Semiclassical Einstein equations: a bridge between quantum field theory and cosmology

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Plan of the talk

- Motivations
- Regularization and Backreaction
  - Semiclassical Einstein equation
  - Simple solutions
- Existence of states with good UV properties
  - Special geometry under investigation
  - Interplay between different field theories (bulk, boundaries)
  - Pullback of some states
  - Their microlocal spectral properties

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Models of the Universe: Geometry

- In first approximation: **homogeneous** and **isotropic**.
- The universe is modelled by a spacetime $M = (I \times S, g)$
  - I is the interval of the *"cosmological time"*
  - S is a 3d manifold: the *"space"*, it has an high symmetry.
- The metric g is of Friedmann Robertson Walker type

$$g = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\mathbb{S}^2(\theta,\varphi)\right].$$

- Knowing $a(t)$ is like knowing the "story" of the universe.
- Recent observations
  - $\kappa \simeq 0 \Longrightarrow$ Conformally Flat.
  - $a(t) \simeq Ae^{Ht}$, H is the Hubble parameter *(de Sitter Universe)* (very small but not zero).

Models of the Universe: Matter

- It takes the simple form $T_a{}^b = diag(-\rho, P, P, P)$
- Like a classical fluid (apart the equation of state).
- Einstein's equations become FRW equations $H = \frac{\dot{a}}{a}$

$$3H^2 = 8\pi\rho - \frac{3\kappa}{a^2}, \qquad 3\dot{H} + 3H^2 = -4\pi\left(\rho + 3P\right)$$

- Eventually we shall use

$$-R = 8\pi T, \qquad \nabla^a T_{ab} = 0$$

are equivalent up to an initial condition.

Cosmological scenario: Observation

- If we use **Radiation**, **Dust** and **cosmological constant** to model the present day observations:
  - Radiation is less important. $\rho_R \sim a(t)^{-4}$
  - We look for a mixture of $\rho_M \sim a(t)^{-3}$ and $\rho_\Lambda \sim C$

We have a problem

in modeling CMB and Supernovae red-shift observation:

Total **Energy density** is:
$\sim$ **74%** *Cosmological constant*, $\sim$ **26%** *Dust*.

Known matter: only $\sim$ **4%**.

What is the role of **quantum physics** ?

Towards quantum gravity?

- We would like to have a quantum theory of gravity and matter

  No satisfactory description.

- But we can understand how that theory should look like analyzing *some particular regimes* *[Hawking]*.
  - Quantum Fields on **fixed** curved spacetimes *(Hawking Radiation, Particle Creation)* good for the description of the metric fluctuations.
  - Backreaction in a semiclassical fashion

$$G_{ab} = 8\pi \langle T_{ab} \rangle .$$

    good for the description of "evolution" in cosmological models.

- It should work: when fluctuations of $\langle T_{ab} \rangle$ are negligible.
- Analogy in atomic physics: quantum mechanical electron with external classical field.

Semiclassical approximation

- In $G = 8\pi\langle T\rangle$ we need to compute T in some class of states.

But: in QM T_{ab} tends to be singular.

$$:T_{ab}(x): = T_{ab}(x) - \omega_0(T_{ab}(x))$$

- We need a renormalization prescription for T_{ab} on CST.
- Wald axioms $\Longrightarrow$ meaningful semiclas approx. *[Wald 77, 78]*

(1.) It must agree with **formal results** for T_{ab}
(2.) $:T_{ab}:$ in Minkowski is "normal ordering".
(3.) **Conservation**: $\nabla^a\langle T_{ab}\rangle = 0$.
(4.) **Causality**: $\langle T_{ab}\rangle$ at p depends only on $J^-(p)$.
(5.) T_{ab} depends on derivatives of the metric up to the second order (or third).

- **Generally covariant quantum field theory**.
[Brunetti Fredenhagen Verch 2003] [Hollands Wald 2003]

- The fifth, is the most problematic.

