9. Space charge effects

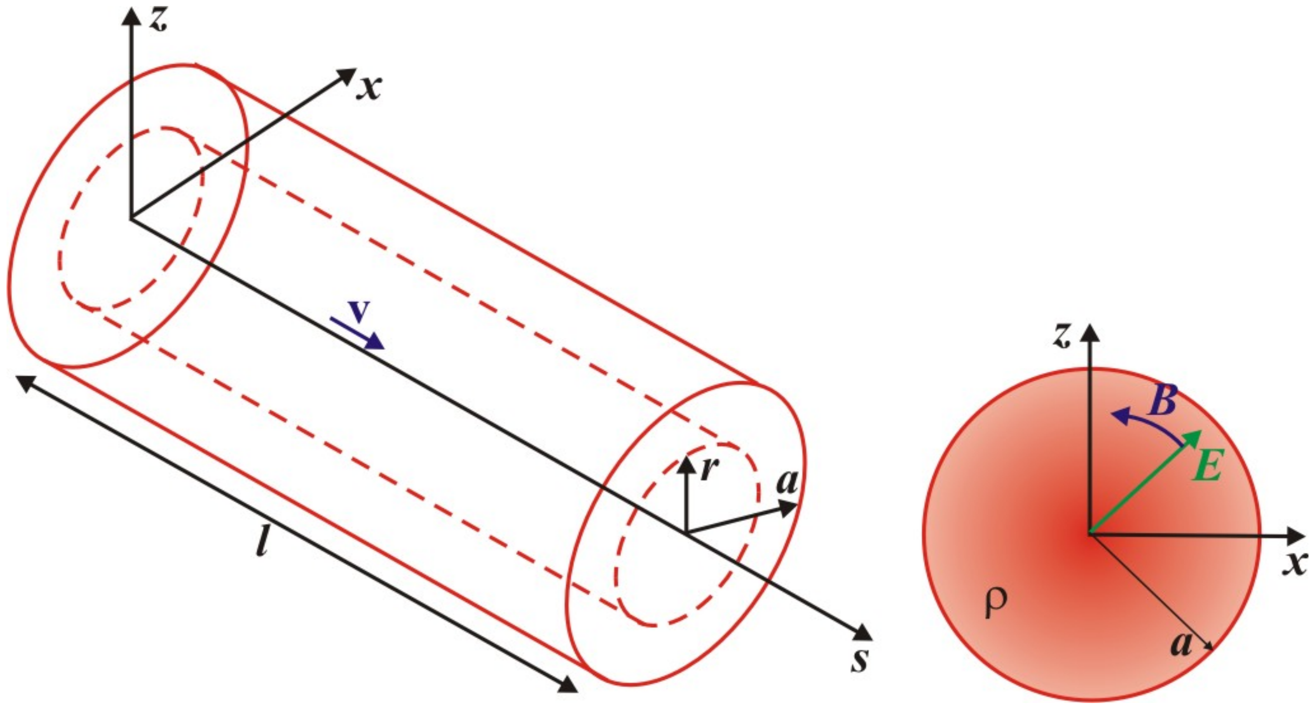
In this chapter effects shall be discussed which originate in the space charge of particle beams. To that effect single as well as colliding beams shall be considered.

9.1. Beam transport not dominated by space charge

In the following we assume that effects due to space charge are weak and can be treated as a small perturbation.

9.1.1. Direct tune shift

For simplification matters we start with a non-bunched (so-called "coasting beam"), round beam with radius a and homogeneous charge distribution:



In the distance $r \le a$ it generates the following electric and magnetic fields:

$$\oiint_{\partial V} \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} \iiint_V \rho \cdot d^3 r \quad \Rightarrow \quad 2\pi r\, l\, E_r(r) = \frac{\pi r^2 l}{\varepsilon_0} \rho$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \iint \vec{j} \cdot d\vec{A} \quad \Rightarrow \quad 2\pi r\, B_\varphi(r) = \mu_0 \pi r^2 j_z$$

With the beam current $I = \pi a^2 \beta c \rho$ we thus get

$$E_r(r) = \frac{\rho}{2\varepsilon_0}\cdot r = \frac{I}{2\pi\varepsilon_0\beta c}\cdot\frac{r}{a^2}$$

$$B_\varphi(r) = \frac{\beta\rho}{2\varepsilon_0 c}\cdot r = \frac{I}{2\pi\varepsilon_0 c^2}\cdot\frac{r}{a^2}$$

On a particle in the beam with radial displacement *r* the following force is exerted:

$$\vec{F}(r) = e\left(\vec{E}(r)+\vec{v}\times\vec{B}(r)\right) = \frac{e\rho}{2\varepsilon_0\gamma^2}\cdot\vec{r} = \frac{eI}{2\pi\varepsilon_0\beta c\gamma^2}\cdot\frac{\vec{r}}{a^2}$$

This force implies an additional defocusing of the beam which is written as a quadrupole disturbance δk. With

$$\gamma m_0\ddot{x} = F_x \qquad \gamma m_0\ddot{z} = F_z$$

and

$$\frac{d^2x}{ds^2}\cdot(\beta c)^2 = \ddot{x} \qquad \frac{d^2z}{ds^2}\cdot(\beta c)^2 = \ddot{z}$$

as well as

$$x'' + \delta k_x \cdot x = 0 \qquad z'' + \delta k_z \cdot z = 0$$

we have:

$$\delta k_{x,z} = -\frac{e\,\rho}{2\varepsilon_0\, m_0 \left(\beta c\right)^2 \gamma^3} = -\frac{e\,I}{2\pi\,\varepsilon_0\, m_0\, a^2 \left(\beta c\right)^3 \gamma^3}$$

The change of the tunes is obtained by integration over the whole path length (watch out: the beta function $\beta_{x,z}$ is indexed, the Lorentz factor β not!):

$$\Delta Q_{x,z} = \frac{1}{4\pi}\oint \delta k_{x,z} \cdot \beta_{x,z} \cdot ds = -\oint \frac{e\,I\,\beta_{x,z}}{8\pi^2 \varepsilon_0\, m_0\, a^2 \left(\beta c\right)^3 \gamma^3} \cdot ds$$

Because of $a = \sqrt{\varepsilon_{x,z} \cdot \beta_{x,z}}$ this can be expressed through the horizontal and vertical emittance respectively which is constant in the equilibrium case. Therefore we have:

$$\Delta Q_{x,z} = -\frac{e}{8\pi^2 \varepsilon_0\, m_0 \left(\beta c\right)^3 \gamma^3} \cdot \frac{I}{\varepsilon_{x,z}} \cdot L$$

The incoherent tune shift thus depends on the emittance (and not on the beta function) and scales with $1/\gamma^3$!

The effect is practically completely negligible for $\beta \to 1$.

The result can be generalized for the case of **elliptical beams** with **semi-axes *a* and *b*** (this we do without detailed calculations). For the fields we have:

$$\vec{E}(x,z) = \frac{I}{\pi\varepsilon_0\,\beta c\,(a+b)} \cdot \left(\frac{x}{a}\hat{e}_x + \frac{z}{b}\hat{e}_z\right)$$

$$\vec{B}(x,z) = \frac{\mu_0\,I}{\pi(a+b)} \cdot \left(-\frac{z}{b}\hat{e}_x + \frac{x}{a}\hat{e}_z\right)$$

The space charge force on a particle with displacements *x* und *z* then is

$$\vec{F}(x,z) = e\left(\vec{E} + \vec{v}\times\vec{B}\right) = \frac{e\,I}{\pi\varepsilon_0\,\beta c\gamma^2(a+b)} \cdot \left(\frac{x}{a}\hat{e}_x + \frac{z}{b}\hat{e}_z\right)$$

and with the emittance coupling $\kappa = \varepsilon_z/\varepsilon_x$ it causes the following tune shifts:

$$\Delta Q_x = \frac{e}{4\pi^2 \varepsilon_0 m_0 (\beta c)^3 \gamma^3} \cdot \frac{I}{\varepsilon_x} \cdot \oint \frac{ds}{1+\sqrt{(\kappa\beta_z)/\beta_x}}$$

$$\Delta Q_z = \frac{e}{4\pi^2 \varepsilon_0 m_0 (\beta c)^3 \gamma^3} \cdot \frac{I}{\varepsilon_z} \cdot \oint \frac{ds}{1+\sqrt{\beta_x/(\kappa\beta_z)}}$$

In the case of bunched beams the longitudinal charge distribution is gaussian. The transverse forces perceived by a particle at the position s_0 are predominantly generated by charges in the range $\Delta s \le a/\gamma$. If the current $I(s)$ does only slightly change over Δs we have

$$\Delta Q_{x,z}(s-s_0) = -\frac{e}{8\pi^2 \varepsilon_0 m_0 (\beta c)^3 \gamma^3} \cdot \frac{I(s-s_0)}{\varepsilon_{x,z}} \cdot L$$

The tune shift thus depends on the longitudinal position and together with synchrotron oscillations leads to a smearing of the tunes!

9.1.2. Wall effects

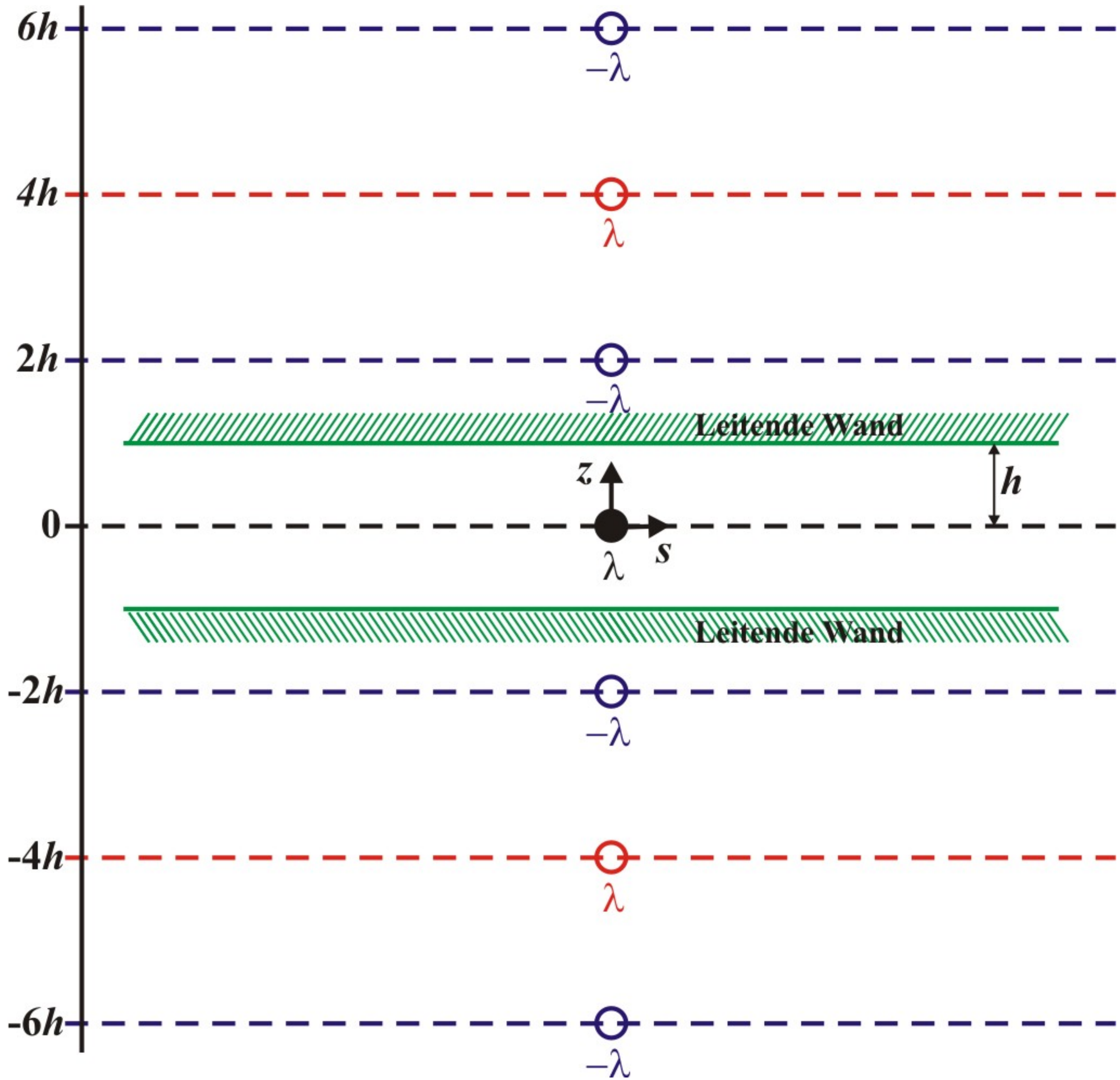
We begin with the effect of the **influence** of the **vacuum chamber** on the **electric fields**. To this end we restrict ourselves to the case of two perfectly conducting plates in vertical distance $\pm h$ to the beam which horizontally extend to infinity. In the following we neglect the expansiveness of the beam which due to $a \ll h$ is well justified, generally. Doing so, we will define the line charge density by

$$\lambda = \iint_A \rho(\vec{r})\, dA \approx \pi a^2 \rho$$

which is linked to the total beam current by

$$I = \iint_A \vec{j} \cdot d\vec{A} = \iint_A \rho \vec{\mathrm{v}} \cdot d\vec{A} = \beta c \lambda$$

On the plates the parallel electric fields have to vanish. This can be achieved by an arrangement of "mirror" line charge densities $\lambda = \pi a^2 \rho = I/(\beta c)$ with the distance $\pm 2nh$:


6h
−λ
4h
λ
2h
−λ
Leitende Wand
z
h
0
λ
s
Leitende Wand
-2h
−λ
-4h
λ
-6h
−λ

At an displacement z in close proximity to the beam the nth line charge density pair generates the field

$$E_{z,n}^{\text{ind}} = \frac{(-1)^n \cdot \lambda}{2\pi\varepsilon_0} \cdot \left[\frac{1}{2nh+z} - \frac{1}{2nh-z} \right] \approx \frac{(-1)^n \cdot \lambda}{2\pi\varepsilon_0} \cdot \frac{-2z}{(2nh)^2} = \frac{-\lambda z}{4\pi\varepsilon_0 h^2} \cdot \frac{(-1)^n}{n^2}$$

Summing over all mirror charge densities due to $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$ yields

$$\boxed{E_z^{\text{ind}}(z) = \frac{\lambda z}{4\pi\varepsilon_0 h^2} \cdot \frac{\pi^2}{12}}$$

The horizontal field component can be obtained from $\operatorname{div} \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0$:

$$\frac{\partial E_x^{\text{ind}}}{\partial x} = -\frac{\partial E_z^{\text{ind}}}{\partial z} = \frac{-\lambda}{4\pi\varepsilon_0 h^2} \cdot \frac{\pi^2}{12} \quad \Rightarrow \quad \boxed{E_x^{\text{ind}}(x) = -\frac{\lambda x}{4\pi\varepsilon_0 h^2} \cdot \frac{\pi^2}{12}}$$

This results in an additional space charge force with no B-field contribution, therefore both fields do not cancel mutually by $1/\gamma^2$:

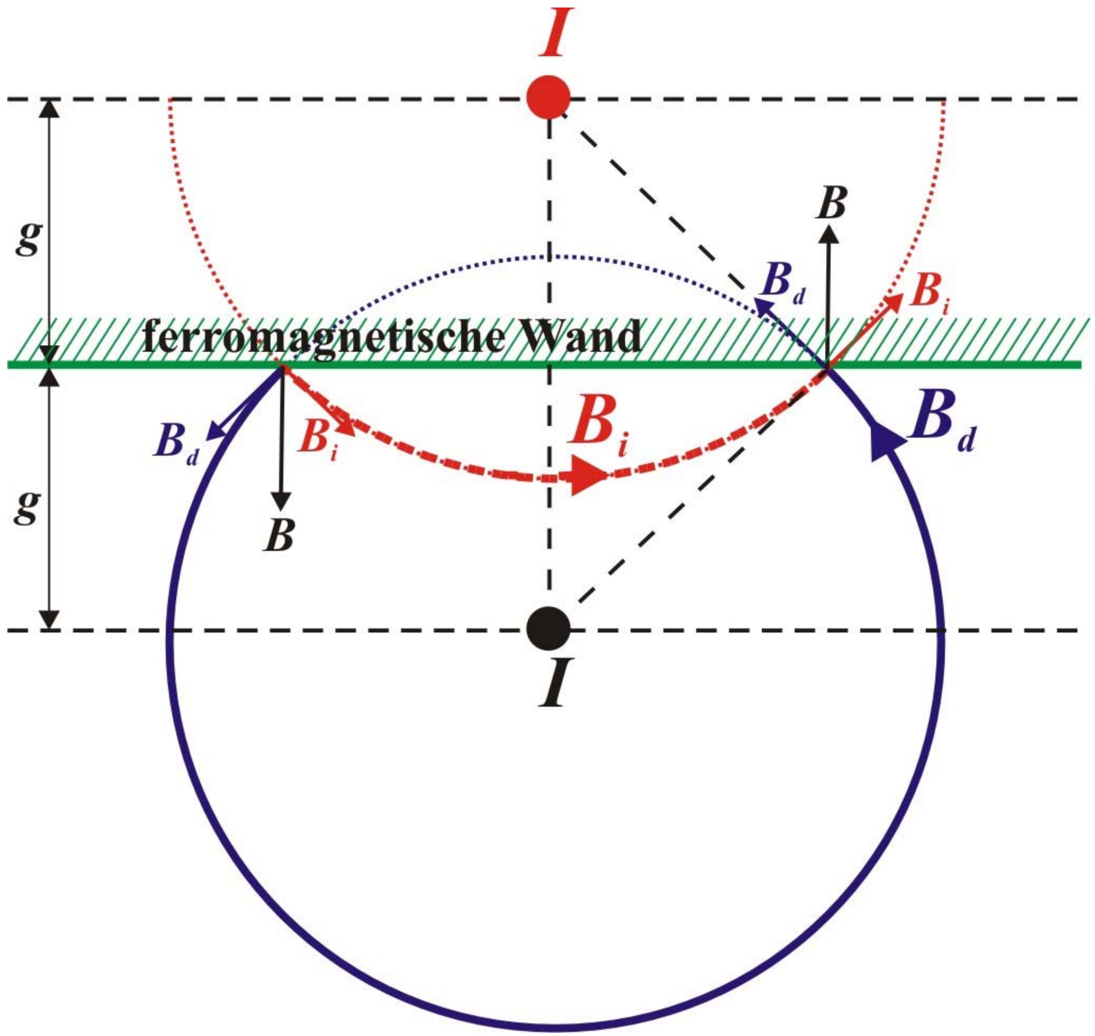
$$\vec{F}^{\,\text{ind}}(x,z) = \frac{e\lambda}{\pi\varepsilon_0} \cdot \frac{\pi^2}{48h^2} \cdot \left(-x\hat{e}_x + z\hat{e}_z\right)$$

This force causes the following tune shift, depending on the beam current $I = \beta c \cdot \lambda$:

$$\Delta Q_x^{\text{el}} = +\frac{eI \cdot \oint \beta_x \, ds}{4\pi^2 \varepsilon_0 m_0 (\beta c)^3 \gamma} \cdot \frac{\pi^2}{48h^2}$$

$$\Delta Q_z^{\text{el}} = -\frac{eI \cdot \oint \beta_z \, ds}{4\pi^2 \varepsilon_0 m_0 (\beta c)^3 \gamma} \cdot \frac{\pi^2}{48h^2}$$

Alongside an influence on the electric fields we do expect, too, an **influence on the magnetic fields** by the **ferromagnetic pole shoes** of the magnets. In complete analogy to the electric case we again do restrict ourselves to two ferromagnetic plates in the vertical distance of $\pm g$ to the beam, which horizontally extend to infinity. On the plates the parallel magnetic fields have to vanish, which, in analogy to the electric case, can be performed by "mirror" currents:



With $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ we obtain for the horizontal magnetic field component

$$B_x^{\text{ind}} = \frac{\mu_0 I}{2\pi} \cdot \sum_{n=1}^{\infty} \left(\frac{1}{2ng - z} - \frac{1}{2ng + z} \right) \approx \frac{\mu_0 I z}{4\pi g^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \boxed{B_x^{\text{ind}}(z) = \frac{\mu_0 I z}{4\pi g^2} \cdot \frac{\pi^2}{6}}$$

The vertical field component is gained via $\left(\text{rot}\,\vec{B}\right)_s = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = 0$:

$$\frac{\partial B_z^{\text{ind}}}{\partial x} = \frac{\partial B_x^{\text{ind}}}{\partial z} = \frac{\mu_0 I}{4\pi g^2} \cdot \frac{\pi^2}{6} \Rightarrow \boxed{B_z^{\text{ind}}(x) = \frac{\mu_0 I x}{4\pi g^2} \cdot \frac{\pi^2}{6}}$$

From this and $\vec{F} = e c \vec{\beta} \times \vec{B}$ results an additional space charge force with no E-field contribution and therefore again without γ -dependency:

$$\vec{F}^{\text{ind}}(x,z) = \frac{e I}{\pi \varepsilon_0 \beta c} \cdot \frac{\pi^2 \beta^2}{24 g^2} \cdot \left(-x \hat{e}_x + z \hat{e}_z \right)$$

That force leads to the following tune shift:

$$\Delta Q_x^{\text{mag}} = +\frac{e\,I\cdot\oint\beta_x\,ds}{4\pi^2\varepsilon_0\,m_0\,(\beta c)^3\,\gamma}\cdot\frac{\pi^2\beta^2}{24\,g^2}$$

$$\Delta Q_z^{\text{mag}} = -\frac{e\,I\cdot\oint\beta_z\,ds}{4\pi^2\varepsilon_0\,m_0\,(\beta c)^3\,\gamma}\cdot\frac{\pi^2\beta^2}{24\,g^2}$$

All effects can be summarized (in the case of a thin conducting vacuum chamber inside the deflecting magnets) as follows:

$$\Delta Q_x^{\text{inc}} = -\frac{e\,I\cdot\oint\beta_x\,ds}{4\pi^2\varepsilon_0\,m_0\,(\beta c)^3\,\gamma}\cdot\left\{\frac{1}{2a^2\gamma^2}-\frac{\pi^2}{48\,h^2}-\frac{\pi^2\beta^2}{24\,g^2}\right\}$$

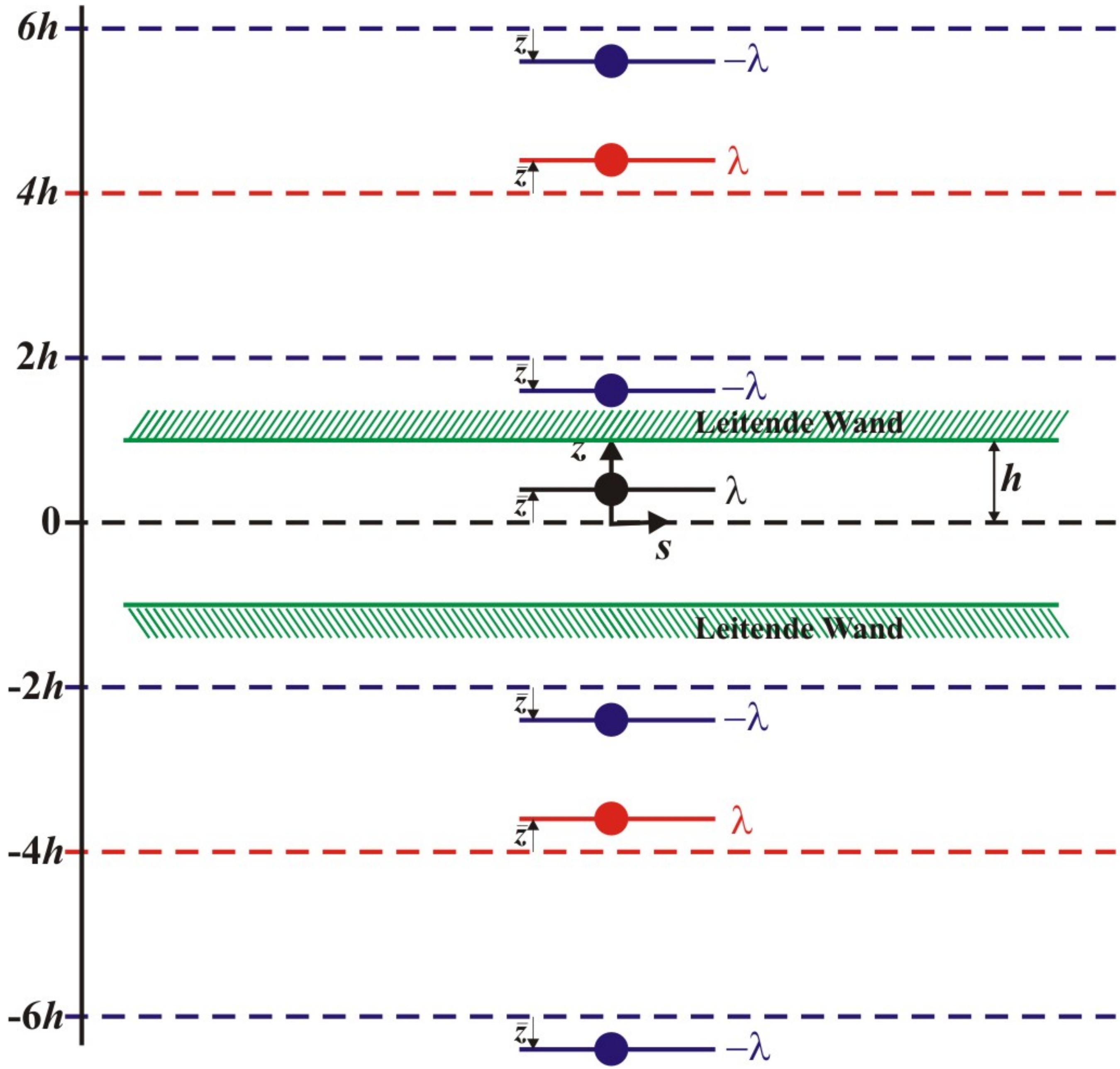
$$\Delta Q_z^{\text{inc}} = -\frac{e\,I\cdot\oint\beta_z\,ds}{4\pi^2\varepsilon_0\,m_0\,(\beta c)^3\,\gamma}\cdot\left\{\frac{1}{2a^2\gamma^2}+\frac{\pi^2}{48\,h^2}+\frac{\pi^2\beta^2}{24\,g^2}\right\}$$

Here some "exemplary values" of incoherent tune shifts at the accelerator facility ELSA for $I = 10 \text{ mA}$:

	Particle source: $E = 50 \text{ keV}$	**Synchrotron:** $E = 25 \text{ MeV}$	**ELSA:** $E = 1{,}2 \text{ GeV}$
$\Delta Q^{\text{inc}}(10 \text{ mA}) =$	$\boxed{0{,}05}$	$1{,}3 \cdot 10^{-3}$	$1{,}5 \cdot 10^{-4}$

9.1.3. Coherent tune shift

Up to now space charge effects on incoherently oscillating beam particles have been considered while the beam barycentre remained unchanged. In case of coherent oscillations the space charge fields of the mirror charges and currents are modulated and retroact on the beam dependent on the phase relation. We again do consider the approximation of two conducting plates. The oscillation of the beam centre causes a displacement of the mirror line charge densities:



6h
4h
2h
0
-2h
-4h
-6h
−λ
λ
−λ
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If the beam barycentre is shifted vertically by $\overline{z}$ then for the field of the *n*th line charge pair we do have:

$$E_z^{(n)} = \frac{(-1)^n \lambda}{2\pi\varepsilon_0} \cdot \left\{ \frac{1}{2nh + \overline{z}\cdot\left[1-(-1)^n\right]} - \frac{1}{2nh - \overline{z}\cdot\left[1-(-1)^n\right]} \right\}$$

$$= -\frac{(-1)^n \lambda \cdot \overline{z}}{4\pi\varepsilon_0 h^2} \cdot \left[\frac{1}{n^2} - \frac{(-1)^n}{n^2} \right]$$

Summation over all line charges yields

$$E_z^{\text{koh}} = \frac{\lambda \cdot \overline{z}}{4\pi\varepsilon_0 h^2} \cdot \left[-\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} \right] = \frac{\lambda \cdot \overline{z}}{4\pi\varepsilon_0 h^2} \cdot \left(\frac{\pi^2}{12} + \frac{\pi^2}{6} \right)$$

and generates the space charge force

$$\boxed{F_z^{\text{coh}} = \frac{e\lambda\,\overline{z}}{\pi\varepsilon_0 h^2} \cdot \frac{\pi^2}{16}}$$

Therefrom the following coherent tune shift for vertical beam oscillations results:

$$\Delta Q_x^{\mathrm{el,coh}} = 0$$

$$\Delta Q_z^{\mathrm{el,coh}} = -\frac{e I \cdot \oint \beta_z \, ds}{4\pi^2 \varepsilon_0 \, m_0 \left(\beta c\right)^3 \gamma} \cdot \frac{\pi^2}{16\, h^2}$$

For the magnetic case, we get in complete analogy

$$B_x^{(n)} = \frac{\mu_0 I}{2\pi} \cdot \sum_{n=1}^{\infty} \left\{ \frac{1}{2 n g - \bar{z} \cdot \left[1 - \left(-1\right)^n\right]} - \frac{1}{2 n g + \bar{z} \cdot \left[1 - \left(-1\right)^n\right]} \right\}$$

$$\approx \frac{\mu_0 I \cdot \bar{z}}{4\pi g^2} \cdot \sum_{n=1}^{\infty} \left[\frac{1}{n^2} - \frac{\left(-1\right)^n}{n^2} \right] = \frac{\mu_0 I \cdot \bar{z}}{4\pi g^2} \cdot \frac{\pi^2}{4}$$

and again via calculations the Lorentz force

$$F_z^{\mathrm{koh}} = \frac{e I \bar{z}}{\pi \varepsilon_0 \, \beta c \, g^2} \cdot \frac{\pi^2}{16}$$

which leads to the following coherent tune shift for vertical beam oscillations:

$$\Delta Q_x^{\text{mag,coh}} = 0$$

$$\Delta Q_z^{\text{mag,coh}} = -\frac{e I \cdot \oint \beta_z \, ds}{4\pi^2 \varepsilon_0 m_0 (\beta c)^3 \gamma} \cdot \frac{\beta^2 \pi^2}{16 \, h^2}$$

9.1.4. Laslett coefficients

The effusions of the hitherto performed calculations can be expressed in a generalized form through the so-called **Laslett coefficients** ε and ξ. One then obtains with the bunch factor $B = \langle I \rangle / I_{\max}$ for bunched beams

$$\Delta Q^{\text{inc}} = -\frac{e I \cdot \oint \beta_{x/z} \, ds}{4\pi^2 \varepsilon_0 m_0 (\beta c)^3 \gamma} \cdot \left\{ \frac{(1-\beta^2-\eta)}{B} \frac{\varepsilon_{sc}}{a^2} + \left[\beta^2 + \frac{(1-\beta^2-\eta)}{B} \right] \frac{\varepsilon_1}{h^2} + \beta^2 \frac{\varepsilon_2}{g^2} \right\}$$

$$\Delta Q^{\text{coh}} = -\frac{e I \cdot \oint \beta_{x/z} \, ds}{4\pi^2 \varepsilon_0 m_0 (\beta c)^3 \gamma} \cdot \left\{ \left[\beta^2 + \frac{(1-\beta^2-\eta)}{B} \right] \frac{\xi_1}{h^2} + \beta^2 \frac{\xi_2}{g^2} \right\}$$

where η specifies the degree of beam neutralization by ion trapping (cp. 8.4). For the Laslett coefficients we dependent on the geometry obtain:

coefficient	round	elliptic	parallel	
$\varepsilon_{sc,x}$	$1/2$	$b^2/a(a+b)$	-	**direct space charge**
$\varepsilon_{sc,z}$	$1/2$	$b/a+b$	-	
$\varepsilon_{1,x/z}$	0	$\pm\frac{h^2}{12d^2}\left[\left(1+k'^2\right)\left(\frac{2K}{\pi}\right)^2-2\right]$	$\pm\pi^2/48$	**incoherent electric**
$\varepsilon_{2,x/z}$	-	-	$\pm\pi^2/24$	**incoherent magnetic**
$\xi_{1,x}$	$1/2$	$\frac{h^2}{4d^2}\left[\left(\frac{2K}{\pi}\right)^2-1\right]$	0	**coherent electric**
$\xi_{1,z}$	$1/2$	$\frac{h^2}{4d^2}\left[1-\left(\frac{2Kk'}{\pi}\right)^2\right]$	$\pi^2/16$	
$\xi_{2,x}$	-	-	0	**coherent magnetic**
$\xi_{2,z}$	-	-	$\pi^2/16$	

Here $K(k)$ is the first complete elliptic integral

$$K(k) = \int_0^{2\pi} \frac{d\varphi}{\sqrt{1-k^2 \sin\varphi}}$$

which satisfies the relation $e^{-\pi K'/K} = \dfrac{w-h}{w+h}$ with *w* as half the chamber width and h as half the chamber height and $K' = K(k')$ where $k' = \sqrt{1-k^2}$.

