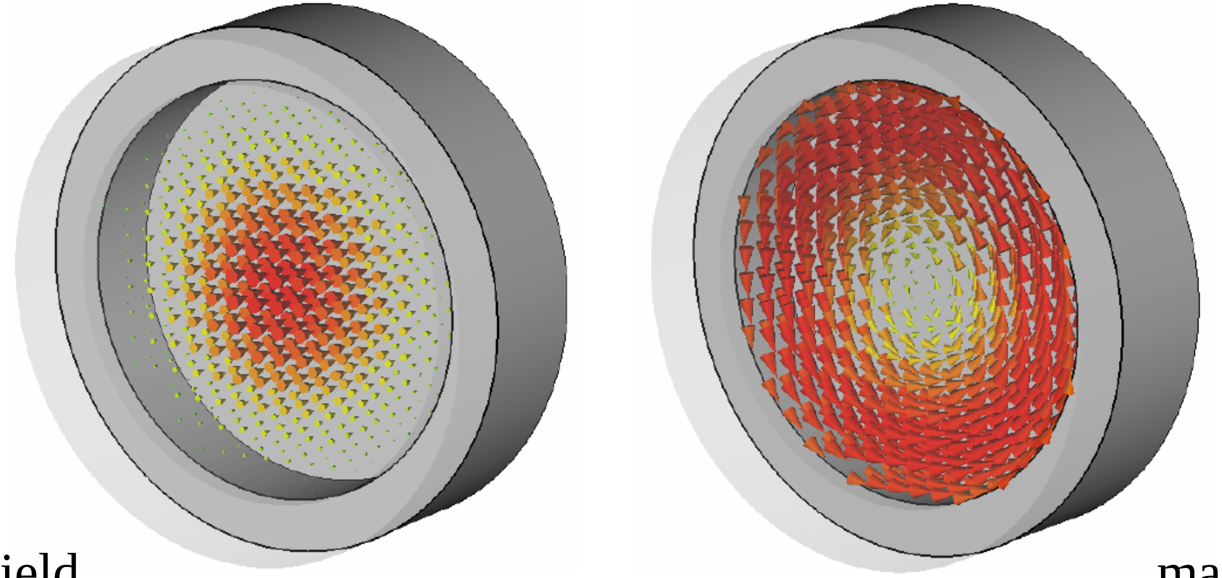
$$\Delta U = \int_0^d F_z^{SG}\cdot dz = \underbrace{\gamma\vec{\mu}^*\bullet\vec{B}^*\Big|_0^d}_{=0} - \frac{\vec{\mu}^*}{\beta c}\bullet\int_0^d \frac{\partial}{\partial t}\left[\frac{\gamma}{1+G}\left\{\left(G+\frac{1}{\gamma}\right)\vec{B}_\perp - \left(G+\frac{1}{1+\gamma}\right)\frac{\vec{\beta}}{c}\times\vec{E}\right\} + \vec{B}_\parallel\right]dz$$

→ **Energy transfer to the cavity:**

$$\Delta U = \int_C F_z^{SG}\cdot dz = -\frac{\vec{\mu}}{\beta c}\bullet\frac{\partial}{\partial t}\int_C\left\{\underbrace{\frac{G+{}^{1}\!/_{\gamma}}{1+G}}_{=\xi_B}\vec{B}_\perp - \underbrace{\left(\frac{G}{1+G}+\frac{1}{(1+G)(1+\gamma)}\right)}_{=\xi_E}\frac{\vec{\beta}}{c}\times\vec{E} + \frac{1}{\gamma}\vec{B}_\parallel\right\}dz$$

