On the backreation of quantised Dirac fields on curved backgrounds

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Motivation

Quantum fields on curved spacetimes

- Quantum Field Theory on Curved Spacetimes (QFT on CST): approximate solution to the problem of formulating a quantum theory of both gravity and matter
- Matter: quantum fields
- Spacetime: arbitrary but fixed classical curved background, non-dynamical in particular

Curved spacetimes from quantum fields

- Back-reaction of the quantum field on the (curvature of) spacetime

$$G_{\mu\nu}(x) = 8\pi G\omega(:T_{\mu\nu}(x):)$$

- This can be formally derived by expanding around a vacuum solution, keeping "one-loop" ($\hbar^1$) terms of the quantum matter and "tree" ($\hbar^0$) terms of the quantum metric ...

- ... and can thus only make sense for special states or as a model equation.

- It also seems necessary to quantise matter "on all spacetimes at once".

In this talk I

- How can one sensibly define a r.h.s. for $G_{\mu\nu}(x) = 8\pi G\omega(:T_{\mu\nu}(x):)$?

- We will see that in the case of Dirac spinor fields

  1. a modified version of the classical stress-energy tensor,
  2. regularised by point-splitting and subtraction of the Hadamard singularity
  3. and evaluated on Hadamard states gives a satisfactory result.

In this talk II

- How do the solutions of $G_{\mu\nu}(x) = 8\pi G\omega(:T_{\mu\nu}(x):)$ look like?
- We will see that
  1. normal ordering in CST is ambiguous. (There is no cosmological constant problem!)
  2. The regularisation freedom can be exploited to obtain solutions which are stable and de Sitter at late times.
  3. All types of (free) quantum matter display the same behaviour.
  4. Quantum effects are strong! The picture of strong classical effects and small quantum perturbations seems incomplete.

Outline of the talk

1. The quantisation of free Dirac fields on curved spacetimes
2. The microlocal spectrum and Hadamard states
3. The expected stress-energy tensor
4. Stable cosmological solutions of $G_{\mu\nu}(x) = 8\pi G\omega(:T_{\mu\nu}(x):)$
5. Conclusions & outlook

The quantisation of free Dirac fields on curved spacetimes

The classical Dirac field on curved spacetimes I

- Spacetime: (M, g) is a fourdimensional, globally hyperbolic, oriented and time oriented, smooth manifold M with Lorentzian metric g of signature $(-, +, +, +)$.

- γ-matrices: $\{\gamma_a\}_{a=0..3} \subset M(4, \mathbb{C})$ constitute a complex irreducible representation of $Cl(3, 1)$, i.e.,

$$\{\gamma_a, \gamma_b\} \doteq \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}I_4.$$

- We choose (i times the) Dirac representation and set $\beta = -i\gamma_0$.

- There exist global vector frames e_a which can be lifted to global spinor frames E_A $\quad\rightarrow\quad$ $\psi : M \rightarrow \mathbb{C}^4$.

The classical Dirac field on curved spacetimes II

- Define a covariant derivative on spinors by lifting the Levi-Civita connection.
- Spin connection coefficients: $\sigma_b = \frac{1}{4}\Gamma^a_{bc}\gamma_a\gamma^c$
- Covariant derivative on spinor-tensors, e.g.,

$$\nabla_a \gamma^A_{bB} \doteq \gamma^A_{bB;a} = \partial_a \gamma^A_{bB} - \sigma^A_{aC}\gamma^C_{bB} + \sigma^C_{aB}\gamma^A_{bC} - \Gamma^c_{ab}\gamma^A_{cB} = 0$$

- Spin curvature tensor: $C_{ab} = \frac{1}{4} R_{abcd}\gamma^c\gamma^d$

Section spaces and Dirac conjugation

- Spaces of smooth sections (with compact support): $C^\infty(M,\mathbb{C}^4)$, $C^\infty(M,\mathbb{C}^{4*})$, $C_0^\infty(M,\mathbb{C}^4)$, $C_0^\infty(M,\mathbb{C}^{4*})$

- Global pairing of $C_0^\infty(M,\mathbb{C}^4)$ and $C^\infty(M,\mathbb{C}^{4*})$ or $C^\infty(M,\mathbb{C}^4)$ and $C_0^\infty(M,\mathbb{C}^{4*})$

$$\langle\psi'\psi\rangle \doteq \int_M d^4x\sqrt{|g|}\,\psi'(x)\psi(x)$$

- Dirac conjugation

$$^\dagger : C^\infty(M,\mathbb{C}^4) \to C^\infty(M,\mathbb{C}^{4*}), \quad \psi^\dagger(x) \doteq \psi(x)^*\beta$$
$$^\dagger : C^\infty(M,\mathbb{C}^{4*}) \to C^\infty(M,\mathbb{C}^4), \quad \psi'^\dagger(x) \doteq \beta\psi'(x)^*$$

The Dirac equations

- Feynman slash notation $\not{v} \doteq v^a \gamma_a$
- Dirac operators

$$D^{(\prime)} : C^\infty(M, \mathbb{C}^4) \to C^\infty(M, \mathbb{C}^4), \quad D^{(\prime)} : C^\infty(M, \mathbb{C}^{4*}) \to C^\infty(M, \mathbb{C}^{4*}),$$
$$D \doteq -\not{\nabla} + m, \quad D' \doteq \not{\nabla} + m$$

- Dirac equations for $\psi \in C^\infty(M, \mathbb{C}^4)$, $\psi' \in C^\infty(M, \mathbb{C}^{4*})$

$$D\psi = 0, \quad D'\psi' = 0 \quad (1)$$

- Solutions of (1) solve the spinorial Klein-Gordon equation [Lichnerowicz]

$$P\psi^{(\prime)} = 0, \quad P \doteq -D'D = -DD' = \nabla_a \nabla^a - \frac{R}{4} - m^2.$$

Algebraic Quantum Field Theory I

- No preferred states in QFT on CST $\longrightarrow$ algebraic approach makes it possible to formulate QFT without having recourse to a particular state or Hilbert space

- One seeks to define a net of *-algebras $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\subset M}$ with

  1. $\mathcal{A}(\mathcal{O})$ represents the physical observables localised in $\mathcal{O}$,
  2. $\mathcal{O}\subset\mathcal{O}'\Rightarrow\mathcal{A}(\mathcal{O})\subset\mathcal{A}(\mathcal{O}')$,
  3. $[\mathcal{A}(\mathcal{O}),\mathcal{A}(\mathcal{O}')]=0$ if $\mathcal{O}$ and $\mathcal{O}'$ are spacelike separated,
  4. ...

Algebraic Quantum Field Theory II

- Given a *-algebra $\mathcal{A}$, a state ω is a positive, normalised, linear functional on $\mathcal{A}$, i.e.,

$$\omega : \mathcal{A} \to \mathbb{C},$$

$$\omega(A^* A) \geq 0 \; \forall A \in \mathcal{A}, \quad \omega(\mathbf{1}) = 1.$$

- The relation to the Hilbert space formalism is provided by the *GNS*-representation, s.t. ω is represented as a "vacuum" vector and elements of $\mathcal{A}$ as linear operators.

- Conversely, any normalised Hilbert space vector constitutes a state on the algebra of linear operators with the *-operation given by the Hermitian adjoint.

The causal propagator of D

- To quantise, we need (anti)commutation relations, usually given by the causal propagator (commutator function) of the field theory.
- Unique fundamental solutions of D [Dimock]
  1. $S^{\pm} : C_0^{\infty}(M, \mathbb{C}^4) \to C^{\infty}(M, \mathbb{C}^4) \quad DS^{\pm} = S^{\pm} D = id_{C_0^{\infty}(M,\mathbb{C}^4)}$
  2. supp $(S^{\pm} f) \subset J^{\pm}(\text{supp } f) \ \forall f \in C_0^{\infty}(M, \mathbb{C}^4)$
  3. Causal propagator $S \doteq S^+ - S^-$

The algebra of Dirac fields

- *Borchers-Uhlmann algebra* of Dirac fields: $\mathcal{F}(M)$ is generated by the unit $\mathbf{1}$ and finite sums of products of symbols $\psi(f)$, $\psi^\dagger(g)$ with $f \in C_0^\infty(M, \mathbb{C}^{4*})$, $g \in C_0^\infty(M, \mathbb{C}^4)$ such that

  1. $f \mapsto \psi(f)$ and $g \mapsto \psi^\dagger(g)$ are $\mathbb{C}$-linear,
  2. $\psi(f)^* = \psi^\dagger(f^\dagger)$,
  3. $D\psi(f) \doteq \psi(D'f) = 0$ and $D'\psi^\dagger(g) \doteq \psi^\dagger(Dg) = 0$,
  4. $\{\psi(g), \psi^\dagger(f)\} = -i \langle g\, S(f) \rangle \mathbf{1}$ and all other anticommutators vanish.

- Analogously, one can define $\mathcal{F}(\mathcal{O})$ for $\mathcal{O} \subset M$ starting from $C_0^\infty(\mathcal{O}, \mathbb{C}^4)$ and $C_0^\infty(\mathcal{O}, \mathbb{C}^{4*})$.

The algebra of observables

- Let supp f and supp g as well as supp $f_1 \cup$ supp g_1 and supp $f_2 \cup$ supp g_2 be spacelike separated:

$$\{\psi(g), \psi^\dagger(f)\} = -i \langle g\, S(f) \rangle \mathbf{1} = 0, \quad \text{but, e.g.,}$$

$$[\psi^\dagger(f_1)\psi(g_1), \psi^\dagger(f_2)\psi(g_2)] = \cdots = 0.$$

- Possible algebras of observables

$$\mathcal{A}(\mathcal{O}) \doteq \text{even subalgebra of } \mathcal{F}(\mathcal{O})$$

- But $\mathcal{A}(\mathcal{O})$ is both "too large" and "too small", one needs to include Wick polynomials and restrict to "gauge-invariant" elements.

Locality and general covariance

- Locally covariant QFT [..., Dimock, Kay, Hollands & Wald, Verch, Brunetti & Fredenhagen & Verch, Fewster, Sanders, ...]

- The Dirac field ψ ($\psi^\dagger$) is locally covariant [Sanders]. Essentially, let

$$\chi : (M_1, g_1) \to (M_2, g_2)$$

be a map which

1. corresponds to an isometric embedding of (M_1, g_1) into (M_2, g_2),
2. preserves space and time orientation as well as causal relations,
3. and respects the spin structure,

then $\exists$ an injective *-homomorphism $\alpha_\chi : \mathcal{F}(M_1) \to \mathcal{F}(M_2)$ s.t. ψ can be understood as a collection of maps

$$\psi_M : C_0^\infty(M, \mathbb{C}^{4*}) \to \mathcal{F}(M), \quad \alpha_\chi \circ \psi_{M_1} = \psi_{M_2} \circ \chi_*.$$

The microlocal spectrum and Hadamard states

Quasifree states

- Quasifree, gauge-invariant state ω on $\mathcal{F}(M)$

$$\omega\left(\psi^\dagger(f_1)\cdots\psi^\dagger(f_m)\psi(g_1)\cdots\psi(g_n)\right)$$
$$=\delta_{mn}\sum_{\pi_m\in S_m}\prod_{i=1..m}\mathrm{sign}(\pi_m)\omega\left(\psi^\dagger(f_i)\psi(g_{\pi_m(i)})\right)$$

- Motivation: The Hilbert space obtained by a GNS construction out of a quasifree state is unitarily equivalent to a Fock space.

- $$\omega^+(f,g)\doteq\omega\left(\psi(g)\psi^\dagger(f)\right)\quad\omega^-(f,g)\doteq\omega\left(\psi^\dagger(f)\psi(g)\right)$$

- Positivity implies: $\omega^+(f,f^\dagger)\geq 0$, $\omega^-(f,f^\dagger)\geq 0$

Preferred states

- Minkowski: isometry group (Poincaré group) & spectrum condition $\Rightarrow$ unique vacuum state
- generic CST: trivial isometry group & microlocal spectrum condition (μSC) $\Rightarrow$ Hadamard state<u>s</u>
- Properties of Hadamard states:
  1. same UV behaviour as the Minkowski vacuum [Radzikowski, Köhler, Kratzert, Hollands, Sahlmann & Verch]
  2. ...
  3. well-suited for normal ordering, e.g., a definition of $\omega(:T_{\mu\nu}(x):)$ [Wald]

Why does normal ordering work in Minkowski? I

- For real scalar fields: $:\phi^2(x): \doteq \lim_{x\to y} \{\phi(x)\phi(y) - \omega(\phi(x)\phi(y))\}$
- $\Rightarrow \omega(:\phi^2(x)::\phi^2(y):) = 2\omega(\phi(x)\phi(y))^2$
- $\Rightarrow \omega_2(x,y) \doteq \omega(\phi(x)\phi(y))$ is singular, but regular enough to have a well-defined square!
- This follows from the spectrum condition:

$$\omega_2(x,y) = \int \frac{d\vec{k}}{2\omega_{\vec{k}}} e^{i\vec{k}(\vec{x}-\vec{y})-i\omega_{\vec{k}}(x_o-y_o)} = \int d^4k\ \Theta(k_0)\delta^4(k^2-m^2)e^{-ik(x-y)}$$

$$= \int d^4k\ \delta^+(k)e^{-ik(x-y)} \qquad \text{with supp}\delta^+ \subset V^+$$

Why does normal ordering work in Minkowski? II

- We can define

$$\omega_2^2(x,y) = \int d^4k \, (\delta^+ \star \delta^+)(k) e^{-ik(x-y)}.$$

- Well defined on f, $g \in C_0^\infty(M, \mathbb{R})$:

$$\begin{aligned}\omega_2^2(f,g) &= \int d^4k \, (\delta^+ \star \delta^+)(k) \hat{f}(k) \hat{g}(k) \\ &= \int d^4k \int d^4q \, \delta^+(k-q) \delta^+(q) \hat{f}(k) \hat{g}(k)\end{aligned}$$

- The integral converges since there are no large (w.r.t. Euclidean norm) k, $q \in \text{supp}\delta^+$ with $k + q = 0$.

- We need a way to say if two distributions can be multiplied on CST!

The wave front set I

- Define the *wave front set* ("the microlocal spectrum") $WF(u) \subset \mathbb{R}^n \times \mathbb{R}^n$ of $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R})'$ as follows [Hörmander]

  1. for every $x \in \mathbb{R}^n$ where u is singular, choose a test function $f \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ with $f(x) \neq 0$.
  2. $(x, k) \in WF(u)$ iff $\widehat{fu}(k)$ is <u>not</u> rapidly decreasing in the direction of $k \neq 0$ for some f.

- This definition is local and covariant under coordinate transformations. It thus generalises to CST (in contrast to the Fourier transform)!

- For $u \in C_0^\infty(M, \mathbb{R})$, $WF(u) \in T^*M \setminus \{0\}$. For vector-valued distributions, take the component-wise union.

The wave front set II

- The pointwise product of two distributions u, v is well-defined if there are no $(x,k) \in WF(u)$, $(x,q) \in WF(v)$ with $k+q=0$.

- $$WF(\omega_2(x,y)) = \{(x,y,k_x,k_y) \,|\, k_x = -k_y,\ k_x || (x-y),\ k_x^2 = 0,\ (k_x)_0 > 0\} \cup \{(x,x,k,-k) \,|\, k^2 = 0,\ k_0 > 0\}$$

  $\Rightarrow \omega_2^2$ is well-defined!

- $WF(\delta^n(x)) = \{0\} \times \mathbb{R}^n \setminus \{0\}$

  $\Rightarrow (\delta^n)^2$ is not (necessarily) well-defined!

The Hadamard condition

- Hadamard states have to be specified by constraining the singularity structure of $\omega^{\pm}(x,y)$. There are two ways to do this.

- Recall $\quad \omega^{+}(x,y) = \omega\left(\psi(y)\psi^{\dagger}(x)\right), \quad \omega^{-}(x,y) = \omega\left(\psi^{\dagger}(x)\psi(y)\right)$

- A state ω on $\mathcal{F}(M)$ fulfils the *Hadamard condition* iff

$$WF(\omega^{\pm}) = \left\{(x, k_x, y, -k_y) \in (T^*M)^{\boxtimes 2} \setminus \{\mathbf{0}\}, \mid (x, k_x) \sim (y, k_y),\ k_x \underset{\triangleright}{\triangleleft} 0\right\}$$

- $\rightarrow$ We can define normal ordering by subtracting the two point functions of Hadamard states, since $(\omega^{\pm})^2$ and $\omega^{+}(y,x)\omega^{-}(x,y)$ are well-defined distributions!

The Hadamard form

- The Hadamard condition is very powerful, but for calculation a more explicit criterion for Hadamard states is needed.

- The half squared geodesic distance $\sigma(x, y)$

- $\omega^{\pm}(x, y)$ are said to be of *Hadamard form* iff $\exists$ smooth U, V and W (the *Hadamard coefficients*), s.t.

$$\omega^{\pm}(x,y) = \pm\frac{1}{8\pi^2} D'_y \left(H^{\pm}(x,y) + W(x,y)\right),$$

$$H^{\pm}(x,y) \doteq \frac{U(x,y)}{\sigma(x,y)} + V(x,y)\ln\left(\frac{\sigma(x,y)}{\lambda^2}\right), \quad V(x,y) = \sum_n V_n(x,y)\sigma^n$$

- ω fulfils the Hadamard condition iff $\omega^{\pm}$ are of Hadamard form. [Kratzert, Hollands, Sahlmann & Verch]

Hadamard normal ordering

- Possible: definition of $:\cdot:$ by subtraction of some Hadamard state
- But this is not local and covariant (because states are not)!
- $\rightarrow$ Subtract only (appropriate derivatives of) the Hadamard singularity.
- Ambiguities:
  1. geometric ambiguities of $:\cdot:$ (scale λ)
  2. state ambiguities of $\omega(:\cdot:)$ as no preferred state exists
- All Wickpolynomials (e.g. $:T_{\mu\nu}:$) have finite fluctuations due to the Hadamard wave front set.

Determining the Hadamard coefficients

- $D'_x \omega^\pm = D_y \omega^\pm = 0 \quad \Rightarrow \quad D'_x D'_y H$, $P_y H$ smooth (H denotes either H^+ or H^-)
- We have also been able to show that $(D'_x - D_y)H$ and $P_x H$ are smooth (but non-vanishing).
- These data yield recursive differential equations for U, V and W.
- Starting with $\lim_{x \to y} U(x, y) = I_4$, one can show that U and V depend only on the local curvature and m, while W depends on the state ω.

Coinciding point limits of H

- For the calculation of the stress energy tensor we will need coinciding point limits of (derivatives) of the Hadamard distribution H.
- Notation: $[B(x,y)] \doteq \lim_{x \to y} B(x,y)$, primed indices denote vector indices at y, Tr denotes taking the trace over spinor indices, we switch from the frame basis to a coordinate basis.
- Several months of calculations ($[\sigma_{\alpha\beta\gamma\delta\varepsilon\phi\lambda}] = -\frac{1}{6} R_{\alpha\beta\gamma\delta;\varepsilon\phi\lambda} + 779$ terms) yield:

$$[P_x H] = 6[V_1]$$

$$[V_1] = \left(\frac{m^4}{8} + \frac{m^2 R}{48} + \frac{R^2}{1152} - \frac{\Box R}{480} - \frac{R_{\mu\nu} R^{\mu\nu}}{720} + \frac{R_{\mu\nu\rho\tau} R^{\mu\nu\rho\tau}}{720} \right) I_4 + \frac{C_{\mu\nu} C^{\mu\nu}}{48}$$

... and many more

The expected stress-energy tensor

The classical stress-energy tensor

- Action functional of Dirac fields

$$S[\psi] = \int_M d^4x \sqrt{|g|} L(\psi) = \int_{M^4} d^4x \sqrt{|g|} \left[\frac{1}{2} \psi^\dagger \left(D\psi \right) + \frac{1}{2} \left(D' \psi^\dagger \right) \psi \right]$$

- Cassical stress-energy tensor of Dirac fields

$$T_{\mu\nu} \doteq \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{2} \left(\psi^\dagger \gamma_{(\mu} \psi_{;\nu)} - \psi^\dagger_{;(\mu} \gamma_{\nu)} \psi \right) - \frac{1}{2} L(\psi) g_{\mu\nu}$$

- Dirac equations $\Rightarrow$

$$\nabla^\mu T_{\mu\nu} = 0 \qquad g^{\mu\nu} T_{\mu\nu} = -m\psi^\dagger \psi$$

Definition of $\omega(:T_{\mu\nu}(x):)$

- One could enlarge $\mathcal{A}(M)$ to include Wick polynomials, but here we employ a direct definition of $\omega(:T_{\mu\nu}(x):)$.

- Point-splitting along a geodesic

$$T_{\mu\nu}(x,y) \doteq \frac{1}{2}\left(\psi^\dagger(x)\gamma_{(\mu}g_{\nu)}^{\nu'}\psi(y)_{;\nu'} - \psi^\dagger(x)_{;(\mu}\gamma_{\nu)}\psi(y)\right)$$

- Subtraction of the singularity, conninciding point limit

$$\omega(:T_{\mu\nu}(x):) \doteq Tr\left[\omega(T_{\mu\nu}(x,y)) - T_{\mu\nu}^{sing}(x,y)\right]$$
$$\doteq Tr\left[D_{\mu\nu}^0\left(\omega^-(x,y) + \frac{1}{8\pi^2}D_y'H\right)\right] \doteq \frac{1}{8\pi^2}Tr\left[D_{\mu\nu}W(x,y)\right]$$

- Canonical but unsatisfactory choice of $D_{\mu\nu}^0$, $D_{\mu\nu}$

$$D_{\mu\nu}^{0,can} \doteq \frac{1}{2}\gamma_{(\mu}\left(g_{\nu)}^{\nu'}\nabla_{\nu'} - \nabla_{\nu)}\right) \qquad D_{\mu\nu}^{can} \doteq -D_{\mu\nu}^{0,can}D_y'$$

Wald's axioms I

- (A1) Given ω_1 and ω_2, such that $\omega_1^-(x,y) - \omega_2^-(x,y)$ is smooth,

$$\omega_1(:T_{\mu\nu}(x):) - \omega_2(:T_{\mu\nu}(x):) = Tr\left[D^{0,can}_{\mu\nu}\left(\omega_1^- - \omega_2^-\right)\right].$$

- (A2) $\omega(:T_{\mu\nu}(x):)$ is locally covariant: Let

$$\chi : (M_1, g_1) \mapsto (M_2, g_2),$$

$$\alpha_\chi : \mathcal{A}(M_1) \to \mathcal{A}(M_2)$$

as before. If two states ω_1 and ω_2 on $\mathcal{A}(M_1)$ and $\mathcal{A}(M_2)$ are related by $\omega_1 = \omega_2 \circ \alpha_\chi$, then

$$\omega_2(:T_{\mu_2\nu_2}(x_2):) = \chi_*\left(\omega_1(:T_{\mu_1\nu_1}(x_1):)\right).$$

Wald's axioms II

- (A3) $\nabla^\mu \omega(:T_{\mu\nu}(x):) = 0$
- (A4) On Minkowski spacetime and in the Minkowski vacuum state, $\omega_{Mink}(:T_{\mu\nu}(x):) = 0$. (drop this for cosmological applications)
- (A5) $\omega(:T_{\mu\nu}(x):)$ does not contain derivatives of the metric of order higher than two.

Uniqueness of Wald's $\omega(:T_{\mu\nu}(x):)$

- Any $\omega(:T_{\mu\nu}(x):)$ fulfilling the five axioms is unique up to a conserved local curvature term (A(4): that vanishes in locally flat regions of M). [Wald]
- Requiring appropriate scaling and analyticity in m [Hollands & Wald]: the only sensible choices are $m^4 g_{\mu\nu}$ (if we drop A(4)), $m^2 G_{\mu\nu}$, and

$$
\begin{aligned}
I_{\mu\nu} &\doteq \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g_{\mu\nu}} \int_M R^2 d\mu_g \\
&= g_{\mu\nu}\left(\frac{1}{2}R^2 - 2\Box R\right) + 2R_{;\mu\nu} - 2RR_{\mu\nu} \\
J_{\mu\nu} &\doteq \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g_{\mu\nu}} \int_M R_{\rho\tau}R^{\rho\tau} d\mu_g \\
&= \frac{1}{2}g_{\mu\nu}(R_{\mu\nu}R^{\mu\nu} - \Box R) + R_{;\mu\nu} - \Box R_{\mu\nu} - 2R_{\rho\tau}R^{\rho\ \ \tau}_{\ \mu\ \ \nu}.
\end{aligned}
$$

Which $D_{\mu\nu}$?

- $D'_y H$ does not satisfy the Dirac equations, thus $D^{can}_{\mu\nu}$ yields neither a conserved nor a traceless $\omega(:T_{\mu\nu}(x):)$.
- Possible solution (scalar case: [Moretti]): Add multiples of $L(\psi)$ to $T_{\mu\nu}$.
- This amounts to the choice

$$D^{c}_{\mu\nu} \doteq D^{can}_{\mu\nu} - \frac{c}{2} g_{\mu\nu} \left(D'_x + D_y\right) D'_y .$$

- It turns out that one can not assure both conservation and vanishing trace in the conformally invariant case!

The winner is $c = -\frac{1}{6}$.

- If we take $c = -\frac{1}{6}$, the resulting $\omega(:T_{\mu\nu}(x):)$ fulfils the first four of Wald's axioms (for a suitable choice of λ)!
- Furthermore, it exhibits the following trace (anomaly)

$$g^{\mu\nu}\omega(:T_{\mu\nu}(x):)$$

$$\begin{aligned}
&= -\frac{1}{\pi^2}\left(\frac{1}{1152}R^2 + \frac{1}{480}\Box R - \frac{1}{720}R_{\mu\nu}R^{\mu\nu} - \frac{7}{5760}R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau}\right) \\
&\quad - \frac{1}{\pi^2}\left(\frac{m^4}{8} + \frac{m^2 R}{48}\right) + mTr\left[D'_y W(x,y)\right] \\
&= \frac{1}{2880\pi^2}\left(\frac{7}{2}C_{\mu\nu\rho\tau}C^{\mu\nu\rho\tau} + 11\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) - 6\Box R\right) \\
&\quad - \frac{1}{\pi^2}\left(\frac{m^4}{8} + \frac{m^2 R}{48}\right) + mTr\left[D'_y W(x,y)\right].
\end{aligned}$$

Sketch of the proof

- Leaving c unspecified, one computes

$$8\pi^2\nabla^\mu\omega(:T_{\mu\nu}(x):) = (1+6c)\,Tr[V_1(x,y)]_{;\nu}$$

and $\quad 8\pi^2 g^{\mu\nu}\omega(:T_{\mu\nu}(x):) = 6(4c+1)\,Tr[V_1(x,y)] + m\,Tr\left[D'_y W^-(x,y)\right].$

This gives (A3) and the trace.

- (A1) holds for Hadamard states ω, since adding multiples of $L(\psi)$ to $T_{\mu\nu}$ amounts to adding multiples of $Tr[V_1]$ to $\omega(:T_{\mu\nu}(x):)$.

- (A2) holds since $\omega(:T_{\mu\nu}(x):)$ is constructed entirely out of ω^- and H; these are preserved by χ.

Comments

- Scalar fields: Similar results are available. [Moretti]
- Dirac fields: Trace anomaly has already been computed, though based on a non-rigorous "heat-kernel-expansion". [Christensen & Duff]
- $\lambda \to \lambda' \quad \Rightarrow \quad \omega(:T_{\mu\nu}(x):)$ changes by multiples of

$$Tr[D_{\mu\nu}^{-\frac{1}{6}} V] = \frac{m^4}{2} g_{\mu\nu} - \frac{m^2}{6} G_{\mu\nu} + \frac{1}{60}\left(I_{\mu\nu} - 3J_{\mu\nu}\right)$$

- Assuring (A5) therefore seems impossible for $m = 0$, but is possible for the trace.
- Different point of view: Defining both $:T_{\mu\nu}(x):$ and $:\nabla^\mu T_{\mu\nu}(x):$ as locally covariant quantum fields and using the renormalisation freedom (via further requirements) to assure $:\nabla^\mu T_{\mu\nu}(x): \equiv 0$. [Hollands & Wald]

Stable cosmological solutions of $G_{\mu\nu}(x) = 8\pi G\omega(:T_{\mu\nu}(x):)$

Conclusions & outlook

Conclusions & Outlook

- We have been able to define an (almost) sensible sourceterm for the semiclassical Einstein equation.
- In Robertson-Walker spacetimes one can [Dappiaggi, Fredenhagen, Pinamonti]
  1. re-express $G_{\mu\nu}(x) = 8\pi G\omega(:T_{\mu\nu}(x):)$ as an equation for the traces
  2. and obtain solutions, stable at late times, which offer a potential description of "dark energy".
  3. How do these solutions look like for interacting fields?
- Maybe one can fulfil (A5) in the general case for special states?

Thank you for your attention!