QFT: The scalar case

$$P\phi = 0\,, \qquad P = -\Box + \xi R + m^2$$

- Algebra of Fields: *Borchers Uhlmann algebra*

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}) = \bigoplus_n C_0^\infty(\mathcal{O})^{\otimes_s n}$$

Mod out I: the ideal containing the eq. of motion and the commutation relation

$$Pf = 0\,, \qquad [f,h] = i\Delta(f,h)$$

NB: we can quantize **simultaneously** on every globally hyperbolic spacetime *[Brunetti Frendehagen Verch 2003]*

- We extend it to more singular objects like $\delta(x-y)$:

$$\mathcal{O} \to \mathcal{F}(\mathcal{O}) = \bigoplus_n \mathcal{E}'(\mathcal{O})^{\otimes_s n} \qquad .....$$

- ... but we have to restrict the class of **states** *(positive linear functionals)*

Hadamard states and μSC

In $\mathbb{M}$ vacuum ω_0 is chosen selecting positive frequency

Our choice in CST is: Quasifree states that satisfy the μSC
[Kay Wald 1991] [Radzikowski 1995] [Brunetti Fredenhagen Köhler 1996]

$$WF(\omega) = \left\{((x,k_x),(y,k_y)) \in (T^*M)^2 \setminus 0 \mid (x,k_x) \sim (y,-k_y), k_x \triangleright 0\right\},$$

It is equivalent to the **Hadamard** condition

$$\omega(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y)\log\left(\frac{\sigma_\epsilon(x,y)}{\lambda}\right) + W(x,y)$$

- **Physically:** The fluctuations of the fields are always finite on Hadamard states.
- "States" that look like the vacuum in Minkowksi.

Quantum Anomalies of the Stress tensor for scalar field

On the regularized state

$$\langle \phi(x)\phi(y)\rangle_\omega := \omega(x,y) - H(x,y).$$

T_{ab} built on it has anomalies. *[Wald, Hollands Wald, Moretti]*

$$8\pi^2\langle \phi P \phi\rangle_\omega = 6[v_1], \qquad 8\pi^2\langle (\nabla_a\phi)(P\phi)\rangle_\omega = 2\nabla_a[v_1]$$

Conservation equations for T_{ab} are satisfied quantum mechanically

$$\nabla_a \langle T^a{}_b\rangle_\omega = 0$$

but (un)-fortunately the trace is different from the classical one.

$$\langle T\rangle_\omega := \frac{2[v_1]}{8\pi^2} + \left(-3\left(\frac{1}{6}-\xi\right)\Box - m^2\right)\langle\phi^2\rangle_\omega .$$

Remaining freedom

... or a look in the literature (for example *[Fulling]*) gives ($\xi = 1/6$)

$$2[v_1] = \frac{1}{360}\left(C_{ijkl}C^{ijkl} + R_{ij}R^{ij} - \frac{R^2}{3} + \Box R\right) + \frac{m^4}{4}.$$

In the trace $\Box R$. Wald's fifth axiom does not hold!

- **Other** reg. methods give different stress-energy tensors.
- **Difference:** A conserved t_{ab} build out of the metric, m and ξ.
- It must behave as $\langle T_{ab}\rangle$ under *"scale"* transformations.
- Some possibilities arises from the variation of

$$t_{ab} = \frac{\delta}{\delta g^{ab}} \int \sqrt{g}\left(C\ R^2 + D\ R_{ab}R^{ab}\right)$$

- The trace ${t_a}^a$ is proportional to $\Box R$
- We use this freedom to cancel the $\Box R$ term from $\langle T \rangle$.

Some Remarks:

- Wald's fifth axiom partially holds for $\langle T'_{ab}\rangle = \langle T_{ab}\rangle - t_{ab}$.
- **General principle of local covariance**: When regularization freedom is fixed in a region, is fixed in every spacetime. *[Brunetti Fredenhagen Verch 2003]*.
- The remaining freedom is $\langle\phi^2\rangle'_\omega = \langle\phi^2\rangle_\omega + A\ m^2 + B\ R$.
- But we can **not** cancel $[v_1]$ from $\langle T\rangle_\omega$ completely.
- Similarities with $f(R)$ gravity, but t_{ab} alone does not guaranty stable solutions.

With $\kappa = 0$ and $\xi = 1/6$, the equation $-R = 8\pi\langle T\rangle$ becomes

$$-6\left(\dot{H} + 2H^2\right) = -8\pi G m^2 \langle\phi^2\rangle_\omega + \frac{G}{\pi}\left(-\frac{1}{30}\left(\dot{H}H^2 + H^4\right) + \frac{m^4}{4}\right)$$

Massive model

Important: The quantum states enter in the equation via $\langle \phi^2 \rangle$

Physical input: We would like to use **"vacuum states"**

Impossible: Adiabatic states, have similar properties
[Parker, Parker Fulling, Lüders Roberts, Junker Schrohe, Olbermann]

- Minimize the particle creation rate. *[Parker]*
- Minimal smeared energy in the sense of Fewster. *[Olbermann]*

The ω_2 of a quasifree *(vacuum like)* state looks like

$$\omega_2(x_1, x_2) = C \int d\mathbf{k}\ e^{i\mathbf{k}(x_1 - x_2)} \frac{\overline{\chi}_k(\tau_1)}{a(\tau_1)} \frac{\chi_k(\tau_2)}{a(\tau_2)}$$

$$\chi_k(\tau)'' + (m^2 a(\tau)^2 + k^2)\chi_k(\tau) = 0$$

where τ is the conformal time: $ds^2 = a^2\left(-d\tau^2 + d\mathbf{x}^2\right)$

Adiabatic states

- The two point function ω_2 can be found in an approximated way (WKB for $\chi_k(\tau)$), as a sequence of ω_n.
- We expand it in powers of $1/m^2$

$$\langle\phi^2\rangle_{(n)} = Am^2 + BR + O\left(\frac{R}{m^2}\right)$$

- The regime $m^2 >> R$ is what we need. If $m = 1GeV$ $\frac{m^2}{R} \sim 10^{82}$

We have three parameters A, B, m.

$$\dot{H}\left(H^2 - H_c^2\right) = -H^4 + 2H_c^2 H^2 + M$$

where $H_c(B, m)$ and $M(A, m)$ are two constants

$$H_c^2 = \frac{180\pi}{G} - 1440\pi^2 m^2 B, \qquad M = \frac{15}{2} m^4 - 240\pi^2 m^4 A$$

At most two fixed stable points (**de Sitter** phases)

$$H_\pm^2 = \left(H_c^2 \pm \sqrt{H_c^4 + M}\right).$$

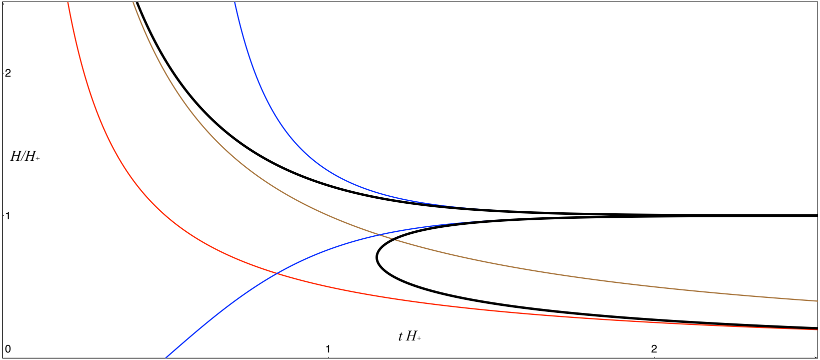
We want to have **Minkowski** $H_- = 0, \implies A = (32\pi^2)^{-1}$.

Freedom in m and B to *"Fine tune"* H_+.

The full solution

$$Ce^{4t} = e^{2/H} \left| \frac{H + H_+}{H - H_+} \right|^{1/H_+}$$

Clearly $H = 0$ and $H = H_+$ are stable solutions.



- ($m = 0$) a length scale is introduced (proportional to G). Two fixed points instead of one.
- Quantum effects are hardly negligible. [Starobinsky 80, Vilenkin 85]
- ($m \neq 0$) H_+ can be *"fine tuned"* to model the present expansion of the universe.

Classical or Quantum model?

- $\langle T_\mu{}^\nu \rangle = T_\mu{}^\nu(classic) + A_\mu{}^\nu$
- $A_\mu{}^\nu = diag(-\rho, P, P, P)$, $\qquad H := \partial_t \log a(t) = \dot{a}/a$

$$\rho = \frac{C}{4} H^4 \,, \qquad P = -\frac{C}{3}\dot{H}H^2 - \frac{C}{4}H^4$$

- it is **not** a perfect fluid (due to trace anomaly):

$$P = -\left(1 + \frac{4}{3}\frac{\dot{H}}{H^2}\right) \rho$$

Notice that

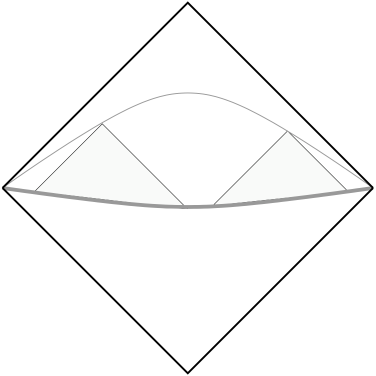
it is not a simple mixture of **dust**, **radiation** and **dark energy**

Form of the initial singularity

Question

Where is the singularity t_0 in the Penrose diagram?

$$ds^2 = a^2\left(-d\tau^2 + d\mathbf{x}^2\right).$$



- **Classical solution**
  Radiation dominated:
  $\tau = \tau_0 + A(t - t_0)^{1/2} \longrightarrow \tau_0$
  for $t \longrightarrow t_0$
  Horizon problem.

- **Quantum Correction**
  $\rho = 1/a(t)^2 \quad :$
  $\tau = \tau_0 + \log(t - t_0) \longrightarrow -\infty$
  for $t \longrightarrow t_0$
  Singularity is light like.

What have we done?

We have tried to solve the present coupled system of equations:

$$F(H,\dot{H}) = Cm^2 \int_0^\infty k^2 \frac{\overline{\chi}_k \chi_k(\tau)}{a(\tau)^2} - \frac{k^2}{a(\tau)^2\sqrt{k^2 + a(\tau)^2 m^2}} dk,$$

$$\chi_k(\tau)'' + (k^2 + a(\tau)^2 m^2)\chi_k(\tau) = 0$$

- We have found approximate solutions ($m \neq 0$).
- To improve the result: we need to characterize **unambiguously** the states...
- .. by fixing suitable initial conditions. (At which time?)
- The choice must lead to Hadamard states.

Fields on fixed background: Hadamard states on FRW

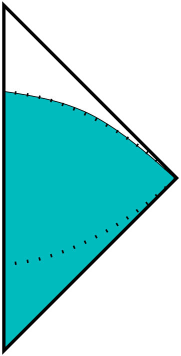
Questions:

- Do Hadamard states really exist on cosmological models?
- How to define them only employing "initial" conditions?

- Although the procedure works for $a(\tau) \simeq e^{c\tau}, \tau \to -\infty$.
- Let us choose spacetimes that **"looks like"** de Sitter (only for convenience)

$$a(t) \simeq e^{H\,t} \simeq -\frac{1}{H\tau}, \quad t \to -\infty \text{ or } \tau \to -\infty$$

- Conformal null infinity $\Im^{-}$ corresponds to the **cosmological horizon**.



QFT in the spacetime

- $\mathcal{S}(M)$ formed by real solutions $\phi_f = Ef$, $f \in C_0^\infty$

$$P\phi_f = 0 \,, \qquad P = -\Box + \xi R + m^2 \,,$$

  generated by compactly supported initial data on Cauchy surf.
- The symplectic structure $(\mathcal{S}(M), \sigma_M)$.

$$\sigma_M(\phi_f, \phi_h) = \int_\Sigma d\Sigma \, (\phi_h \nabla_n \phi_f - \phi_f \nabla_n \phi_h) = E(f, h)$$

- The Weyl operators associated to $(\mathcal{S}(M), \sigma_M)$ generate the C^*-algebra of local observables $\mathcal{W}(M)$.

Quantization on the Horizon

$\Im^-$ topologically equivalent to $\mathbb{R}\times\mathbb{S}^2$, coordinates (ℓ,θ,φ).

The symplectic space of real wavefunctions $(\mathcal{S}(\Im^-),\sigma)$:

$$\mathcal{S}(\Im^-)=\left\{\psi\in C^\infty(\mathbb{R}\times\mathbb{S}^2)\mid\psi\in L^\infty,\partial_\ell\psi\in L^1,\widehat{\psi}\in L^1,k\widehat{\psi}\in L^\infty\right\},$$

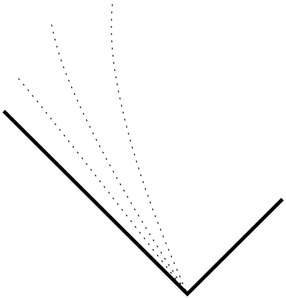
$$\sigma(\psi,\psi')=\int_{\mathbb{R}\times\mathbb{S}^2}\left(\psi\frac{\partial\psi'}{\partial\ell}-\psi'\frac{\partial\psi}{\partial\ell}\right).\quad\forall\psi,\psi'\in\mathcal{S}(\Im^-)$$

It forms symplectic structure, with data on the null surface, we can employ Weyl quantization to obtain a C^* algebra $\mathcal{W}(\Im^-)$

Preferred state on the null surface (Horizon)

- ∂_τ restricted on the Horizon $H^{-1}\partial_\ell$.
- Positive frequencies w.r.t. ∂_ℓ.

$$\widehat{\psi}(k,\theta,\varphi) = \int_{\mathbb{R}} \frac{e^{ik\ell}}{\sqrt{2\pi}} \psi(\ell,\theta,\varphi) d\ell.$$



defines a pure gaussian state

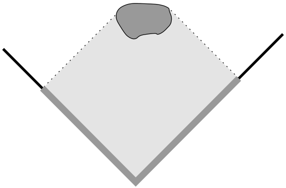
$$\lambda(W(\psi)) = e^{\frac{\mu(\psi,\psi)}{2}},$$

$$\mu(\psi,\psi') = \int_{\mathbb{R}\times\mathbb{S}^2} 2k\Theta(k)\overline{\widehat{\psi}(k,\theta,\varphi)}\widehat{\psi}'(k,\theta,\varphi)\, dk\, d\mathbb{S}^2(\theta,\varphi),$$

Projection on the horizon and pull back of the states

Analysis of modes.

$$\gamma : \mathcal{S}(M) \to C^\infty(\Im^-), \qquad \gamma(\Phi) = \Omega\Phi|_{\Im^-}$$



Theorem

The restriction γ preserves the symplectic form

Theorem

γ generates $*-$homomorphism (embedding)

$$\imath : \mathcal{W}(M) \to \mathcal{W}(\Im^-)$$

Pullback of states

Any state $\omega : \mathcal{W}(\Im^-) \to \mathbb{C}$, can be pulled back to $\mathcal{W}_M$ with $\imath^*(\omega)$.

- The preferred state

$$\lambda_M(a) := \lambda(\imath(a)). \quad \forall a \in \mathcal{W}(M)$$

- In the de Sitter spacetime, λ_M is the Bunch-Davies state.

- That state is the one considered in cosmology as the "ground state" for the analyses of **perturbations**.

- If $m \sim 0$ and $\xi \sim 0$ we have on Σ_τ

$$\lambda_M(x,y) \sim \int e^{i\mathbf{k}(x-y)} P(k) d\mathbf{k}^3 \ , \qquad P(k) \sim \frac{\alpha}{|\mathbf{k}|^{\sim 3}} + \frac{\beta}{|\mathbf{k}|^{\sim 1}}$$

Hadamard property for these states

The integral kernel of λ_M has the form

$$\lambda_M(x_1, x_2) = C \int d\mathbf{k}\; e^{i\mathbf{k}(x_1 - x_2)} \frac{\overline{\chi}_k(\tau_1)}{a(\tau_1)} \frac{\chi_k(\tau_2)}{a(\tau_2)}$$

with

$$\chi_k(\tau) \sim \frac{e^{-ik\tau}}{\sqrt{2k}}\,, \qquad \text{for } \tau \to -\infty$$

Theorem

λ_M is a distribution that satisfies the μSC

$$WF(\lambda_M) = \Gamma =$$

$$= \left\{((x, k_x), (y, k_y)) \in (T^*M)^2 \setminus 0 \mid (x, k_x) \sim (y, -k_y), k_x \triangleright 0\right\},$$

hence it is Hadamard

▸ Sketch of the proof.

Conclusion and open questions

- If $a(\tau) \sim e^{c\tau}$ for $\tau \to -\infty$ we can similarly obtain a state.
- We do not need any information about the spacetime ($a(\tau)$) to define those states.
- The initial conditions for our coupled system:

$$F(H, H') = Cm^2 \int_0^\infty k^2 \frac{\overline{\chi}_k \chi_k(\tau)}{a(\tau)^2} - \frac{k^2}{a(\tau)^2 \sqrt{k^2 + a(\tau)^2 m^2}} dk,$$
$$\chi_k(\tau)'' + (k^2 + a(\tau)^2 m^2)\chi_k(\tau) = 0$$

are

$$a(\tau) \sim e^{C\tau}, \qquad \chi_k(\tau) \sim \frac{e^{-ik\tau}}{\sqrt{2k}}, \qquad \text{for } \tau \to -\infty$$

Summary

- Quantum anomalies have an interesting backreaction effect
- Semiclassical solutions of Einstein's equation can be found
- Preferred states can be defined in cosmological models
- They have interesting properties:
  - "Positive frequency" w.r.t. the conformal time
  - Good singular behavior
  - They can be used as "initial condition" for the semiclassical problem
- It could be a companion tool in the falsification of the several proposed cosmological models

Open Questions

- What happens considering more realistic models? Different fields?
- Origin of R^2 terms in the action? Quantum gravity?
- Relation with theory of cosmological fluctuations.
- What happens quantizing $a(\tau)$?

Analysis of the "modes"

Back to theorem.

$\phi \in \mathcal{S}(M)$ can be decomposed in modes ($\mathbf{k} \in \mathbb{R}^3$, $k = |\mathbf{k}|$,)

$$\phi(\tau, \vec{x}) = \int_{\mathbb{R}^3} d^3\mathbf{k} \left[\phi_{\mathbf{k}}(\tau, \vec{x}) \widetilde{\Phi}(\mathbf{k}) + \overline{\phi_{\mathbf{k}}(\tau, \vec{x}) \widetilde{\Phi}(\mathbf{k})} \right],$$

with respect to the functions

$$\phi_{\mathbf{k}}(\tau, \vec{x}) = \frac{1}{a(\tau)} \frac{e^{i\mathbf{k}\cdot\vec{x}}}{(2\pi)^{\frac{3}{2}}} \chi_{\mathbf{k}}(\tau),$$

$\chi_{\mathbf{k}}(\tau)$, is solution of the differential equation

$$\chi_{\mathbf{k}}'' + (V_0(\mathbf{k}, \tau) + V(\tau))\chi_{\mathbf{k}} = 0,$$

$$V_0(\mathbf{k}, \tau) := k^2 + \left(\frac{1}{H\tau} \right)^2 \left[m^2 + 2H^2 \left(\xi - \frac{1}{6} \right) \right], \quad V(\tau) = O(1/\tau^3).$$

With the normalization $\overline{\chi_{\mathbf{k}}}' \chi_{\mathbf{k}} - \overline{\chi_{\mathbf{k}}} \chi_{\mathbf{k}}' = i$.

Perturbative solutions in the general case

$\chi_{\mathbf{k}}'' + (V_0(\mathbf{k},\tau) + V(\tau))\chi_{\mathbf{k}} = 0$. Normalization $\overline{\chi_{\mathbf{k}}}'\chi_{\mathbf{k}} - \overline{\chi_{\mathbf{k}}}\chi_{\mathbf{k}}' = i$.

- V perturbation potential over the de Sitter solution $\rho_{\mathbf{k}}$
- Then the general solutions $\chi_{\mathbf{k}}$

$$\chi_{\mathbf{k}}(\tau) = \rho_{\mathbf{k}}(\tau)$$
$$+(-1)^n \sum_{n=1}^{+\infty} \int_{-\infty}^{\tau} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n S_{\mathbf{k}}(\tau, t_1) S_{\mathbf{k}}(t_1, t_2) \cdots$$
$$S_{\mathbf{k}}(t_{n-1}, t_n) V(t_1) V(t_2) \cdots V(t_n) \rho_{\mathbf{k}}(t_n),$$

Absolute convergence

if $|Re\nu| < 1/2$ and $V = O(\tau^{-3})$ or
if $|Re\nu| < 3/2$ and $V = O(\tau^{-5})$

With: $\nu = \sqrt{\frac{9}{4} - \left(\frac{m^2}{H^2} + 12\xi\right)}$, $\rho_{\mathbf{k}}(\tau) \simeq \frac{e^{-i\tau k}}{\sqrt{2k}} \quad \tau \to -\infty$

Back to theorem.

Sketch of the proof. $\supset$

▸ Back to conclusions.

Having

$$\lambda_M(f, Pg) = \lambda_M(Pf, g) = 0, \qquad \lambda_M(f, g) - \lambda_M(g, f) = E(f, g),$$

then the inclusion $\supset$ descends from:
Proposition 6.1 Strohmaier Verch Wollenberg (2002).

Sketch of the proof. $\subset$

- The state can be seen as a "composition" of distribution

$$\lambda_M(f,g) = \langle T(Ef)_{\restriction \Im^-}, (Eg)_{\restriction \Im^-} \rangle.$$

- The **restriction** of one entry of E on $\Im^-$ is meaningful

$$WF(E)_{\Im^-} = \emptyset \quad \Longrightarrow \quad \widetilde{E} := E_{\restriction \Im^-} \in \mathcal{D}'(\Im^- \times M)$$

- $WF'(T) \cap WF\left(\widetilde{E} \otimes \widetilde{E}\right)_{\Im^- \times \Im^-} = \emptyset$ we can **multiply** them.
- Consider the distribution $K \in \mathcal{D}'(\Im^- \times \Im^- \times M \times M)$

$$K = (T \otimes I) \cdot \left(\widetilde{E} \otimes \widetilde{E}\right),$$

  K is the kernel of the following map

$$\mathcal{K} : C_0^\infty(\Im^- \times \Im^-) \to \mathcal{D}'(M \times M)$$

- We would like to give sense to the following expression, and to control its wave front set

$$\lambda_M(f,g) \sim \text{“}\mathcal{K}(1\otimes 1)(f\otimes g)\text{”}$$

- $\chi(\ell) \in C_0^\infty(\mathbb{R})$ such that $\chi(0) = 1$ and

$$\chi_n(\ell,\theta,\varphi) = \chi\left(\frac{\ell}{n}\right). \quad \forall n \in \mathbb{N}$$

Hence we can define the following sequence

$$\lambda_n = \mathcal{K}(\chi_n(\ell)\chi_n(\ell')) \in \mathcal{D}'(M\times M).$$

We have that

$$WF(\lambda_n) \subset \Gamma =$$

$$= \left\{((x,k_x),(y,-k_y)) \in T^*M^2 \setminus 0 \mid (x,k_x) \sim (y,k_y), k_x \triangleright 0\right\},$$

Theorem

λ_n tends to λ_M in the Hörmander topology $\mathcal{D}'_\Gamma(M \times M)$:

1. *In the topology of $\mathcal{D}'(M \times M)$*

$$\lambda_n \to \lambda_M$$

2. $$\sup_n \sup_{k \in V} |k|^N |\widehat{\lambda_n(\cdot\, \phi)}| < \infty, \qquad N = 1, 2, 3, \ldots$$

   $\phi \in C_0^\infty(M \times M)$, *The closed cone* $V \cap \Gamma = \emptyset$.

Hence $WF(\lambda_M) \subset \Gamma$

Proof: ... long and tedious computations.

Back to conclusions.

Form of the spacetime models we are considering

- If $a(t) = e^{Ht}$ we have **de Sitter** spacetime. (or $a(\tau) = -\frac{1}{H\tau}$).
- Let's assume

$$a(\tau) = -\frac{1}{H\,\tau} + O\left(\tau^{-2}\right)\ , \qquad \frac{da(\tau)}{d\tau} = \frac{1}{H\,\tau^2} + O\left(\tau^{-3}\right)\ ,$$

$$\frac{d^2a(\tau)}{d\tau^2} = -\frac{2}{H\,\tau^3} + O\left(\tau^{-4}\right)\,.$$

- For $\tau \to -\infty$ the space time **"looks like"** de Sitter. (Positive cosmological constant), exponential acceleration in the proper time t.
- **Cosmological horizon** $(\tau \to -\infty)$.