9.2. Colliding beams

In the following we discuss the additional effects occurring at colliders. To this end luminosity shall be defined at first as an important parameter.

9.2.1. Luminosity

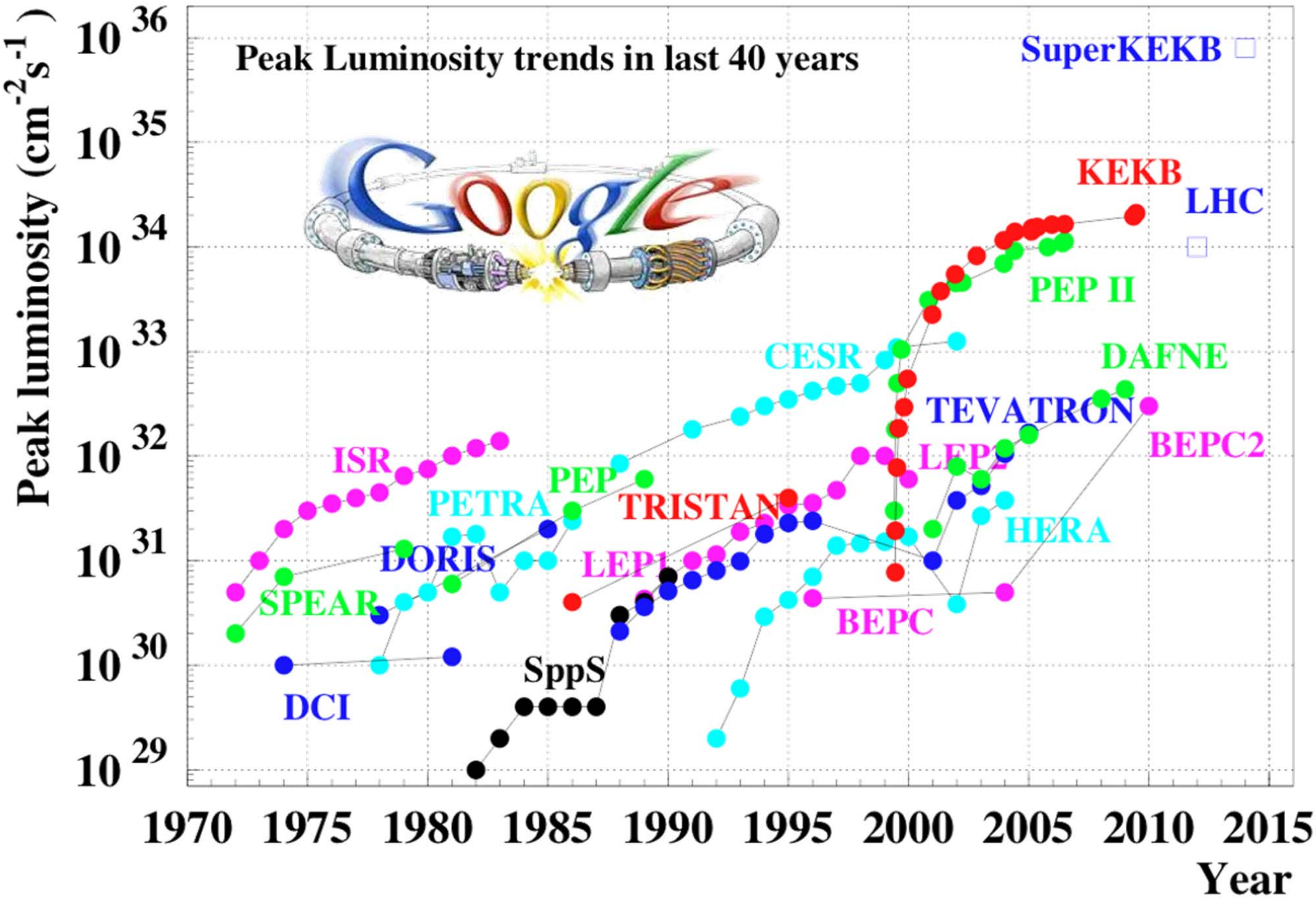
The event rate $\dot{N}$ of a scattering experiment depends on the cross section of the considered reaction in the following simple fashion:

$$\boxed{\dot{N} = \sigma \cdot \mathfrak{L}}$$

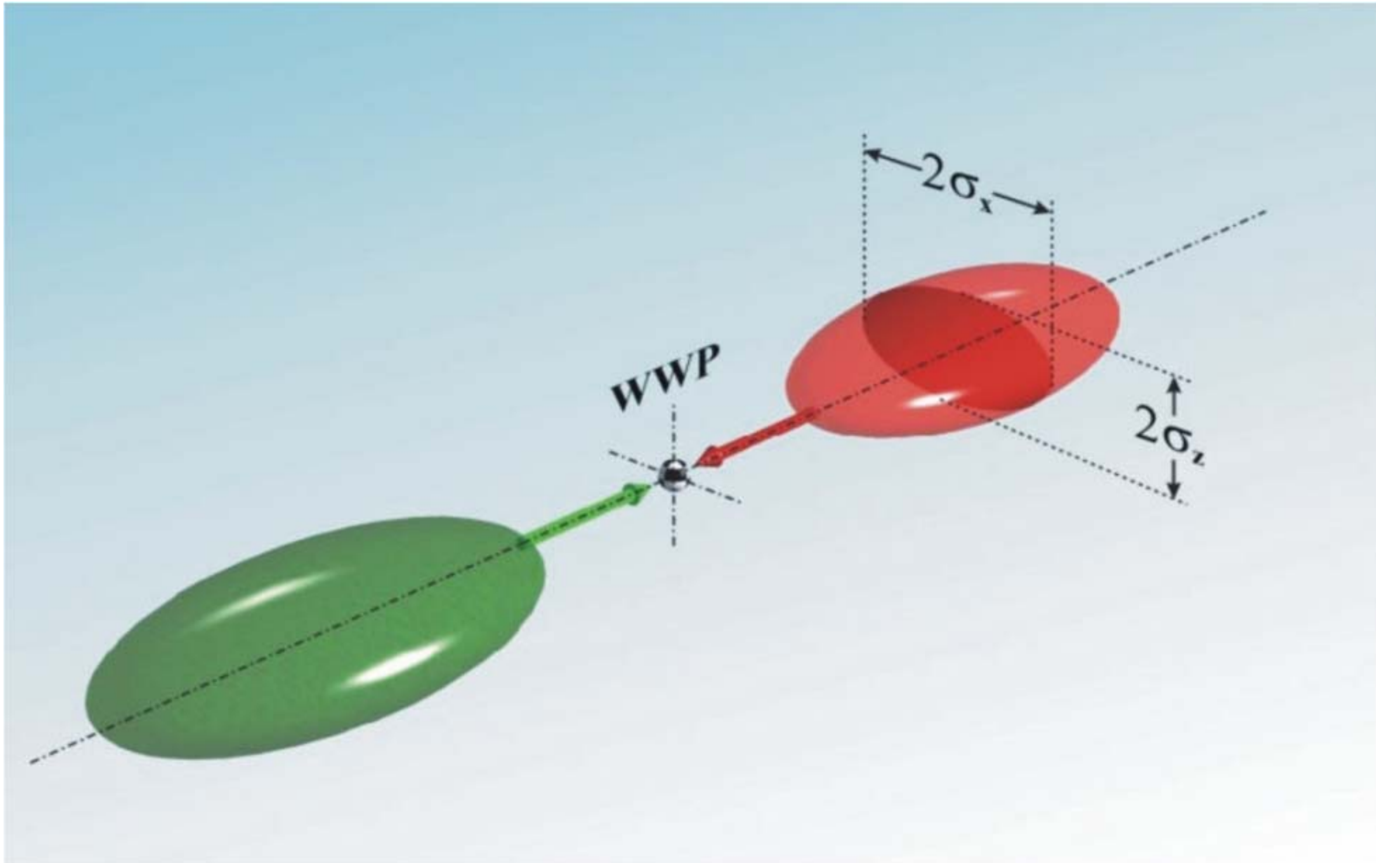
The quantitiy $\mathfrak{L}$ is called luminosity. The total number of reactions N within a given measurement time can be determined out of the integrated luminosity $\mathfrak{I}$

$$N = \sigma \cdot \int_{\text{meas. time}} \mathfrak{L} \cdot dt = \sigma \cdot \mathfrak{I}$$

and is specified mostly in inverse nanobarn (nb^{-1}).


Peak Luminosity trends in last 40 years
Peak luminosity (cm^{-2}s^{-1})
10^{36}
10^{35}
10^{34}
10^{33}
10^{32}
10^{31}
10^{30}
10^{29}
1970
1975
1980
1985
1990
1995
2000
2005
2010
2015
Year
Google
SuperKEKB
KEKB
LHC
PEP II
CESR
DAFNE
TEVATRON
LEP2
BEPC2
ISR
PETRA
PEP
TRISTAN
HERA
DORIS
LEP1
SPEAR
BEPC
SppS
DCI

When two particle beams collide every particle of one bunch flies through the entire other bunch. For calculating the event rate one therefore can integrate over the longitudinal intensity distribution (projection on the cross sectional area):



The total number N of the particles of a bunch follow a gaussian distribution:

$$n_2(x,z) = \frac{N_2}{2\pi\,\sigma_{2,x}\,\sigma_{2,z}} \cdot e^{-\frac{x^2}{2\sigma_{2,x}}-\frac{z^2}{2\sigma_{2,z}}}$$

$$n_1(x,z) = \frac{N_1}{2\pi\,\sigma_{1,x}\,\sigma_{1,z}} \cdot e^{-\frac{x^2}{2\sigma_{1,x}}-\frac{z^2}{2\sigma_{1,z}}}$$

where $\sigma_{i,x}$ is half of the horizontal and $\sigma_{i,z}$ is half of the vertical rms beam width in the interaction zone. The probability that a particle of bunch 1 with the displacement (x,z) will hit a particle in bunch 2 amounts to

$$dP(x,z) = \sigma \cdot n_2(x,z)$$

This has to be integrated over all particles in bunch 1 and yields an event rate per bunch collision of

$$\dot{N}_{b,b} = \sigma \cdot \iint n_1(x,z) \cdot n_2(x,z) \cdot dx\,dz$$

If j bunches revolve with the frequency f_0 one obtains the event rate

$$\dot{N} = \sigma \cdot \frac{j \cdot f_0 \cdot N_1 \cdot N_2}{(2\pi)^2 \cdot \sigma_{1,x}\sigma_{2,x} \cdot \sigma_{1,z}\sigma_{2,z}} \cdot \iint e^{-x^2\left(\frac{1}{2\sigma_{1,x}^{\;2}}+\frac{1}{2\sigma_{2,x}^{\;2}}\right)-z^2\left(\frac{1}{2\sigma_{1,z}^{\;2}}+\frac{1}{2\sigma_{2,z}^{\;2}}\right)} \cdot dx\,dz$$

and, after integration, the luminosity

$$\mathfrak{L} = \frac{j \cdot f_0 \cdot N_1 \cdot N_2}{2\pi \cdot \sqrt{\left(\sigma_{1,x}^{\;2} + \sigma_{2,x}^{\;2}\right) \cdot \left(\sigma_{1,z}^{\;2} + \sigma_{2,z}^{\;2}\right)}}$$

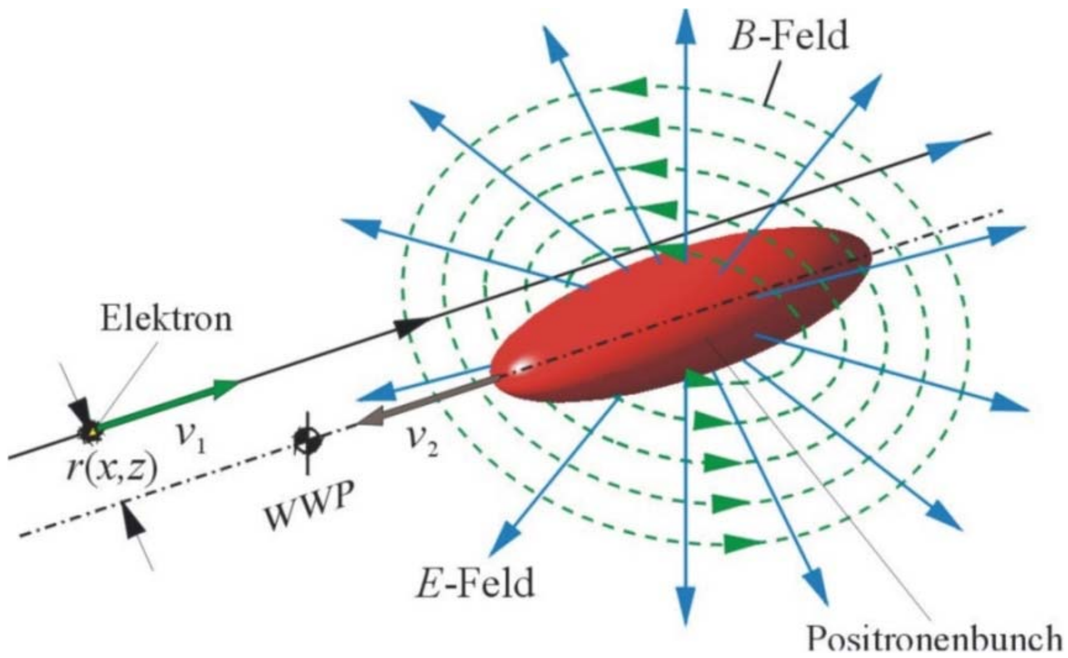
Assuming equal beam cross sections and expressing it through the beam currents $I_i = e \cdot j \cdot N_i \cdot f_0$ we have:

$$\boxed{\mathfrak{L} = \frac{1}{4\pi e^2 \, f_0 \, j} \cdot \frac{I_1 \cdot I_2}{\sigma_x \cdot \sigma_z}}$$

Therefore, a high luminosity is obtained through high beam currents and small beam cross sections. In case of strongly differing beam cross sections there is nothing to be gained in shrinking the smaller one further and further!

9.2.2. Tune shift

The mutual attraction / repulsion of the bunches in the interaction zone leads to a shift in the betatron tune which in interplay with the optical resonances does limit the maximum beam current. ***Amman* und *Ritson*** have been the first to investigate this effect. In the following we shall calculate the situation under the simplifying approximation of round beams. For that purpose we consider e.g. a single electron which flies in the distance $\vec{r} = x\,\hat{e}_x + z\,\hat{e}_z$ from the closed orbit through an oncoming bunch of positrons:



In the centre of momentum frame of the positrons there only does exist the electric field $\vec{E}^*$. When being transformed into the laboratory frame the transverse fields are boosted and a magnetic field emerges:

$$\vec{E}_{\|}^{L} = \vec{E}_{\|}^{*} \qquad B_{\|}^{L} = 0$$

$$\vec{E}_{\perp}^{L} = \gamma \cdot \vec{E}_{\perp}^{*} \qquad B_{\perp}^{L} = \frac{1}{c} \cdot \gamma \, \vec{\beta}_2 \times E_{\perp}^{*}$$

The electron is affected by the following focusing through the Lorentz force:

$$\begin{aligned} \vec{F}_{\perp} &= -e \cdot \left(\vec{E}_{\perp}^{L} + c\,\vec{\beta}_1 \times \vec{B}_{\perp}^{L} \right) \\ &= -e \cdot \left[\gamma \cdot \vec{E}_{\perp}^{*} + \vec{\beta}_1 \times \left(\gamma\, \vec{\beta}_2 \times \vec{E}_{\perp}^{*} \right) \right] \\ &= -e \cdot \left(1 - \vec{\beta}_1 \cdot \vec{\beta}_2 \right) \cdot \vec{E}_{\perp}^{(L)} \end{aligned}$$

In case of highly relativistic particles and a so-called "head-on collision" we have

$$\vec{F}_{\perp} = -2e\vec{E}_{\perp}^{L}$$

For the charge density distribution of a round bunch ($\sigma_x = \sigma_z \equiv \sigma_r$) we obtain in its centre of momentum frame:

$$\rho^*(r,s) = \frac{e\,N}{\sqrt{2\pi}^3\left(\sigma_r^*\right)^2\sigma_s^*}\cdot e^{-\frac{1}{2}\left(\frac{s^*}{\sigma_s^*}\right)^2}\cdot e^{-\frac{1}{2}\left(\frac{r^*}{\sigma_r^*}\right)^2}$$

When a particle flies through that bunch with a transverse displacement *r*, it experiences according to Gauss's theorem an electric field $\vec{E}^*$ in the COM frame of the bunch which is generated by the charges of the bunch located within *r*:

$$\oiint\limits_{r'<r^*} \vec{E}^*(r',s^*)\cdot d\vec{A}' = \frac{1}{\varepsilon_0}\cdot\iiint\limits_{r'<r^*}\rho^*(r',s)\cdot d^3r'$$

