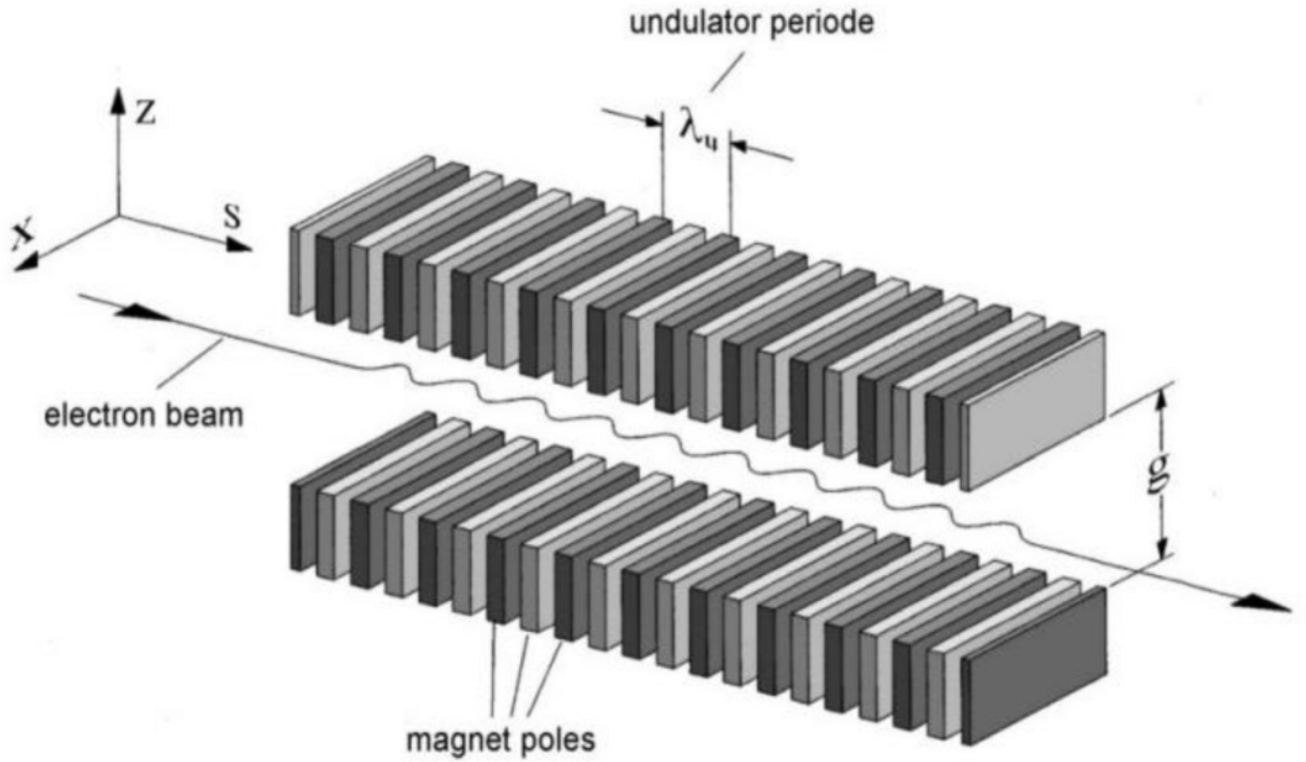
11. Free Electron Laser (FEL)

In this chapter a short insight into the mode of operation of FELs will be delivered. We will pay particular attention to the descriptive comprehension of the processes, especially compared to the mode of operation of conventional lasers using the here „well-known“ formalism (population inversion etc.). For this purpose a quantum mechanical description of coherent states in the formalism of the Glauber theory is advantageous.

11.1. Introductive survey

11.1.1. Undulators

Let us consider a planar undulator, consisting of an arrangement of small dipole magnets with alternating magnetic fields:



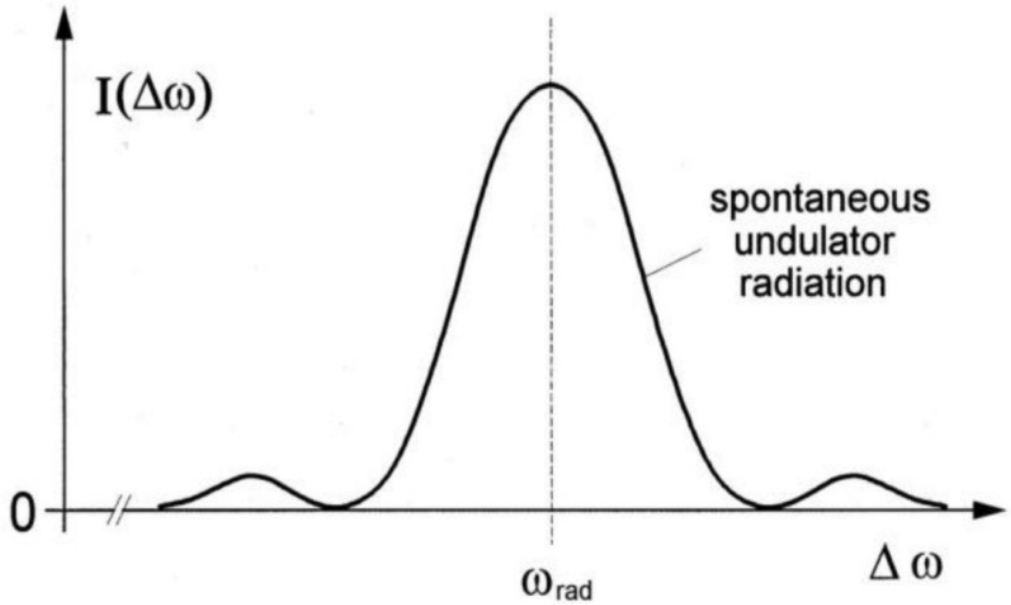
The spectrum of the radiation consists of discrete lines (harmonics) whose line widths are given by the emission period T of the undulator radiation with the wave length λ and the frequency ω observed in the laboratory frame:

$$T = \frac{N_u \cdot \lambda}{c} = 2\pi \frac{N_u}{\omega}.$$

One obtains the intensity spectrum in the center of mass frame of the electrons using the Fourier transformation

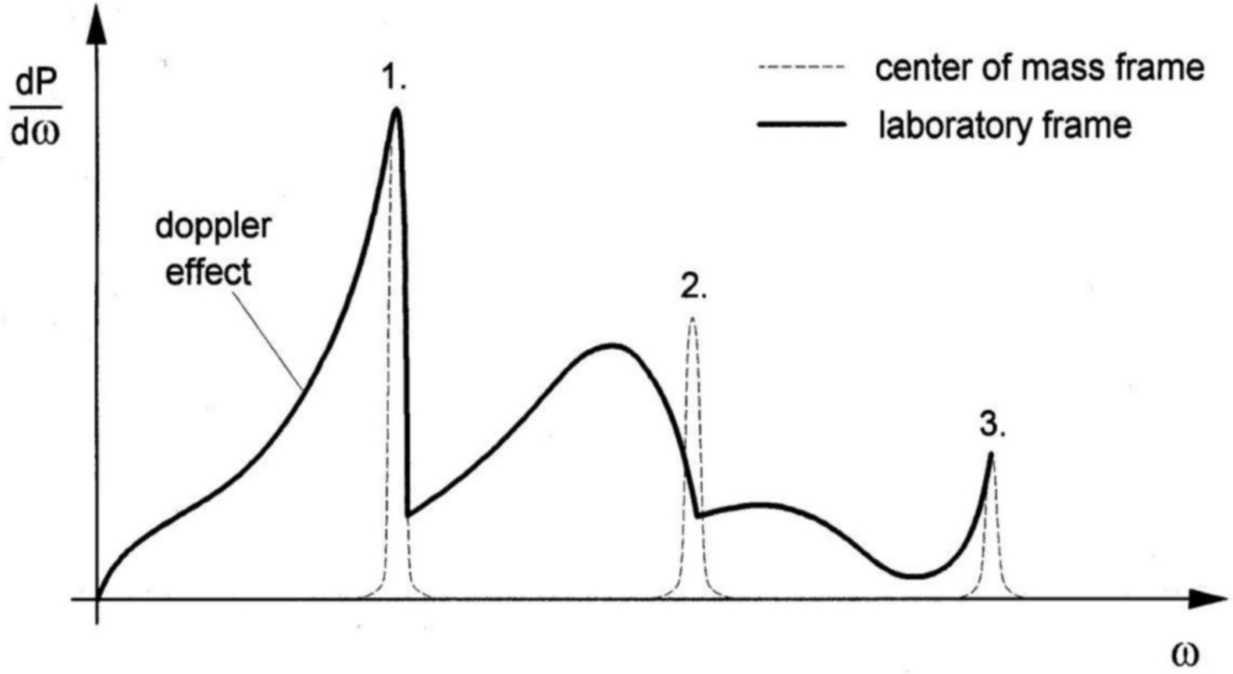
$$I(\omega) = \left|E(\omega)\right|^2 = \left|\frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} \left[E_0 \cdot e^{i\omega_u t}\right] \cdot e^{-i\omega t} \cdot dt\right|^2 = \left|\frac{E_0 \cdot T}{\sqrt{2\pi}} \cdot \frac{\sin\left(\Delta\omega \cdot T/2\right)}{\Delta\omega \cdot T/2}\right|^2$$

and it displays the well-known behaviour:



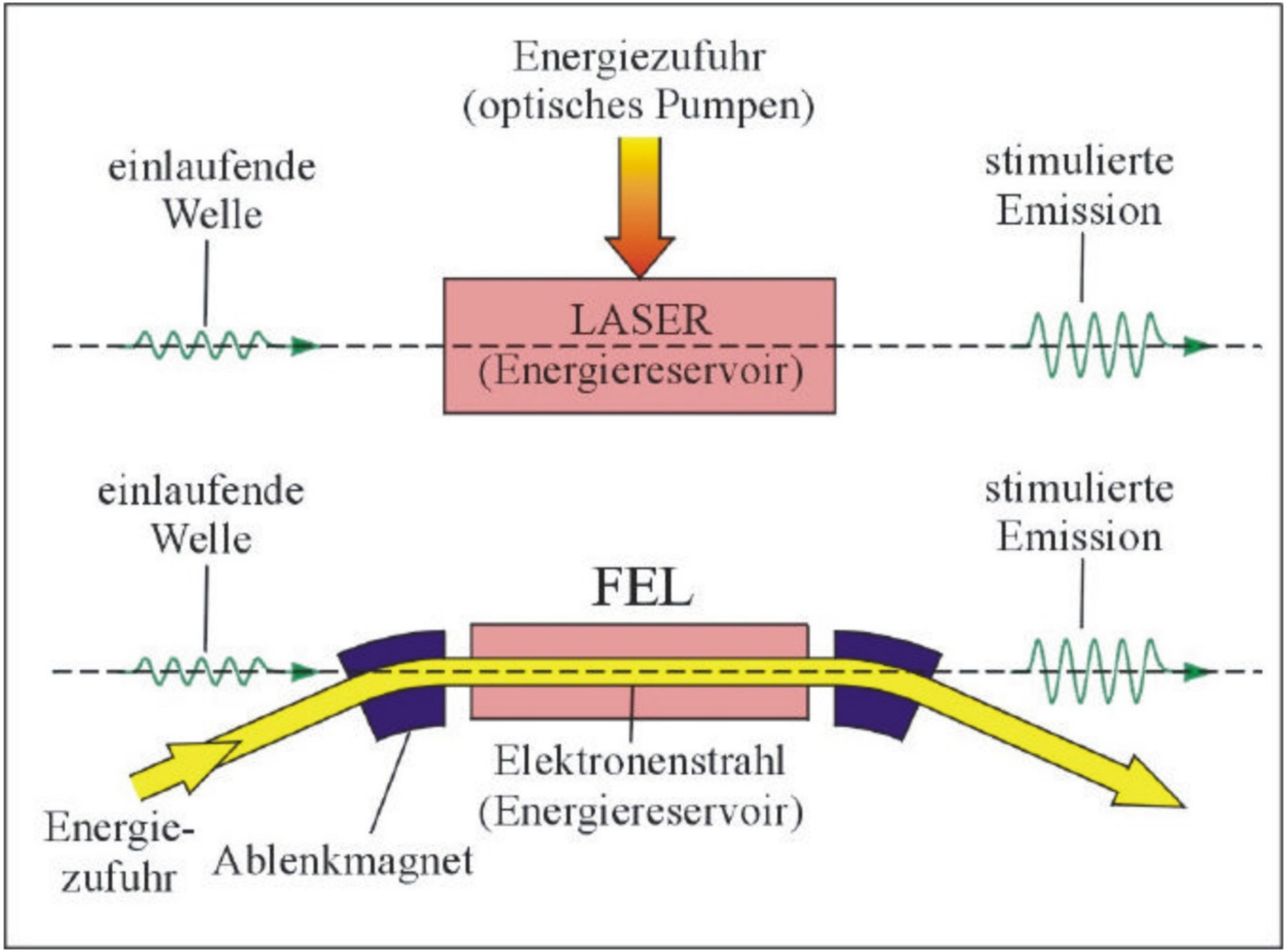
The full width half maximum will be $\Delta\omega \approx \omega / N_U$.

