Towards a Rigorous Functional Framework for Classical Field Theory: Algebraic Structure and Dynamics

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Introduction

Nowadays the framework of classical field theory most commonly employed by physicists is based on (formal) functional methods, tailored to the needs of (path-integral-based) quantum field theory.

- Heuristic infinite-dimensional generalisation of Lagrangian mechanics;
- To make it mathematically rigorous is possible, but technically quite complicated at the level of generality stated above – usually done in the (restricted) setting of Banach spaces, doing it at the (more natural) level of Fréchet spaces (e.g. smooth field configurations) precludes the use of many local analytic tools.

Common (albeit subtle) conceptual misunderstanding lies in the 1st. point. Namely, we emphasize:

> *"Classical field theory is not as 'infinite dimensional' as it might appear!"*

The underlying reason is locality (in spacetime).

Several reasons to emphasize locality:

1. Euler-Lagrange equations of motion are PDE's (put in another way: variational principle is local);
2. Physical classical field theories are hyperbolic $\Rightarrow$ propagation speed of dynamical effects is finite (relativistic microcausality);
3. Locality and relativistic microcausality play a pivotal role also in rigorous approaches to quantisation.

Let's be more precise about all this. Let $(\mathscr{M}, g)$ be a globally hyperbolic Lorentzian manifold, with volume element $\mathrm{d}\mu_g = \sqrt{|\det g|}\mathrm{d}x$, $\mathscr{E} \xrightarrow{p} \mathscr{M}$ a (vector) bundle over $\mathscr{M}$ endowed with a connection ∇, i.e. a global section of the 1st order jet bundle $J^1\mathscr{E} \xrightarrow{t} \mathscr{E}$. We'll call $\Gamma^\infty(\mathscr{M}, \mathscr{E})$ a(n off-shell) field configuration space.

Kinematics and observables of classical field theory

$\Gamma^\infty(\mathscr{M},\mathscr{E})$ is endowed with the usual Fréchet topology. We'll single out our class of observable quantities:

Definitions

We say that a function(al) F defined on $\Gamma^{infty}(\mathscr{M},\mathscr{E})$ is:

Smooth if for all k the k-th order derivatives

$$F^{(k)}[\phi](\delta\phi^{\otimes k}) \doteq \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\restriction_{\lambda=0} F(\phi+\lambda\delta\phi)$$

exist as jointly continuous maps from $\Gamma^\infty(\mathscr{M},\mathscr{E})^{k+1}$ to $\mathbb{R}$. In particular, $F^{(k)}[\phi]$ is a distribution (density) of compact support;

Additive if for all $\phi_1,\phi_2,\phi_3 \in \Gamma^\infty(M,\mathscr{E})$ such that $\mathrm{supp}\phi_1 \cap \mathrm{supp}\phi_3 = \varnothing$, then

$$F(\phi_1+\phi_2+\phi_3) = F(\phi_1+\phi_2) - F(\phi_2) + F(\phi_2+\phi_3);$$

Local if it's additive and, for all $\phi \in \Gamma^\infty(\mathscr{M}, \mathscr{E})$, $WF(F^{(k)}[\phi]) \perp T\Delta^k(\mathscr{M})$, where $\Delta^k(\mathscr{M}) \doteq \{(x, \dots, x) \in \mathscr{M}^k : x \in \mathscr{M}\}$ is the thin diagonal of $\mathscr{M}^k$. In particular, $F^{(1)}[\phi]$ is a smooth function for each fixed ϕ.

Remarks:

- If we define the spacetime support of a functional F as

  $$\text{supp}F \doteq \mathscr{M} \backslash \{x \in \mathscr{M} : \exists U \ni x \text{ open s.t. } \forall \phi, \psi, \text{supp}\phi \subset U, F(\phi + \psi) = F(\psi)\},$$

  additivity implies that any functional with compact spacetime support can be decomposed as a finite sum of functionals of arbitrarily small support.

- For smooth, spacetime compactly supported functionals, additivity also entails that $\text{supp}F^{(k)}[\phi] \subset \Delta^k(\mathscr{M})$, and locality, that F must be of the form

  $$F(\phi) = \int_{\mathscr{M}} j^m\phi^* \mathscr{L}(x) \mathrm{d}\mu_g(x),$$

  with $\mathscr{L} \in \mathscr{C}_c^\infty(J^m\mathscr{E})$ for some m.