With the restriction to transverse fields one can simplify this to the integration over a circular disc with radius *r* and longitudinal displacement *s*. In polar coordinates we have:

$$\oint \vec{E}_\perp^* \cdot r^* \cdot d\varphi = 2\pi r^* \cdot \vec{E}_\perp^* = \frac{1}{\varepsilon_0} \cdot \iint_{r' < r^*} \rho^* \cdot r' \cdot dr' \, d\varphi' = \frac{2\pi}{\varepsilon_0} \cdot \int_0^{r^*} \rho^* \cdot r' \cdot dr'$$

After executing the integration one obtains

$$E_\perp^*(r^*, s^*) = \frac{e\,N}{\sqrt{2\pi}^3 \, \varepsilon_0 \, \sigma_s^*} \cdot e^{-\frac{1}{2}\left(\frac{s^*}{\sigma_s^*}\right)^2} \cdot \frac{1 - e^{-\frac{1}{2}\left(\frac{r^*}{\sigma_r^*}\right)^2}}{r^*}$$

This is transformed into the laboratory frame in the usual fashion and yields, because of $E_\perp^L = \gamma E_\perp^*$ and $\gamma \sigma_s = \sigma_s^*$ as well as $\gamma s = s^*$,

$$\boxed{E_\perp^L(r, s) = \frac{e\,N}{\sqrt{2\pi}^3 \, \varepsilon_0 \, \sigma_s} \cdot e^{-\frac{s^2}{2\sigma_s}} \cdot \frac{1 - e^{-\frac{r^2}{2\sigma_r^2}}}{r}}$$

A Taylor expansion for electrons close to the axis ($r \ll \sigma_r$) results with a linear approximation in:

$$E_{\perp}^{L}(r,s) = \frac{e\,N}{\sqrt{2\pi}^{3}\,\varepsilon_0\,\sigma_s} \cdot e^{-\frac{s^2}{2\sigma_s}} \cdot \frac{r}{2\sigma_r^{\,2}}$$

This field causes a change of the angle of the orbit which results from the integration over the interaction time:

$$\Delta x' = \frac{\Delta p_x}{p} = \int_{WW} \frac{F_x}{p} \cdot dt = \int_{-\infty}^{\infty} \frac{F_x}{p} \cdot \frac{ds}{2c} \approx \frac{e}{p} \cdot \int_{-\infty}^{\infty} E_x^L \cdot \frac{ds}{c}$$

$$\Delta z' = \frac{\Delta p_z}{p} = \int_{WW} \frac{F_z}{p} \cdot dt = \int_{-\infty}^{\infty} \frac{F_z}{p} \cdot \frac{ds}{2c} \approx \frac{e}{p} \cdot \int_{-\infty}^{\infty} E_z^L \cdot \frac{ds}{c}$$

After executing the integration we obtain with $r_e = e^2 / \left(4\pi\,\varepsilon_0\,m_0 c^2\right)$ being the classic electron radius:

$$\Delta x' = -2\,N\,r_e \cdot \frac{1}{\gamma} \cdot \frac{x}{2\sigma_r^{\,2}} \overset{\sigma_x \neq \sigma_z}{=} -2\,N\,r_e \cdot \frac{1}{\gamma} \cdot \frac{x}{(\sigma_x + \sigma_z)\sigma_x}$$

$$\Delta z' = -2\,N\,r_e \cdot \frac{1}{\gamma} \cdot \frac{z}{2\sigma_r^{\,2}} \overset{\sigma_x \neq \sigma_z}{=} -2\,N\,r_e \cdot \frac{1}{\gamma} \cdot \frac{z}{(\sigma_x + \sigma_z)\sigma_z}$$

With $\Delta x' = k \cdot l \cdot x$ and $\Delta Q = \dfrac{1}{4\pi}\int \beta k \cdot ds \approx \dfrac{\langle \beta \rangle}{4\pi} \cdot k \cdot l$ where the beta function has been averaged over the bunch length *l*, this can be written as a change in the betatron tune:

$$\Delta Q_x = -\frac{N r_e}{2\pi} \cdot \frac{1}{\gamma} \cdot \frac{\langle \beta_x \rangle}{(\sigma_x + \sigma_z)\sigma_x}$$

$$\Delta Q_z = -\frac{N r_e}{2\pi} \cdot \frac{1}{\gamma} \cdot \frac{\langle \beta_z \rangle}{(\sigma_x + \sigma_z)\sigma_z}$$

This can be expressed with $I_i = e \cdot j \cdot N_i \cdot f_0$ and $\sigma = \sqrt{\beta \varepsilon}$ in dependency on the beam current and the emittances.

$$\Delta Q_x = -\frac{e I}{8\pi^2 \varepsilon_0 j f_0} \cdot \frac{1}{E} \cdot \frac{\langle \beta_x \rangle}{\left(\sqrt{\langle \beta_x \rangle \varepsilon_x} + \sqrt{\langle \beta_z \rangle \varepsilon_z}\right) \cdot \sqrt{\langle \beta_x \rangle \varepsilon_x}}$$

$$\Delta Q_z = -\frac{e I}{8\pi^2 \varepsilon_0 j f_0} \cdot \frac{1}{E} \cdot \frac{\langle \beta_z \rangle}{\left(\sqrt{\langle \beta_x \rangle \varepsilon_x} + \sqrt{\langle \beta_z \rangle \varepsilon_z}\right) \cdot \sqrt{\langle \beta_z \rangle \varepsilon_z}}$$

If one furthermore considers that the emittance scales with E^2 and if one introduces a normalized emittance $\varepsilon = \gamma^2\,\tilde{\varepsilon}_0$, one gets with the emittance coupling κ

$$\kappa = \frac{\varepsilon_z}{\varepsilon_x} \quad \Rightarrow \quad \varepsilon_x = \gamma^2 \frac{\tilde{\varepsilon}_0}{1+\kappa}, \quad \varepsilon_z = \gamma^2 \frac{\kappa \cdot \tilde{\varepsilon}_0}{1+\kappa}$$

If tolerating a maximum tune shift of $\Delta Q_{\max}$ which generally should not exceed some few hundredths (tune diagram up to the 16th order), one obtains from the above relation an upper limit for the acceptable maximum current (in order to avoid confusion the electric constant is expressed here through $\varepsilon_0 = 1/\mu_0 c^2$):

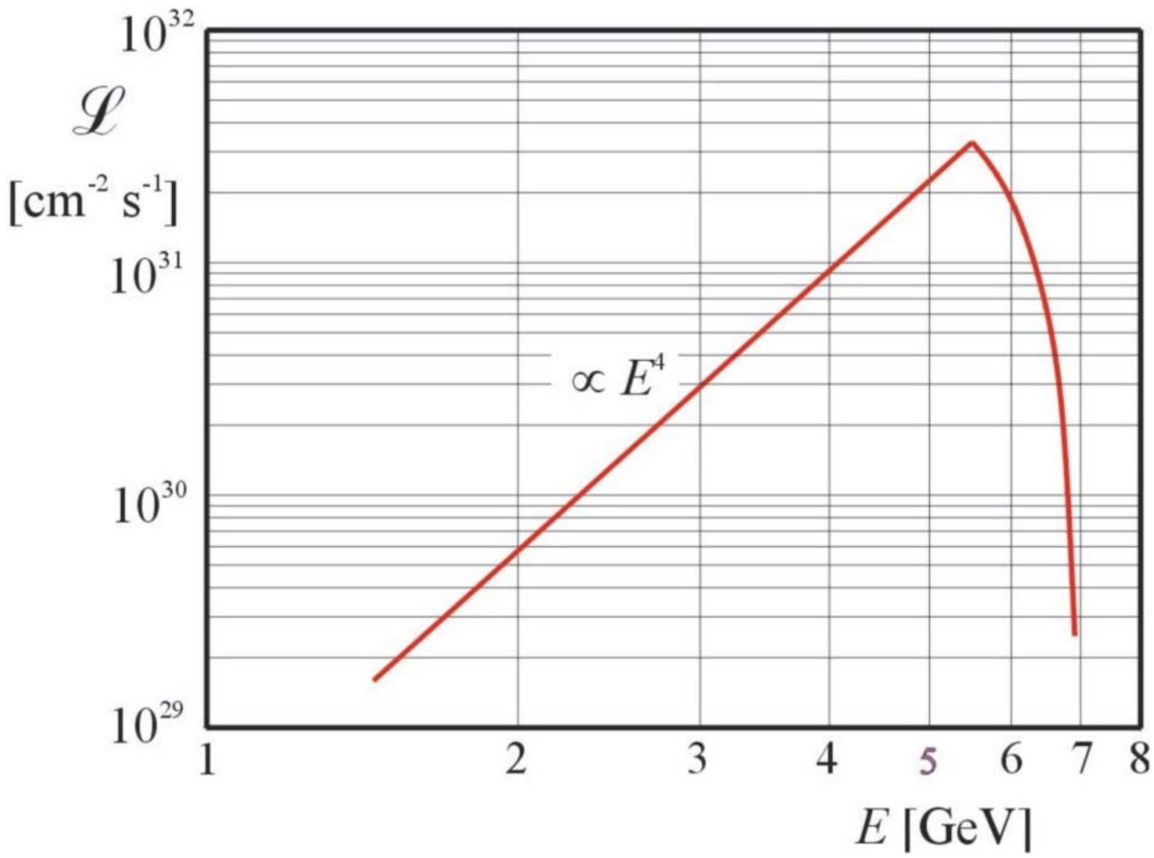
$$I_{\max,x} = -\frac{8\pi^2 m_0\, j\, f_0}{e\,\mu_0} \cdot \gamma^3 \tilde{\varepsilon}_0 \cdot \frac{1}{1+\kappa} \cdot \frac{\left(\sqrt{\langle\beta_x\rangle} + \sqrt{\kappa\langle\beta_z\rangle}\right)}{\sqrt{\langle\beta_x\rangle}} \Delta Q_{\max,x}$$

$$I_{\max,z} = -\frac{8\pi^2 m_0\, j\, f_0}{e\,\mu_0} \cdot \gamma^3 \tilde{\varepsilon}_0 \cdot \frac{\sqrt{\kappa}}{1+\kappa} \cdot \frac{\left(\sqrt{\langle\beta_x\rangle} + \sqrt{\kappa\langle\beta_z\rangle}\right)}{\sqrt{\langle\beta_z\rangle}} \Delta Q_{\max,z}$$

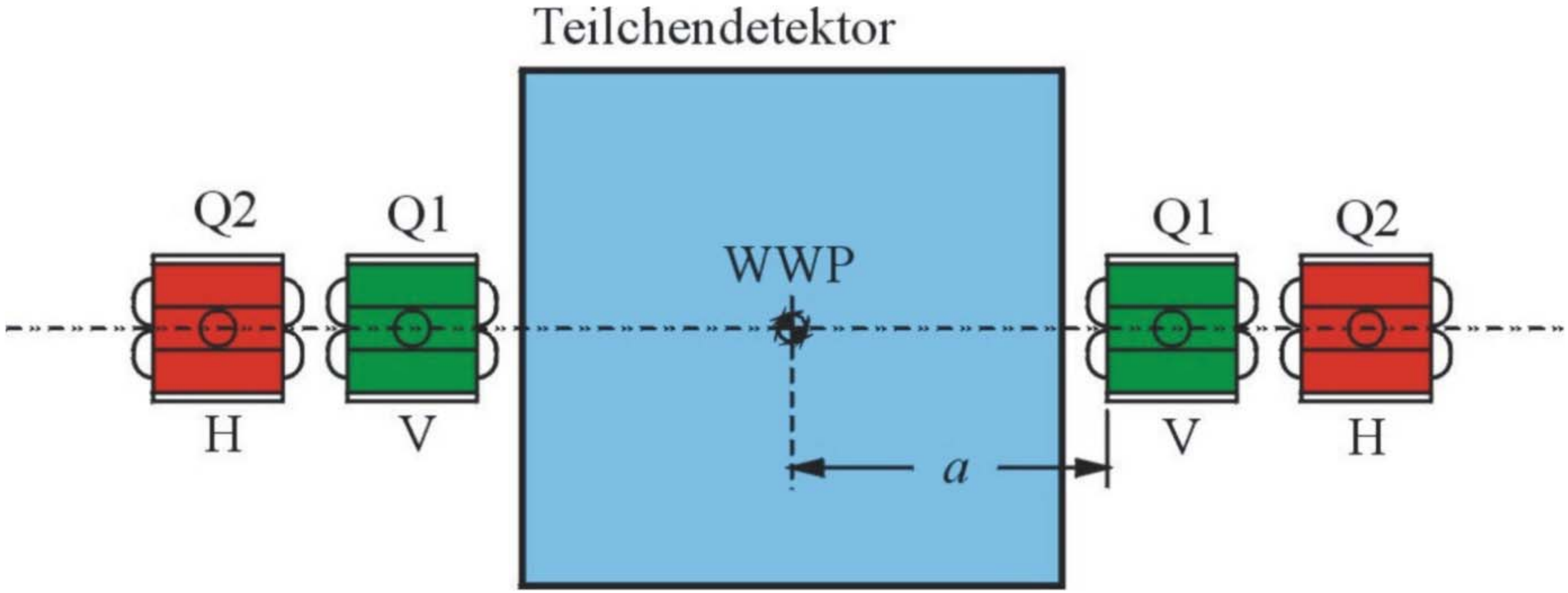
As a general rule the maximum current is limited by the vertical condition. Assuming equal currents $I_1 = I_2 = I_{\max}$ of the colliding beams, for the maximum possible luminosity we do have

$$\mathfrak{L}_{\max} = \frac{16\pi^3\, j\, f_0\, m_0^{\,2}}{\mu_0^{\,2}\, e^4} \cdot \gamma^4 \tilde{\varepsilon}_0 \cdot \frac{\sqrt{\kappa}}{1+\kappa} \cdot \frac{\left(\sqrt{\langle\beta_x\rangle} + \sqrt{\kappa\langle\beta_z\rangle}\right)^2}{\sqrt{\langle\beta_x\rangle \cdot \langle\beta_z\rangle^3}} \left(\Delta Q_{\max}\right)^2$$

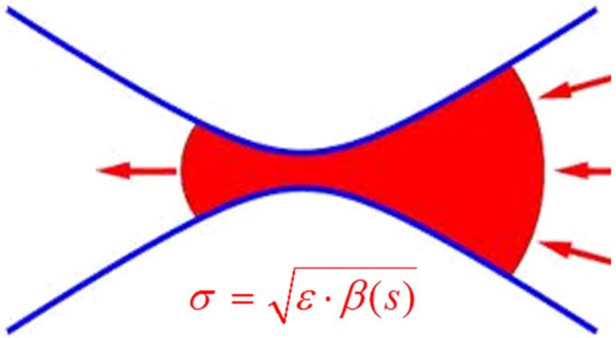
In principle the luminosity can be increased drastically ($\sim E^4$!) by elevating the beam energy – though only if the beam currents can be increased ($\sim E^3$!) simultaneously:



At any given beam energy only "tampering with" the ΔQ_{max} remains and foremost the decrease of the beta functions in the interaction zone. To that end the quadrupoles have to be spaced as tightly as possible in order to constrain chromaticity within reasonable limits (**mini beta principle)**:



Squeezing the beta function below $\beta_{min} < \sigma_s$ will not result in significantly enhanced lumi values because

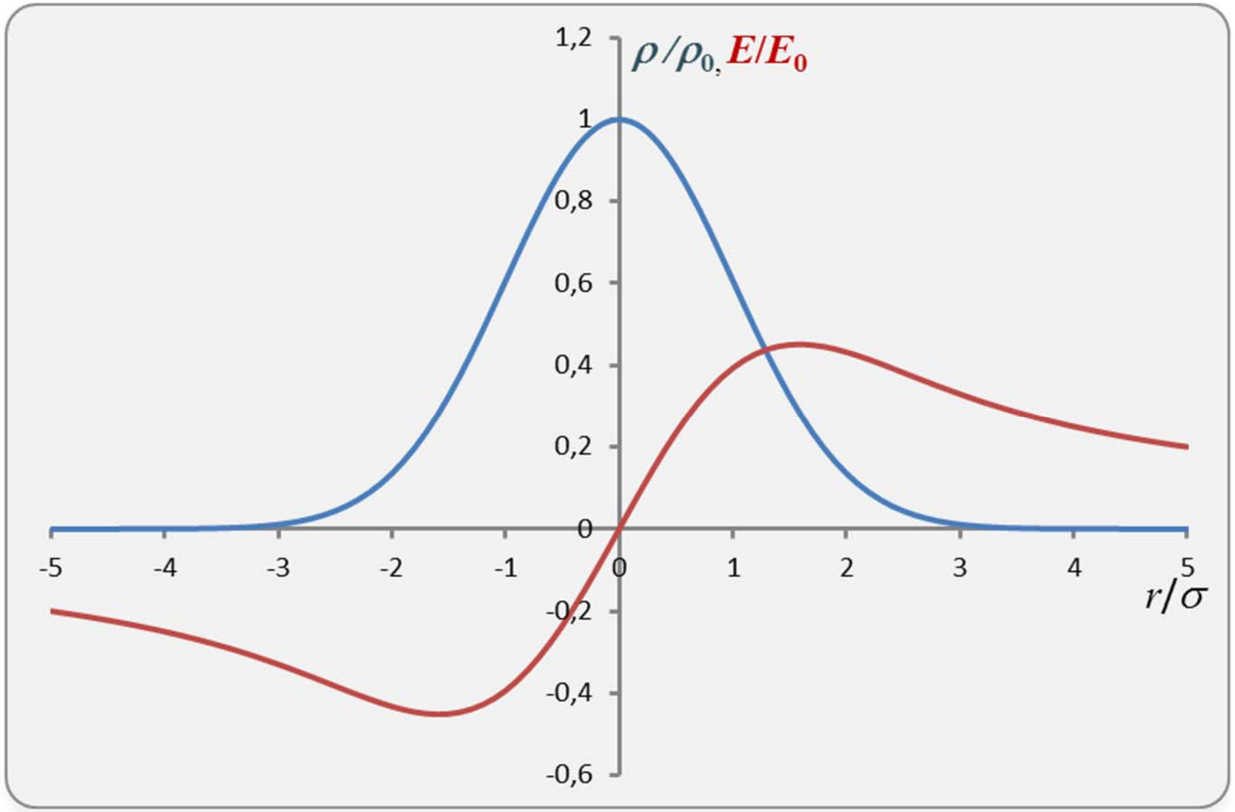


$$\beta(s) = \beta_{min} + \frac{s^2}{\beta_{min}} \quad \rightarrow \quad \langle \beta \rangle = \beta_{min} + \frac{\sigma_s^2}{3\beta_{min}}$$

indicates that the beta function will “explode” before and after the IR – which is called the ***hourglass effect***.