One gets for the spectral description of the emitted power via a Lorentz boost into the laboratory frame:

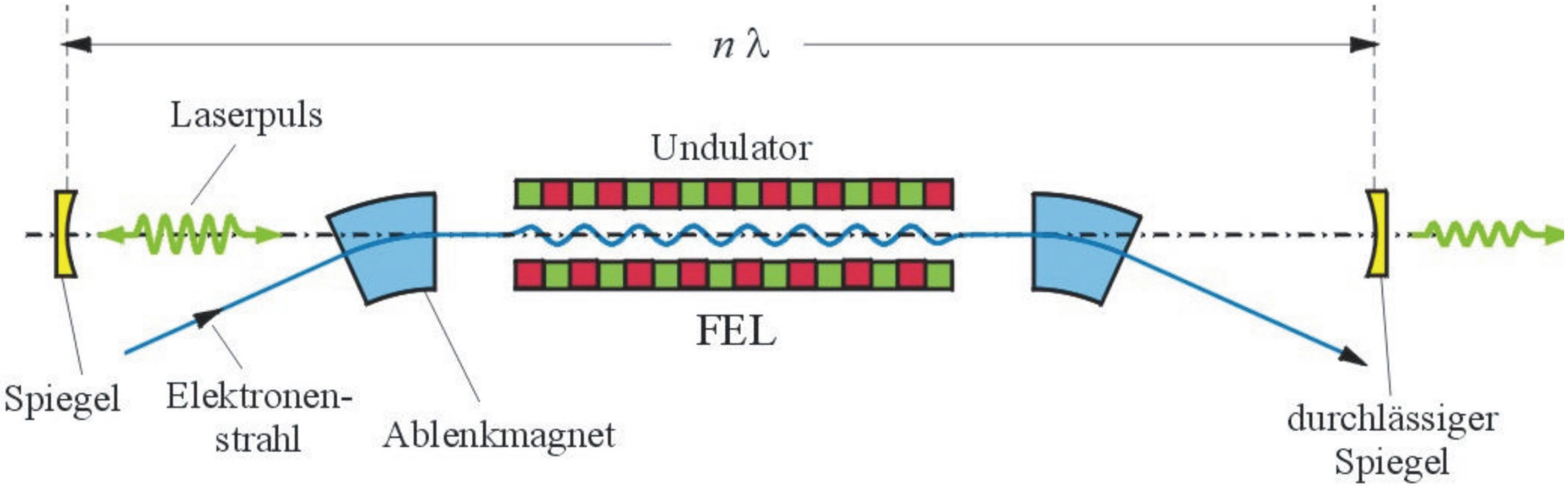


11.1.2. The FEL principle (classical)

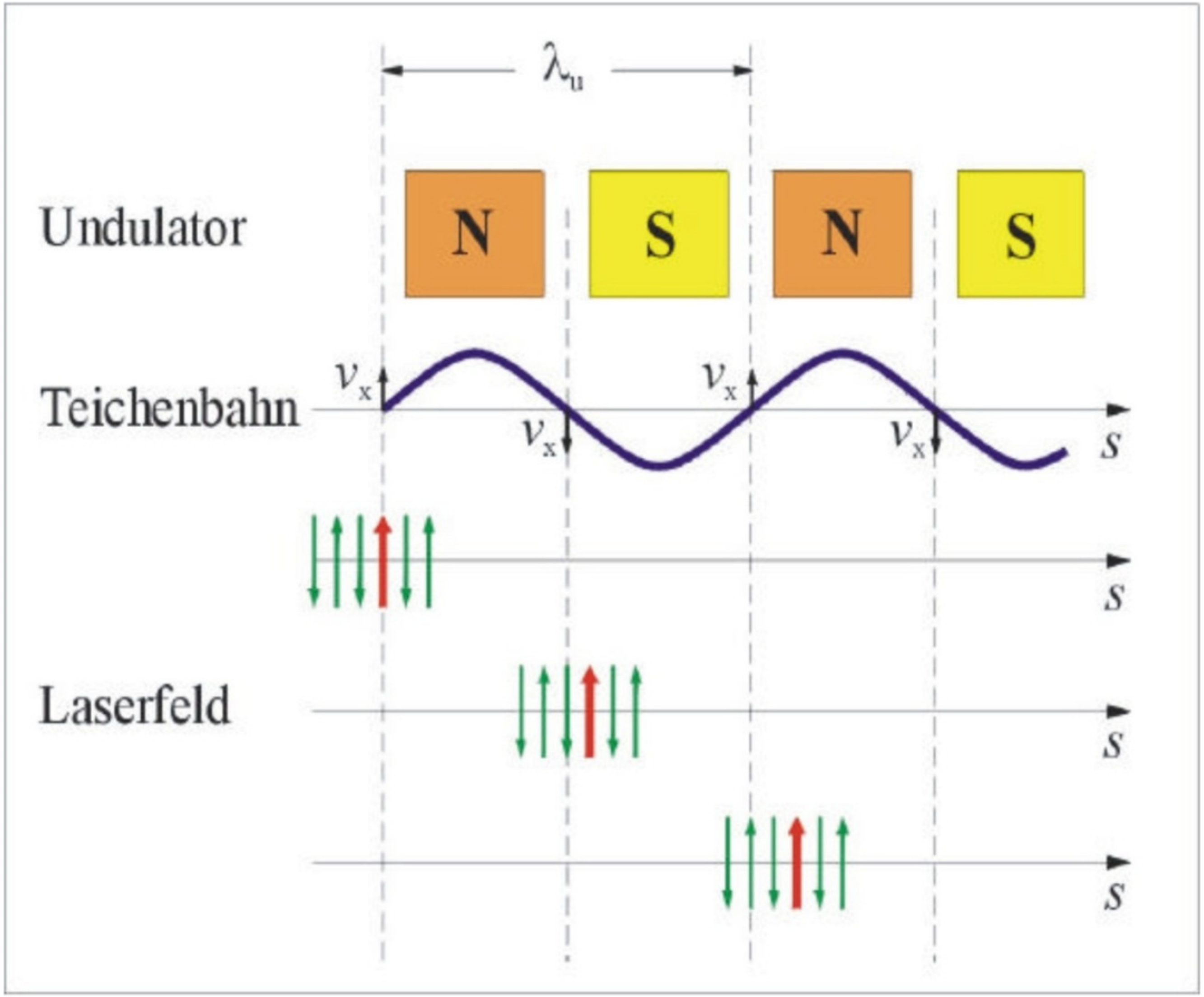
As in the case of a conventional laser the stimulated emission shall be utilized. We need a radiation field yet which has to be amplified and we have – totally analogous to the optical laser – the following set-up:


Energiezufuhr
(optisches Pumpen)
einlaufende
Welle
stimulierte
Emission
LASER
(Energiereservoir)
einlaufende
Welle
stimulierte
Emission
FEL
Elektronenstrahl
(Energiereservoir)
Energie-
zufuhr
Ablenkmagnet

Therefore a "conventional" FEL still needs an optical resonator:



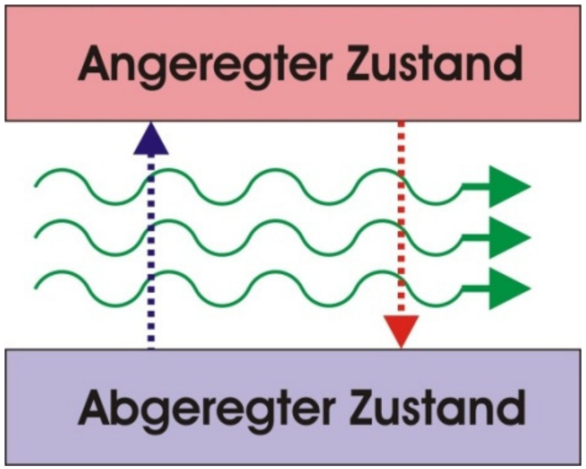
From a classical point of view, the electrons sustain an additional energy loss in the electric field of the incoming radiation:



Certain frequencies of the laser field are needed for the correct phase shift of λ_L per undulator period. Hereby the line spectrum of the emission is determined (so-called coherence condition). The relative phasing to the laser field decides if energy from the beam is transferred into the radiation field or vice versa. Hence "micro bunches" being shorter than half the laser wave length must be generated for the laser emission.

11.1.3. The FEL principle (quantum mechanical)

For an atom or a molecule laser the emission induced by the radiation field is exploited. Normally a 2-level-system is investigated, consisting of an excited and a ground state:



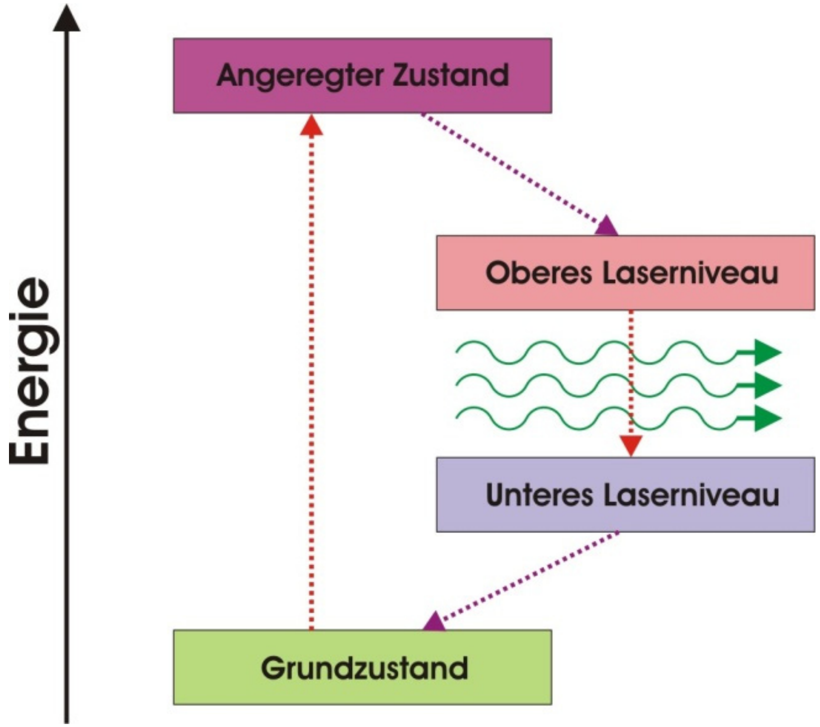
The factor n of the Bose statistics is utilized at the transition (see next chapter). However this holds true for both directions:

$$|\mathbf{M}_{\text{if}}|^2 = \left|\langle f|\hat{O}|i\rangle\right|^2 = \left|\langle f|-\frac{e}{mc}\vec{A}(r,t)\cdot\vec{P}|i\rangle\right|^2 \sim |\alpha|^2$$

- emission: $\sim N_{\text{i}}$ (population number of the upper state)
- absorption: $\sim N_{\text{f}}$ (population number of the lower state)

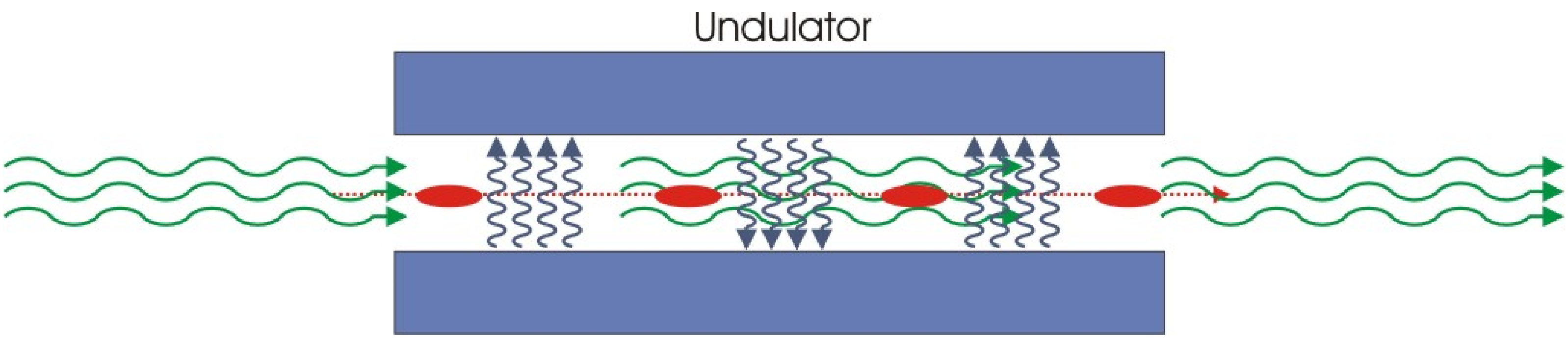
$\Rightarrow$ **Population inversion necessary!**

At an optical laser, this is often achieved by a 4-level-system:



The simplest set-up consists of an active medium surrounded by an optical resonator:

We have a 2-level-system at the FEL. The de-excited state is the undulator field, the excited state the laser field:



The transition persists by Compton scattering off the according photons:

1. **emission:**

2. absorption:

Population inversion is produced by phasing!!!

The status of the electrons relatively to the laser field phase is essential!

So called micro bunching ($\Delta l \approx \lambda_{Laser}$) is necessary!

This is produced in the FEL so far!

If all electrons are involved in the laser process, in the amplification formula one gets the „duplex *N*²" ($N_e^2 \cdot N_\gamma^2$) as a factor!

11.2. Some Glauber theory

In the following we try to get grip on the idea of a coherent photon radiation in the quantum mechanical way. Classically, coherence is linked with the availability of a fixed phase relation. But what is the ”phase“ of a photon actually?

11.2.1. The electric field in 2nd quantization

Firstly we describe the electric field strength as a Fourier integral:

$$E(\vec{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\vec{r},\omega) \cdot e^{-i\omega t} d\omega$$

A splitting into parts of positive and negative frequencies passes to:

$$E^{(+)}(\vec{r},t) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} A(\vec{r},\omega) \cdot e^{-i\omega t} d\omega$$

and since E is a real number, it applies $A(\vec{r},-\omega) = A^*(\vec{r},\omega)$ and therewith

$$E^{(-)}(\vec{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} A(\vec{r},\omega) \cdot e^{-i\omega t} d\omega = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} A^{*}(\vec{r},\omega) \cdot e^{i\omega t} d\omega = E^{(+)*}(\vec{r},t) .$$

We write for a monochromatic wave:

$$E(\vec{r},t) = A(\vec{r}) \cdot e^{-i\omega t} + A^{*}(\vec{r}) \cdot e^{i\omega t}$$

This can be translated quantum mechanically into

$$\hat{E}(\vec{r},t) = \hat{E}^{(+)}(\vec{r},t) + \hat{E}^{(-)}(\vec{r},t) = \underline{\underline{\varepsilon(\vec{r}) \cdot e^{-i\omega t} \cdot \hat{a} + \varepsilon^{*}(\vec{r}) \cdot e^{i\omega t} \cdot \hat{a}^{\dagger}}} ,$$

whereas we have split the factor $\varepsilon(\vec{r})$ at the spatial part and we have expressed the amplitude of the oscillation by the creation and annihilation operators $\hat{a}$ and $\hat{a}^{\dagger}$. These operators act on states in the Fock space. We have already got to know them at the harmonic oscillator:

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}}\left\{\sqrt{m\omega}\hat{X} - \frac{i}{\sqrt{m\omega}}\hat{P}\right\}$$

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}\left\{\sqrt{m\omega}\hat{X} + \frac{i}{\sqrt{m\omega}}\hat{P}\right\}$$

Application of the operators „→“ creation / annihilation of photons:

$$\hat{a}^\dagger \left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$$

$$\hat{a} \left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle$$

One gets the total energy of the system by application of

$$\hat{H} = \hbar\omega\left\{\hat{a}^\dagger\hat{a} + \frac{1}{2}\cdot\mathbf{1}\right\},$$

what leads us to the following definition of the particle number operator:

$$\hat{N} = \hat{a}^\dagger\hat{a}$$

In the case of a linearly polarized wave we have:

$$\hat{E}(\vec{r},t) = E_0\left\{e^{i(\vec{k}\cdot\vec{r}-\omega t)}\hat{a} + e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\hat{a}^{\dagger}\right\}.$$

In the real representation, we obtain via Euler's formula

$$\hat{E}(\vec{r},t) = \frac{1}{\sqrt{2}}E_0\left\{\hat{x}\cdot\cos(\omega t-\vec{k}\cdot\vec{r}) + \hat{p}\cdot\sin(\omega t-\vec{k}\cdot\vec{r})\right\}.$$

The new operators $\hat{x}$ und $\hat{p}$ correspond to the real resp. imaginary part of the classical complex amplitude A and can be identified as position and momentum of a one-mode-field by comparing with a harmonic oscillator:

$$\hat{x} = \frac{1}{\sqrt{2}}\left(\hat{a}^{\dagger}+\hat{a}\right) = \sqrt{\frac{m\omega}{\hbar}}\cdot\hat{X}$$

$$\hat{p} = \frac{i}{\sqrt{2}}\left(\hat{a}^{\dagger}-\hat{a}\right) = \sqrt{m\omega\hbar}\cdot\hat{P}$$

Thus we have separated the electric field strength into one part oscillating in phase and into another part oscillating not in phase. For a general phase Θ it follows with

$$\hat{E}(\vec{r},t) = \frac{1}{\sqrt{2}} E_0 \left\{ \hat{x}_\Theta \cdot \cos(\omega t - \vec{k}\cdot\vec{r} - \Theta) + \hat{p}_\Theta \cdot \sin(\omega t - \vec{k}\cdot\vec{r} - \Theta) \right\}$$

for the operators

$$\hat{x}_\Theta = \frac{1}{\sqrt{2}}\left(e^{i\Theta}\cdot\hat{a}^\dagger + e^{-i\Theta}\cdot\hat{a}\right)$$

$$\hat{p}_\Theta = \frac{i}{\sqrt{2}}\left(e^{i\Theta}\cdot\hat{a}^\dagger - e^{-i\Theta}\cdot\hat{a}\right)$$

and also

$$\begin{pmatrix} \hat{x}_\Theta \\ \hat{p}_\Theta \end{pmatrix} = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix}$$

For completeness: The commutation relations apply

$$\left[\hat{a},\hat{a}^\dagger\right] = \mathbf{1} \qquad \text{and} \qquad \left[\hat{x},\hat{p}\right] = i\mathbf{1}\,.$$

11.2.2. Coherent states

Using the creation and the annihilation operators we can define states with known ("sharp") photon number. We begin with a system of 2 identical bosons in the states 1 and 2. The total wave function has to be symmetric for permutation of the particles, therefore

$$|2\rangle = \frac{1}{\sqrt{2}}\{\Psi_1(1)\cdot\Psi_2(2)+\Psi_1(2)\cdot\Psi_2(1)\}$$

One has to sum up over all permutations for n particles and we obtain as correct ($\langle n|n\rangle = 1$ for distinguishable particles) normalized wave function:

$$|n\rangle = \frac{1}{\sqrt{n!}}\left\{\sum_{\text{permutations}}\Psi_1(1)\cdot\Psi_2(2)\cdots\Psi_n(n)\right\}$$

The creation resp. annihilation of photons transforms a wave function $|n\rangle$ normalized to 1 to a wave function $|n+1\rangle$ resp. $|n-1\rangle$ normalized to 1 as well. We obtain automatically

$$|n\rangle = \frac{1}{\sqrt{n!}} \cdot \left(\hat{a}^\dagger\right)^n \cdot |0\rangle \qquad \text{and} \qquad \begin{aligned} \hat{a}|n\rangle &= \sqrt{n} \cdot |n-1\rangle \\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1} \cdot |n+1\rangle \end{aligned}$$

So the probability to find a system of ***n*** identical bosons in the same quantum state is increased by a factor of ***n*** than in a system of ***n*** distinguishable bosons:

undistingishable particles	distinguishable particles
$\frac{1}{\sqrt{n!}} \left\| \sum_{P(k)} \prod_j \Psi_j(k) \right\rangle \overset{k=\xi\ \forall k}{=} \frac{1}{\sqrt{n!}} \cdot n \cdot \prod_j \Psi_j(\xi)$	$\|\Psi\rangle = \left\| \prod_j \Psi_j(j) \right\rangle \overset{\text{state } \xi}{=} \left\| \prod_j \Psi_j(\xi) \right\rangle$
$\|\Psi\|^2 = \langle \Psi \| \Psi \rangle = n \cdot \prod_j \|\Psi_j(\xi)\|^2$	$\|\Psi\|^2 = \langle \Psi \| \Psi \rangle = \prod_j \|\Psi_j(\xi)\|^2$

Let us apply this to electric fields. The expectation value of the field strength vanishes for such a sharp state since

$$\langle n-1|n\rangle = \langle n+1|n\rangle = 0\,.$$

Conclusion:

At a sharp state, the "phases of the individual photons" are distributed arbitrarily!

The „individual fields" cancel each other!

Expedient:

1. Claim for a **minimum medium fluctuation** of the electric field strength:

$$\overline{\Delta E^2(\vec{r},t)} = \overline{\left\langle \hat{E}^2(\vec{r},t)\right\rangle - \left\langle \hat{E}(\vec{r},t)\right\rangle^2} = \min$$

2. Claim for a **defined medium energy** (photon number):

$$N_\gamma = \left\langle \hat{N} \right\rangle = \left\langle \hat{a}^\dagger \hat{a} \right\rangle = \overline{N} .$$

We obtain for the fluctuation:

$$\overline{\Delta E^2(\vec{r},t)} = 2\left|\varepsilon(\vec{r})\right|^2 \left\{ \left\langle \hat{a}^\dagger \hat{a} \right\rangle - \left\langle \hat{a}^\dagger \right\rangle \left\langle \hat{a} \right\rangle + \frac{1}{2} \right\} = \min ,$$

which is minimized for a fixed $\left\langle \hat{a}^\dagger \hat{a} \right\rangle = \overline{N}$, if

$$\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle = \langle \hat{a} \rangle^* \langle \hat{a} \rangle = \left| \langle \hat{a} \rangle \right|^2 = \max .$$

Using the general ansatz for the searched state

$$|\Psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \qquad \text{with} \qquad \langle \Psi | \Psi \rangle = \sum_{n=0}^{\infty} |c_n|^2 = 1$$

we obtain for the expectation value of the operator

$$\langle \Psi | \hat{a} | \Psi \rangle = \sum_{n=0}^{\infty} c_n^* \cdot c_{n+1} \cdot \sqrt{n+1} .$$

According to Schwarz's inequality it applies

$$\left| \langle \Psi | \hat{a} | \Psi \rangle \right|^2 = \left| \sum_{n=0}^{\infty} c_n^* \cdot c_{n+1} \cdot \sqrt{n+1} \right|^2 \le \sum_{n=0}^{\infty} |c_n|^2 \cdot \sum_{n=0}^{\infty} |c_{n+1}|^2 (n+1) = \overline{N} ,$$

since it applies for a medium photon number

$$\overline{N} = \langle \Psi | \hat{a}^\dagger \hat{a} | \Psi \rangle = \sum_{n=0}^{\infty} n \cdot |c_n|^2 .$$

The inequality becomes an equality if both "vectors" (c_n) and $\left(\sqrt{n+1}\cdot c_{n+1}\right)$ are parallel. Then it holds for the components

$$\sqrt{n+1}\cdot c_{n+1} = \alpha\cdot c_n \qquad \Rightarrow \qquad \boxed{c_n = \frac{\alpha^n}{\sqrt{n!}}\cdot c_o}$$

with an arbitrary factor α. Including the normalization of c_n we obtain thereby for the searched state:

$$|\alpha\rangle = \sum_{n=0}^{\infty} e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} |n\rangle .$$

Such states are called Glauber states. They have not any sharp defined photon number!

It holds: $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle, \quad \langle\alpha|\,\hat{a}^\dagger = \alpha^*\langle\alpha|, \quad \langle\alpha|\,\hat{N}\,|\alpha\rangle = \langle\alpha|\,\hat{a}^\dagger\hat{a}\,|\alpha\rangle = |\alpha|^2$

$|\alpha\rangle$ **are not eigen states of any Hermitian operator!**

Application on the electric field strength results:

$$\left\langle \alpha \middle| \hat{E}(\vec{r},t) \middle| \alpha \right\rangle = \varepsilon(\vec{r})e^{-i\omega t}\alpha + \varepsilon^*(\vec{r})e^{i\omega t}\alpha^*$$

$$\left\langle \alpha \middle| \hat{E}^2(\vec{r},t) \middle| \alpha \right\rangle = \varepsilon^2(\vec{r})e^{-2i\omega t}\alpha^2 + \varepsilon^{*2}(\vec{r})e^{2i\omega t}\alpha^{*2} + 2\left|\varepsilon(\vec{r})\right|^2 \left\{ \left|\alpha\right|^2 + \frac{1}{2} \right\}$$

Except for the additional term $\left|\varepsilon(\vec{r})\right|^2$ describing the vacuum fluctuations, we obtain exactly the classical expressions. Particularly we read the following:

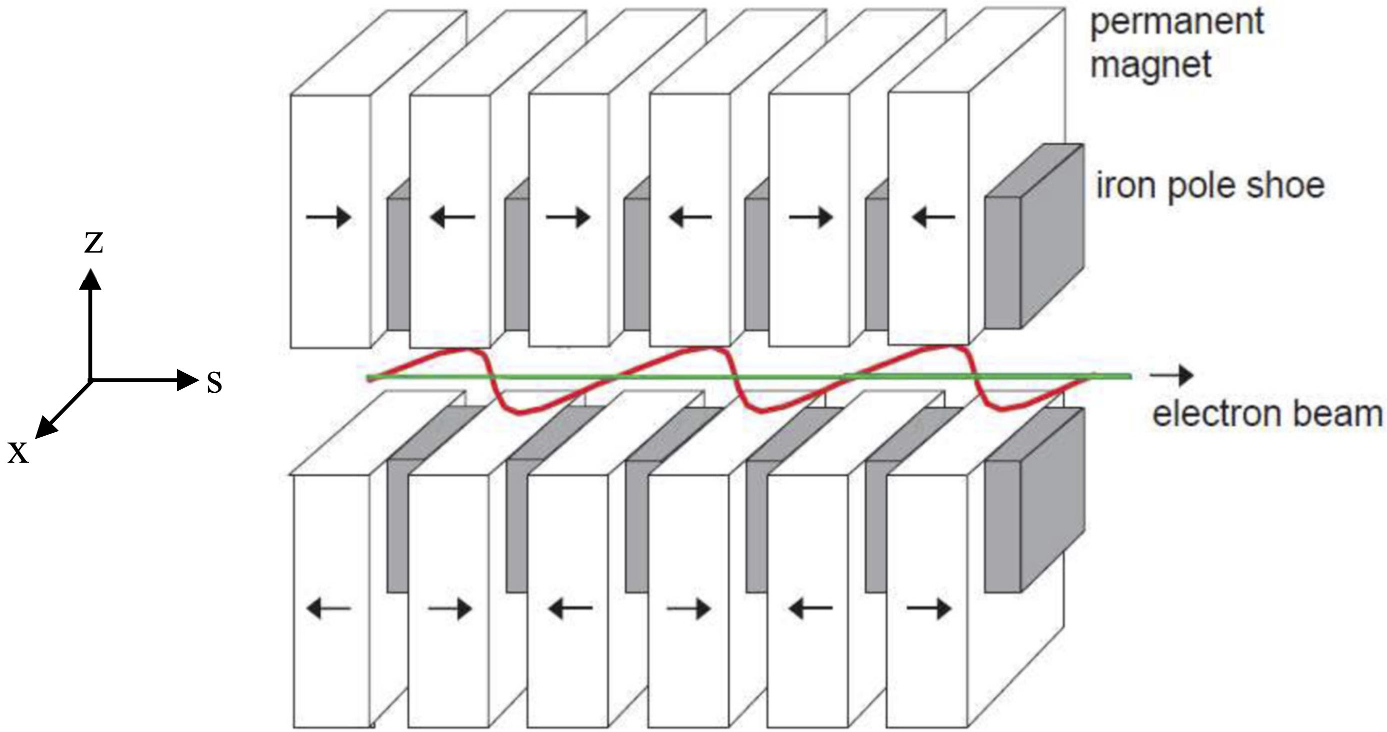
α corresponds to the classical complex amplitude of the wave!

11.3. Undulator radiation

In the following we deduce the coherence condition for the undulator radiation. We do this in the quantum mechanical as well as in the classical way. From the classical treatment we obtain additionally the trajectories of the electrons.

11.3.1. Magnetic field of a planar undulator

A planar undulator is characterized by the period λ_U of the magnet arrangement and the maximum magnetic field strength B_0 on its central axis, determined by the distance of the magnetic poles, the undulator gap. In case of a large width of the pole shoes we can neglect the horizontal dependence of the magnetic field along the electron's path.