Cavity Modes: TM

TM$_{010}$

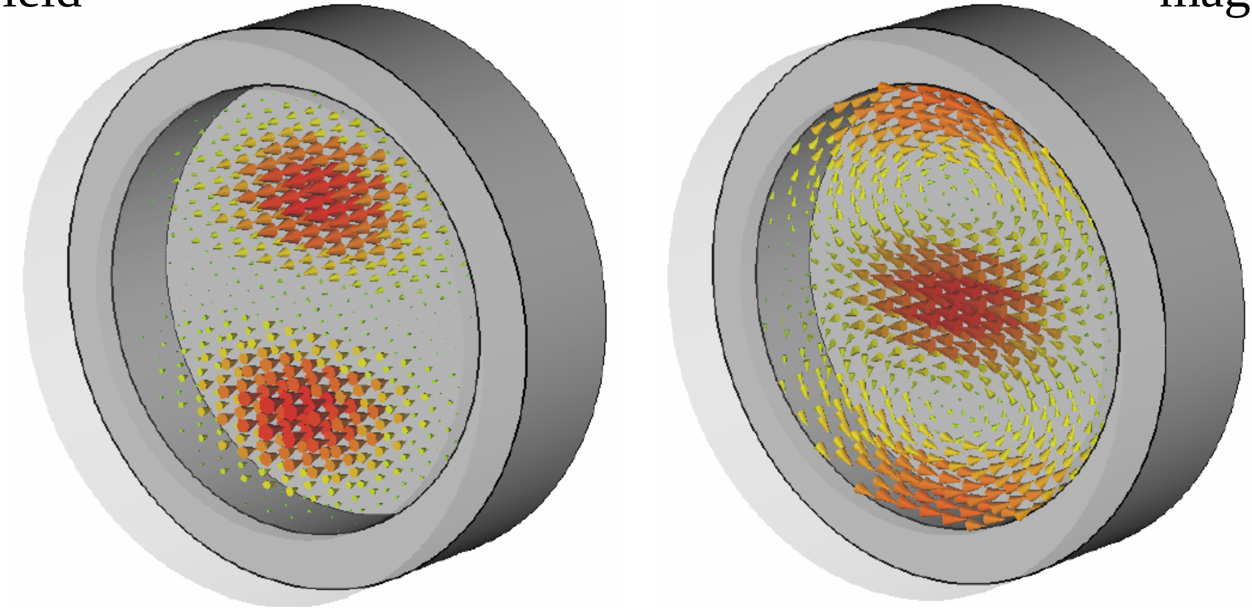


☹

electric field

magnetic field

TM$_{110}$



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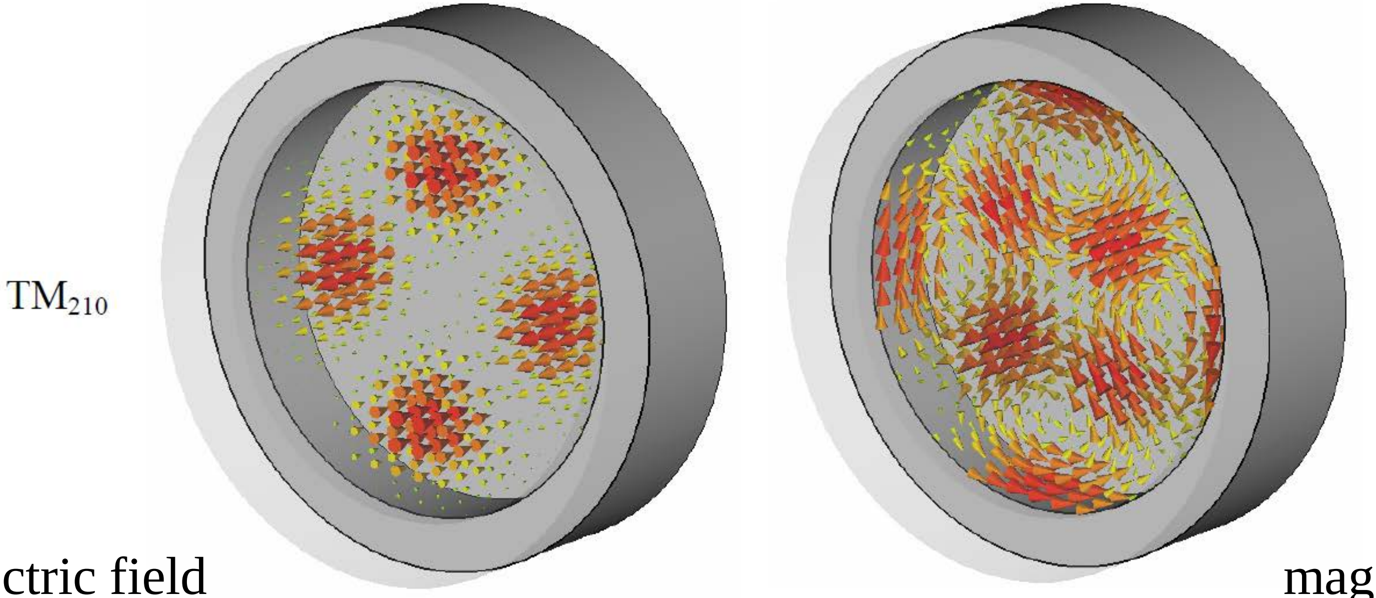
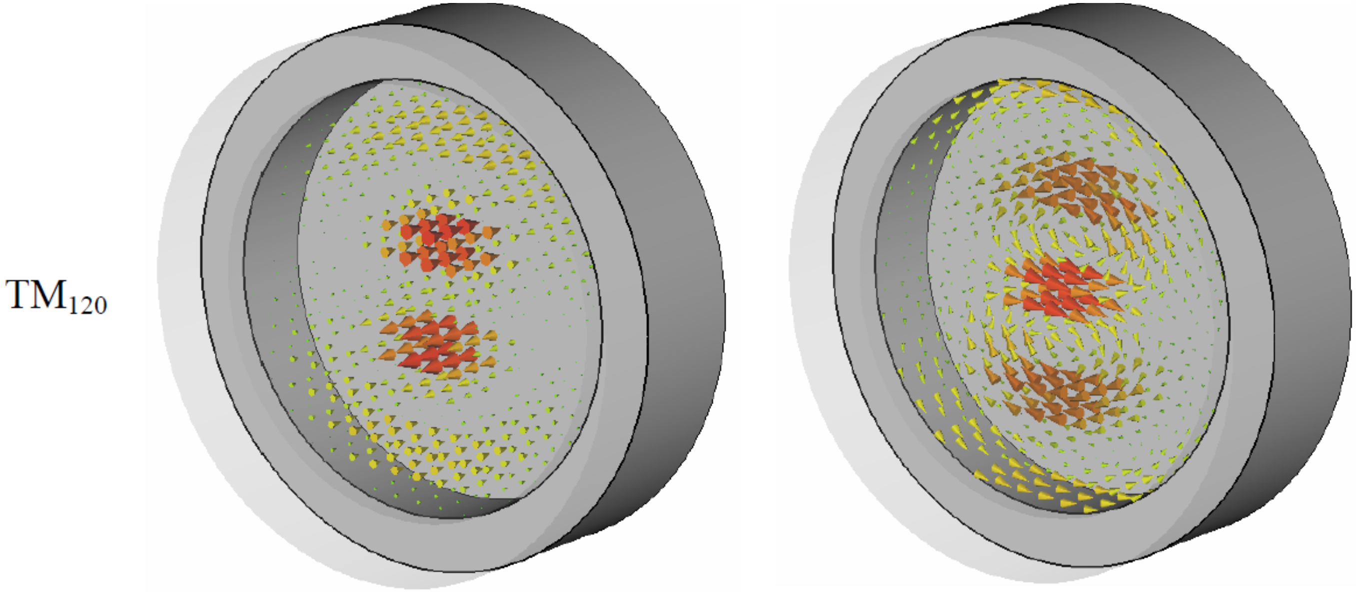
Cavity Modes: TM



☹

electric field

magnetic field



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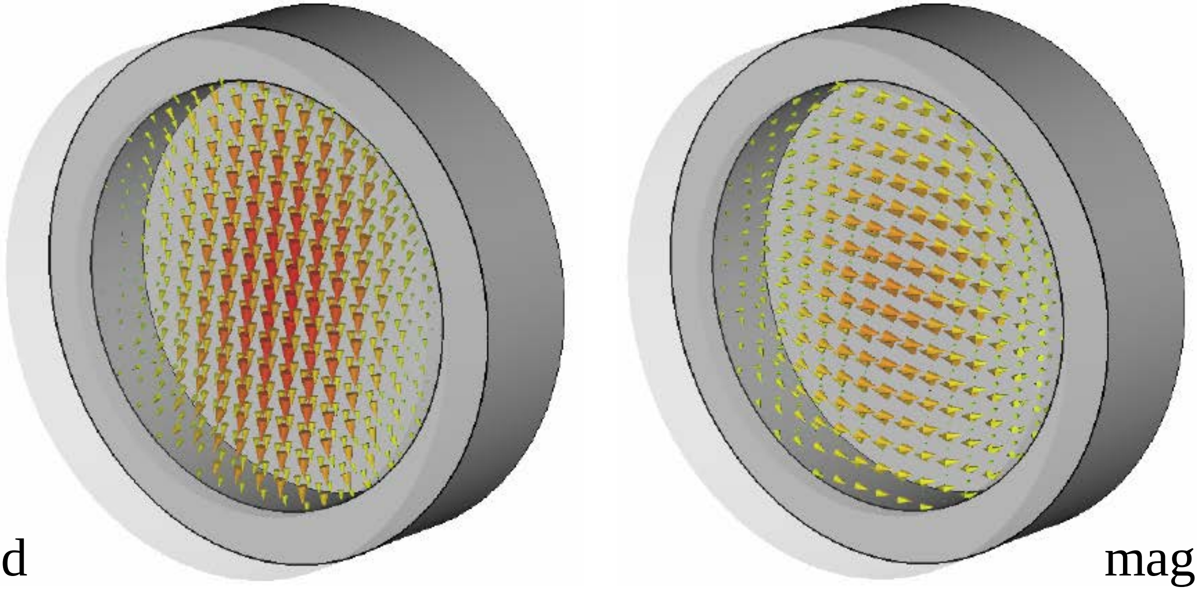
Cavity Modes: TE

TE$_{111}$

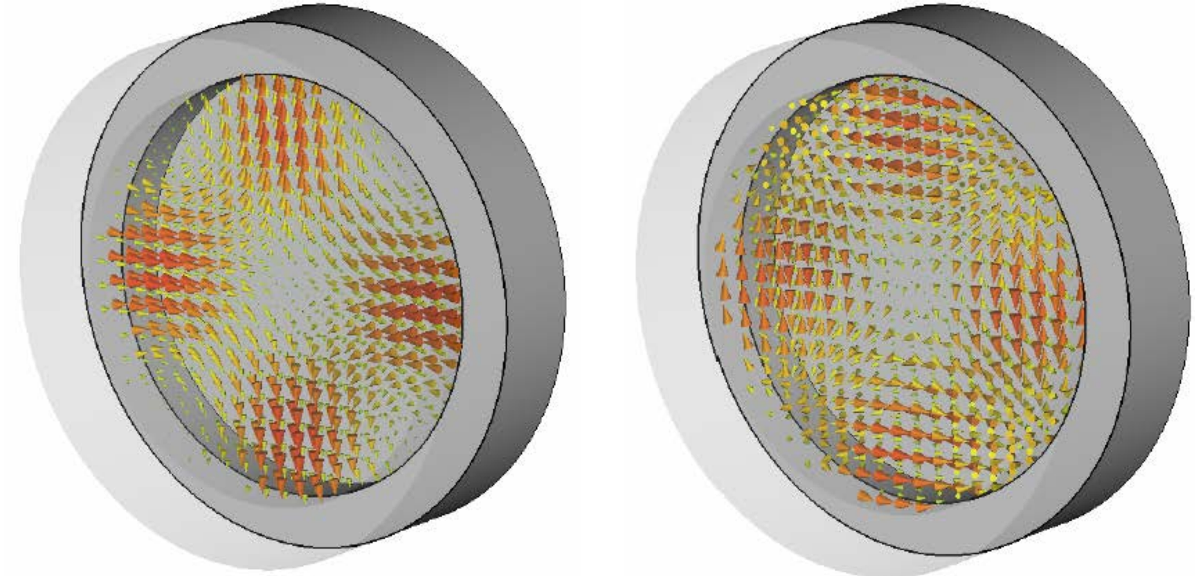


electric field

magnetic field

🙂

TE$_{211}$



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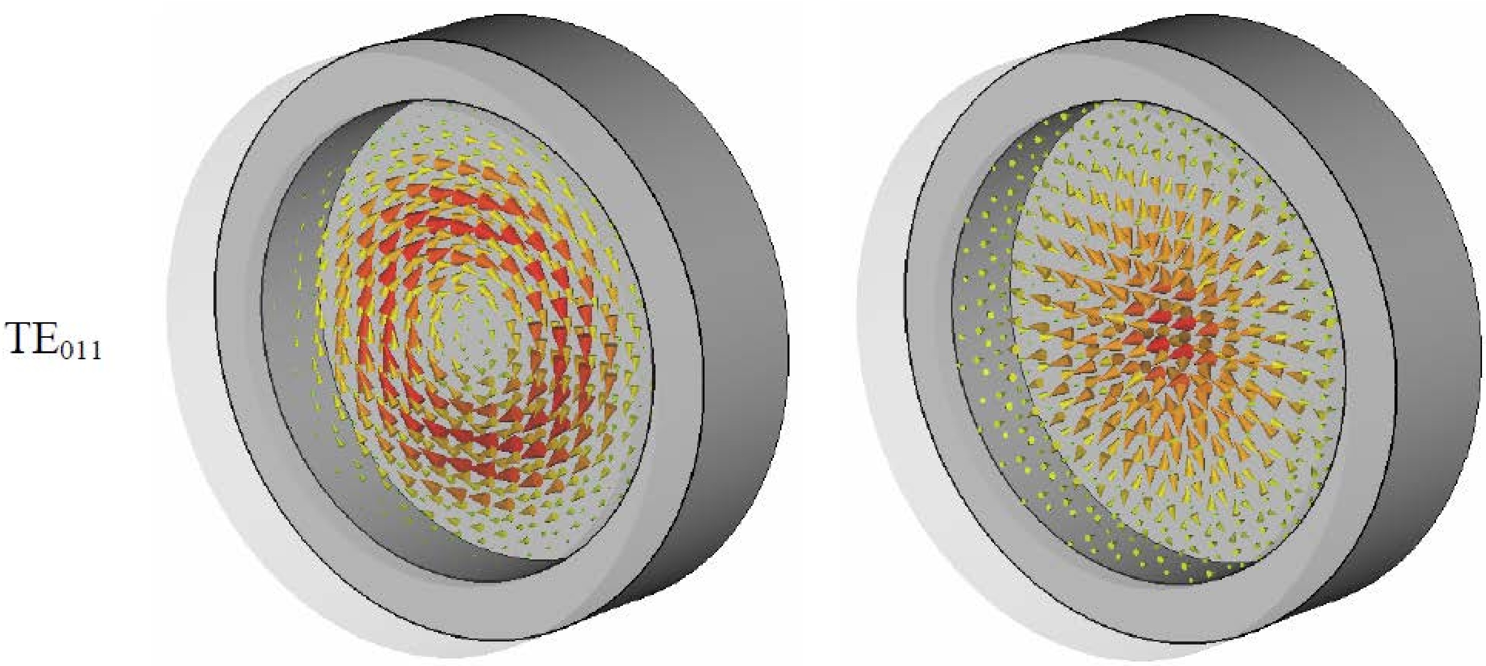
Cavity Modes: TE

and longitudinal:



electric field

magnetic field

General Findings and Set-Up

TM$_{mnp}$ and TE$_{mnp}$ modes, on-axis fields = 0 for *m* > 1!!!

odd longitudinal *p*:

$$\vec{B}_\perp(z,t) = \vec{B}_\perp^0 \cdot \sin\left(\frac{p\pi z}{L}\right) \cdot \cos(\omega t + \phi)$$

$$B_z(z,t) = B_z^0 \cdot \cos\left(\frac{p\pi z}{L}\right) \cdot \cos(\omega t + \phi)$$

$$\vec{E}_\perp(z,t) = \vec{E}_\perp^0 \cdot \cos\left(\frac{p\pi z}{L}\right) \cdot \sin(\omega t + \phi)$$

$$E_z(z,t) = E_z^0 \cdot \sin\left(\frac{p\pi z}{L}\right) \cdot \sin(\omega t + \phi)$$

even longitudinal *p*:

$$\vec{B}_\perp(z,t) = \vec{B}_\perp^0 \cdot \cos\left(\frac{p\pi z}{L}\right) \cdot \cos(\omega t + \phi)$$

$$B_z(z,t) = B_z^0 \cdot \sin\left(\frac{p\pi z}{L}\right) \cdot \cos(\omega t + \phi)$$

$$\vec{E}_\perp(z,t) = \vec{E}_\perp^0 \cdot \sin\left(\frac{p\pi z}{L}\right) \cdot \sin(\omega t + \phi)$$

$$E_z(z,t) = E_z^0 \cdot \cos\left(\frac{p\pi z}{L}\right) \cdot \sin(\omega t + \phi)$$

Origin of coordinate system at the center of the cavity!

Single Particle Energy Transfer

Integration of the Stern-Gerlach force:

- **odd longitudinal *p*:**

$$\Delta U_{\perp} = \frac{-2\cos\phi}{1-\left(\beta/\beta_{ph}\right)^2}\sin\left(\frac{p\pi}{2}\right)\cos\left(\frac{p\pi\beta_{ph}}{2\beta}\right)\vec{\mu}\bullet\left\{\xi_B\vec{B}_{\perp}^{0}\cdot+\xi_E\frac{\beta}{\beta_{ph}}\left(\hat{e}_z\times\frac{\beta}{c}\vec{E}_{\perp}^{0}\right)\right\}$$

$$\Delta U_{\|} = -\frac{2}{\gamma}\mu_z B_z^0\frac{\sin\phi}{1-\left(\beta/\beta_{ph}\right)^2}\frac{\beta}{\beta_{ph}}\sin\left(\frac{p\pi}{2}\right)\cos\left(\frac{p\pi\beta_{ph}}{2\beta}\right)$$

- **even longitudinal *p*:**

$$\Delta U_{\perp} = \frac{2\sin\phi}{1-\left(\beta/\beta_{ph}\right)^2}\cos\left(\frac{p\pi}{2}\right)\sin\left(\frac{p\pi\beta_{ph}}{2\beta}\right)\vec{\mu}\bullet\left\{\xi_B\vec{B}_{\perp}^{0}-\xi_E\frac{\beta}{\beta_{ph}}\left(\hat{e}_z\times\frac{\beta}{c}\vec{E}_{\perp}^{0}\right)\right\}$$

$$\Delta U_{\|} = \frac{2}{\gamma}\mu_z B_z^0\frac{\cos\phi}{1-\left(\beta/\beta_{ph}\right)^2}\frac{\beta}{\beta_{ph}}\cos\left(\frac{p\pi}{2}\right)\sin\left(\frac{p\pi\beta_{ph}}{2\beta}\right)$$

Signal Power

Energy transfer: $P_+ = \frac{I}{e}\cdot\Delta U$ **, bunch factor:** $\eta_b = \int \rho(s)\cdot\cos\left(\frac{\omega s}{\beta c}\right)\cdot ds$

Stored energy: $W_C = \frac{1}{2\mu_0}\int_V B^2 dV = \frac{1}{2\varepsilon_0}\int_V E^2 dV = \upsilon_b\cdot B_0^2 = \upsilon_e\cdot E_0^2$

→ **Energy transfer:** $dW_C = P_+\cdot dt = \frac{I}{e}\cdot\eta_b\cdot\Delta U\cdot dt = \frac{I}{e}\cdot\eta_b\cdot s_\mu\cdot B_0\cdot dt = \varsigma\cdot\sqrt{W_C}\cdot dt$

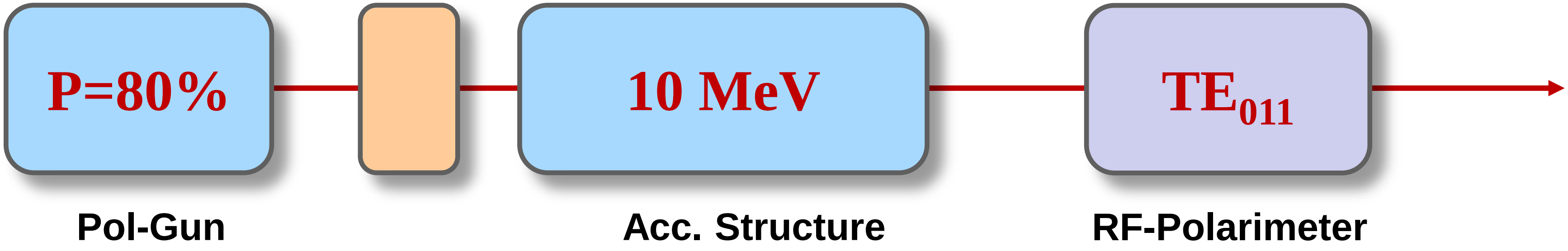
Energy dissipation: $P_- = \frac{\omega}{Q_l}\cdot W_C = \frac{1+\kappa}{Q_0}\cdot\omega\cdot W_C = \frac{1}{\tau}\cdot W_C$

Build-up of stored energy: $\frac{d}{dt}W_C = \varsigma\cdot\sqrt{W_C} - \frac{1}{\tau}\cdot W_C \quad\rightarrow\quad W_C(t) = \left(\varsigma\tau\right)^2\cdot\left(1-e^{-\frac{t}{2\tau}}\right)$

Steady state conditions: $W_C^\infty = \left(\varsigma\tau\right)^2 = \frac{I^2\cdot\eta_b^2\cdot s_\mu^2}{e^2\cdot\upsilon}\cdot\frac{Q_0^2}{\left(1+\kappa\right)^2}\cdot\frac{1}{\omega^2}$

Signal Power: $P_S = \kappa\cdot P_-^C = \kappa\cdot\frac{\omega\cdot W_C}{Q_0} = \frac{I^2\cdot\eta_b^2\cdot s_\mu^2}{e^2\cdot\upsilon}\cdot\frac{\kappa}{\left(1+\kappa\right)^2}\cdot\frac{Q_0}{\omega}$

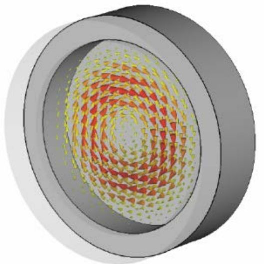
Experiment @ JLAB:

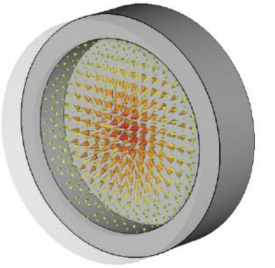


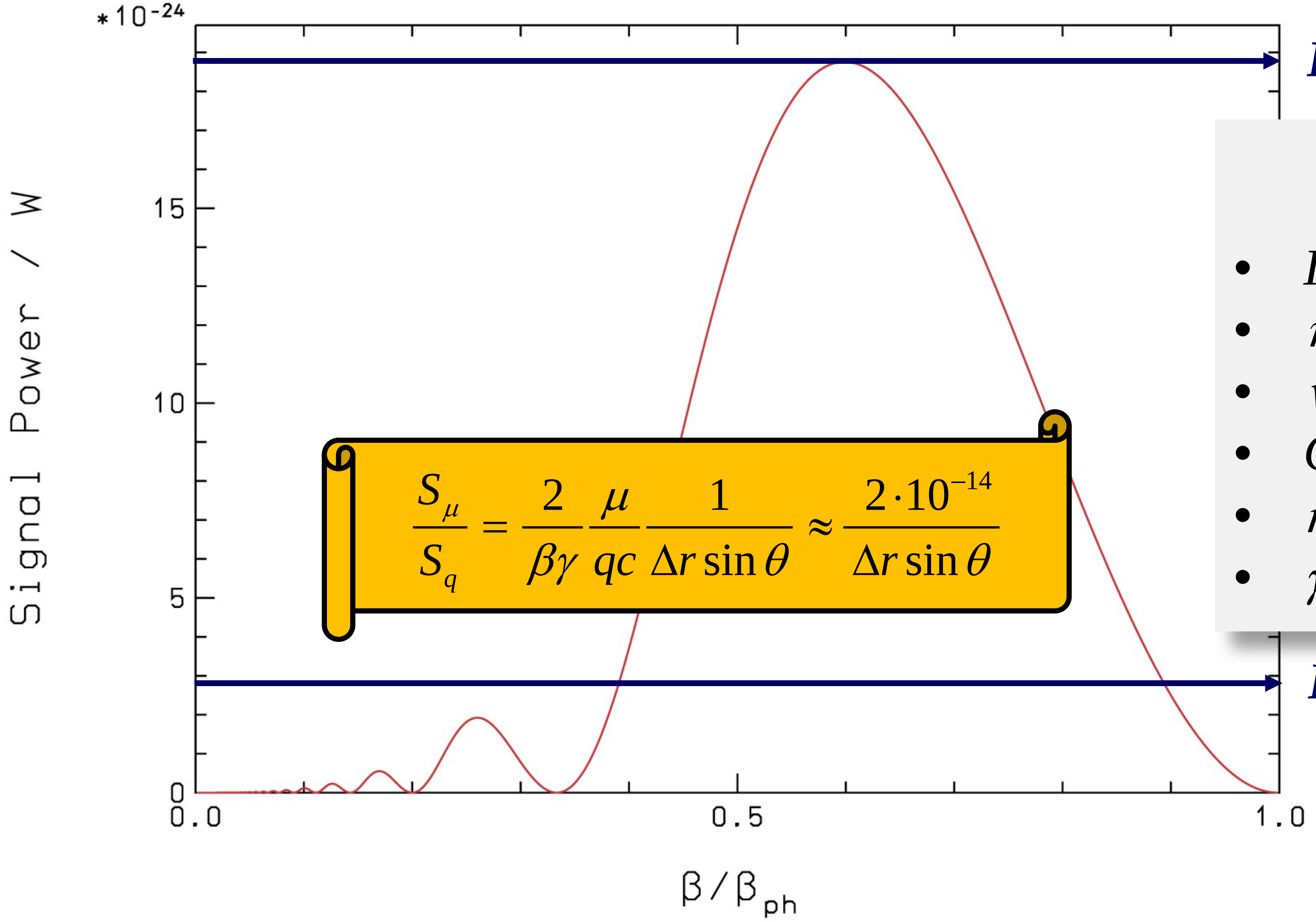
PoP Test at the injector:

- Longitudinal polarisation ↔ long. magn. field
- Low Lorentz gamma
- Flip helicity with Pockels cell
- Tune cavity to bunch repetition frequency
- Use TE mode with no long. electric fields
- Phase locking of polarimeter signal to RF

Longitudinal: TE_{011}







P_{noise} @ 0.6 Hz Bwdth

P_{noise} @ 0.1 Hz Bwdth

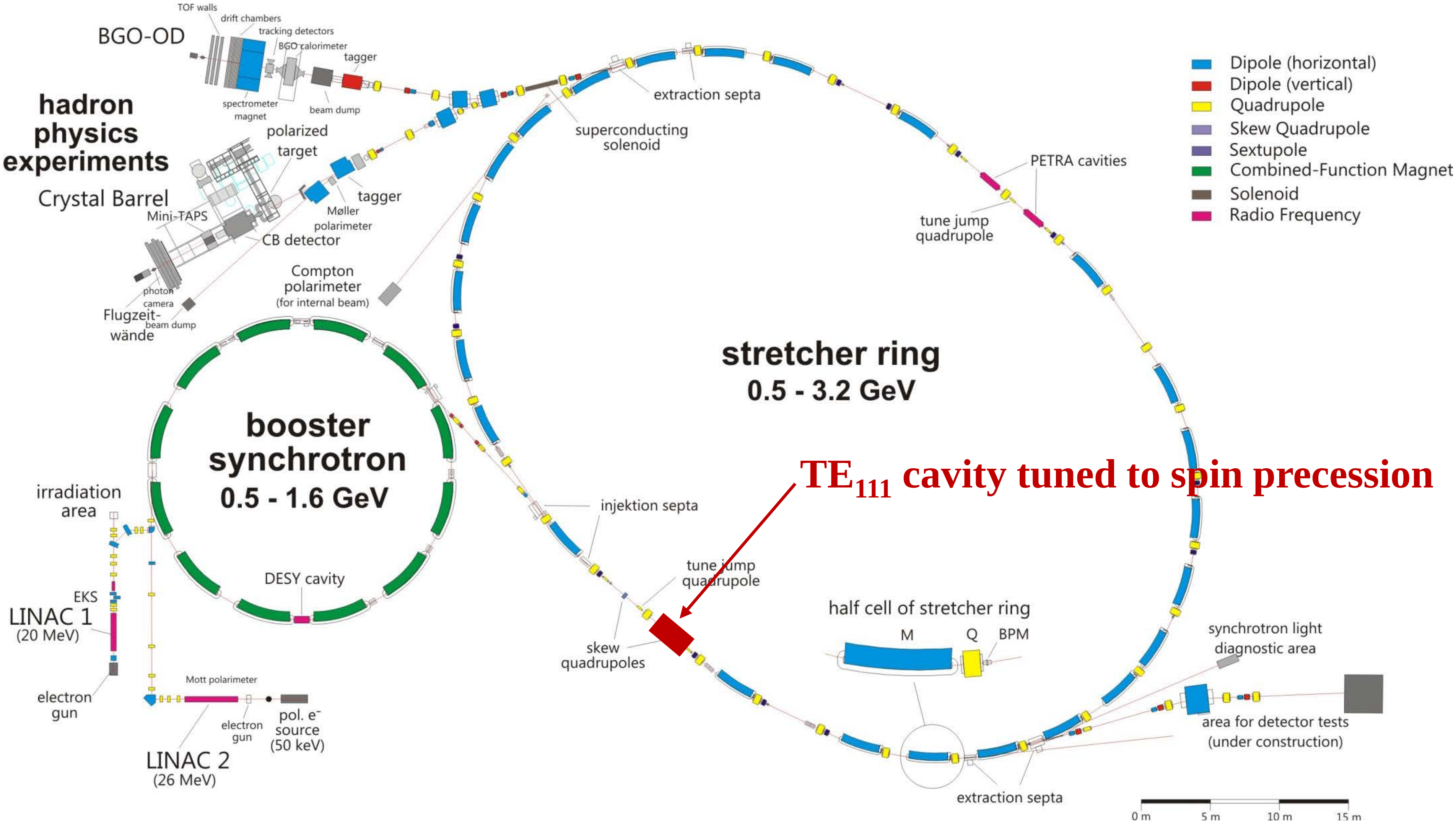
$$\frac{S_\mu}{S_q} = \frac{2}{\beta\gamma}\,\frac{\mu}{qc}\,\frac{1}{\Delta r\sin\theta} \approx \frac{2\cdot 10^{-14}}{\Delta r\sin\theta}$$

Parameters:

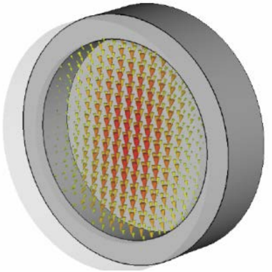
- I_e = 100 μA
- η_b = 0.7
- ν_C = 1.5 GHz
- Q_0 = 10^{10}
- κ = 10
- γ = 20

Expected Signal Power: $P_S = \left(\frac{I\cdot\eta_b}{e}\right)^2 \cdot \frac{16\mu_0\mu_e^2}{\pi^2 c^3} \cdot \frac{f\left(\beta_{ph}\right)}{F\left(j_{11}\right)} \cdot \frac{\kappa Q_0}{\left(1+\kappa\right)^2} \cdot \left(\frac{\omega_C}{\gamma}\right)^2$

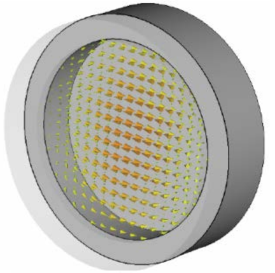
Experiment @ ELSA

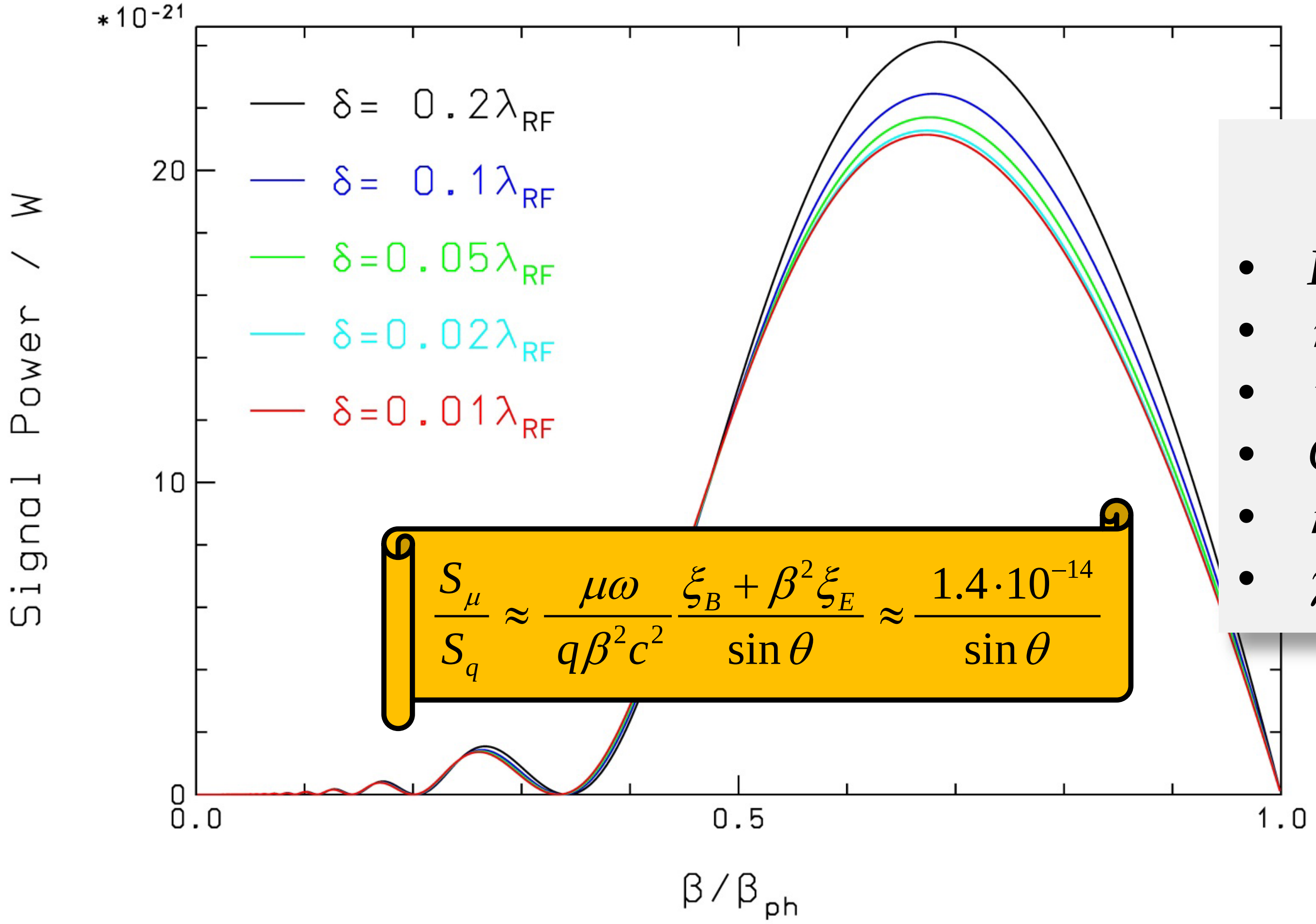


TE_{111} cavity tuned to spin precession



Transverse: TE_{111}





$$\frac{S_\mu}{S_q} \approx \frac{\mu\omega}{q\beta^2c^2}\frac{\xi_B+\beta^2\xi_E}{\sin\theta} \approx \frac{1.4\cdot10^{-14}}{\sin\theta}$$

Parameters:

- I_e = 50 mA
- η_b = 0.7
- ν_C = 1.5 GHz
- Q_0 = 10^{10}
- κ = 10
- γ = 2000

Expected Signal Power: $P_S \approx \left(\frac{I\cdot\eta_b}{e}\right)^2\cdot\frac{32\mu_0\mu_e^2}{\pi^2c^3}\cdot\frac{f\left(\beta_{ph}\right)}{F\left(j'_{11}\right)}\cdot\frac{\kappa Q_0}{\left(1+\kappa\right)^2}\cdot\left(G\cdot\omega_C\right)^2$

Conclusions

- Expected signal power is extremely low!
- sc cavities ($Q_0 \approx 10^{10}$) with weak coupling essential!
- Phase-lock techniques required
- Coupling to charge is about 14 orders of magnitude greater!

PoP will be a really hard task but doable?!

LIGO demonstrated: ultimate precision can be achieved!

Stern-Gerlach

May the force be with us!