9.2.3. Beam-beam parameters

So far, we have calculated the tune shift ΔQ. This could in principle be compensated. According to the non-linearity (we have only considered the first order Taylor expansion of the intensity profile!), we will expect an additional tune spread from the tails of the beam



which cannot be derived analytically. Instead, the **beam-beam parameters** are defined, which assume the same amount of tune spread as has been calculated for the tune spread:

$$\xi_x = -\frac{Nr_e}{2\pi} \cdot \frac{1}{\gamma} \cdot \left\langle \frac{\beta_x^*}{\left(\sigma_x^* + \sigma_z^*\right)\sigma_x^*} \right\rangle_{2\sigma_s}$$

$$\xi_z = -\frac{Nr_e}{2\pi} \cdot \frac{1}{\gamma} \cdot \left\langle \frac{\beta_z^*}{\left(\sigma_x^* + \sigma_z^*\right)\sigma_z^*} \right\rangle_{2\sigma_s}$$

Using these parameters (and taking use of the fact, that in lepton colliders luminosity is mostly limited by ξ_z), the maximum achievable luminosity can be expressed as follows:

$$\mathfrak{L} = \frac{1}{4\pi e^2 f_0 j} \cdot \frac{I_1 \cdot I_2}{\sigma_x \cdot \sigma_z} \quad \rightarrow \quad \mathfrak{L} = \frac{\gamma}{2 e r_e}\left(1 + \frac{\sigma_z^*}{\sigma_x^*}\right) \cdot \frac{I \cdot \xi_z}{\beta_z^*} \cdot \frac{R_L}{R_{\xi_z}}$$

where the reduction due to the hourglass effect and a finite beam crossing angle is parameterized by the factors R_L and $R_{\xi z}$, respectively. There are the following approaches to maximize the luminosity in circular colliders:

- increase of the beam current (→ beam instabilities)
- increase of the number of bunches (→ long range interactions)
- crab crossing
- squeeze of vertical beta function (→ for head-on collisions $\beta_z > \sigma_s$ required!)
- nano beam scheme
- electron lenses (hadron beams)

In linear colliders, we don't have to care about the beam-beam parameters! In order to increase the luminosity, we will try to

- increase the beam current
- squeeze the beam size, e.g. the emittance and the beta function

9.3. Space charge dominated beam transport

If the effects due to space charge cannot be treated as a small perturbation another formalism has to be applied, which under the approximation of paraxial beams leads to a system of coupled differential equations – the **paraxial differential equation for KV distributions**. Thereunto the following approximations are made:

- cylindrical beam with homogeneous charge distribution
- beam of infinite length, not bunched
- changes of the envelope are small when compared to the beam diameter
- laminar flow

The cylindrical symmetry finally (without a detailed calculation) is abandoned in favor of a KV distribution.

8.3.1. Space charge force

In chapter 8.1.1. we obtained for the radially acting space charge force in case of a cylindrically symmetric beam with radius *R*:

$$F_r(r) = \frac{e\,I}{2\pi\,\varepsilon_0\,\beta c\,\gamma^2\,R^2}\cdot r$$

The change in the beam envelope thereby can be described by the following force law:

$$\gamma\, m_0\, \ddot{R} = F_r(R)$$

With $d/dt = \beta c \cdot d/ds$ we have $\ddot{R} = (\beta c)^2 \cdot R''$ and wir obtain for the envelope:

$$\boxed{R'' = \underbrace{\frac{e\,I}{2\pi\,\varepsilon_0\, m_0\,(\beta\gamma c)^3}}_{\text{generalised perveance } K}\cdot\frac{1}{R} = \frac{K}{R}}$$

The generalized perveance introduced here is – if no energy change occurs – a beam constant!

9.3.2. Emittance force

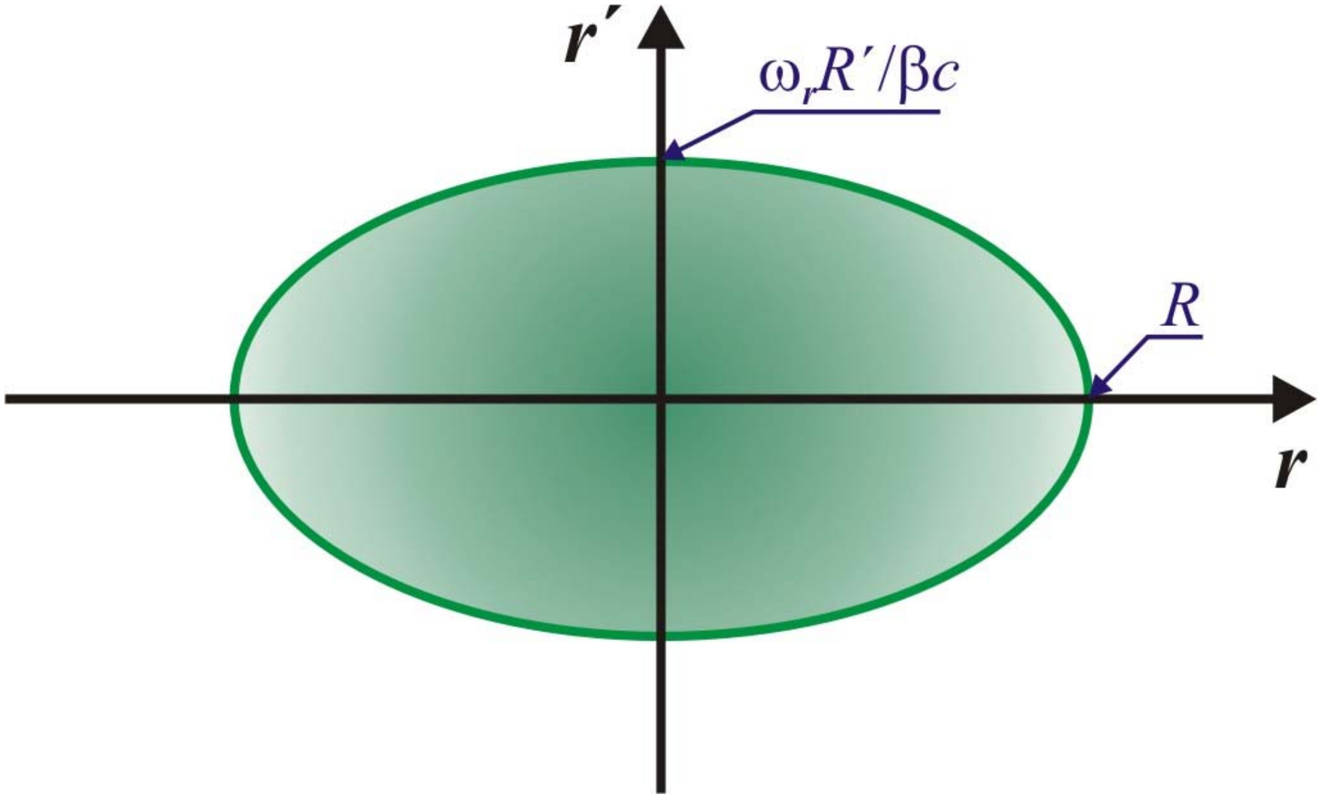
Not only the space charge force causes a widening of the beam – the inertial force of the beam, caused by the statistical distribution of the single trajectories, does so, too. It is described by the beam emittance parameter. Hence a finite emittance results in a "widening force". If we do consider the equilibrium case, the defocusing by the emittance force and the focusing by an external radially acting force $F_0 \cdot r/R_0$ just cancel each other out:

$$\gamma m_0 \ddot{r} = -F_0 \cdot \frac{r}{R_0} \qquad \Rightarrow \qquad \ddot{r} + \underbrace{\frac{F_0}{\gamma m_0 R_0}}_{=\omega_r^2} \cdot r = 0$$

We do get a harmonic oscillation as the solution which can be written due to $t = s/\beta c$ and $d/dt = \beta c \cdot d/ds$ as

$$r(s) = R \cdot \sin\frac{\omega_r s}{\beta c}, \qquad r'(s) = \frac{\omega_r R}{\beta c} \cdot \cos\frac{\omega_r s}{\beta c} \ .$$

The following phase space picture ensues:



In well-known fashion we therewith obtain the emittance ε_r:

$$\boxed{\varepsilon_r = \frac{\omega_r R^2}{\beta c}}$$

With this relation the inertial force can be expressed through the emittance:

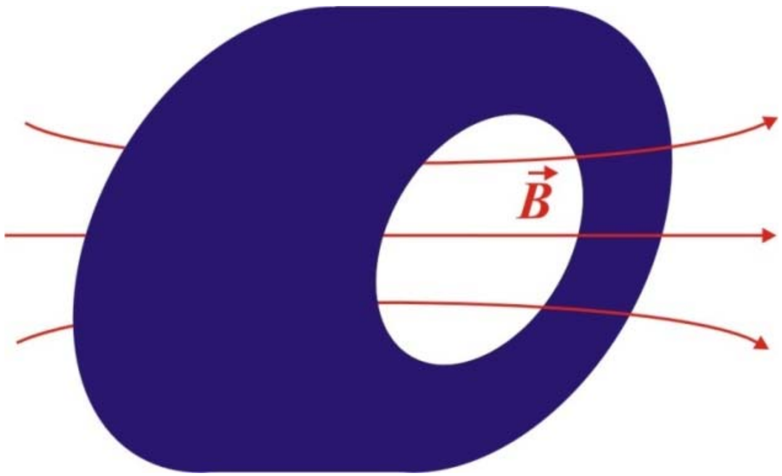
$$F_r(R) = F_0 \cdot \frac{R}{R_0} = \omega_r^{\;2} \gamma m_0 R = \frac{\varepsilon_r^{\;2} \gamma (\beta c)^2 m_0}{R^3} = \gamma m_0 \ddot{R}$$

Thus an emittance term arises in the differential equation:

$$\boxed{R'' = \frac{\varepsilon_r^{\;2}}{R^3}}$$

9.3.3. Focusing by solenoid fields

We now do want to focus the beam by magnetic fields. For low energy beams solenoid magnets do lend themselves for that purpose because they do not disrupt the cylindrical symmetry. In a solenoid only magnetic field components in *r*- and *s*-direction occur:



For the φ-component of the Lorentz force we do have in that case

$$F_\varphi = -e\cdot\left(\dot{r}\,B_s - \dot{s}\,B_r\right) = \frac{1}{r}\frac{d}{dt}\left(\gamma\, m_0\, r^2\,\dot{\varphi}\right)$$

The magnetic flux can be obtained through integration over the circular area:

$$\Phi_B = \iint\limits_{\text{circle}} \vec{B}\cdot d\vec{A} = 2\pi\cdot\int_0^{R_0} r\,B_s\,dr$$

Because of their movement the beam's particles experience a temporal change of the flux on their way through the solenoid:

$$\frac{d}{dt}\Phi_B = 2\pi \cdot \int_0^{R_0} \left(\frac{\partial(r\,B_s)}{\partial r} \cdot \frac{d\,r}{d\,t} + \frac{\partial(r\,B_s)}{\partial s} \cdot \frac{d\,s}{d\,t} \right) \cdot dr$$

which because of

$$\operatorname{div} \vec{B} = \frac{1}{r} \cdot \frac{\partial(r\,B_r)}{\partial r} + \frac{\partial B_s}{\partial s} = 0 \qquad \Rightarrow \qquad \frac{\partial(r\,B_r)}{\partial r} = -r \cdot \frac{\partial B_s}{\partial s}$$

can be written in the form $\displaystyle \frac{d}{dt}\Phi_B = 2\pi\,r\left(B_s \cdot \dot{r} - B_r \cdot \dot{s}\right)$.

If we insert this into the Lorentz force, we obtain the so-called **Busch theorem**:

$$\dot{\varphi} = \frac{-e}{2\pi\,\gamma\,m_0\,r^2} \cdot \left(\Phi_B - \Phi_{B,0}\right)$$

The magnetic flux through the solenoid can be expressed through the φ-component of the magnetic vector potential:

$$\Phi_B = \iint_A \vec{B} \cdot d\vec{A} = \iint_A \operatorname{rot} \vec{A} \cdot d\vec{A} = \oint_{\partial A} \vec{A} \cdot d\vec{s} = \int_0^{2\pi} A_\varphi \cdot r\,d\varphi = 2\pi\,r\,A_\varphi$$

Hence the well-known conservation of the azimuthal component of the canonical momentum in case of cylindrical symmetry applies:

$$p_\varphi = \gamma m_0 r^2 \dot{\varphi} + e r A_\varphi = \text{const.}$$

The constant is defined by the initial conditions and generally equals to zero. If we express the vector potential via the divergence condition through the longitudinal *B*-field, we have:

$$B_s = \frac{1}{r}\frac{\partial}{\partial r}\left(r A_\varphi\right) \qquad \Rightarrow \qquad A_\varphi = \frac{r}{2} \cdot B_s$$

Therewith we obtain out of $p_\varphi = 0$ with $\mathrm{v}_\varphi = r \cdot \dot{\varphi}$:

$$0 = \gamma m_0 r^2 \dot{\varphi} + e \frac{r^2}{2} B_s \qquad \Rightarrow \qquad \boxed{\mathrm{v}_\varphi = -\frac{e B_s}{2 m_0 \gamma} \cdot r}$$

For the corresponding radial component of the Lorentz force $F_r = e \cdot \mathrm{v}_\varphi \cdot B_s$ we have

$$F_{\text{Sol}} = -\frac{(e B_s)^2}{2\gamma m_0} \cdot r$$

If we insert this into the equation of motion we then have to be careful with the inertial term:

$$\gamma m_0 \ddot{R} - \gamma m_0 R \dot{\varphi}^2 = \gamma m_0 (\beta c)^2 R'' - \frac{\gamma m_0}{R} \left(\frac{e B_s}{2\gamma m_0} \right)^2 R^2 = -\frac{(e B_s)^2}{2\gamma m_0} R$$

After permutation we finally obtain:

$$\boxed{R'' = -\left(\frac{e B_s}{2 \beta \gamma c m_0} \right)^2 \cdot R}$$

9.3.4. Paraxial differential equation in cylindrical symmetry

If we summarize the results from the preceding sections we can describe the development of the beam envelope in cylindrical symmetry by the following equation:

$$R'' - \frac{K}{R} - \frac{\varepsilon_r^{\,2}}{R^3} + S \cdot R = 0$$

where the following parameters have been defined:

- generalized perveance $K = eI \big/ 2\pi\varepsilon_0 m_0 (\beta\gamma c)^3$
- radial beam emittance ε_r
- solenoid strength $S = \left(eB_s / 2\beta\gamma c\, m_0\right)^2$

Considering the ratio of the inertial forces and the space charge forces, one can obtain an estimate of whether the space charge forces dominate:

$$\frac{\varepsilon_r^{\,2}}{K R^2} = \frac{2\pi\varepsilon_0 (\beta\gamma c)^3 m_0}{e} \cdot \frac{\varepsilon_r^{\,2}}{I R^2}$$

For a low energy electron beam (ELSA: E=50 keV, I = 100 mA,

$\varepsilon_r = 10\ \pi \cdot \mathrm{mm} \cdot \mathrm{mrad}$, R = 5 mm) we e.g. have $\quad \dfrac{\varepsilon_r^{\ 2}}{K\,R^2} \approx 794 \cdot \dfrac{\varepsilon_r^{\ 2}}{I\,R^2} \approx 0{,}03$

9.3.5. Paraxial differential equation in KV distribution

When the beam is being focused in the quadrupoles, the cylindrical symmetry is broken because the two transverse components experience different focusing strengths each:

$$x'' = -k_x \cdot x \qquad \text{with} \quad k_x = \frac{e}{p} \cdot \frac{\partial B_z}{\partial x} = \frac{e}{\beta \gamma m_0 c} \cdot \frac{\partial B_z}{\partial x}$$

$$z'' = -k_z \cdot z \qquad \text{with} \quad k_z = \frac{e}{p} \cdot \frac{\partial B_x}{\partial z} = \frac{e}{\beta \gamma m_0 c} \cdot \frac{\partial B_x}{\partial z}$$

Therefore the paraxial differential equation has to be extended to elliptical beams with the semiaxes X and Z. In that case one is dealing with the so-called KV distribution and obtains the following coupled system:

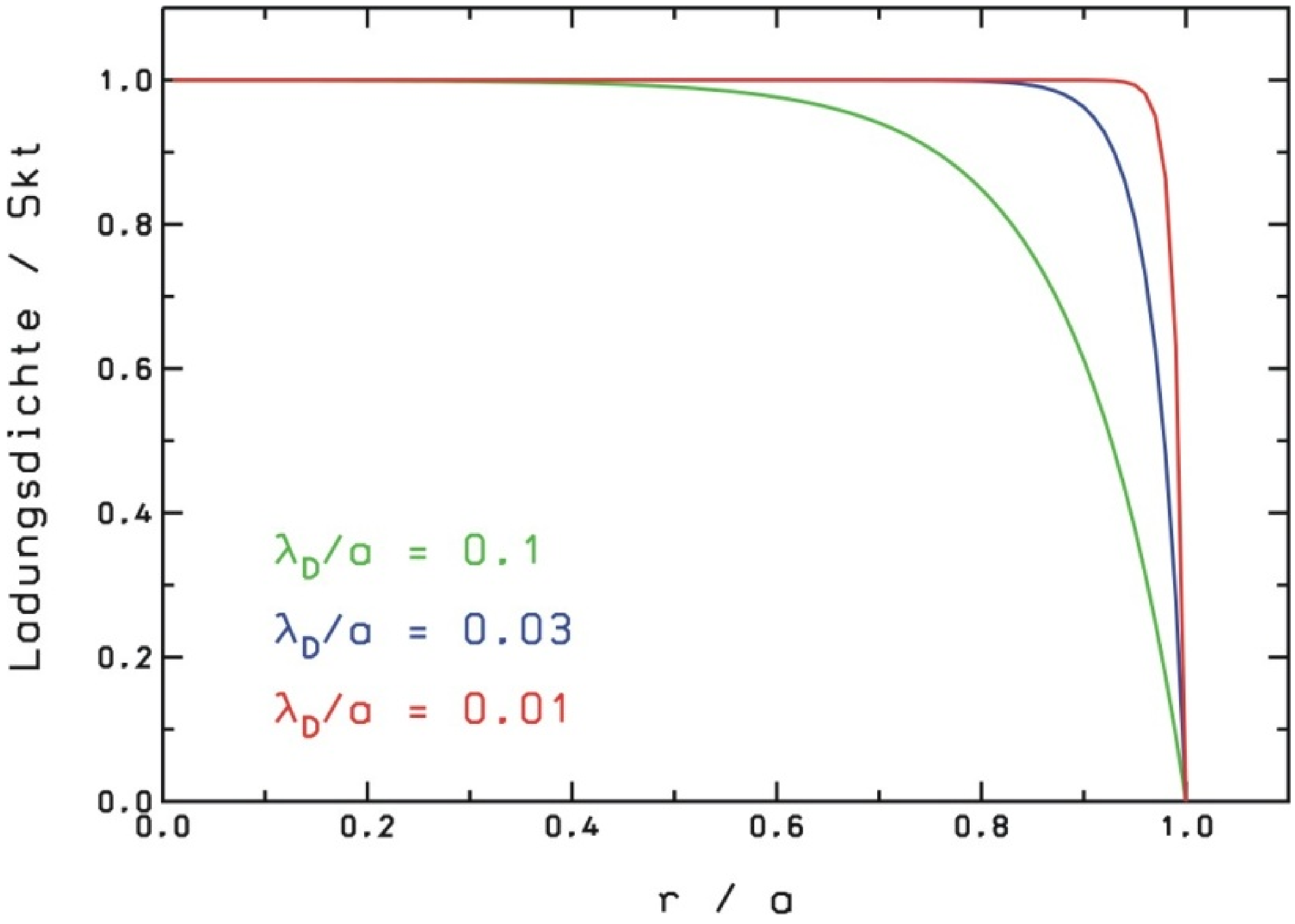
$$X'' - \frac{2K}{X+Z} - \frac{\varepsilon_x{}^2}{X^3} + k_x \cdot X = 0$$

$$Z'' - \frac{2K}{X+Z} - \frac{\varepsilon_z{}^2}{Z^3} + k_z \cdot Z = 0$$

9.3.6. Stationary intensity distribution

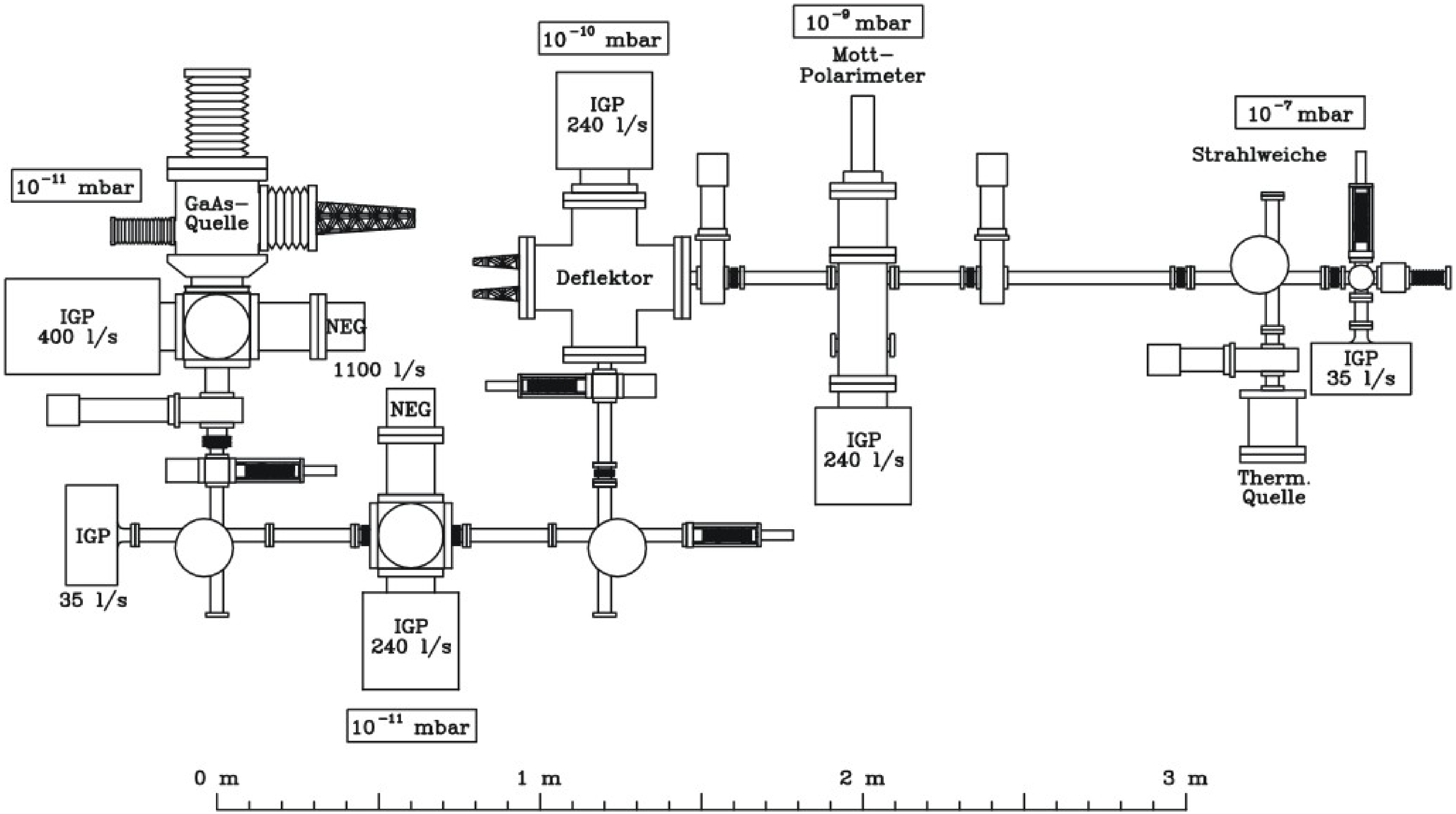
In the equilibrium case dominated by space charge we have a charge density distribution according to:

$$\rho(r) = \rho_0 \cdot \left(1 - \sqrt{\frac{a}{r}} \cdot e^{\left({r-a}/{\lambda_D} \right)} \right):$$

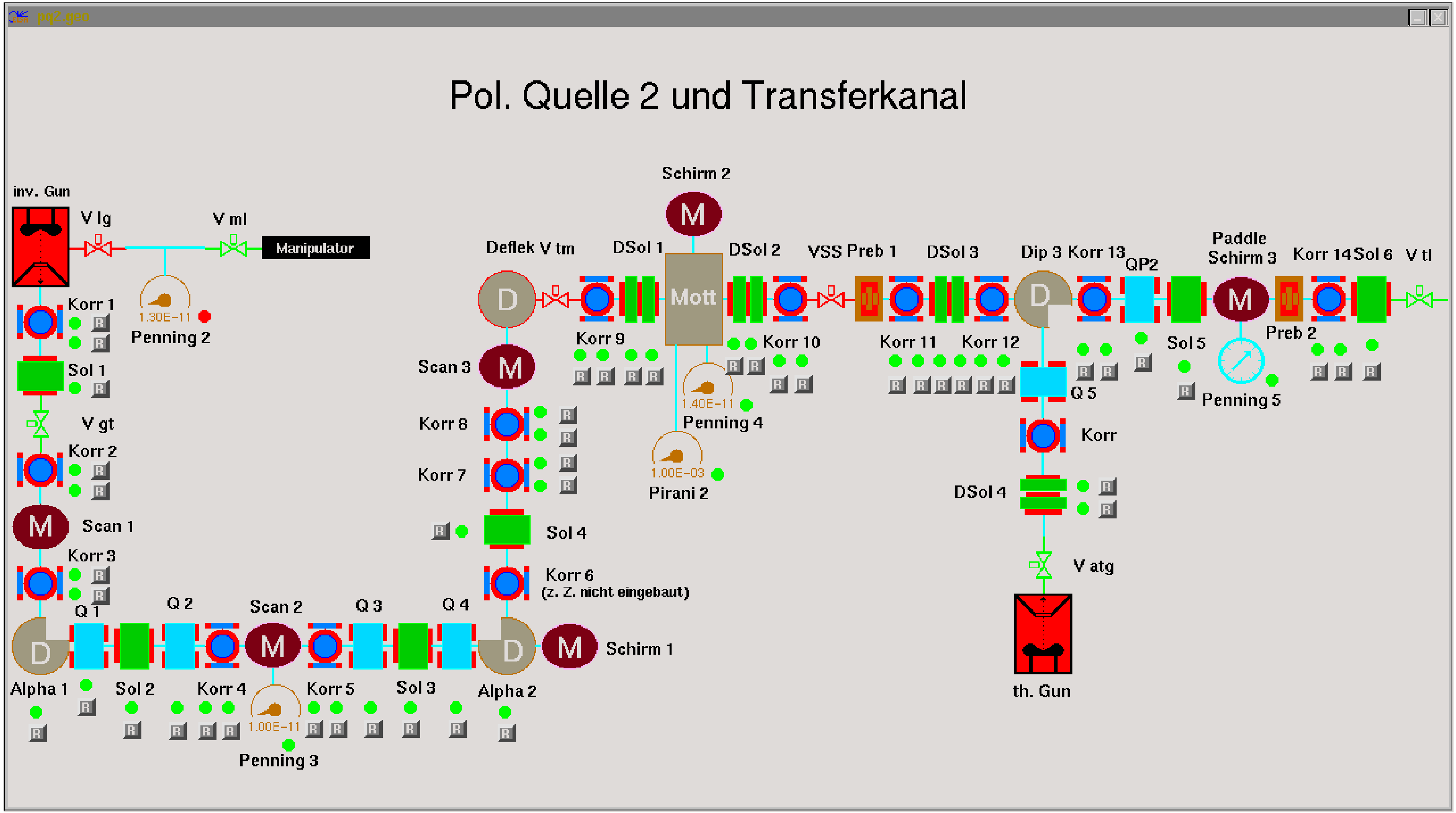

Ladungsdichte / Skt
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
r / a
λD/a = 0.1
λD/a = 0.03
λD/a = 0.01

9.3.7. Example: transfer line source 2

Mechanical setup of the beam line:



Magnet system:



- 9 solenoid magnets
- 4 quadrupole magnets
- 2 alpha magnets and 1 electrostatic deflector
- 13 horizontal and vertical correctors

Simulation of beam transport by means of the paraxial differential equation:

$$\frac{d^2x}{ds^2} + \left[k_x(s) + S(s) + T(s)\right] \cdot x - \frac{\varepsilon^2}{x^3} - \frac{2K}{x+z} = 0$$
$$\frac{d^2z}{ds^2} + \left[k_z(s) + S(s) + T(s)\right] \cdot z - \frac{\varepsilon^2}{z^3} - \frac{2K}{x+z} = 0$$

where:

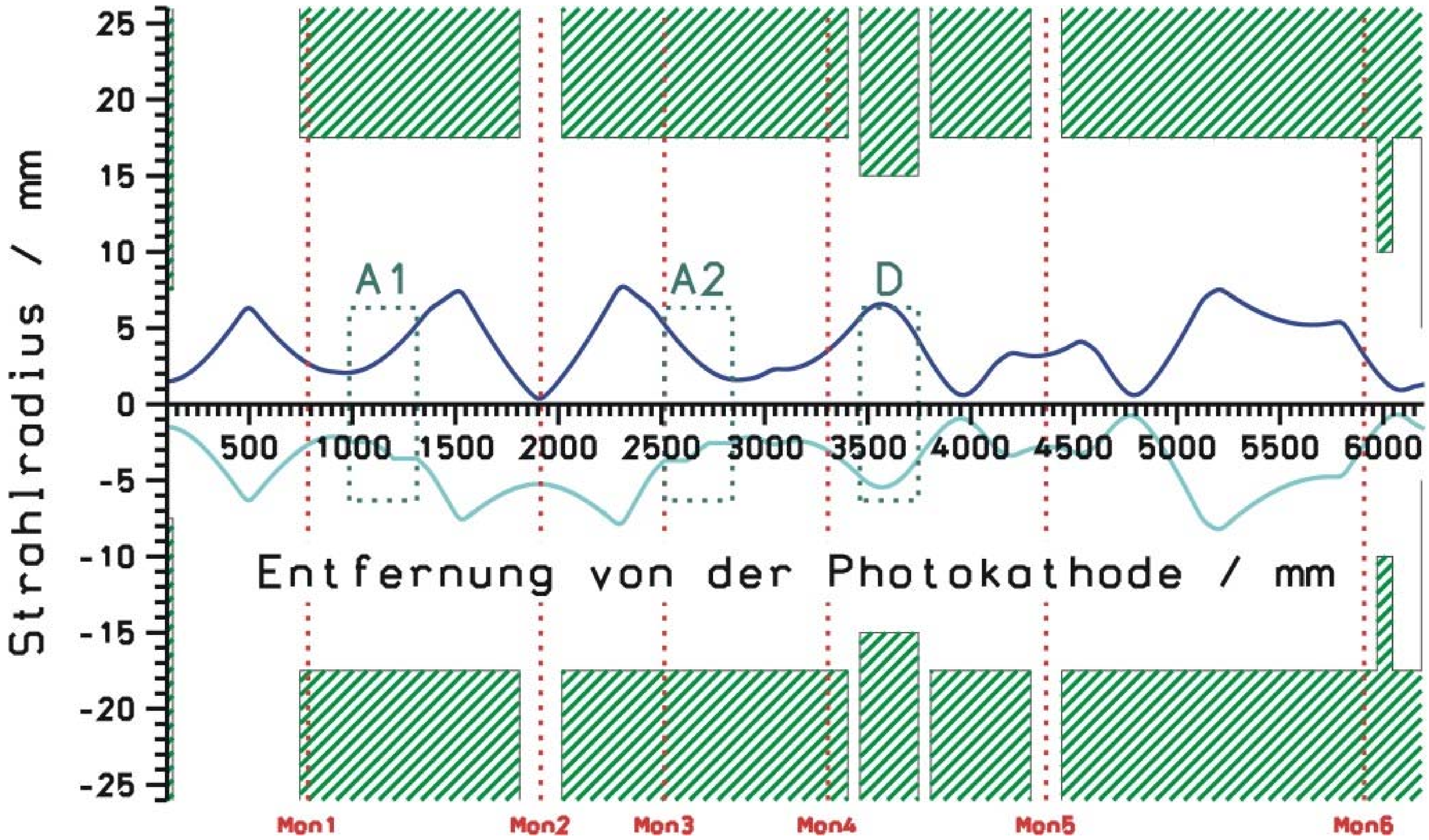
- **solenoids:** $S(s) = \left(\frac{e}{p}\frac{B_s(s)}{2}\right)^2$ and $\frac{1}{f} = \int S \cdot ds$,

  phase space rotation: $\theta(s) = \frac{e}{p}\frac{B_s(s)}{2} = \sqrt{S(s)}$

- **quadrupoles:** $k_x(s) = \frac{e}{p}\frac{\partial B_z}{\partial x}$, $k_z(s) = \frac{e}{p}\frac{\partial B_x}{\partial z}$ and $\frac{1}{f_{x,z}} = \int k_{x,z} \cdot ds$,

- **toroidal capacitor:** $T(s) = \frac{1}{r \cdot R}$ (inside) and $\frac{1}{f} = \int T \cdot ds$,

- **alpha magnet:** ≈ drift space (with different length hor. and vert.!)



9.4. Beam neutralization

The revolving particle beam collides with the molecules of the residual gas in the vacuum chamber and generates positive ions which in turn can be trapped by a particle beam with negative charge. This leads up to a shielding of the electric fields by partial neutralization of the beam's charge and thus to a

9.4.1. Tune shift

To begin with, we assume again a round electron beam of homogeneous charge as well as a round vacuum chamber in which the beam is centered. Taking the number N_i of generated ions into account we obtain the **beam neutralization**

$$\eta = \frac{N_i}{N_e} \qquad \text{or} \qquad \eta(s) = \frac{2\pi R}{N_e} \frac{d N_i}{d s}$$

The line charge λ of the beam with

$$\lambda = \frac{dN_e}{ds} = \frac{I}{e\beta c}$$

generates an electric and magnetic field (cp. chapter 8.1) and a space charge force

$$\vec{F} = \frac{e}{\gamma^2} \cdot \vec{E}$$

which in case of partial neutralization by shielding of the electric field changes into:

$$\boxed{\vec{F} = e\left(\frac{1}{\gamma^2} - \eta\right) \cdot \vec{E}}$$

Depending on the neutralization we thus obtain the following change in tune:

$$\Delta Q_{x,z}^{\text{ions}} = -\frac{e}{8\pi^2 \varepsilon_0 m_0 (\beta c)^3} \cdot \frac{2\pi R}{\gamma} \cdot \frac{I}{\varepsilon_{x,z}} \cdot \left(\frac{1}{\gamma^2} - \eta\right)$$

For ultrarelativistic electrons we get if simplifying:

$$\Delta Q_{x,z}^{\text{ions}} \overset{\gamma \gg 1}{\approx} 9{,}3 \cdot \frac{L[\text{m}] \cdot I[\text{A}]}{\gamma \cdot \varepsilon_{x,z}[\text{mm} \cdot \text{mrad}]} \cdot \eta$$

and for the accumulation of a beam current of $I = 100 \text{ mA}$ into the ELSA stretcher ring at $E = 1{,}2 \text{ GeV}$ we obtain

$$\Delta Q_{100\text{mA}\,@1{,}2\text{GeV}}^{ELSA} \approx 0{,}5 \cdot \eta$$

Neutralization degrees of a few percent thus already cause considerable changes of tune!

9.4.2. Ionization of the residual gas

The average time a revolving particle requires for generating an ion amounts to:

$$\tau_i = \frac{1}{n_i \sigma_i \beta c}, \qquad \tau_{\text{tot}} = \sum_i \frac{1}{\tau_i}$$

The cross section σ_i for the ionization is given by the Bethe Bloch formula:

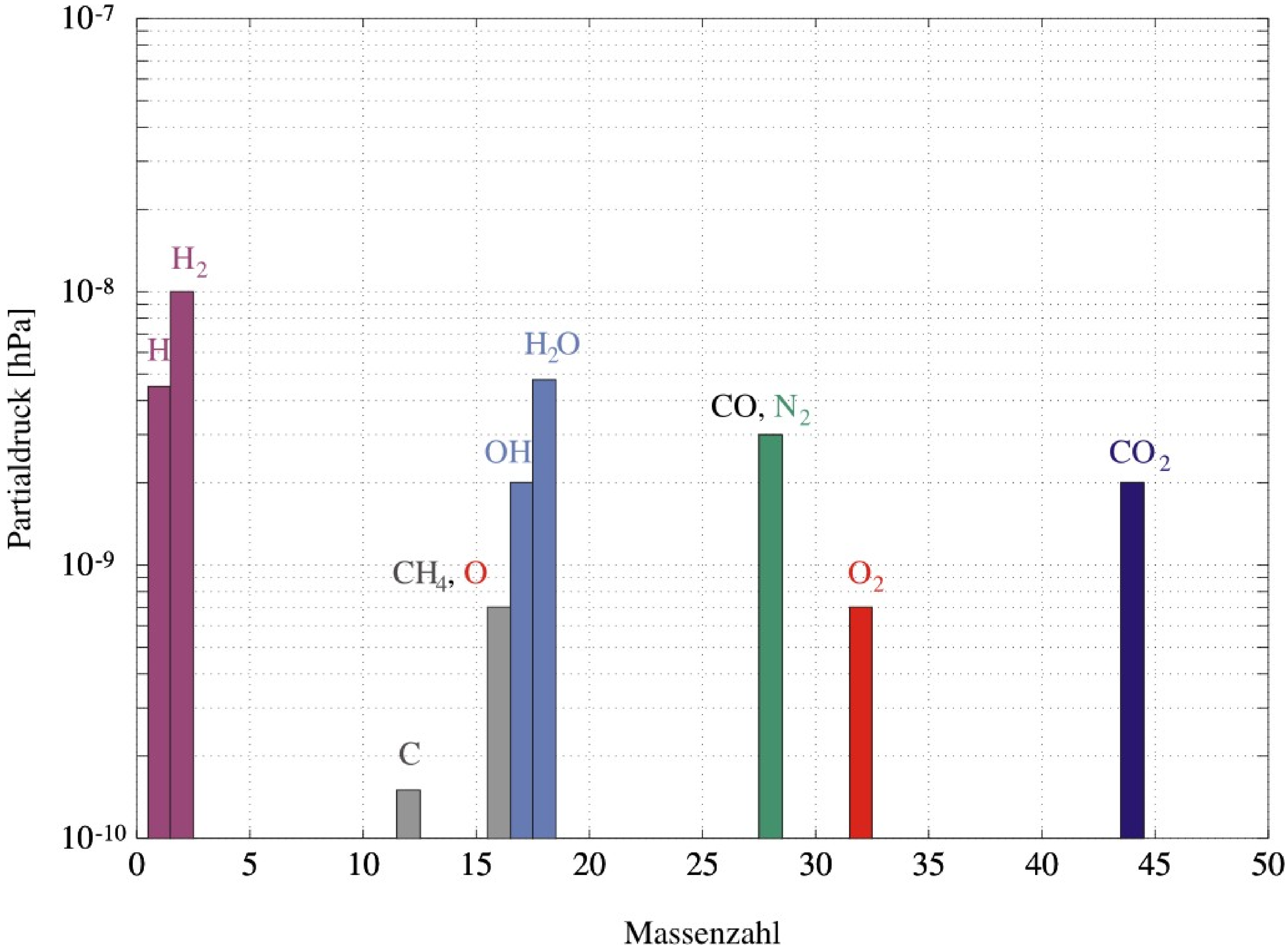
$$\sigma_i = 4\pi\left(\frac{\hbar}{m_e c}\right)^2 \cdot \left\{M_i^{\,2}\left[\frac{2}{\beta^2}\ln(\beta\gamma)-1\right]+\frac{C_i}{\beta^2}\right\} \overset{\beta\approx 1}{\approx} 1{,}874\cdot 10^{-24}\,\mathrm{m}^2\left\{A_i\cdot\ln\gamma + B_i\right\}$$

The actual time periods until a given neutralization degree is reached yet depend, in the case of a bunched beam, on the so-called "bunching factor"

$$B = \frac{\text{bucket spacing}}{\text{bunch length}}$$

$$\Delta t_\eta = \eta B \tau = \eta B \cdot \sum_i \frac{1}{n_i \sigma_i \beta c}$$

For a typical mass spectrum (ELSA: $I = 50\mathrm{mA} @ E = 2{,}3\mathrm{GeV}$):


10-7
10-8
10-9
10-10
Partialdruck [hPa]
H
H2
C
CH4, O
OH
H2O
CO, N2
O2
CO2
0
5
10
15
20
25
30
35
40
45
50
Massenzahl

one obtains the following cross sections and neutralization times ($B \approx 10$):

Molecule	A	B	σ_i [10^{-23} m^2]	Partial pressure [10^{-9} mbar]	Proportion [%]	τ_i [s]	$\Delta t_i(1\%)$ [ms]
H_2	1,0	7,6	3,0	10,0	36	0,46	46
N_2	7,4	31,1	17,5	1,5	5,5	0,52	52
CO	7,4	31,4	17,6	1,5	5,5	0,52	52
O_2	8,4	34,6	19,7	0,8	2,5	0,87	87
H_2O	6,4	29,1	15,5	4,8	17	0,18	18
CO_2	11,5	50,2	27,5	2,0	7,2	0,25	25

In the case of bunched beams not all ion species will be trapped, though:

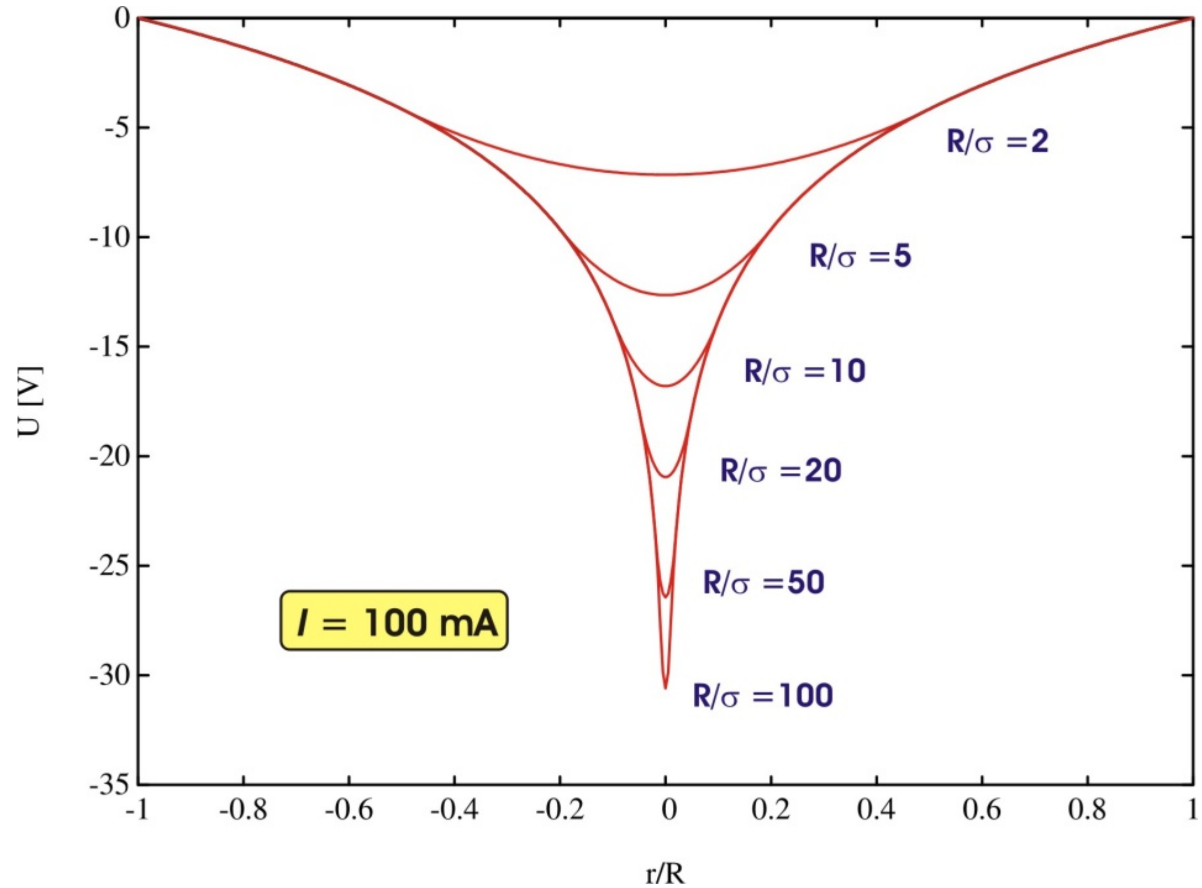
9.4.3. Ion movement

To begin with, we again restrict ourselves to a cylindrical, homogeneous beam with radius σ in a round chamber with radius R_0. The former generates the electric field

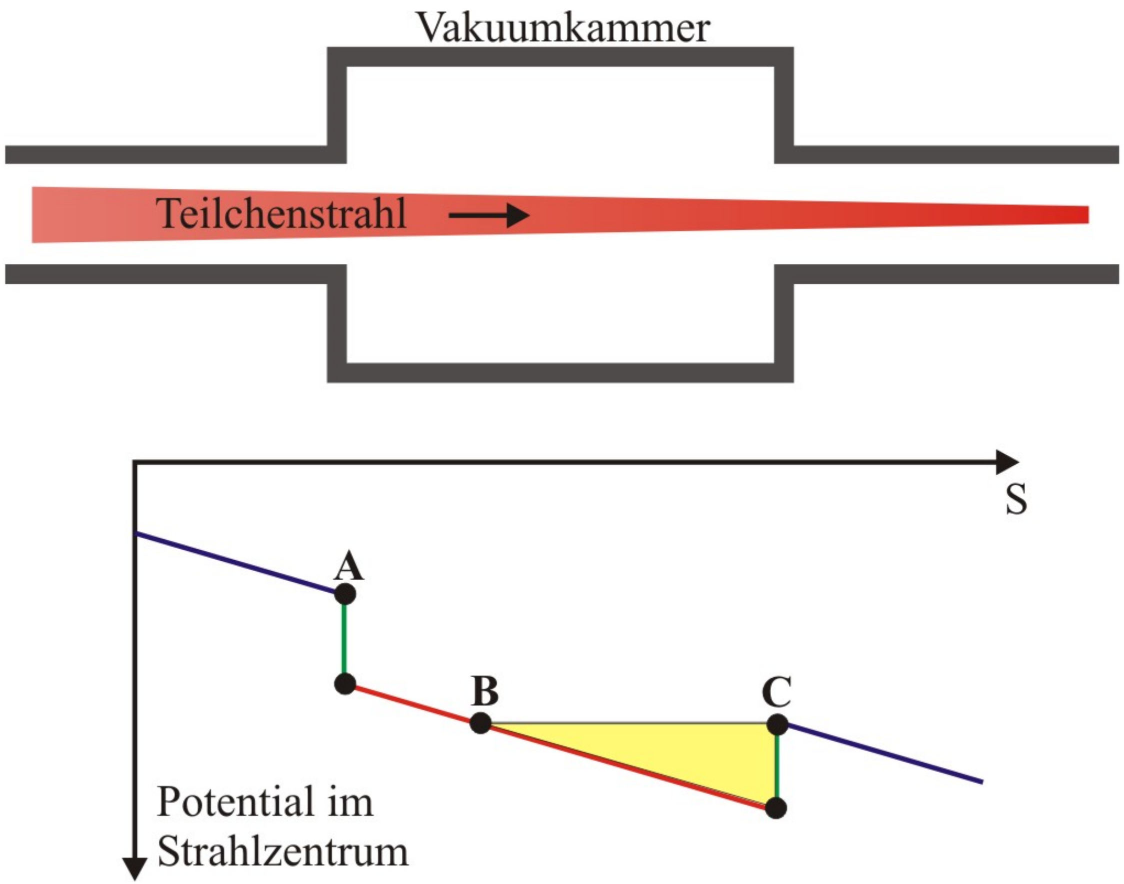
$$E_r(r) = -\frac{I}{2\pi\varepsilon_0\beta c}\cdot\begin{cases}\dfrac{r}{\sigma^2}, & \text{if } r\le\sigma\\[2mm] \dfrac{1}{r}, & \text{if } r\ge\sigma\end{cases}$$

wherefrom the following potential results:

$$U(r) = \frac{I}{2\pi\varepsilon_0\beta c}\cdot\begin{cases}\left(\dfrac{r^2}{2\sigma^2}-\dfrac{1}{2}-\ln\left(\dfrac{R_0}{\sigma}\right)\right), & \text{if } r\le\sigma\\[2mm] -\ln\left(\dfrac{R_0}{r}\right), & \text{if } r\ge\sigma\end{cases}$$