In the undulator gap, we have $\vec{\nabla}\times\vec{B}=0$. As a consequence, we can express the fields as the gradient of a scalar potential with

$$\vec{B}=-\vec{\nabla}\Phi_{\text{mag}} \quad\rightarrow\quad \Delta\Phi_{\text{mag}}=0\,.$$

On the central axis, the field is to very good approximation harmonic. With the ansatz

$$\Phi_{\text{mag}}(z,s)=f(z)\cdot\sin(k_U s) \quad\rightarrow\quad f''-k_U^2 f=0$$

we get the general solution

$$f(z)=a\cdot\sinh(k_U z)+b\cdot\cosh(k_U z)$$

from which the vertical field can be derived by

$$B_z=\frac{\partial\Phi_{\text{mag}}}{\partial z}=-k_U\cdot\left\{a\cdot\cosh(k_U z)+b\cdot\sinh(k_U z)\right\}\cdot\sin(k_U s)\,.$$

The vertical field B_z has to be symmetric with respect to the mid plane $z = 0$ giving $b = 0$. We set $k_U a = B_0$ and obtain for the potential

$$\Phi_{\text{mag}}=\frac{B_0}{k_U}\sinh(k_U z)\cdot\sin(k_U s)$$

and for the fields:

$$B_x = 0$$
$$B_z = -B_0 \cdot \cosh(k_U z) \cdot \sin(k_U s)$$
$$B_s = -B_0 \cdot \sinh(k_U z) \cdot \cos(k_U s)$$

Note, that for $z \neq 0$ always a longitudinal field component is present! In the mid plane $z = 0$ we then have the idealized field

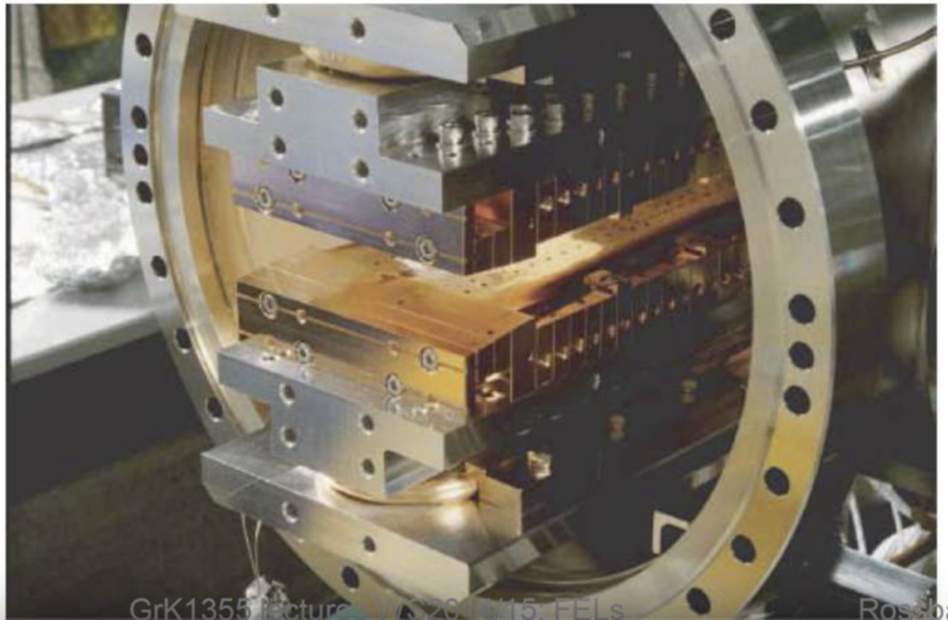
$$\vec{B} = -B_0 \cdot \sin(k_U s) \cdot \hat{e}_s$$

11.3.2. Undulator design concepts

Basically, there exist 3 different design concepts for undulators:

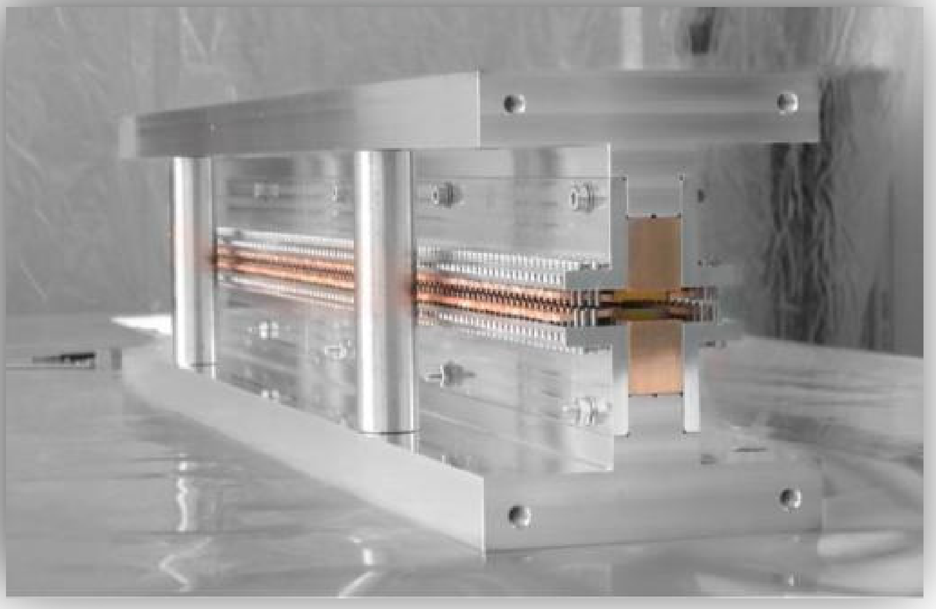
- permanent magnet outside the beam vacuum
- permanent magnet inside the vacuum
- superconducting magnet

Examples:

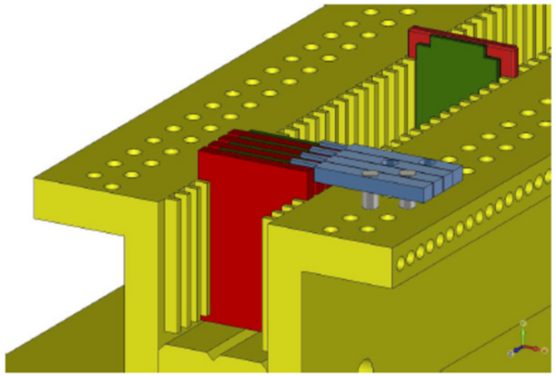


In-vacuum undulator at Spring-8
(Kitamura et al.)

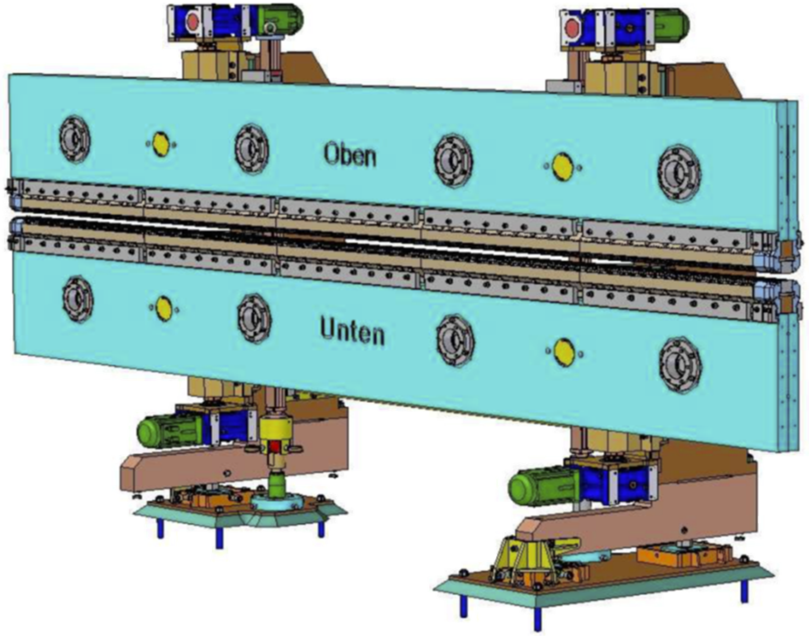
Magnet array covered with thin Cu sheet
for impedance reduction

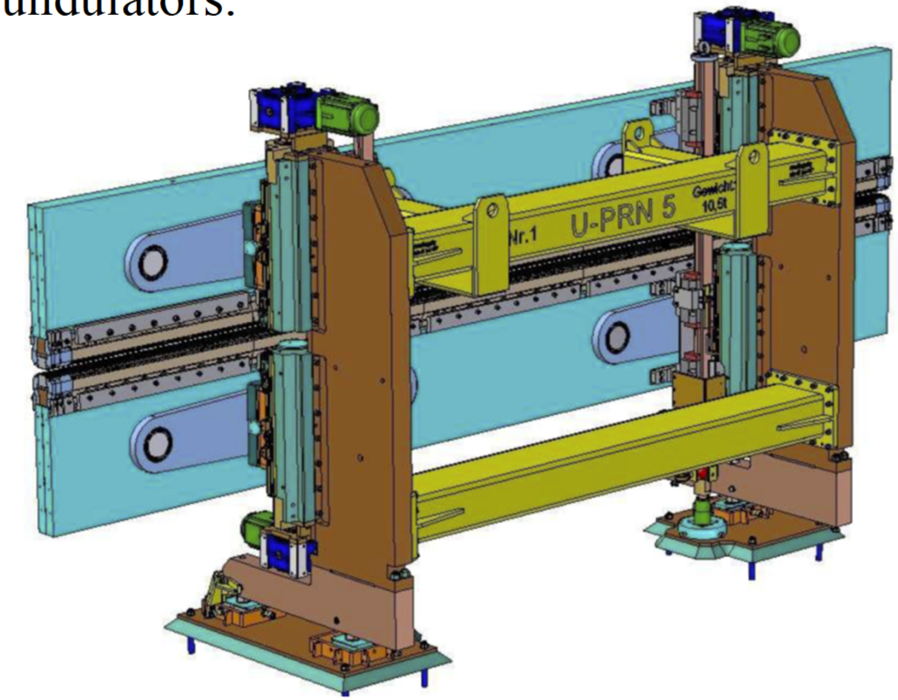


BEAST II for LUX
hybrid in-vacuum undulator
Gap = 2 mm, Period λ_U = 5, #=100



Variable gap size (micrometer precision!) undulators:



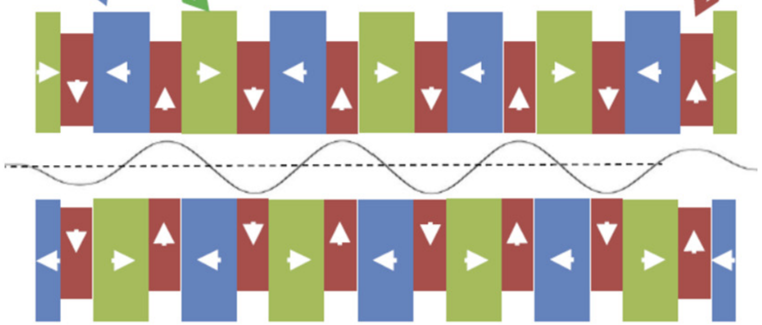


Permanent magnets: NdFeB

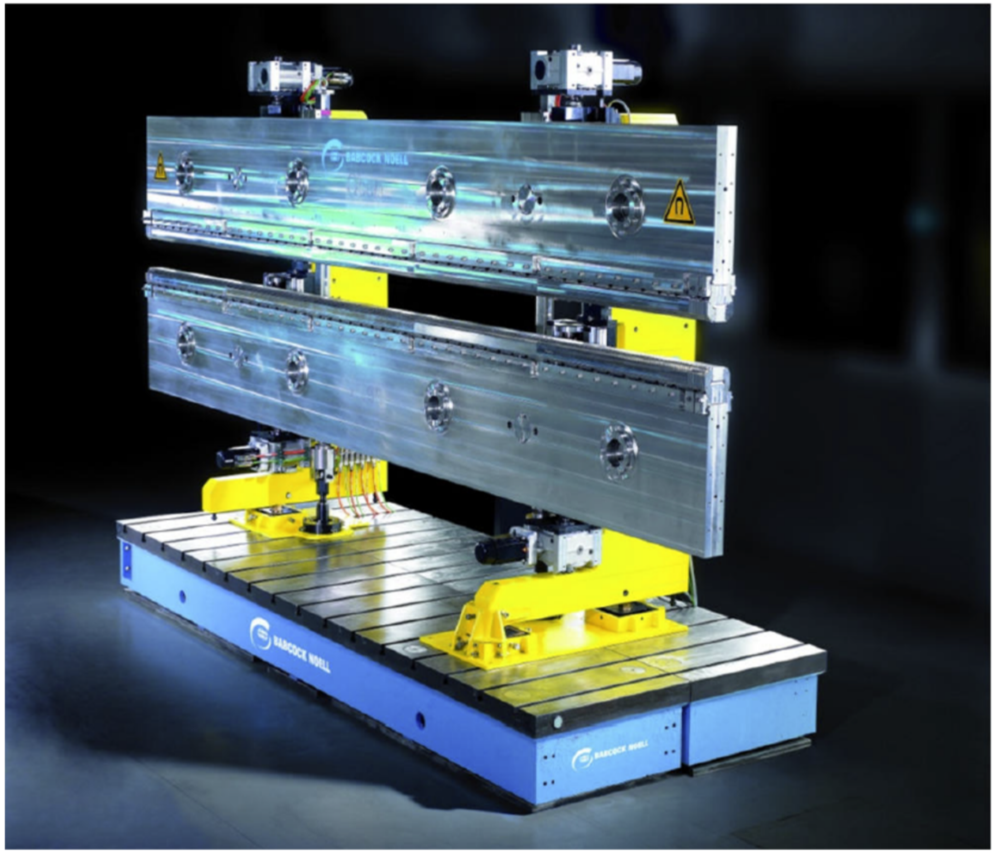
Remanent field: 1.25 (T)

Coercivity >2300 (kA/m)

Poles: Permendur



Prototype undulators for XFEL:



5 m

2m

11.3.3. Compton scattering

We investigate the scattering of the virtual photons of the undulator field off the electron beam. It is described best in the rest frame of the electrons. For this purpose a Lorentz transformation is necessary so that the virtual photons become effectively real and thus the scattering process can be described by Compton scattering. In the following λ_U denotes the "wave length" of the undulator field. For simplification purposes we define the normalized wave numbers:

$$\boxed{\mathrm{K} = \frac{a}{\lambda_U}} \qquad \text{with} \qquad a = \frac{h\,c}{m_e\,c^2} = 2\pi\frac{197\ \mathrm{MeV}\cdot\mathrm{fm}}{0.511\ \mathrm{MeV}} \approx 2.4\cdot 10^{-12}\ \mathrm{m}$$

We calculate the scattering process in three steps:

- **Transformation to the moved reference system:**

$$\boxed{\mathrm{K}_i^* = \gamma\cdot\mathrm{K}_i}$$

- **Compton scattering off the static electrons:**

$$\mathrm{K}_f^* = \frac{1}{1-\cos\theta^* + \dfrac{1}{\mathrm{K}_i^*}}$$

- **Back transformation to the laboratory system:**

$$\mathrm{K}_f = \gamma\left(1-\beta\cos\theta^*\right)\cdot \mathrm{K}_f^*$$

Taking all together it follows for the correlation of the wave lengths:

$$\frac{a}{\lambda_f} = \frac{\gamma\left(1-\beta\cos\theta^*\right)}{1-\cos\theta^* + \dfrac{\lambda_U}{\gamma a}}.$$

Now we have to transform the scattering angle. Considering the appropriate coordinate system definition we obtain:

$$1-\beta\cos\theta^* = \frac{1}{\gamma^2}\frac{1}{1-\beta\cos\theta}, \qquad 1-\cos\theta^* = \frac{1}{2\gamma^2}\frac{1+\cos\theta}{1-\beta\cos\theta}.$$

Inserting this, it follows

$$\frac{\lambda_f}{a} = \gamma\left(1-\beta\cos\theta\right)\left\{\frac{1}{2\gamma^2}\cdot\left(1-\frac{1+\cos\theta}{1-\beta\cos\theta}\right)+\frac{\lambda_U}{\gamma a}\right\}.$$

A Taylor expansion for small angles up to the 2nd order implies:

$$\lambda_f \approx -\frac{a}{2\gamma}\left(1+\beta\right)\cdot\left\{1-\frac{\theta^2}{2}\right\} + \left\{1-\beta+\frac{\theta^2}{2}\right\}\cdot\lambda_i .$$

Since $1-\beta\approx\frac{1}{2\gamma^2}$ and $\frac{a}{\lambda_i}\approx 10^{-10}$, it implies in excellent approximation: $\frac{a}{\gamma}\ll\left(1-\beta\right)\cdot\lambda_i$.

Thereby the coherence condition of the undulator radiation follows:

$$\lambda_f \approx \left\{1-\beta+\frac{\theta^2}{2}\right\}\cdot\lambda_i$$

Important: βc is here the average velocity of the electrons in the undulator and

is not equal to the β of the electron beam outside!!!

11.3.4. Motion of the electrons in the undulator

The transverse acceleration due to the Lorentz force $\gamma m_e \dot{\vec{v}} = -e\vec{v} \times \vec{B}$ caused by the the B field of the undulator $\vec{B}(s) = -B_0 \cdot \sin(k_u s) \cdot \hat{e}_z$ is:

$$\left.\begin{aligned} \ddot{x} &= +\dot{s}\frac{e}{m_e\gamma}B_z(s) \\ \ddot{s} &= -\dot{x}\frac{e}{m_e\gamma}B_z(s) \end{aligned}\right\} \xrightarrow{\dot{s}\approx\beta c} \boxed{\begin{aligned} x'' &= \frac{\ddot{x}}{(\beta c)^2} = \frac{eB_0}{m_e\beta\gamma c}\sin(k_u s) \\ s'' &= \frac{\ddot{s}}{(\beta c)^2} = 0 \end{aligned}}$$

With the initial conditions $x(0) = 0,\; x'(0) = (eB_0)/(m_e\beta\gamma c k_u)$ we obtain the

<u>First-order solution:</u>

$$x(s) = \frac{K}{\beta\gamma k_u}\cdot\sin(k_u s)$$

$$x'(s) = \frac{K}{\beta\gamma}\cdot\cos(k_u s)$$

with

$$\boxed{K = \gamma\cdot\theta_{\max} = \frac{\lambda_u e B_0}{2\pi m_e c}}$$

undulator parameter

or in practical units:

$$\boxed{K = 0.934\cdot B_0[\mathrm{T}]\cdot\lambda_u[\mathrm{cm}]}$$

Second-order solution:

Projection of the motion on the s axis gives with $\beta^2 = 1 - 1/\gamma^2$:

$$\dot{s} = \sqrt{(\beta c)^2 - \dot{x}^2} = c\sqrt{1 - \left(1/\gamma^2 + \dot{x}^2/c^2\right)} \approx c\left\{1 - 1/2\gamma^2 \cdot \left(1 + \gamma^2\,\dot{x}^2/c^2\right)\right\}$$

Inserting the solution $\dot{x} = \beta c \cdot x' = Kc/\gamma \cos(\omega_u t)$ results in

$$\dot{s} = c\left\{1 - \frac{1}{2\gamma^2}\left[1 + \frac{K^2}{2}\left(1 - \cos(2\omega_u t)\right)\right]\right\} \qquad \text{with} \quad \omega_u = \bar{\beta} c \cdot k_u .$$

Thus the electrons move with the average normalized velocity

$$\bar{\beta} = 1 - \frac{1}{2\gamma^2}\left\{1 + \frac{K^2}{2}\right\}$$

and describe the following trajectory in second order:

$$x(t) = \frac{K}{\gamma k_u} \cdot \sin(\omega_u t)$$

$$s(t) = \bar{\beta} c t - \frac{K^2}{8\gamma^2 k_u} \cdot \sin(2\omega_u t)$$

Lorentz transformation to a reference frame (*) moving with $\bar{\beta}c$ gives:

$$\bar{\gamma} = \frac{1}{\sqrt{1-\bar{\beta}^2}} \approx \frac{1}{2\sqrt{1-\bar{\beta}}} = \frac{\gamma}{\sqrt{1+K^2/2}}$$

$$t^* = \bar{\gamma}\left(t - s\,\bar{\beta}/c\right) \approx \bar{\gamma}t\left(1-\bar{\beta}^2\right) = t/\bar{\gamma} = t\Big/\sqrt{1+K^2/2}$$

$$x^* = x = \frac{K}{\gamma k_u}\sin\left(\omega_u t\right)$$

$$s^* = \bar{\gamma}\left(s - \bar{\beta}ct\right) \approx -\frac{K^2}{8\gamma k_u\sqrt{1+K^2/2}}\sin\left(2\omega_u t\right)$$

With the transformed frequency

$$\omega^* = \bar{\gamma}\omega_u = \bar{\gamma}\bar{\beta}ck_u \approx \frac{\gamma c k_u}{\sqrt{1+K^2/2}}$$

and the amplitude

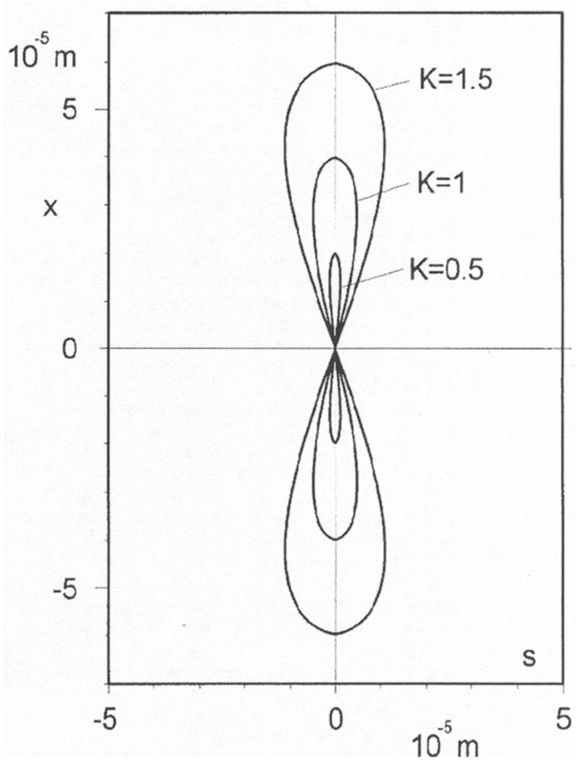
$$a = \frac{K}{\gamma k_u}$$

We obtain the electron orbit in the moving frame $(\bar{\gamma} \approx \gamma)$

$$x^*(t) = a \cdot \sin\left(\omega^* t^*\right)$$

$$s^*(t) \approx -a\frac{K}{8} \cdot \sin\left(2\omega^* t^*\right)$$

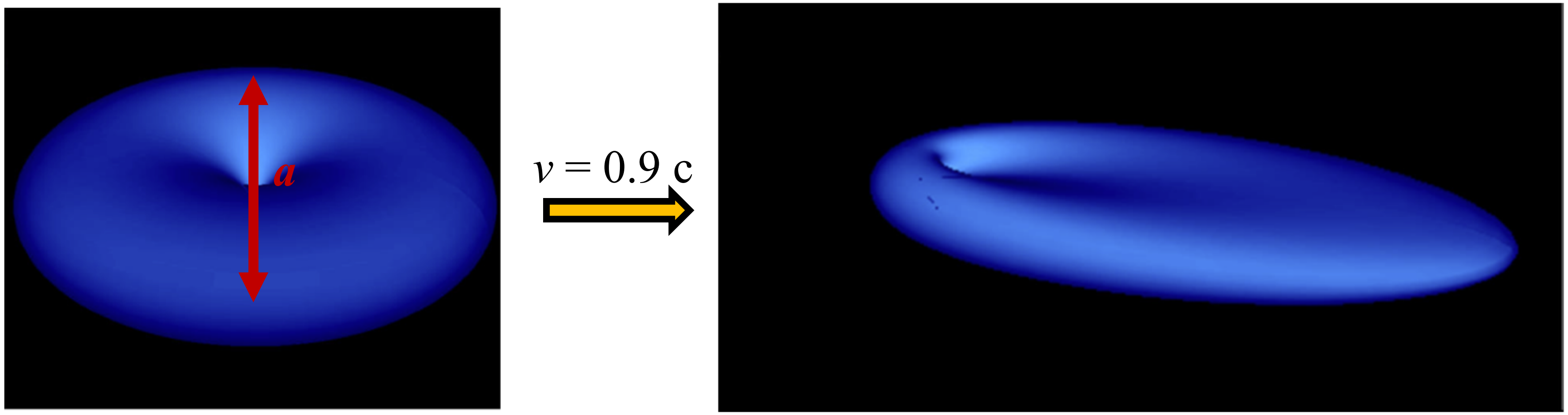
which turns out to have the shape of the number 8:



If we ignore the longitudinal oscillation, the electrons will emit dipole radiation in the moving system with the frequency $\omega^* = \bar{\gamma}\omega_u$ and the wavelength $\lambda^* = \lambda_u / \bar{\gamma}$!

11.3.5. Coherent emission

The radiation characteristics of an oscillating dipole chances when it moves at relativistic speed: with increasing γ it becomes more and more concentrated in the forward direction:



To compute the wavelength of the emitted light in the laboratory frame we have to apply a Lorentz transformation

$$\hbar\omega^{*} = \bar{\gamma}\left\{E_{ph} - \bar{\beta}c \cdot p_{ph} \cdot \cos\theta\right\} = \bar{\gamma}\hbar\omega_{L}\left(1 - \bar{\beta}\cos\theta\right)$$

with $E_{ph}^* = \hbar\omega^*$ in the moving reference system and the energy $E_{ph} = \hbar\omega_L$ and the momentum $\vec{p}_{ph} = \hbar\omega_L / c\left(\cos\theta\cdot\hat{e}_s + \sin\theta\cdot\hat{e}_x\right)$ in the laboratory frame.

Hereby we obtain:

$$\omega^* = \bar{\gamma}\omega_u = \bar{\gamma}\left(1-\bar{\beta}\cos\theta\right)\cdot\omega_L \quad\rightarrow\quad \lambda_L = \left(1-\bar{\beta}\cos\theta\right)\cdot\lambda_u\,.$$

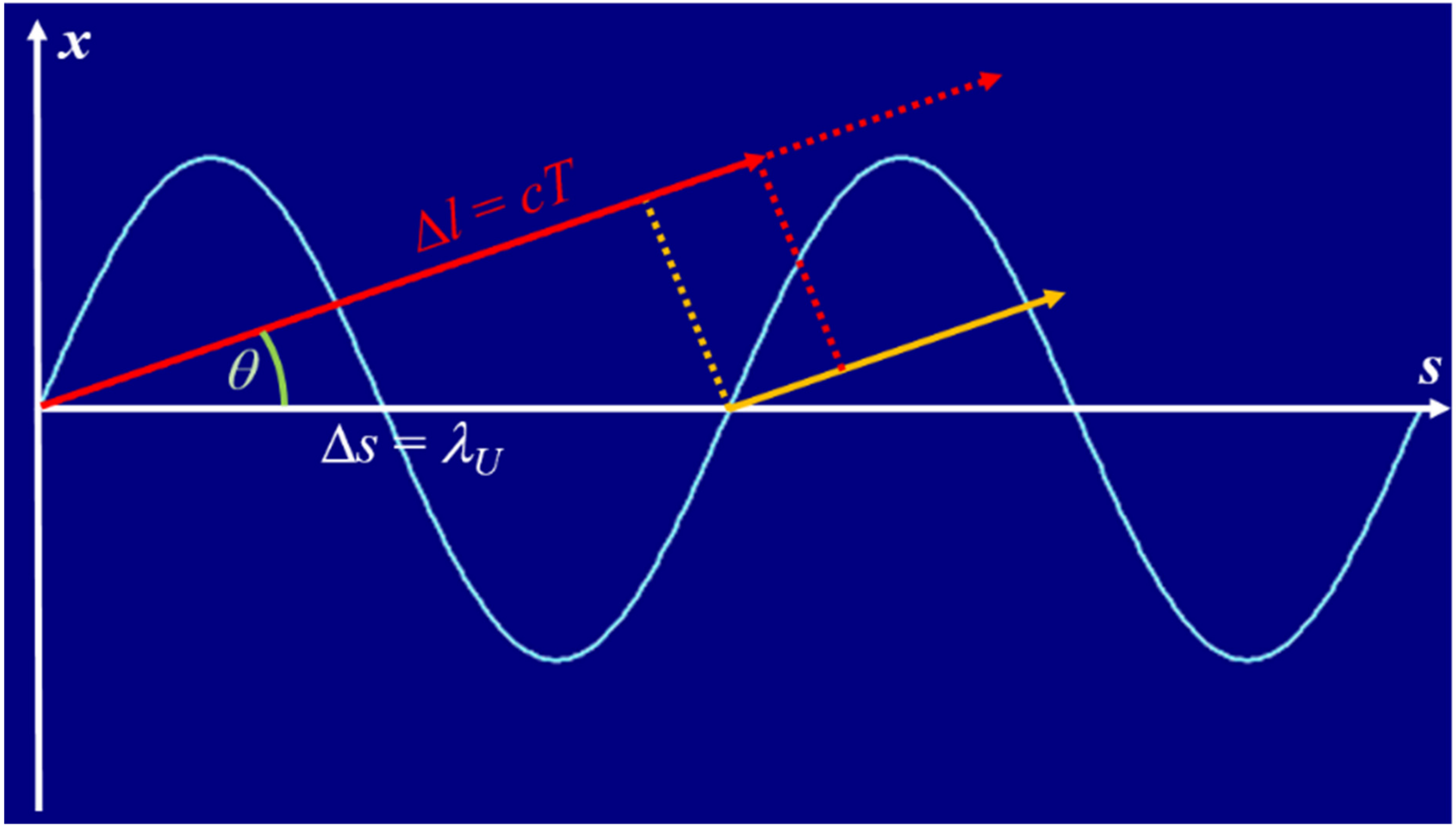
We insert $\bar{\beta} = 1 - 1/2\gamma^2\cdot\left(1+K^2/2\right)$, expand the cosine up to the 2nd order and obtain the following **coherence condition:**

$$\lambda = \left(1+\frac{K^2}{2}+\gamma^2\theta^2\right)\cdot\frac{\lambda_u}{2\gamma^2}\,.$$

Comparing this with the relation in chapter 11.3.3., these formulas match under the approximation $\bar{\beta}\approx 1$ assumed there, since

$$1-\bar{\beta} = \frac{1}{2\gamma^2}\left\{1+\frac{K^2}{2}\right\}.$$

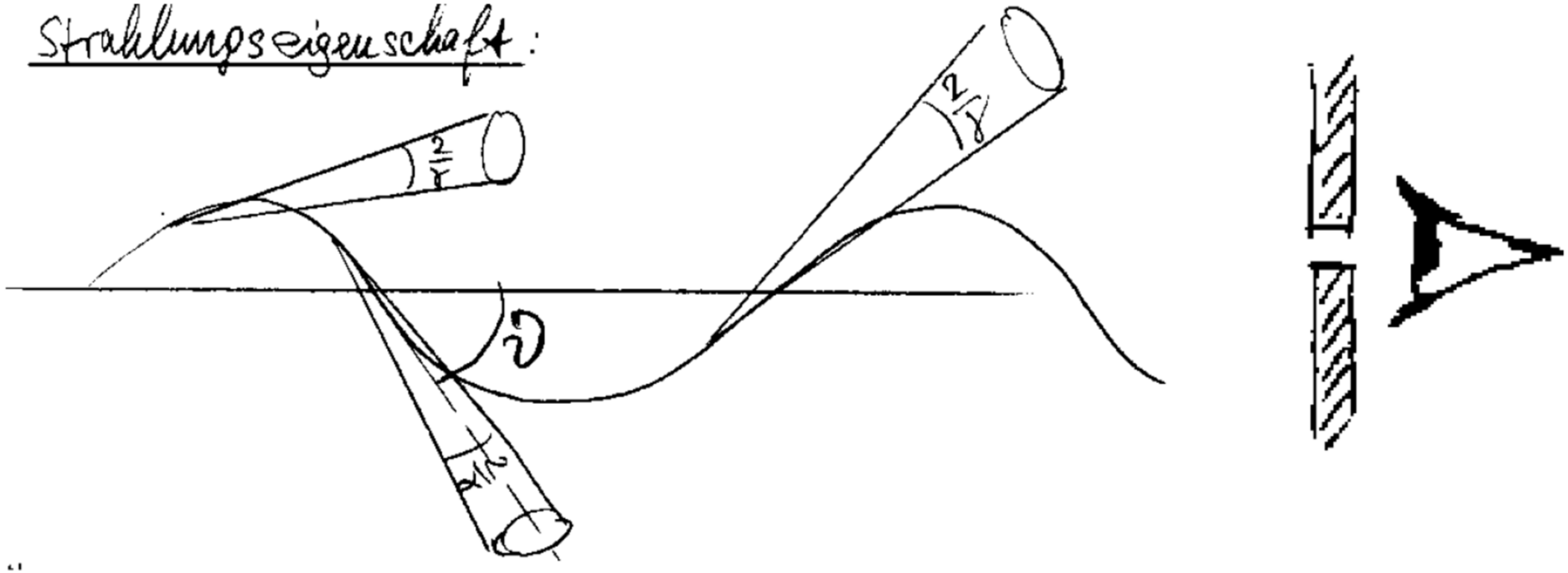
Why is this called a coherence condition? This comes from the fact, that this condition describes the requirements for constructive interference of the radiation emitted at different longitudinal positions in the undulator:



While the electron travels a distance $\lambda_U = \bar{\beta}T$, the photon emitted one period earlier travels a path length of $\Delta l = cT$. We will have constructive interference if

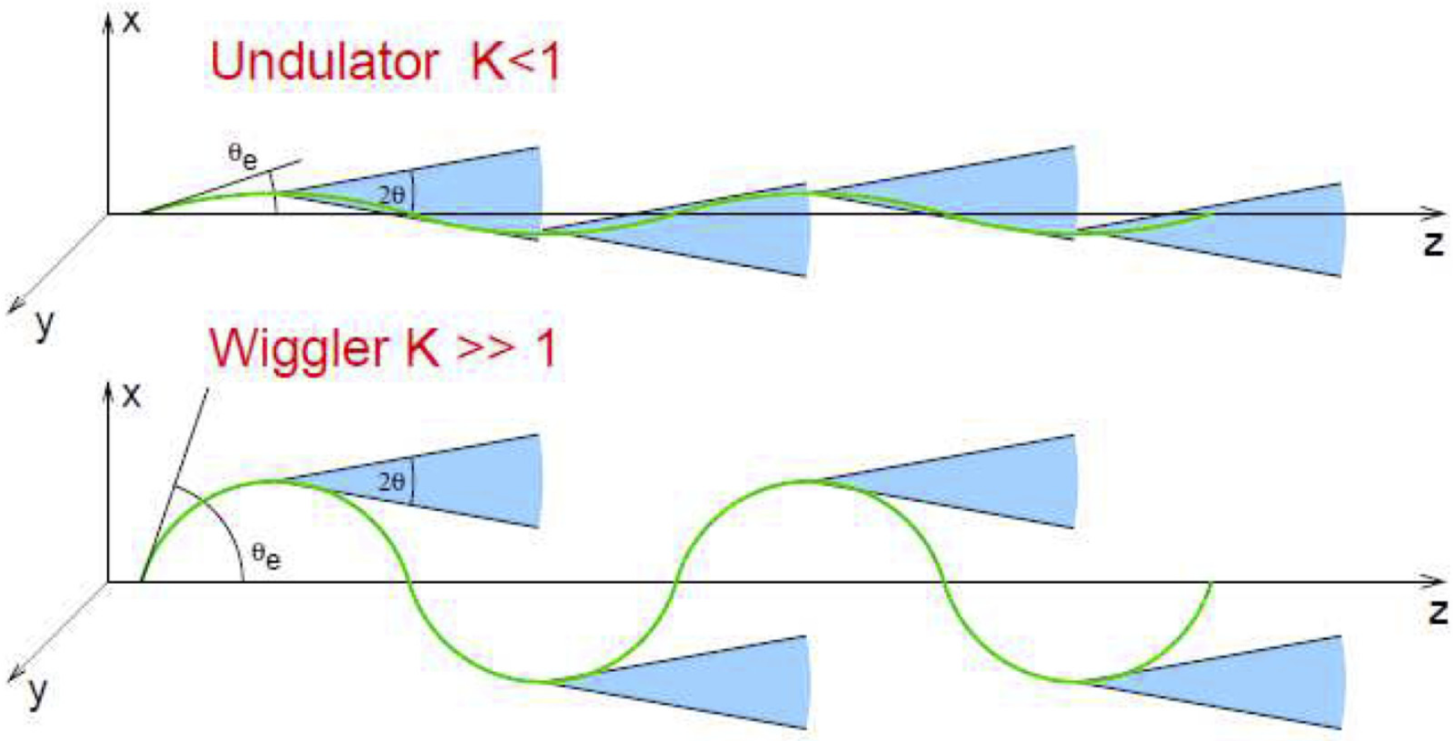
$$\lambda \;=\; \Delta l - \lambda_U \cos\theta \approx \lambda_U \left(\frac{1}{\bar{\beta}} - 1 + \frac{\theta^2}{2} \right) \approx \lambda_U \left(\bar{\beta} - 1 + \frac{\theta^2}{2} \right) = \frac{\lambda_U}{2\gamma^2} \left(1 + \frac{K^2}{2} + \gamma^2\theta^2 \right)$$

But: constructive coherence requires full overlap of the emitted radiation cones!



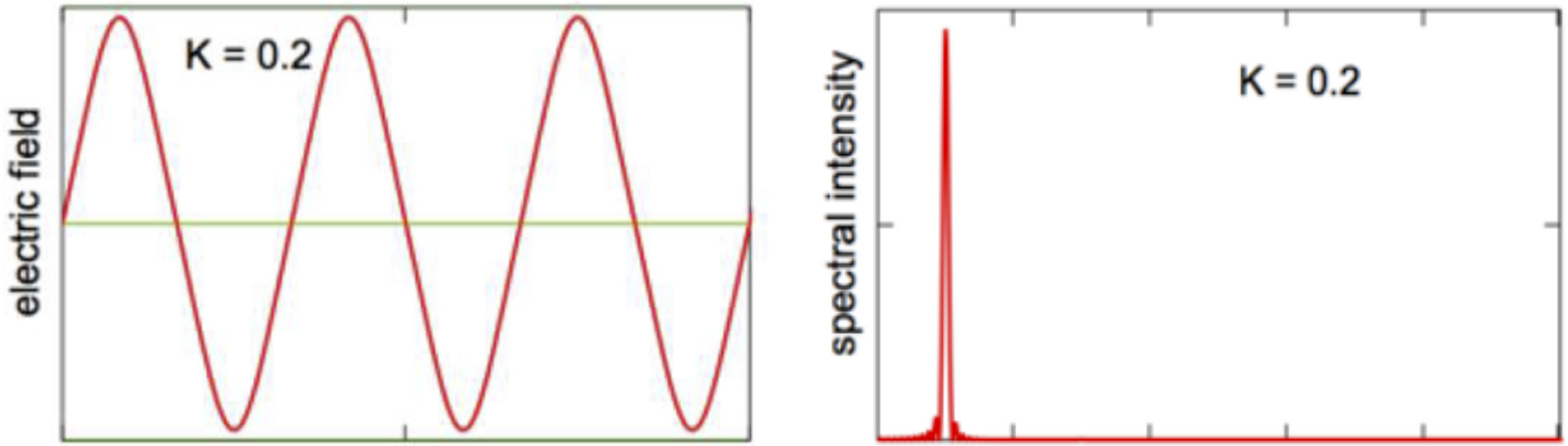
The comparison between the maximum bending angle of the electron orbit $\Theta_{max} = K/\gamma$ and the typical aperture angle $\vartheta = 1/\gamma$ of the emitted radiation results in the differentiation of wigglers and undulators:

- **wiggler:** **K > 1, emits „incoherent“ radiation**
- **undulator:** **K < 1, emits „coherent“ radiation**

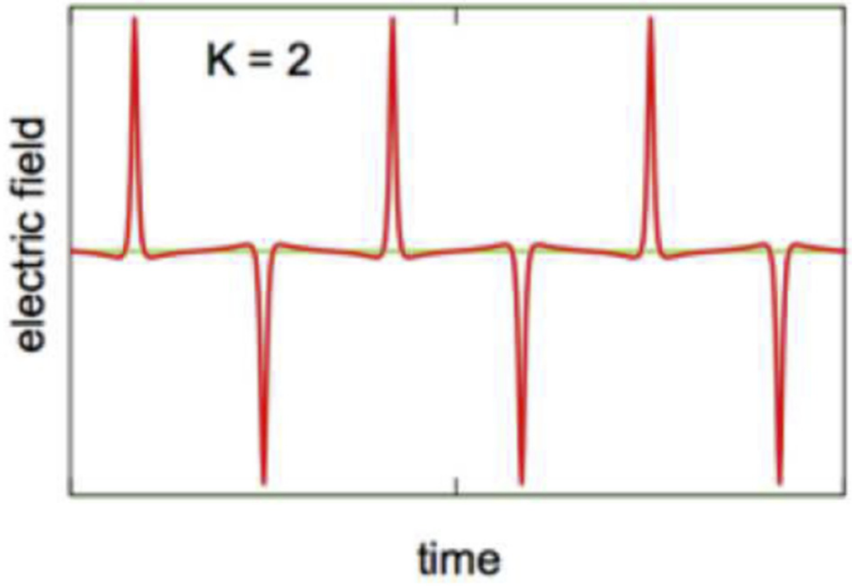


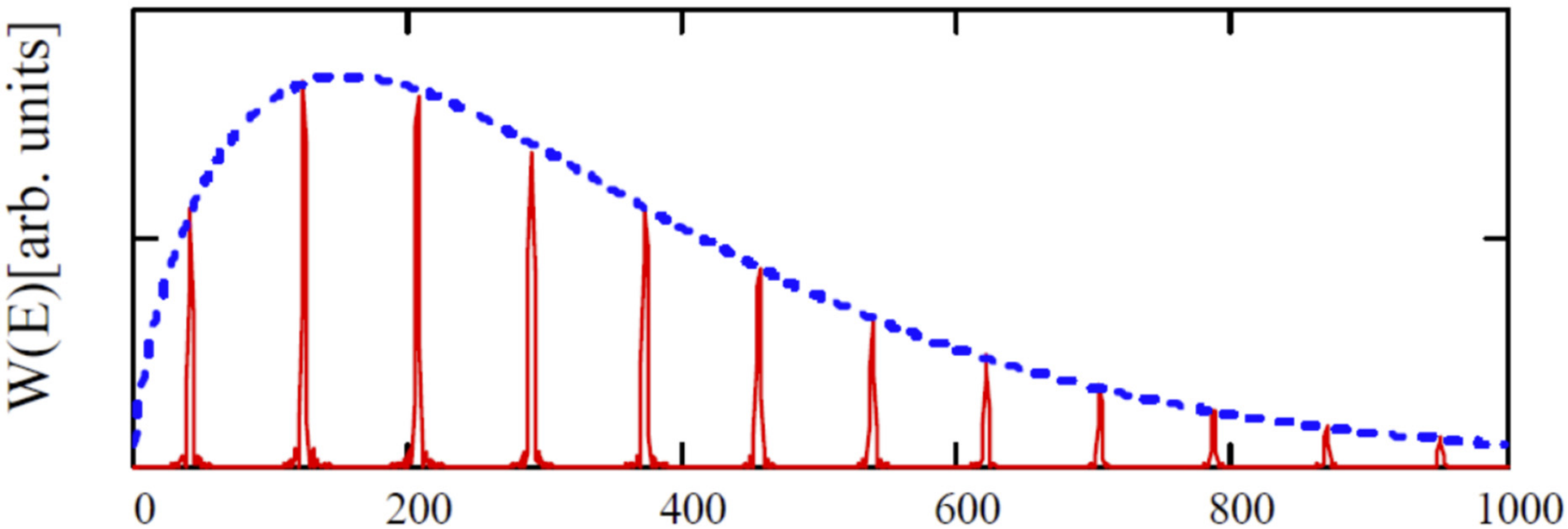
11.3.6. Higher harmonics

If the undulator parameter is small, the radiation will always point towards the detector and the whole trajectory will contribute. A pure sinusoidal electric field at the fundamental frequency is observed.



In case of a large undulator parameter only part of the electron trajectory will contribute and the radiation consists of narrow pulses. The frequency spectrum therefore contains many higher harmonics:


K = 2
electric field
time


W(E)[arb. units]
0
200
400
600
800
1000

11.3.7. Radiation power

We will concentrate on the first harmonic only! The radiation power in the co-moving frame is can be extracted from the Lamor formula (cf. Chapter 6)

$$P = \frac{e^2}{6\pi\varepsilon_0 c^3}\dot{\vec{v}}^2$$

With

$$\dot{v}_x^* = \ddot{x}^* = -\frac{K}{\gamma k_u}\omega^{*2}\sin\left(\omega^* t^*\right) = -\frac{K\gamma c^2 k_u}{1+K^2/2}\sin\left(\omega^* t^*\right)$$

we obtain a time-averaged square of the acceleration of

$$\left\langle \dot{v}^2 \right\rangle = \frac{1}{2}\frac{K^2\gamma^2 c^4 k_u^2}{\left(1+K^2/2\right)^2} \qquad \rightarrow \qquad \boxed{P^* = \frac{e^2 c\gamma^2 K^2 k_u^2}{12\pi\varepsilon_0\left(1+K^2/2\right)^2} = P}$$

The total power, summed over all harmonics and angles, is equal to that emitted in a bending magnet whose field strength is $B = B_0/\sqrt{2}$:

$$\boxed{P_{\text{spont}} = \frac{e^4\gamma^2 B_0^2}{12\pi\varepsilon_0 c m_e^2} = \frac{e^2 c\gamma^2 K^2 k_u^2}{12\pi\varepsilon_0}}$$

11.4. The pendulum equation

In the following, we want to neglect the longitudinal oscillation completely in order to achieve the aim (understanding!) preferably simply and fast. For a correct treatment, we then would have to modify the *K* parameter accordingly to (without proof):

$$K \to K_{JJ} = K\left\{J_0\left(\frac{K^2}{4+2K}\right) - J_1\left(\frac{K^2}{4+2K}\right)\right\}$$

One obtains for the energy variation of the electron beam in the laser field

$$\dot{\gamma} = \frac{d}{dt}\left(\frac{W}{m_e c^2}\right) = \frac{\vec{F}\cdot\vec{\mathrm{v}}}{m_e c^2} = -\frac{e E_0 \dot{x}}{m_e c^2}\cos\left(\omega_L t - k_L s + \phi_L\right).$$

Inserting the derived transverse oscillation

$$\dot{x} = c\cdot\frac{K}{\gamma}\cos\left(k_u s\right), \qquad \bar{\dot{s}} = \bar{\beta} c = \left[1 - \frac{1}{2\gamma^2}\left\{1 + \frac{K^2}{2}\right\}\right]c$$

and using $\cos\alpha\cdot\cos\beta = \frac{1}{2}\left\{\cos\left(\alpha-\beta\right)+\cos\left(\alpha+\beta\right)\right\}$ we get

$$\boxed{\dot{\gamma} = -\frac{eKE_0}{2\gamma m_e c}\left(\cos\psi + \cos\chi\right)}$$

with $\psi = \left(k_L + k_U\right)s - \omega t + \phi_L$ (slowly varying)

and $\chi = \left(k_L - k_U\right)s - \omega t + \phi_L$ (fast oscillating)

ψ is called the "ponderomotive phase", which is constant for in case of resonance

$$\dot{\psi}\Big|_{res} = \left(k_L + k_U\right)\bar{\dot{s}} - \omega_L = -k_L c\left(1-\bar{\beta}\right) + k_U \bar{\beta} c = 0 \quad \text{if} \quad k_L = \left(\frac{\bar{\beta}}{1-\bar{\beta}}\right)k_U \approx \left(\frac{1}{1-\bar{\beta}}\right)k_U$$

since for forward scattering $(\theta = 0)$ the **coherence condition** gives

$$\lambda_L = \frac{\lambda_U}{2\gamma_r^{\,2}}\left(1+\frac{K^2}{2}\right) = \left(1-\bar{\beta}\right)\lambda_U \qquad \leftrightarrow \qquad k_L = \left(\frac{1}{1-\bar{\beta}}\right)k_U$$

We mark the resonant energy by γ_r, define the relative energy deviation from the resonance energy by

$$\eta = \frac{\gamma - \gamma_r}{\gamma_r}, \quad \text{and get if } \eta \ll 1 \ (\gamma \approx \gamma_r): \quad \dot{\eta} \approx -\frac{eKE_0}{2\gamma_r^2 m_e c}\cos\psi$$

In the following, we will redefine the ponderomotive phase by

$$\theta = \psi + \frac{\pi}{2} \ ,$$

yielding no net energy transfer for $\theta = 0, \pi$ and save further writing by setting

$$\varepsilon = \frac{eKE_0}{2\gamma_r^2 m_e c^2} \ .$$

We now have to calculate the derivative of the slowly varying ponderomotive phase.

Using

$$k_U = \frac{1}{2\gamma_r^{\ 2}}\left(1+\frac{K^2}{2}\right)k_L, \quad \left(1-\bar{\beta}\right) = \frac{1}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$$

we obtain for small energy deviations $\eta \ll 1$

$$\dot{\theta} = -k_L c\left(1-\bar{\beta}\right) + k_U\bar{\beta}c = \left(\bar{\beta} - \frac{\gamma_r^{\ 2}}{\gamma^2}\right)ck_U \approx \left(1 - \frac{\gamma_r^{\ 2}}{\gamma^2}\right)ck_U = \left\{1 - \frac{1}{\left(1+\eta\right)^2}\right\}ck_U \approx 2\eta ck_U$$

The 2 canonical variables η and θ, representing the motion in the longitudinal phase space, are linked together by two coupled differential equations, the famous

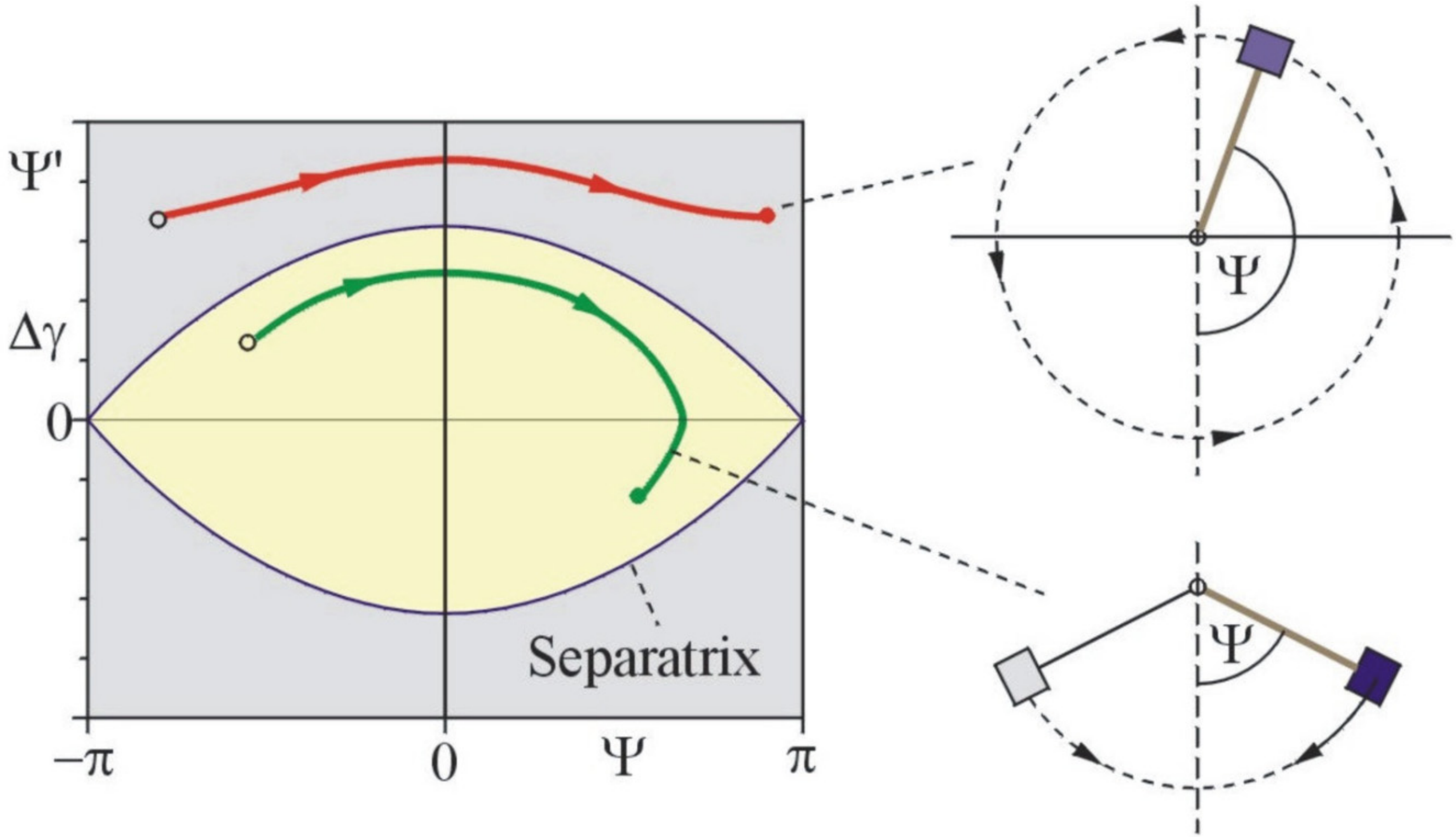
pendulum equations:

$$\eta'(s) = -\varepsilon \cdot \sin\theta(s), \qquad \theta'(s) = 2k_U \cdot \eta(s)$$

or

$$\theta'' + \Omega^2 \sin\theta = 0 \qquad \text{with} \qquad \Omega^2 = \frac{e^2 E_L B_0}{m_e^2 c^3 \gamma_r^2}$$

This equation reminds us deeply on the longitudinal phase space and the synchrotron oscillation! Consequently we have the following illustration:



The electron executes stable oscillations at small oscillation amplitudes and intense laser fields. The limitation of the stable motion is given by the separatrix $\eta_{sep} = \sqrt{\varepsilon\left(1+\cos\theta\right)/k_U}$, the maximum allowed η is given by $\eta_{sep} = \sqrt{2\varepsilon/k_U}$.

11.5. The FEL amplification

At the amplification process, we distinguish two domains by comparing the "frequency" Ω with the undulator length $L_u = N_u \cdot \lambda_u$:

1. **Short signal range:** $\Omega^2 \ll 1/L_U^2$
2. **Long signal range:** $\Omega^2 \le 1/L_U^2 \rightarrow$ **saturation!**

An analytic solution for the FEL amplification is possible only for the short signal range. For this purpose we compute the energy transfer in the laser field:

$$\Delta W_L = -m_e c^2 \cdot \Delta\gamma, \qquad \text{with} \qquad \Delta\gamma = \gamma(L_u) - \gamma_0$$

The amplification factor G is the energy gain normalized to the energy $W_0 = \frac{\varepsilon_0}{2} E_{L,0}^2 \cdot V$ stored in the laser field, thus

$$G = \frac{\Delta W_L}{W_L} = -\frac{2m_e c^2}{\varepsilon_0 E_{L,0}^2 V} \cdot \Delta\gamma \,.$$

We integrate the phase relation

$$\theta'(s) - \theta_0' = \int_0^s \theta''(s) \cdot ds = \frac{2}{\gamma_r} k_u \int_0^s \gamma'(s) \cdot ds = 2\frac{\gamma(s) - \gamma_0}{\gamma_r} k_u$$

and obtain in the case of resonance with

$$0 = \theta'(s)\Big|_{\text{resonance}} = \theta_0' + 2 \cdot \frac{\gamma_r - \gamma_0}{\gamma_r} k_u \qquad \Leftrightarrow \qquad \theta_0' = 2 \cdot \frac{\gamma_0 - \gamma_r}{\gamma_r} k_u$$

the following important relation:

$$\theta'(s) = 2 \cdot \frac{\gamma(s) - \gamma_r}{\gamma_r} k_u \quad .$$

For this reason the energy variation $\Delta\gamma$ occurring during every passage of the undulator can directly be expressed by the variation of the ”phase velocity“ $\Delta\theta'$:

$$\Delta\gamma = \gamma(L_u) - \gamma_0 = \frac{\gamma_r}{2k_u}\left(\theta'(L_u) - \theta'_0\right) = \frac{\gamma_r}{2k_u}\Delta\theta'$$

We obtain this quantity from the integrated pendulum equation

$$\theta'^2(s) - \theta_0'^2 \;=\; 2\Omega^2 \cdot \left(\cos\theta(s) - \cos\theta_0\right)$$

by re-writing the phase variation with this equation as follows:

$$\theta'(s) \;=\; \sqrt{\theta_0'^2 + 2\Omega^2\left(\cos\theta - \cos\theta_0\right)}\,.$$

Inserting of $\theta_0' = 2\left(\gamma_0 - \gamma_r\right)/\gamma_r \cdot k_u$ results in

$$\theta'(s) \;=\; 2\frac{\gamma_0 - \gamma_r}{\gamma_r}k_u \cdot \sqrt{1 + \frac{1}{2}\cdot\left\{\frac{\Omega\cdot\gamma_r}{\left(\gamma_0 - \gamma_r\right)\cdot k_u}\right\}^2 \cdot\left(\cos\theta(s) - \cos\theta_0\right)}\,.$$

In **0th order** the phase variation is constant and we get

$$\theta' \;=\; 2\frac{\gamma_0 - \gamma_r}{\gamma_r}k_u \qquad \Rightarrow \qquad \theta - \theta_0 \;=\; 2\frac{\gamma_0 - \gamma_r}{\gamma_r}k_u \cdot s\,.$$

Therefore holds $\Delta\theta' = \theta'(L_u) - \theta_0' = 0$ and the amplification factor $G = 0$, too!

Hence, we have to look at the higher orders! Now a nasty calculation goes off in which we solve the stuff iteratively. It is time to define the following abbreviatory:

$$\frac{\xi_0}{L_u} = \frac{\gamma_0 - \gamma_r}{\gamma_r} k_u, \qquad \Delta\theta = \theta(L_u) - \theta_0$$

and thereby to simplify the root expression:

$$\theta'(s) = 2\frac{\xi_0}{L_u} \cdot \sqrt{1 + \frac{1}{2} \cdot \frac{\Omega^2 \cdot L_u^2}{\xi_0^{\,2}} \cdot \left[\cos(\theta) - \cos\theta_0\right]}$$

By inserting the phase trend in 0th order we obtain in **1st order**

$$\theta(L_u)\big|_0 = \theta_0 + \Delta\theta\big|_0 = \theta_0 + 2\xi_0$$

and expansion of the root up to the 1st order ($\sqrt{1+x} \approx 1 + x/2$):

$$\Delta\theta'\big|_1 = \frac{\Omega^2 \cdot L_u}{2 \cdot \xi_0} \cdot \left\{\cos(2\xi_0 + \theta_0) - \cos\theta_0\right\},$$

what we have to average over all initial phases θ_0 because of the large spatial spread of the electron bunch. This results again in $G = 0$!

For the calculation of the **2nd order**, initially we have to calculate the phase trend $\theta(s)$ in 1st order. We integrate

$$\left(\theta(L_u)-\theta_0\right)\Big|_1 = \int_0^{L_u}\left\{\frac{2\xi_0}{L_u}+\frac{\Omega^2\cdot L_u}{2\xi_0}\cdot\left[\cos\left(\theta_0+\frac{2\xi_0}{L_u}s\right)-\cos\theta_0\right]\right\}\cdot ds$$

and obtain

$$\Delta\theta\Big|_1 = 2\xi_0 + \frac{\Omega^2\cdot L_u^2}{4\cdot\xi_0^{\,2}}\cdot\left\{\sin\left(2\xi_0+\theta_0\right)-\sin\theta_0-2\xi_0\cos\theta_0\right\}.$$

Now we expand the root up to the 2nd order

$$\theta' = 2\frac{\xi_0}{L_u}\cdot\left\{1+\frac{1}{2}\frac{\Omega^2\cdot L_u^2}{2\xi_0^{\,2}}\left[\cos(\theta)-\cos\theta_0\right]-\frac{1}{8}\left(\frac{\Omega^2\cdot L_u^2}{2\xi_0^{\,2}}\right)^2\left[\cos(\theta)-\cos\theta_0\right]^2\right\}$$ and re-

place in the term of the 1st order $\theta(L_u) \approx \theta_0+\Delta\theta\big|_1$,

but in the term of 2nd order only $\theta(L_u) \approx \theta_0+\Delta\theta\big|_0$.

Further on we use $\Delta\theta\big|_1-\Delta\theta\big|_0 \ll 1$ and hereby approximate

$$\cos\left(\theta_0 + \Delta\theta|_1\right) \approx \cos\left(\theta_0 + \Delta\theta|_0\right) - \left(\Delta\theta|_1 - \Delta\vartheta|_0\right)\cdot\sin\left(\theta_0 + \Delta\theta|_0\right)\cdot$$

Hence it follows

$$\Delta\theta'|_2 = \frac{\Omega^2\cdot L_u}{2\xi_0}\cdot\left[\cos\left(\theta_0 + 2\xi_0\right) - \cos\theta_0\right]$$

$$-\frac{\Omega^4\cdot L_u^3}{8\xi_0^{\;3}}\cdot\left[\sin^2\left(\theta_0 + 2\xi_0\right) - \sin\theta_0\sin\left(\theta_0 + 2\xi_0\right) - 2\xi_0\cos\theta_0\sin\left(\theta_0 + 2\xi_0\right)\right]$$

$$-\frac{\Omega^4\cdot L_u^3}{16\xi_0^{\;3}}\cdot\left[\cos^2\left(\theta_0 + 2\xi_0\right) + \cos^2\theta_0 - 2\cos\theta_0\cos\left(\theta_0 + 2\xi_0\right)\right]$$

and finally, using the addition theorems

$$\sin\alpha\cdot\sin\beta = 1/2\left\{\cos\left(\alpha-\beta\right) - \cos\left(\alpha+\beta\right)\right\}$$

$$\cos\alpha\cdot\cos\beta = 1/2\left\{\cos\left(\alpha-\beta\right) + \cos\left(\alpha+\beta\right)\right\}$$

$$\sin\alpha\cdot\cos\beta = 1/2\left\{\sin\left(\alpha-\beta\right) + \sin\left(\alpha+\beta\right)\right\}$$

we get the following relation:

$$\Delta\theta'\big|_2 = \frac{\Omega^2 \cdot L_u}{2\xi_0}\cdot\left[\cos\left(\theta_0 + 2\xi_0\right) - \cos\theta_0\right]$$
$$-\frac{\Omega^4 \cdot L_u^3}{16\xi_0^{\ 3}}\cdot\Big[1 + \sin^2\left(\theta_0 + 2\xi_0\right) + \cos^2\theta_0 - 2\cos\left(2\xi_0\right) - 2\xi_0\sin\left(2\xi\ \right)$$
$$-2\xi_0\sin\left(2\theta_0 + 2\xi_0\right)\Big]$$

After averaging over all initial phases remains:

$$\left\langle\Delta\theta'\right\rangle_{\theta_0} = -\frac{\Omega^4 \cdot L_u^3}{8\xi_0^{\ 3}}\cdot\left\{1 - \cos\left(2\xi_0\right) - \xi_0\sin\left(2\xi_0\right)\right\} = \frac{\Omega^4 \cdot L_u^3}{8}\cdot\frac{d}{d\xi}\left(\frac{\sin\xi}{\xi}\right)^2 .$$

For the amplification of the FEL follows:

$$G = -\frac{e^4 B^2 N_u^3 \lambda_u^4}{16\pi\varepsilon_0 m_e^3 c^4 V\gamma^3}\cdot\frac{d}{d\xi}\left(\frac{\sin\xi}{\xi}\right)^2 .$$

Having a closer look we realize moreover that the parameter w is linked with the relative line width. We obtain with

$$\omega = \frac{2\gamma^2}{1+K^2/2}\cdot\omega_u \quad \Rightarrow \quad \Delta\omega = \frac{\partial\omega}{\partial\gamma}\Delta\gamma = \frac{2\omega}{\gamma}\Delta\gamma = \frac{2\xi}{L_u k_u}\cdot\omega \quad \Rightarrow \quad \boxed{\xi = \pi N_u \frac{\Delta\omega}{\omega}}$$

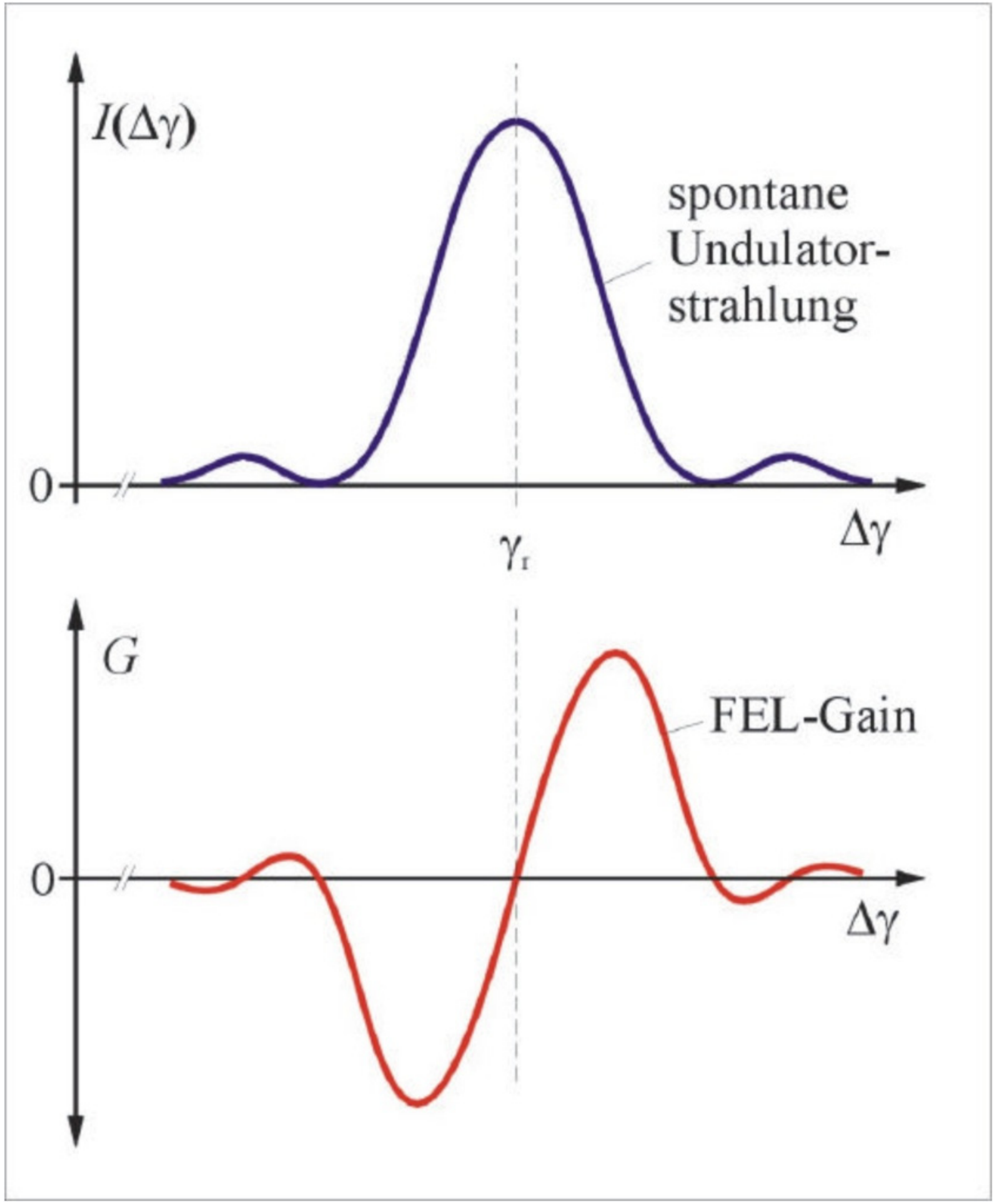
Furthermore, for the intensity spectrum helds

$$I(\omega) \sim \left(\frac{\sin\left(\pi N_u \dfrac{\Delta\omega}{\omega}\right)}{\pi N_u \dfrac{\Delta\omega}{\omega}} \right)^2 .$$

So we obtain the fundamental **Madey theorem** linking the amplification factor with the derivative of the intensity spectrum of the spontaneous undulator radiation:

$$G = -\frac{d}{d\left(\pi N_u \Delta\omega/\omega\right)} I\left(\omega_r + \Delta\omega\right).$$

Therefore an FEL operating at the resonance energy has not any amplification, since no micro bunching occurs. It appears only at positive $\Delta\gamma$!


I(Δγ)
spontane
Undulator-
strahlung
0
γr
Δγ
G
FEL-Gain
0
Δγ