- The idea that additivity should characterise locality for nonlinear functionals goes back to [Kantorovitch-Pinsker '38, '39], in the context of generalised random processes, and it has arisen occasionally in probability [Gel'fand–Vilenkin '64, Rao '71, '80] and in the study of certain nonlinear integral equations [Chacón–Friedman '65, Krasonel'skii '65, Friedman-Katz '69], but has remained unknown in classical field theory.
- Recent developments regarding the renormalisation group in perturbative algebraic quantum field theory [Brunetti–Dütsch–Fredenhagen '09, arXiv:0901.2038; Brunetti–Fredenhagen '09, arXiv:0901.2063] have singled out the class of smooth, compactly spacetime supported local functionals (notation: $\mathfrak{F}_{loc}(\mathscr{M},\mathscr{E})$) as the most relevant one for classical field theory.
- $\mathfrak{F}_{loc}(\mathscr{M},\mathscr{E})$, however, is not closed under (pointwise) products. A slightly larger class is given by the microcausal functionals

$$\mathfrak{F}_{mc}(\mathscr{M},\mathscr{E}) = \{F : \Gamma^{\infty}(\mathscr{M},\mathscr{E}) \to \mathbb{R} \text{ smooth: supp}F \text{ compact},$$

$$WF(F^{(k)}[\phi]) \cap ((\mathscr{M} \times (J^{+}(0)))^{k} \cup (\mathscr{M} \times (J^{-}(0)))^{k}) = \varnothing, \forall\phi\},$$

which is closed under products. Moreover, the latter is also closed under Poisson brackets (to be defined later).

Local dynamics and Main Result

Typical elements of $\mathfrak{F}_{loc}(\mathscr{M},\mathscr{E})$ are the action functionals

$$S(f)[\phi] = \int_{\mathscr{M}} f(x) j^{m-1}\phi^* \mathscr{L}(x) \mathrm{d}\mu_g(x),\ f \in \mathscr{C}_c^\infty(\mathscr{M}).$$

In the remainder of the talk, we shall set $m = 2$ and be interested in the following problem: consider for concreteness...

- A scalar field $\phi \in \Gamma^\infty(\mathscr{M},\mathscr{E}) \doteq \mathscr{C}^\infty(\mathscr{M})$ in a globally hyperbolic spacetime $(\mathscr{M}, g)$, and
- Two (1st-order) action functionals

  $$S_i(f)[\phi] = \int_{\mathscr{M}} f(x) \mathscr{L}_i(x, \phi(x), \partial^1\phi(x)) \mathrm{d}\mu_g(x),\ i = 1, 2$$

  with (semilinear, strictly hyperbolic) Euler-Lagrange derivatives

  $$S_{i(1)}[\phi] = \nabla_a \frac{\partial \mathscr{L}_i}{\partial \nabla_a \phi} - \frac{\partial \mathscr{L}_i}{\partial \phi},$$

  such that

i.) $S_2 \doteq S$ is quadratic ("free"),

ii.)

$$S_1 - S_2 = \lambda F(h) = \lambda \int_{\mathscr{M}} \sqrt{|\det g(x)|} \mathrm{d}x h(x) \mathscr{L}_{int}(x, \phi(x), \partial^1 \phi(x))$$

with $h \in \mathscr{C}_c^\infty(\mathscr{M})$ ("spacetime-cutoff" interaction term), $\lambda > 0$, and

iii.) $F(h)_{(1)}[\phi]$ depends pointwise on ϕ and at most its first derivatives $\nabla\phi$.

We want to

Main Goal & Definition

Prove the existence of a map $r_{S_1,S_2} : \mathscr{C}^\infty(\mathscr{M}) \to \mathscr{C}^\infty(\mathscr{M})$ such that

$$S_{1(1)} \circ r_{S_1,S_2} = S_{2(1)}, \tag{1}$$

$$r_{S_1,S_2}(\phi)(x) = \phi(x), x \notin J^+(\mathrm{supp} h). \tag{2}$$

We call r_{S_1,S_2} the retarded Møller operator of S_1 w.r.t. S_2.

- r_{S_1,S_2} appears naturally in the context of perturbative algebraic QFT (Dütsch–Fredenhagen CMP'03, Brunetti–Fredenhagen *ibid.*, Brunetti–Dütsch–Fredenhagen *ibid.*), where h plays both the role of an IR regulator and of a localization for the algebra of perturbative interacting fields.
- When acting on solutions of $S_{2(1)}[\phi] = 0$ r_{S_1,S_2} can be seen as an intertwiner of (on-shell) covariant phase spaces or, equivalently, as the solution of a "covariant" Cauchy problem.
- (1)–(2) also mean that $r_{S_1,S_2}(\phi)$ solves an inhomogeneous (off-shell) nonlinear hyperbolic PDE with prescribed initial conditions in the past of supp$h$ $\Rightarrow$ very few rigorous well-posedness results exist, qualitative behaviour of solutions can be dramatically changed – (parabolic) example: (incompressible) Navier-Stokes equations

$$\underbrace{\partial_t \mathbf{v} - \nu \triangle \mathbf{v}}_{\text{free (heat) part}} + \underbrace{(\mathbf{v}.\nabla)\mathbf{v}}_{\text{interaction}} = \underbrace{-\nabla p}_{\text{source}} \to \begin{cases} \nabla p = 0 : \text{only laminar flow} \\ \nabla p \neq 0 : \text{turbulence!} \end{cases}$$

Coupling as an off-shell flow parameter $\Rightarrow$ Main Claim

- It's clear that r_{S_1,S_2} exist on shell whenever local well posedness for $S_{1(1)}[\psi] = (S + \lambda F(h))_{(1)}[\psi] = 0$ in a ngb. of supp$h$ holds. More in general, in the future of supp$h$ (1) tells us that $\psi = r_{S_1,S_2}(\phi) - \phi$ solves $S_{(1)}[\psi] = 0 \Rightarrow$ finding r_{S_1,S_2} boils down to finding it locally!
- Before we start...

Caveat 1

Notice that we cannot use (2) and apply the (background-independent) retarded fundamental solution Δ^R_S of $S^{(1)} = S^{(2)}[\psi], \forall\psi$ to (1) directly to obtain the so-called Yang-Feldman equation

$$r_{S+\lambda F(h),S}(\phi) = \phi - \lambda\Delta^R_S \circ F(h)_{(1)}[r_{S+\lambda F(h),S}(\phi)],$$

for $S_{(1)}[\psi]$ doesn't necessarily have compact support or even enough decay at infinity!

- Differentiating (1) w.r.t. λ leads to

$$(S_{(1)}+\lambda F(h)_{(1)})^{(1)}[r_{S+\lambda F(h),S}(\phi)]\circ\frac{\mathrm{d}}{\mathrm{d}\lambda}r_{S+\lambda F(h),S}(\phi)+F_{(1)}[r_{S+\lambda F(h),S}(\phi)]=0. \tag{3}$$

- Now we invoke (2) and apply the retarded fundamental solution $\Delta^R_{S+\lambda F(h)}[r_{S+\lambda F(h),S}(\phi)]$ of the linearised Euler-Lagrange operator $(S+\lambda F(h))^{(1)}_{(1)}[r_{S+\lambda F(h),S}(\phi)]$ around the background $r_{S+\lambda F(h),S}(\phi)$ to the left of both sides of (3):

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}r_{S+\lambda F(h),S}(\phi)=-\Delta^R_{S+\lambda F(h)}[r_{S+\lambda F(h),S}(\phi)]\circ F_{(1)}(h)[r_{S+\lambda F(h),S}(\phi)], \tag{4}$$

which shows that $\psi(\lambda)\doteq r_{S+\lambda F(h),S}(\phi)$ is the unique solution of the flow equation (4) with initial condition $\psi(0)=\phi$.

- Formally integrating (4) w.r.t. λ on both sides and using the initial condition above, we arrive at

$$r_{S+\lambda F(h),S}(\phi) = \phi - \int_0^\lambda \mathrm{d}\lambda' \Delta^R_{S+\lambda' F(h)}[r_{S+\lambda' F(h),S}(\phi)] \circ F_{(1)}(h)[r_{S+\lambda' F(h),S}(\phi)]. \tag{5}$$

- We could keep proceeding formally by iterating (4) and write $r_{S+\lambda F(h),S}(\phi)$ as a formal power series [Dütsch–Fredenhagen *ibid.*]

$$r_{S+\lambda F(h),S} \sim \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \restriction_{\lambda=0} r_{S+\lambda F(h),S}.$$

  This is actually the method used in (perturbative algebraic) quantum field theory when performed after composition with functionals, barring renormalisation issues.

- However, our aim is nonperturbative, and thus achieved by looking at the map

$$\psi(\lambda) \mapsto \phi(\lambda) = \psi(\lambda) + \int_0^\lambda \mathrm{d}\lambda' \Delta^R_{S+\lambda' F(h)}[\psi(\lambda')] \circ F_{(1)}(h)[\psi(\lambda')], \quad (6)$$

  which just defines the inverse $r^{-1}_{S+\lambda F(h),S}$ of $r_{S+\lambda F(h),S}$.
- From now on, for the sake of pedagogy we shall set $(\mathscr{M}, g) = \mathbb{R}^{1,d-1} \ni (x^0 = t, x)$.

Main Claim

The map (6) is invertible in a neighbourhood of zero in $\mathscr{C}^1([0,\Lambda], \mathscr{C}^\infty(\mathscr{M}))$; its inverse satisfies (1), (2).

Towards a proof of Main Claim

- The crucial step in our proof is to obtain a priori estimates on $\Delta^R_{S+\lambda F(h)}[\psi]$ in terms of both the linear and the nonlinear (background) arguments. These are essentially refined energy estimates for $S^{(1)}_{(1)} + \lambda F(h)^{(1)}_{(1)}$ which state explicitly their dependence on the latter's coefficients, and were originally obtained by Klainerman [Klainerman '78–'80–'82].
- Suppose that there exist $0 < T$ such that supp$h$ is contained in the interior of the slab $\{(t,x) : 0 \leq t \leq T\}$, and define the energy norms

$$\|\psi\|_{E^k} \doteq \sup_{t'\in[0,T]} \|\psi(t',.)\|_{H^{(k+1)}_x} + \sup_{t'\in[0,T]} \|\partial_t\psi(t',.)\|_{H^{(k)}_x}.$$

Caveat 2

It feels tempting to apply a fixed-point strategy to (5), but the Proposition below shows that $r_{S+\lambda F(h),S}(\phi) - \phi$ is not a Lipschitz map with respect to any $\|.\|_{E^k}$! (PDE folk wisdom: "hyperbolic equations are not strongly stable w.r.t. perturbations of the coefficients." [Tataru ICM '02])

Proposition (refined energy estimates)

For $\phi, \delta\phi \in E^\infty \doteq \{\psi : \|\psi\|_{E^k} < +\infty, \forall k \geq 0\}$ we have

$$\|\Delta^R_{S+\lambda F(h)}[\phi]\delta\phi\|_{E^0} \leq D_0 \sup_{t' \in [0,T]} \|\delta\phi\|_{L^2_x}, \tag{7}$$

$$\|\Delta^R_{S+\lambda F(h)}[\phi]\delta\phi\|_{E^k} \leq D_k \left(\|\delta\phi\|_{E^{k-1}} + \sup_{t' \in [0,T]} |(hF^{(1)}_{(1)})(t', .)|_{\mathscr{C}^k_x} \|\delta\phi\|_{E^0} \right), \; k \geq 1, \tag{8}$$

where D_k, $k \geq 0$ are constants which depend only on d, T and $\|\phi\|_{\mathscr{C}^1(\mathrm{supp} h)}$.

- Applying Sobolev inequalities and Schauder estimates to the spatial $\mathscr{C}^k$ norms of $(hF^{(1)}_{(1)})(t', .)$ in (8), we arrive at

$$\|\Delta^R_{S+\lambda F(h)}[\phi]\delta\phi\|_{E^k} \leq D'_k[\|\delta\phi\|_{E^{k-1}} + (1 + \|\phi\|_{E^{k+1+[\frac{d+1}{2}]}})\|\delta\phi\|_{E^0}], \tag{9}$$

where $[s]$ gives the integer part of s.

Nash–Moser–Hörmander iteration scheme

- The argument above shows that one loses $1 + [\frac{d+1}{2}]$ derivatives at each iteration when trying to solve (5) by a fixed-point method (see Caveat 2!). This phenomenon has no on-shell counterpart.
- Alternative: combine fixed-point method with a "multiscale" (Paley-Littlewood) smoothing procedure that is gradually removed at each iteration ⇒ the former doesn't converge fast enough to give overall convergence for the above loss of derivatives, but using a Newton iteration scheme instead does. The result is the celebrated

Theorem (Nash–Moser–Hörmander)

Let $\Phi : \mathscr{U} \subseteq E^\infty \cap \{\psi : \|\psi - \psi_0\|_{E^\mu} < R\} \to E^\infty$, $\mu \in \bar{\mathbb{Z}}_+$, $R > 0$ be twice Gâteaux differentiable satisfying for all $k \geq 0$ the tame estimates

$$\|\Phi(\psi)\|_{E^k} \leq C_k(1 + \|\psi\|_{E^{k+r_0}}) \text{ for some } r_0 > 0, \tag{10}$$

$$\|\Phi'(\psi)(\delta\psi)\|_{E^k} \leq C'_k[(1 + \|\psi\|_{E^{k+r_1}})\|\delta\psi\|_{E^{s_1}} + \|\delta\psi\|_{E^{k+s_1}}] \text{ for some } r_1, s_1 > 0, \tag{11}$$

$$\|\Phi''(\psi)(\delta_1\psi,\delta_2\psi)\|_{E^k} \leq C''_k[(1+\|\psi\|_{E^{k+r_2}})\|\delta_1\psi\|_{E^{s_2}}\|\delta_2\psi\|_{E^{t_2}}+\|\delta_1\psi\|_{E^{s_2}}\|\delta_2\psi\|_{E^{k+t_2}}$$
$$+\|\delta_1\psi\|_{E^{k+t_2}}\|\delta_2\psi\|_{E^{s_2}}], \text{ for some } r_2, s_2, t_2 > 0, \tag{12}$$

and such that for all ψ in $\mathscr{V} \subset \{\psi : \|\psi-\psi_0\|_{E^{\mu'}} < R'\}$, $\mu' \in \bar{\mathbb{Z}}_+$, $R' > 0$ there is a right inverse $\Psi(\psi)$ to $\Phi'(\psi)$ w.r.t. the linear factor satisfying for all $k \geq 0$ the tame estimates

$$\|\Psi'(\psi)(\delta\psi)\|_{E^k} \leq C'''_k[(1+\|\psi\|_{E^{k+a_1}})\|\delta\psi\|_{E^{b_1}}+\|\delta\psi\|_{E^{k+b_1}}] \text{ for some } a_1, b_1 > 0. \tag{13}$$

Then, for all k sufficiently large, there is a $R_k > 0$ such that for all $\phi \in E^\infty$ fulfilling $\|\phi\|_{E^{k+b_1}} < R_k$ the equation $\Phi(\psi) = \Phi(\psi_0) + \phi$ has a unique solution $\psi = \psi(\phi)$ such that $\|\psi(\phi)-\psi_0\|_{E^k} \leq R''\|\phi\|_{E^{k+b_1}}$. In particular, if ϕ also belongs to E^∞, so does $\psi(\phi)$.

In our problem, we take $\psi_0 \equiv 0$ and add a dependence in λ.

Tame (Gâteaux) differentiability of $\Delta^R_{S+\lambda F(h)}[\psi]$

To check that Φ_λ fulfills the hypotheses of the Theorem, first we collect some following formulae coming directly from the definition of a fundamental solution ("resolvent formula" [Dütsch–Fredenhagen *ibid.*]):

$$\Delta^{R(1)}_{S+\lambda F(h)}[\psi](\delta\psi) = -\Delta^R_{S+\lambda F(h)}[\psi] \circ F(h)^{(2)}_{(1)}[\psi](\delta\psi, \Delta^R_{S+\lambda F(h)}[\psi]), \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta^R_{S+\lambda F(h)}[\psi] = -\Delta^R_{S+\lambda F(h)}[\psi] \circ F(h)^{(1)}_{(1)}[\psi] \circ \Delta^R_{S+\lambda F(h)}[\psi], \tag{15}$$

$$\begin{aligned} &\Delta^{R(2)}_{S+\lambda F(h)}[\psi](\delta_1\psi, \delta_2\psi) = \\ &= \Delta^R_{S+\lambda F(h)}[\psi] \circ F(h)^{(2)}_{(1)}(\delta_1\psi, \Delta^R_{S+\lambda F(h)}[\psi]) \circ F(h)^{(2)}_{(1)}(\delta_2\psi, \Delta^R_{S+\lambda F(h)}[\psi]) + \\ &+ \Delta^R_{S+\lambda F(h)}[\psi] \circ F(h)^{(2)}_{(1)}(\delta_2\psi, \Delta^R_{S+\lambda F(h)}[\psi]) \circ F(h)^{(2)}_{(1)}(\delta_1\psi, \Delta^R_{S+\lambda F(h)}[\psi]) + \\ &- \Delta^R_{S+\lambda F(h)}[\psi] \circ F(h)^{(3)}_{(1)}(\delta_1\psi, \delta_2\psi, \Delta^R_{S+\lambda F(h)}[\psi]). \end{aligned} \tag{16}$$

Equation (15) shows in particular that $\Delta^R_{S+\lambda F(h)}[\psi]$ is strongly differentiable (hence strongly continuous) in λ, thus allowing all the computations we need.

Tame estimates for iteration map, end of proof

From (14) and (16), one get the following formulae for the first two derivatives of the iteration map Φ_λ (6):

$$\Phi'_\lambda(\psi)(\delta\psi) = \delta\psi + \\ + \int_0^\lambda \mathrm{d}\lambda' \left(\Delta^R_{S+\lambda' F(h)}[\psi] \circ F(h)^{(1)}_{(1)}[\psi](\delta\psi) + \Delta^{R(1)}_{S+\lambda' F(h)}[\psi](\delta\psi) \circ F(h)_{(1)}[\psi] \right), \tag{17}$$

$$\Phi''_\lambda(\psi)(\delta_1\psi, \delta_2\psi) = \int_0^\lambda \mathrm{d}\lambda' \left(\Delta^R_{S+\lambda' F(h)}[\psi] \circ F(h)^{(2)}_{(1)}[\phi](\delta_1\phi, \delta_2\phi) + \right. \\ + \Delta^{R(1)}_{S+\lambda' F(h)}[\psi](\delta_1\psi) \circ F(h)^{(1)}_{(1)}[\phi](\delta_2\phi) + \Delta^{R(1)}_{S+\lambda' F(h)}[\psi](\delta_2\psi) \circ F(h)^{(1)}_{(1)}[\phi](\delta_1\phi) + \\ \left. + \Delta^{R(2)}_{S+\lambda' F(h)}[\psi](\delta_1\psi, \delta_2\psi) \circ F(h)_{(1)}[\phi] \right), \tag{18}$$

where $\Delta^{R(1)}_{S+\lambda' F(h)}[\psi]$ and $\Delta^{R(2)}_{S+\lambda' F(h)}[\psi]$ are respectively given by (14) and (16). Notice that $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi'_\lambda(\psi)$, seen as a linear map acting on $\delta\psi$ for fixed ψ, doesn't lose derivatives, due to the fact that the assumed loss in $F(h)_{(1)}$ is exactly compensated by the smoothing effect of $\Delta^R_{S+\lambda' F(h)}[\psi]$.

- The Proposition, together with Schauder estimates, show that Φ_λ satisfy the tame estimate (10) with $a_0 = [\frac{d+1}{2}] + 1$ for $\sup_{\lambda' \in [0,\lambda]} \|\psi(\lambda')\|_{E^{[\frac{d+1}{2}]+1}} < R$, that is, $\mu = [\frac{d+1}{2}] + 1$.
- Formulae (17)–(18) show that $\Phi'_\lambda(\psi)(\delta\psi)$ and $\Phi''_\lambda(\psi)(\delta_1\psi, \delta_2\psi)$ fulfill resp. the tame estimates (11) and (12) with $r_1 = r_2 = [\frac{d+1}{2}] + 1$ and $s_1 = s_2 = t_2 = 1$.
- Finally, due to (15) and the remark following (18), $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi'_\lambda(\psi)$ is a bounded and uniformly strongly continuous (in λ) linear map $\Rightarrow \Phi'_\lambda(\psi)$ be inverted by means of a Dyson series. Iterating the tame estimate for $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi'_\lambda(\psi])$, together with the argument for the convergence for the Dyson series, leads to the tame estimate (13) with $a_1 = [\frac{d+1}{2}] + 1$, $b_1 = 1$ and $\mu' = [\frac{d+1}{2}] + 2$ for the right inverse.
- Now... Just plug in the data above, run the "Nash–Moser–Hörmander machine", and we get local existence and uniqueness of $r_{S+\lambda F(h),S}$ in E^∞. The intertwining relation (1) shows that actually $r_{S+\lambda F(h),S}(\phi) \in \mathscr{C}^\infty$ for $\phi \in \mathscr{C}^\infty$. □

Structural consequences

The existence and properties of $r_{S+\lambda F(h),S}$ have fundamental implications for the underlying Poisson structure of any classical field theory determined by an action functional S, given by the Peierls bracket

$$\{F, G\}_S \doteq F_{(1)}[.] \circ (\Delta_S^R[.] - \Delta_S^A[.]) \circ G_{(1)}[.]$$

of microcausal functionals F, G. Here $\Delta_S^A[\psi]$ is the advanced fundamental solution of $S_{(1)}^{(1)}[\psi]$ around the background ψ, which is simply the adjoint of $\Delta_S^R[\psi]$.

Corollary $r_{S+\lambda F(h),S}$ is a canonical transformation, i.e. it intertwines the Poisson structures associated to S and $S + \lambda F(h)$:

$$\{.,.\}_{S+\lambda F(h)} \circ r_{S+\lambda F(h)} = \{.,.\}_S .$$

In particular, even off shell does it allow one to put $\{.,.\}_{S+\lambda F(h)}$ in "normal form", i.e. to make it locally background-independent ("Functional Darboux Theorem").

Scholium The space of microcausal functionals vanishing on solutions of $(S + \lambda F(h))_{(1)}[\psi] = 0$ is a multiplicative ideal.

Coda: final considerations

- We've given a characterisation of the classes of functionals of off-shell field configurations $\mathfrak{F}_{loc}(\mathscr{M},\mathscr{E})$ and $\mathfrak{F}_{mc}(\mathscr{M},\mathscr{E})$, relevant for classical field theory. It can be further shown [Brunetti–Fredenhagen–PLR, work in progress] that $\mathfrak{F}_{mc}$ carries the structure of a nuclear topological Poisson algebra for any given dynamics, in a way amenable to quantisation and renormalisation.
- We've shown the existence of r_{S_1,S_2} for "sufficiently small" field configurations around a given one. This latter condition can be controlled in general by adjusting λ (coupling strength) or supp$h$ (lifespan).
- If the Cauchy problem for $S_{1(1)}[\psi]=0$ is well-posed in the large, one can use (2) and the composition property of r_{S_1,S_2}

$$r_{S,S}=\mathbb{1},\, r_{S_2,S_3}\circ r_{S_1,S_2}=r_{S_1,S_3}$$

stemming from (1) to remove the cutoff (i.e. dependence on h) $\Rightarrow$ probably impossible off shell, unless probably in a suitable algebraic sense ("algebraic adiabatic limit" – [Brunetti–Dütsch–Fredenhagen *ibid.*]).

- We illustrated our strategy for the case of a scalar field in $\mathbb{R}^{1,d-1}$, but the argument carries through for arbitrary sections in any globally hyperbolic spacetime $\Rightarrow$ one has a local energy estimate of the same form as (7)–(8) by combining Klainerman's argument with the estimates in [Hawking–Ellis '73]; only the control of the extra error terms due to curvature and the absence of Killing fields is more cumbersome.
- Alas, the more general quasilinear case (e.g. general relativity) seems to pose some new difficulties; the Dyson series argument to invert Φ'_λ w.r.t. the linear factor fails since one then loses one derivative at each order. It seems to be possible, however, to circumvent this issue by means of paradifferential calculus (PLR, work in progress).