Since the beam diameters vary along the ring according to the beta functions and, as the case may be, the cross sections of the vacuum chambers alternate, too, the depth of the potential changes. Longitudinal gradients and potential wells occur:



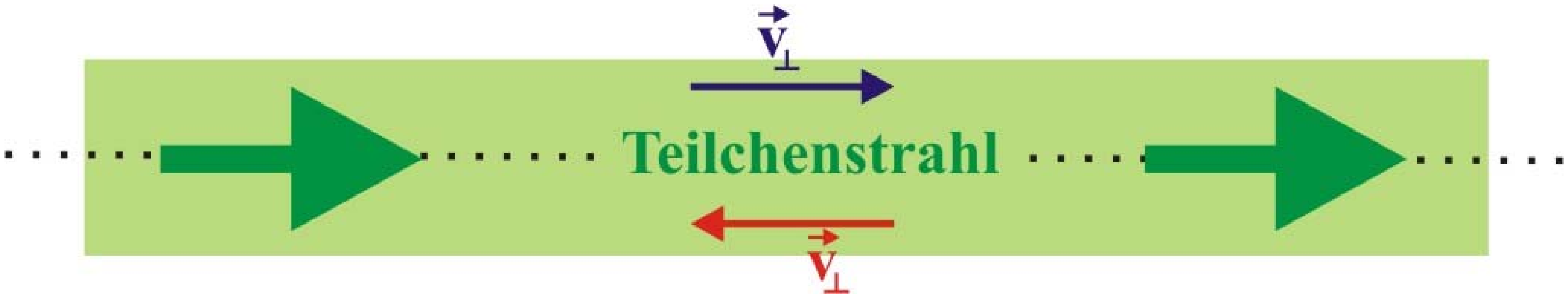
In a **homogeneous magnetic field** (e.g. in dipole magnets) the ions perform a **cyclotron movement**. If one decomposes the velocity in $\vec{v}_{\parallel}$ parallel to and $\vec{v}_{\perp}$ perpendicular to $\vec{B}$, the quantity $\vec{v}_{\parallel}$ is not altered while

$$\mathrm{v}_\perp = \omega_c \cdot r, \qquad \omega_c = \frac{eB}{m_I}, \qquad r = \frac{m\mathrm{v}_\perp}{eB}$$

Under the added influence of the longitudinal electric fields the ions are subjected to an $\vec{E} \times \vec{B}$ – drift. In equilibrium the drift speed leads to a compensation of the electric and magnetic part in the Lorentz force:

$$e \cdot \vec{E} = e \cdot \vec{\mathrm{v}}_\perp \times \vec{B} \qquad \Rightarrow \qquad \mathrm{v}_\perp = E/B,$$

which because of $E < c \cdot B$ is always possible. The drift vanishes in the centre of the beam and features differing signs at the edges:



In a **magnetic field with transverse gradients** (e.g. in quadrupole magnets) we have an additional drift:

With $B_z = B_{z,0} + \frac{\partial B_z}{\partial x} x$ and $\mathrm{v}_s = \mathrm{v}_\perp \cdot \cos\omega_c t$ as well as $x = r \cdot \cos\omega_c t$ we have

$$\mathrm{v}_s = \omega_c r \cdot \cos\omega_c t = r \frac{e B_{z,0}}{m_I} \cos\omega_c t + r \frac{e}{m_I} \frac{\partial B_z}{\partial x} \cdot r \cdot \cos\omega_c t \cdot \cos\omega_c t$$

and thus a non-vanishing average gradient drift

$$\mathrm{v}_D = \langle \mathrm{v}_s \rangle = \frac{1}{2} r^2 \frac{e B_z}{m_I} \frac{1}{B_z} \frac{\partial B_z}{\partial x} = \frac{1}{2\omega_c} \mathrm{v}_\perp^2 \frac{1}{B_z} \frac{\partial B_z}{\partial x} .$$

Ions with a component $\vec{\mathrm{v}}_{\|}$ follow in addition the curved lines of force. This causes an additional drift and we obtain altogether:

$$\mathrm{v}_D = \frac{1}{\omega_c} \left(\mathrm{v}_{\|}^2 + \frac{1}{2} \mathrm{v}_\perp^2 \right) \frac{1}{B_z} \frac{\partial B_z}{\partial x}$$

In magnetic quadrupole fields the ion movement is determined by the electric field since the kinetic energies required for the equilibrium drift cannot be generated by the electric field of the beam.

The **circumstances at bunched beams** resemble those at colliding beams. The approach to calculating the focusing is completely analogous to the one in chapter 9.2.2., only that here, the forces on a slow ($\gamma = 1$) ion are considered. In the formulas the following replacements have to be carried out:

- electron mass $m_e \to \frac{A}{n_q} \cdot m_p$, with A = mass number, n_q = charge number,
- $r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2} \to r_p = \frac{e^2}{4\pi\varepsilon_0 m_p c^2}$ (classic proton radius),
- $dt = \frac{ds}{2c} \to dt = \frac{ds}{\beta c}$ (β of the circulating beam!)
- set $\vec{F}_L = \left(1 - \vec{\beta} \cdot \vec{\beta}_{\text{Ion}}\right) \cdot e\vec{E}_\perp^L \overset{\beta_{\text{Ion}} \ll 1}{\approx} e\vec{E}_\perp^L$ and $\gamma_{\text{Ion}} \approx 1$.

Therewith we obtain for the change in angle when one of the j filled bunches with the charge N_e / j passes by

$$\Delta x' = -\frac{N_e r_p}{j\beta\sigma_r^2}\cdot\frac{n_q}{A}\cdot x \overset{\sigma_x\neq\sigma_z}{=} -\frac{2N_e r_p}{j\beta(\sigma_x+\sigma_z)\sigma_x}\cdot\frac{n_q}{A}\cdot x$$

$$\Delta z' = -\frac{N_e r_p}{j\beta\sigma_r^2}\cdot\frac{n_q}{A}\cdot z \overset{\sigma_x\neq\sigma_z}{=} -\frac{2N_e r_p}{j\beta(\sigma_x+\sigma_z)\sigma_z}\cdot\frac{n_q}{A}\cdot z$$

which in the thin lens approximation can be written in the form of the following focal lengths:

$$\boxed{\begin{aligned}\frac{1}{f_x} &= k_x\cdot l = \frac{\Delta x'}{x} = -\frac{N_e r_p}{j\beta\sigma_r^2}\cdot\frac{n_q}{A} \overset{\sigma_x\neq\sigma_z}{=} -\frac{2N_e r_p}{j\beta(\sigma_x+\sigma_z)\sigma_x}\cdot\frac{n_q}{A}\\ \frac{1}{f_z} &= k_z\cdot l = \frac{\Delta z'}{z} = -\frac{N_e r_p}{j\beta\sigma_r^2}\cdot\frac{n_q}{A} \overset{\sigma_x\neq\sigma_z}{=} -\frac{2N_e r_p}{j\beta(\sigma_x+\sigma_z)\sigma_z}\cdot\frac{n_q}{A}\end{aligned}}$$

The passing of n bunches by an ion can be written as follows in matrix notation:

$$\mathbf{M}_n = \prod_n \mathbf{M}_B\,,$$

where the passage of a single bunch can be written as

$$\mathbf{M}_B = \begin{pmatrix} 1 & \Delta l \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} = \begin{pmatrix} 1+\Delta l/f & \Delta l \\ 1/f & 1 \end{pmatrix} .$$

The movement of the ions thus is stable for

$$-2 < \mathbf{Tr}\left(\mathbf{M}_B\right) = 2 + \frac{\Delta l}{f} < 2 \ .$$

If all j bunches are filled we have $\Delta l = 2\pi R/j$ as the bunch spacing

and we obtain a lower mass limit for the ion trapping:

$$\left.\frac{A_{\text{crit.}}}{\text{n}_\text{q}}\right|_x = \frac{\pi R}{\beta j^2} \cdot \frac{N_e r_p}{\sigma_x{}^2\left(1+\sigma_z/\sigma_x\right)}$$

$$\left.\frac{A_{\text{crit.}}}{\text{n}_\text{q}}\right|_z = \frac{\pi R}{\beta j^2} \cdot \frac{N_e r_p}{\sigma_z{}^2\left(1+\sigma_x/\sigma_z\right)}$$

In most of cases the critical relative ion mass is located between 0.1 and 100 (at ELSA unfortunately as a general rule <10^{-2}).

This explains why electrons ($A/n_q \approx 1/2000$) are neither trapped by bunched proton nor by bunched positron beams!

If some ions are already trapped, they exert a defocusing influence on the ion movement. For small neutralization degrees this leads to a decrease of the critical relative ion mass → **ion conductor**! Only at high neutralization degrees the defocusing prevails and the beam does not trap further ions.

9.4.4. Implications of and countermeasures against ion trapping

The following effects have been observed at varied accelerators:

- **incoherent tune shift** (cp. 9.4.1),

- **local rise of the pressure** *dP* by increased desorption (factor κ) at the walls of the vacuum chamber (cross sectional area *A*) which is not pumped away by the pumps (effective pumping speed *S* at chamber length Δl):

$$\frac{dP}{dt} = \frac{P}{A}\cdot\left(\kappa\cdot I\cdot\frac{\sigma_i}{e}-\frac{S}{\Delta l}\right),$$

- **phase space coupling** by simultaneously occurring horizontal and vertical forces:

$$\begin{pmatrix}\Delta\dot{x}_I\\ \Delta\dot{z}_I\end{pmatrix} = \frac{N_e}{j}\frac{n_q}{A}r_p c\sqrt{\frac{2\pi}{\sigma_x{}^2-\sigma_z{}^2}}\cdot\begin{pmatrix}-\mathrm{Re}\\ \mathrm{Im}\end{pmatrix}\cdot$$

$$\cdot\left\{\frac{2}{\sqrt{\pi}}\int_0^{\frac{x+iz}{\sqrt{2\left(\sigma_x{}^2-\sigma_z{}^2\right)}}} e^{-t^2}\cdot dt - e^{-\left(\frac{x^2}{2\sigma_x{}^2}+\frac{z^2}{2\sigma_z{}^2}\right)}\cdot\frac{2}{\sqrt{\pi}}\int_0^{\frac{x\cdot(\sigma_z/\sigma_x)+iz\cdot(\sigma_x/\sigma_z)}{\sqrt{2\left(\sigma_x{}^2-\sigma_z{}^2\right)}}} e^{-t^2}\cdot dt\right\}$$

- **coherent instabilities** by retroaction of the ion oscillations on the electron beam, such as e.g. for dipole oscillations:

$$\ddot{z}_e + Q_z^2\,\omega_0^2\,z_e = -\,\omega_e^2\cdot\left(z_e - z_I\right) \qquad \ddot{z}_I = -\,\omega_I^2\cdot\left(z_I - z_e\right) \quad \text{with} \quad \omega_I^2 = \frac{2\,\lambda_e\,r_p\,c^2}{A\sigma_z\left(\sigma_x + \sigma_z\right)}, \quad \omega_e^2 = \frac{A m_p}{\gamma\,m_e}\eta\,\omega_I^2$$

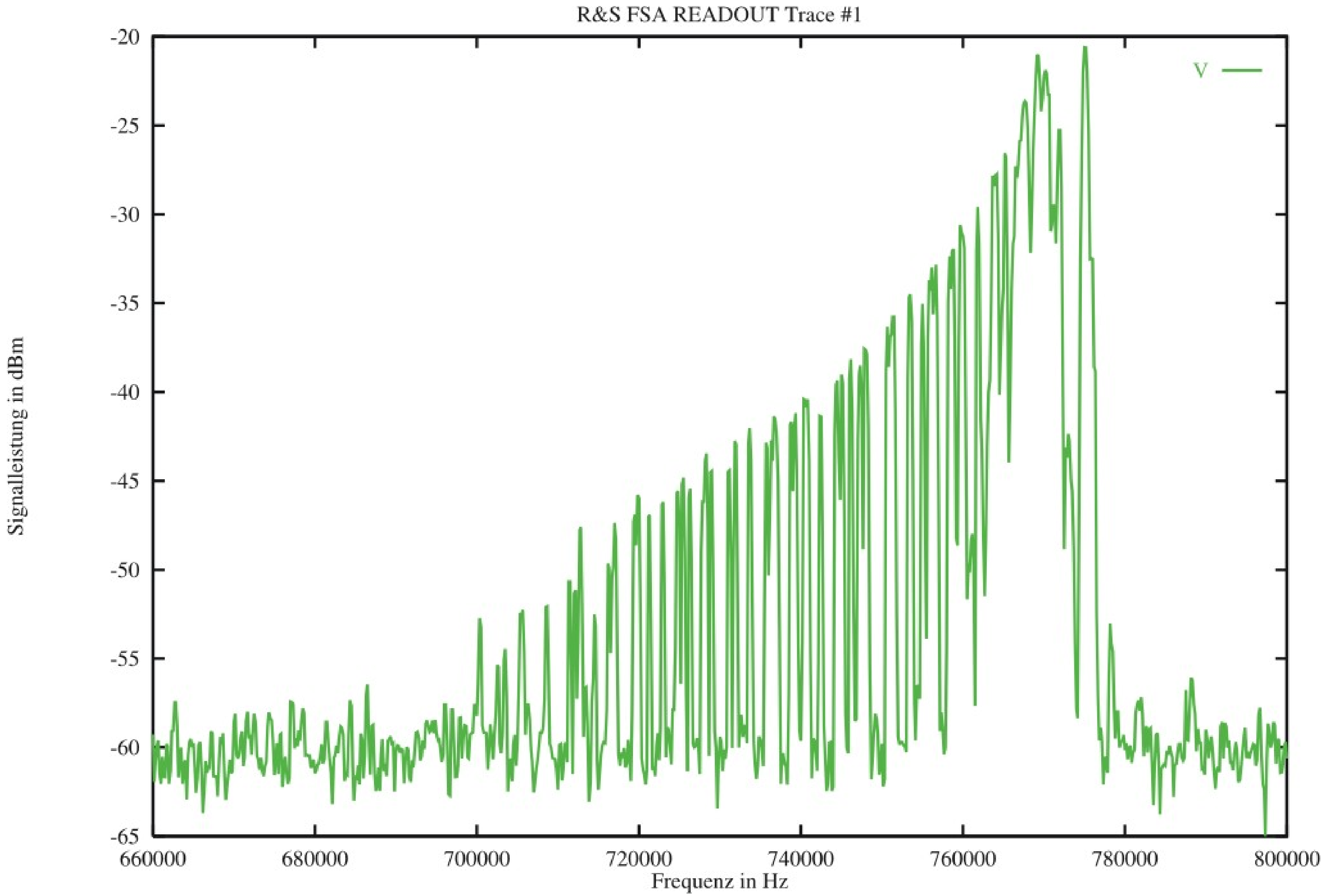
which provoke a periodic increase of the emittance.

Basically there are three measures against ion trapping:

- **inhomogeneous filling pattern,**
- **suction electrodes,**
- **resonant beam excitation.**

Example:

Spectrum of the coherent transverse beam oscillations in ELSA:



Switching on the resonant beam excitation (50 W) at 749 kHz: