Perturbative Methods on the Noncommutative Minkowski Space

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Zusammenfassung

In dieser Arbeit werden zwei verschiedene Ansätze studiert, die es erlauben, unitäre Störungstheorie für Quantenfelder auf dem nichtkommutativen Minkowskiraum auch bei nicht-kommutierender Zeit zu definieren.
Der erste Ansatz basiert auf der üblichen Dyson'schen Störungstheorie mit einem entsprechend definierten Hamiltonoperator. Die Graphentheorie wird angegeben. Wird die Nichtkommutativität nicht als festes Hintergrundfeld betrachtet, sondern an jedem Vertex gemittelt, so sind Theorien mit kubischer (und mit hoher Wahrscheinlichkeit auch solche mit quartischer) Selbstwechselwirkung ultraviolett-endlich. Verwendet man dagegen die sogenannte Quanten-Diagonal-Abbildung (eine neue Definition des Limes zusammenfallender Punkte auf dem nichtkommutativen Minkowskiraum) zur Definition des Wechselwirkungsterms, so ist die resultierende S-Matrix immer ultraviolett-endlich. Das übliche Ultraviolett-Infrarot-Mischungsproblem tritt hier nicht auf.
Der zweite, zum Hamiltonformalismus inäquivalente, störungstheoretische Ansatz basiert auf der Yang-Feldman-Gleichung. Quantenfelder werden als sogenannte q-Distributionen definiert, und die zugehörige Graphentheorie wird angegeben. Der Limes zusammenfallender Punkte wird für diesen Rahmen definiert, und darauf basierend wird ein Kriterium, genannt q-Lokalität, angegeben, welches die erlaubten Gegenterme für die Renormierung auszeichnet. Sodann werden Produkte von Feldern, die sogenannten quasiplanaren Wick-Produkte, dadurch definiert, daß nur q-lokale Gegenterme zugelassen sind. Diese bleiben im Limes zusammenfallender Punkte wohldefiniert. Die resultierende Dispersionsrelation deutet darauf hin, daß das asymptotische Verhalten der Theorie von demjenigen lokaler Theorien auf dem Minkowskiraum stark abweicht.

Abstract

In this thesis, two different approaches are studied which allow for the definition of unitary perturbation theory for quantum fields on the noncommutative Minkowski space, without assuming commutativity in the time-variable.
The first approach is based on the usual perturbation theory according to Dyson, using a suitably defined Hamilton operator. The corresponding graph theory is presented. Theories with cubic (and, most likely, also those with quartic) self-interaction turn out to be ultraviolet finite, if the noncommutativity is averaged at each vertex. On the other hand, if the so-called quantum diagonal map (a suitable definition of the limit of coinciding points on the noncommutative Minkowski space) is used to define interaction terms, the resulting S-matrix is always ultraviolet-finite. The ordinary ultraviolet-infrared mixing problem is not present here.
The second approach to perturbation theory, which is inequivalent to the Hamilton formalism, is based on the Yang-Feldman equation. Quantum fields are defined as so-called q-distributions, and the appropriate graph theory is specified. The limit of coinciding points is defined in this framework, and a criterion, called q-locality, is formulated, which singles out the admitted counterterms. Products of fields, called quasi-planar Wick products, are defined by only admitting q-local counterterms. These products remain well-defined in the limit of coinciding points. The resulting dispersion relation provides evidence that the theory's asymptotic behaviour is considerably changed compared to that of local theories on Minkowski space.

Contents

Introduction

The idea that spacetime might not be continuous down to arbitrarily small scales can be traced back to the early days of quantum theory [74]. By the advent of quantum field theory, a particle's Compton wavelength was generally considered to be the bound as to how accurately its localization should be measurable. In a next step, motivated by the incorporation of gravitational effects into field theory, a fundamental length scale, the Planck length $\lambda_P = \sqrt{G\,\hbar\,c^{-3}} \sim 10^{-33}$cm with G the gravitational constant, beyond which the resolution of different events was expected to be impossible, was introduced and considered as a possible cure to treat the divergences arising in quantum field theory [25, 68]. The idea that the geometry of spacetime should radically differ at distances smaller than the Planck length, or that measurements of such distances should be meaningless, has been further elaborated in theories of quantum gravity [82, 6, 52] and in string theory [5]. On a less fundamental level, lattice structures have proved to be convenient modifications of Minkowski space, mainly motivated by the quest to regularize the ultraviolet divergences in quantum field theory. It was realized early that, contrary to such fixed lattice structures, a spacetime whose coordinates do not commute might provide a covariant means to regularize quantum electrodynamics [76]. An interpretation of the resulting uncertainty relations was not attempted, and probably due to the success of renormalization theory this approach was apparently soon forgotten.

Ideas on the existence of a fundamental length and the modification of spacetime at small scales were taken up in [27], and *relative* bounds on the accuracy of simultaneous measurements of spacetime directions in terms of uncertainty relations were shown to result from the following heuristic argument. By Heisenberg's principle of uncertainty, a very accurate measurement of the position of an event up to uncertainties $\Delta x_0, \ldots, \Delta x_3$ requires an energy-transfer of the order $E \sim \hbar c/a$, where a is the smallest of the uncertainties. Taking general relativity into consideration, this energy, which is concentrated at some time t_0 in a region $\Delta x_1 \cdot \Delta x_2 \cdot \Delta x_3$, should act as a source term $M = E/c^2$ in the Einstein equation. The resulting gravitational potential is the stronger, the larger M is, i.e. the more accurately we try to measure the event's position in spacetime. In an extreme situation, it could become strong enough to trap photons, by building a horizon (of radius $R \sim MG/c^2$), in which case the region of interest would be shielded completely from the observer. If only one direction is measured very accurately, the energy may spread along the other directions and, depending on how accurately these other directions are to be measured, a horizon does not necessarily form. In this sense, arbitrarily accurate localizations become meaningless, and, based on this argument, the following uncertainty relations were derived in [27],

$$\Delta q_0 \cdot (\Delta q_1 + \Delta q_2 + \Delta q_3) \geq \lambda_P^2\,, \qquad \Delta q_1 \cdot \Delta q_2 + \Delta q_1 \cdot \Delta q_3 + \Delta q_2 \cdot \Delta q_3 \geq \lambda_P^2\,.$$

These uncertainty relations provide relative bounds and still allow very accurate measurements of some directions at the cost of the others. In [27], they were taken as a starting point to modify the structure of spacetime itself in such a way that more precise measurements than those complying with the uncertainty relations are impossible. This was achieved by replacing the usual coordinates by noncommuting coordinate operators,

$$[q^\mu, q^\nu] \;=\; i\,Q^{\mu\nu}\,, \qquad \mu,\nu = 0,\ldots,3\,,$$

where the commutators themselves are central,

$$[q_\rho, Q_{\mu\nu}] \;=\; 0\,,$$

and subject to the so-called quantum conditions, which fix the spectrum of the commutators in a manner similar to what is known from the field strength tensor in classical electrodynamics (I the identity),

$$Q_{\mu\nu}Q^{\mu\nu} \;=\; 0\,, \qquad \left(\tfrac{1}{8}\,Q_{\mu\nu}\,Q_{\rho\sigma}\,\epsilon^{\mu\nu\rho\sigma}\right)^2 \;=\; \lambda_P^8\,I\,.$$

These conditions are invariant under the full Poincaré group, including space as well as time reflections. The commutators being central, the problem of finding a representation of the (unbounded) operators q^μ is similar to that known from quantum mechanics. Contrary to the situation there, however, the commutators are not given by a fixed matrix with entries $\theta^{\mu\nu}$, but by operators $Q^{\mu\nu}$ possessing a non-trivial spectrum. This is a consequence of the fact that the quantum conditions are Poincaré-invariant. If fixed values $\theta^{\mu\nu}$ for the commutators were chosen, Lorentz covariance would evidently be broken, since the $\theta^{\mu\nu}$ would have to be the same in all reference frames, providing a modern version of the aether. Allowing for the Poincaré-invariant quantum conditions above, a representation in terms of a noncommutative C^*-algebra was achieved in [27] and the desired uncertainty relations were shown to hold. In particular, functions of the quantum coordinates were defined, with a product given in terms of the so-called twisted convolution in momentum space, where compared to the ordinary convolution (corresponding to the local product of ordinary functions on Minkowski space) an oscillating function of the momenta, the so-called twisting, appears. For an attempt to implement a stronger form of the uncertainty relations, as well as to give up the requirement that the commutators themselves be central in this framework see [26].

The argument leading to the uncertainty relations is not absolutely compulsory in itself. In particular, it is assumed in the derivation that the laws of classical gravity hold at very small distances, extrapolating the theory over many orders of magnitude – an assumption whose correctness Einstein apparently doubted (see for instance, [34], where he expresses doubts on the existence of an initial singularity, claiming that the field equations do not necessarily hold at arbitrarily small distances). In this spirit, it has been proposed in [71], that the strength of gravitation may be scale-dependent and lessen at smaller distances, which provides an alternative resolution to the paradox of arbitrarily accurate localizations.

However, the analysis in [27] starts from two well-established theories and points out that the contradiction arising if both of them hold (at arbitrarily small distances) can be resolved in a covariant way, by assuming that continuous spacetime is to be replaced by a noncommutative C^*-algebra. The hypothesis that the structure of spacetime changes at small distances is sufficiently motivated and certainly worthwhile to be investigated. The modifications to be expected when quantum field theory is formulated on such a noncommutative spacetime should at least in principle be verifiable or falsifiable by experiment, and indeed, the phenomenology of field theory on noncommutative spacetimes has already aroused much interest [54, 1]. Furthermore, the question of whether the noncommutative spacetime provides a natural regularization and a possible cure for ultraviolet divergences in quantum field theory is to be considered.

Related to the approach pursued in [27] is the idea to generalize the Gelfand-Naimark construction from commutative to noncommutative C^*-algebras. In the same manner as the topology of a locally compact space is encoded in the commutative C^*-algebra of functions on this space which vanish at infinity, the topology of "quantum spaces" is supposed to be encoded in the properties of a noncommutative C^*-algebra. This approach had far-reaching consequences in mathematics [20]. Moreover, an elegant motivation of the standard model action including the Higgs mechanism has been formulated in this framework (cf. [23] and [72]) and was unified with the Einstein-Hilbert action in [21]. Unfortunately, the spectral triples employed in this formalism are only defined for compact Riemannian manifolds. Partly motivated in the context of quantum groups, a variety of noncommutative spacetimes were introduced and investigated from different viewpoints (see for instance [66, 67, 81, 65, 49] as well as [45, 46]).

More recently, interest in formulating field theory on noncommutative spacetimes has been aroused by its appearance in the context of string theory [83]. Certain limits of M-theory have led to gauge theories on noncommutative spacetimes [22], and field theories on noncommutative spacetimes were derived as special low-energy limits of open string theories on D-brane configurations in background magnetic fields [75, 73]. The remarkable feature of this limit is that the string tension approaches 0, but the strings do not collapse, as the magnetic field and the open string metric are kept fixed. The resulting theory is interpreted as a field theory on a spacetime whose spatial coordinates do not commute. The situation is different from what was proposed in [27], as in these models the noncommutativity by construction breaks Lorentz covariance, since the background field singles out distinguished directions. See also [29, 79] for reviews.

In this context, the so-called Seiberg-Witten map [75] was introduced, which maps a classical gauge theory defined on the ordinary Minkowski space to a counterpart in a noncommutative framework. The latter is defined as a formal power series in the noncommutativity parameter θ, proportional to the inverse of the magnetic field strength, and as such allows for a wide range of possible commutators, not only central ones as in the C^*-algebra approach investigated here. Structural insight has been gained [58], relying in part on the general theory of deformation quantization (see also [15]). An advantage of this so-called θ-expanded approach is that arbitrary gauge groups can be considered, while it remains a problem in other approaches to implement gauge groups other than $U(N)$. The price to pay is that the theory allows for the calculation of modifications only up to some order in θ, where at every order the theory is an ordinary field theory with derivatives in the interaction. The standard model has been formulated in this context [16], and phenomenological calculations have been performed at tree level. Little is known, however, about the interplay of the formal power series arising from the Seiberg-Witten map and the perturbative expansion in quantum field theory, which is a formal power series itself. Earlier hopes that the resulting effective theories as they stand would be renormalizable order by order in θ have not been met [85].

Apart from the θ-expanded approach, most investigations on field theories on noncommutative spacetimes are based on a set of modified Feynman rules [37, 38]. Here, the commutators are again assumed to possess fixed values, given in terms of a noncommutativity matrix $\theta^{\mu\nu} = -i[q^\mu, q^\nu]$. Formally, the rules, which first appeared in the context of matrix models [33, 44], may be derived from a functional integral approach on a Euclidean noncommutative space. They have also been applied to field theories on the noncommutative Minkowski space with fixed $\theta^{\mu\nu}$, but due to the fact that a generalization of Osterwalder-Schrader positivity is not yet available and that not even the ordinary Wick rotation has been properly defined as yet, the relation between the Euclidean and the Minkowskian regime remains obscure. Effectively, the theory is treated as one defined on the ordinary Minkowski space with a nonlocal interaction, and the ordinary perturbative setup is applied. In particular, Feynman propagators serve as internal lines. The only difference to ordinary quantum field theory is that at every vertex an oscillatory function of the momenta, the so-called twisting, appears.

As first observed in [43, 2], the major drawback of this approach is that it does not lead to a unitary S-matrix for general spacetime-noncommutative structures, i.e. with noncommuting time-variable. In particular, a fixed noncommutativity matrix complying with the quantum conditions given in [27] leads to a non-unitary perturbation theory in this approach. The same was observed for theories derived as low-energy limits of string theories in the presence of background *electro*-magnetic fields, which are interpreted as theories on a general noncommutative spacetime. It seems that such a low-energy limit only yields a unitary field theory, if one allows for tachyonic states with negative norm [3].

Subsequently, it was proposed to consider only noncommutative spacetimes that are consistent with the modified Feynman rules. These are spacetimes with commuting time-variable (space-space-noncommutativity), where the noncommutativity matrix satisfies the equation $\theta^{\mu\nu}n_\nu = 0$ with a timelike vector n, and spacetimes with so-called lightlike noncommutativity [2], where $\theta^{\mu\nu}n_\nu = 0$ with a lightlike vector n, which can be characterized in a Poincaré-invariant way by setting both right-hand sides of the quantum conditions to 0. As an aside, it is mentioned that the latter type of noncommutative spacetimes was already introduced in [27], where it was called "dilation covariant quantum spacetime" and appeared as a large scale limit. No uncertainty relations seem to hold in this case [27, p.199].

One of the main goals of this thesis is to show that the restriction to such spacetimes is not at all necessary. Already in [27], before the modified Feynman rules became popular, a symmetric Hamilton operator was introduced, which by means of the ordinary Dyson series automatically yields a formally unitary S-matrix, regardless of whether lightlike, space-space or time-space-noncommutativity complying with the quantum conditions given in [27] is assumed. In addition to that, it was proposed in [9] to employ the Yang-Feldman approach [86, 59], where the field equation is used as a starting point and the interacting field is constructed perturbatively. Again, the interacting field is Hermitean for lightlike, space-space or time-space-noncommutativity, and the theory is unitary. Both approaches, which, contrary to the case on the ordinary Minkowski space, are in general inequivalent, are investigated in the present thesis and will be analysed in detail. As problems still arise at a very fundamental level, the analysis is confined to scalar self-interacting theories.

Taking as an interaction term a normally ordered product $:\phi^n(q):$ as proposed in [27], where $\phi(q)$ is a suitably defined generalization of the ordinary quantum field to the noncommutative Minkowski space, the Hamiltonian formalism and the framework of the modified Feynman rules are similar to one another, with twistings appearing at the vertices – and until the violation of unitarity in the setting of the modified Feynman rules was found, they were supposed to be equivalent. As pointed out in [9], this violation of unitarity can be linked with the way the time-ordering is implicitly defined in the modified Feynman rules as opposed to the Hamiltonian formalism. In particular, the internal lines in the two unitary approaches, i.e. the Hamiltonian formalism and the Yang-Feldman equation, are not, in general, given by Feynman propagators. In the special cases of lightlike and space-space noncommutativity, where the time-ordering may be defined as usual, the framework of the modified Feynman rules, which is unitary in these cases, coincides with the Hamiltonian as well as with the Yang-Feldman approach.

While the interaction term above is a straightforward generalization of the ordinary local interaction term $:\phi^n(x):$, this is not the only possibility, and the major question of this thesis is how ordinary local interaction terms are to be replaced in the noncommutative setting. Since ordinary quantum field theory is based on the principle of locality, one of the most challenging questions is how to replace this notion on a noncommutative spacetime. While plagued with problems such as the violation of causality, it is shown that field theories on the noncommutative Minkowski space do allow for minimal notions of locality from which suitable generalizations of interaction terms may be derived. Two different approaches are proposed here.

The first is based on re-defining the concept of taking products of fields at the same point. Since, by construction, on the noncommutative Minkowski space strict localization is impossible, it is argued that fields cannot be evaluated "at the same point", but rather only at points which are "close together". Using the notion of states which minimize the uncertainty [27], a so-called quantum diagonal map is introduced which replaces the ordinary concept of coinciding points. Applying this map to define interaction terms, Hamilton operators are found which lead to

ultraviolet finite theories for any ϕ^n-self-interaction [10].
The second approach follows a different line of thought, and is adapted to the Yang-Feldman framework. A precise definition of a field on the noncommutative Minkowski space is given in terms of the so-called q-distributions, which generalize the operator-valued distributions ordinarily arising in local quantum field theory [12]. A criterion, called q-locality, is then given which singles out the admissible counterterms – and interaction terms – as those q-distributions which satisfy a minimal locality requirement. Roughly speaking, a q-distribution is q-local, if it does not increase a testfunction's support. On the basis of this definition, products of fields, the so-called quasiplanar Wick products are defined, much in the same manner as in ordinary field theory, but allowing only q-local counterterms. They provide a suitable generalization of a local interaction term.
Other major results of this thesis are the following. In the framework of the Hamiltonian approach with interaction term $:\phi^n(q):$, it is shown that the ultraviolet behaviour depends on whether the noncommutativity is treated as a fixed background or whether the dependence on the commutators of the noncommuting coordinates is averaged over a certain measure at each vertex. The latter approach in fact leads to an ultraviolet finite theory in the case of a cubic (and, most likely, also for a quartic) self-interaction.
The mixing of ultraviolet and infrared divergences [70], as found in the setting of the modified Feynman rules, has not appeared in the approaches investigated here, at least not in the sense that the insertion of ultraviolet finite graphs into higher order diagrams may produce infrared singularities. It appears, however, that the asymptotic behaviour (i.e. the infrared regime) is considerably modified compared to the ordinary case.
Furthermore, differences and similarities compared to ordinary local field theory and the nonlocal theories considered earlier [87, 60, 11] as a means to regularize the divergences of quantum field theory are investigated in the different approaches, paying particular attention to matters such as causality and covariance. One of the surprising results here is that the cluster decomposition property is satisfied for vacuum expectation values of free fields.

This thesis is structured as follows.
The first chapter is a presentation of known results about the noncommutative Minkowski space defined in [27] and intended to make the reader familiar with the necessary definitions and notations.
In the second chapter, the perturbation theory based on the introduction of a Hamilton operator on the noncommutative Minkowski space is analysed. General properties of the resulting Dyson series are discussed. The graph theory and the rules to calculate expectation values are spelled out explicitly, and emphasis is put on contrasting them with the modified Feynman rules. It is shown how the definition of the time-ordering automatically yields a unitary theory in the Hamiltonian approach. This Hamiltonian approach leads to an ultraviolet finite theory when the twisting is integrated against a certain measure at each vertex.
In the third chapter, the quantum diagonal map is introduced, and it is shown that the resulting Hamilton operators yields a regularized theory which is ultraviolet finite as long as an adiabatic cutoff is employed. The removal of this cutoff requires a mass renormalization which is not Lorentz-invariant.
The fourth chapter contains an introduction to the Yang-Feldman approach on the noncommutative Minkowski space. A precise definition of quantum fields as q-distributions is given and the limit of coinciding points is defined accordingly. Vacuum expectation values of free fields are calculated and compared with ordinary Wightman functions. The graphical rules for the Yang-Feldman approach are specified.
In the last chapter, the notion of a q-local counterterm is introduced. It is established that products of fields, the quasiplanar Wick products, which are defined by subtracting merely q-

local counterterms remain well-defined in the limit of coinciding points. The combinatorics of the resulting Wick theorem is treated and first results regarding the domain of definition are given. The distorted dispersion relation arising in this context is discussed.
An outlook concludes the thesis.

Chapter 1

The noncommutative Minkowski space and free fields

This chapter is a presentation of known results from [27] (see also [28, 40]) and should provide the reader with the necessary definitions and notations. Let us start with an outline of the properties of the quantum spacetime defined by the conditions mentioned in the introduction,

$$[q^\mu, q^\nu] \;=\; i\,Q^{\mu\nu}\,, \tag{1.1}$$

where the commutators themselves are central,

$$[q_\rho, Q_{\mu\nu}] \;=\; 0\,, \tag{1.2}$$

and subject to the so-called quantum conditions

$$Q_{\mu\nu}Q^{\mu\nu} \;=\; 0\,, \qquad \left(\tfrac{1}{8}\,Q_{\mu\nu}\,Q_{\rho\sigma}\,\epsilon^{\,\mu\nu\rho\sigma}\right)^2 \;=\; \lambda_P^8\,I\,. \tag{1.3}$$

The q^μ are referred to as the quantum coordinates or the noncommutative coordinates. Unless explicitly stated otherwise, units will be used such that $\lambda_P = 1$. As emphasized in the introduction, the quantum conditions fix the spectrum of the operators $Q^{\mu\nu}$ in a Poincaré-invariant manner, in the same way as is well-known from classical electrodynamics (the only difference being, that we also retain invariance under space- and time-reflections separately). Introducing the "electric" and "magnetic" components $\vec{E}$ and $\vec{M}$ of Q, where $E_i = Q_{0i}$ and $M_i = \epsilon_{ijk}Q_{jk}$, the quantum condition can be rewritten as

$$\vec{E}^{\,2} = \vec{M}\ , \qquad \tfrac{1}{2}(\vec{E}\cdot\vec{M} + \vec{M}\cdot\vec{E}) = \pm 1\,.$$

The price to pay for not breaking Poincaré-covariance is that the representation theory of the commutation relations is more complicated than that of the canonical commutation relations of ordinary quantum mechanics.

As desired, by application of the ordinary relation for uncertainties of selfadjoint operators A, B in a state ω, $\Delta_\omega(A)\cdot\Delta_\omega(B) \geq \frac{1}{2}|\,\omega(\,[A,B]\,)\,|$, the following uncertainty relations ($\lambda_P \neq 1$) for the noncommutative coordinates were found in [27],

$$\Delta q_0\cdot(\Delta q_1 + \Delta q_2 + \Delta q_3) \;\geq\; \lambda_P^2/2\,, \qquad \Delta q_1\cdot\Delta q_2 + \Delta q_1\cdot\Delta q_3 + \Delta q_2\cdot\Delta q_3 \;\geq\; \lambda_P^2/2\,. \tag{1.4}$$

In the derivation it is important that the commutators are central.

As shown in [27], a C^*-algebra $\mathcal{E}$ to which the quantum coordinates q^μ are affiliated[1] can be constructed in the following way. First it is shown that the $Q^{\mu\nu}$ are affiliated to $C_0(\Sigma)$, the

[1]Following [27], affiliation is used in the sense of [84]: Let $\mathfrak{A}$ be a C^*-algebra with unit. An (unbounded) operator A is affiliated to $\mathfrak{A}$, if there is a $*$-homomorphism (with unbounded support) of $C_0(\mathbb{R})$ into $\mathfrak{A}$, $f \mapsto f(A)$, whose support projection $E \in \mathfrak{A}^{**}$, the double commutant of $\mathfrak{A}$, is central. If $\mathfrak{A}$ does not have a unit, A is said to be affiliated to $\mathfrak{A}$, if it is affiliated to $M(\mathfrak{A})$, the multiplier algebra of $\mathfrak{A}$. This construction is necessary, as the operators q^μ are unbounded.

algebra of continuous functions vanishing at infinity on Σ, where

$$\begin{aligned}\Sigma &= \{\sigma \,|\, (\sigma_{\mu\nu}) \text{ real antisymmetric 2-tensor}, \sigma_{\mu\nu}\sigma^{\mu\nu} = 0, \tfrac{1}{8}\,\sigma_{\mu\nu}\epsilon^{\mu\nu\rho\tau}\sigma_{\rho\tau} = \pm 1\} \\ &= \{\sigma \,|\, \sigma_{\mu\nu} = -\sigma_{\nu\mu} \,:\, \sigma_{0i} = e_i,\, \epsilon_{ijk}\sigma_{jk} = m_i,\, \vec{e}^{\,2} = \vec{m}^2\,,\, \tfrac{1}{2}(\vec{e}\cdot\vec{m} + \vec{m}\cdot\vec{e}) = \pm 1\}\,,\end{aligned}$$

which is homeomorphic to the non-compact manifold TS^2. Σ is the orbit of the standard symplectic matrix $(\sigma^{(0)})^{\mu\nu} = \begin{pmatrix} 0 & -\mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}$ under the action $\sigma \mapsto \Lambda\sigma\Lambda^t$ of the full Lorentz group, where ^{t} denotes the transpose.

Note here, that the dilation-covariant noncommutative spacetime introduced in [27], later called the lightlike noncommutative spacetime [2], is defined by quantum conditions where both invariants are zero, $Q_{\mu\nu}Q^{\mu\nu} = Q_{\mu\nu}\,Q_{\rho\sigma}\,\epsilon^{\mu\nu\rho\sigma} = 0$. As shown in [27], the spectrum of the $Q^{\mu\nu}$ in this case is connected and consists of the orbit of the degenerate matrix $\sigma^{\mu\nu} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ under the restricted Lorentz group. No uncertainty relations seem to hold in this case; when following the analysis in [27], both right-hand sides of (1.4) are zero, cf. [27, p.199].

Obviously, for a spacetime with commuting time (space-space-noncommutativity), no uncertainty relation with respect to the time variable will arise.

Now consider functions from Σ to $L^1(\mathbb{R}^4)$, vanishing at infinity, endowed with a product

$$(F \times G)(\sigma, k) = \int d^4p\, F(\sigma, p)\, G(\sigma, k-p)\, e^{-\frac{i}{2}k^\mu\sigma_{\mu\nu}p^\nu}\,,$$

an involution $(F^*)(\sigma, k) = \overline{F(\sigma, -k)}$ and a norm $||F|| = \sup_\sigma \int dk\, |F(\sigma, k)|$. Completion with respect to the norm $||\cdot||$ yields the Banach *-algebra $\mathcal{E}_0$. Since the commutators are central, von Neumann's uniqueness theorem can be applied and it can be deduced that there is a unique C^*-norm on $\mathcal{E}_0$ [27, Theorem 4.1]. The associated C^*-algebra, denoted $\mathcal{E}$, is isomorphic to $C_0(\Sigma, \mathcal{K})$, where $\mathcal{K}$ is the algebra of compact operators on a fixed separable Hilbert space. In particular, for fixed $\sigma \in \Sigma$ the corresponding C^*-algebra $\mathcal{E}_\sigma$ is isomorphic to $\mathcal{K}$. By construction, the quantum coordinates are elements of the multiplier algebra $M(\mathcal{E})$ of $\mathcal{E}$.[2] In the same manner as $C_0(\mathbb{R}^4)$ encodes the structure of Minkowski space, $\mathcal{E}$ is supposed to encode the structure of a "quantum spacetime", the so-called noncommutative Minkowski space.

This representation can also be understood as follows. The commutation relations are canonical in the sense that the commutators are central. This makes a generalized Weyl correspondence possible, where

$$\mathcal{W}(h \otimes f) = h(Q) f(q) \tag{1.5}$$

with

$$f(q) = \int d^4k\; \check{f}(k) e^{ikq}\,.$$

Here, $f \in \mathcal{F}L^1(\mathbb{R}^4)$, $\check{f} = \mathcal{F}^{-1}f$, and $\mathcal{F}$ denotes the ordinary Fourier transform. h is an element of $C_0(\Sigma)$ and $h(Q)$ is to be understood in the same way as $f(q)$, in the sense of the joint functional calculus of the commutators $Q^{\mu\nu}$. The function $h(\sigma)\, f(x)$ is called the *symbol* of $\mathcal{W}(h \otimes f)$. Since the multiplier algebra of C_0 is C_b, the algebra of bounded functions, $C_b(\Sigma)$ can be identified with the centre $\mathcal{Z}$ of the multiplier algebra $M(\mathcal{E})$. This centre will play an important role throughout, and it is emphasized again that it is nontrivial, because Poincaré invariance was required.

The generalized Weyl correspondence extends to any symbol $F \in \mathcal{C}_0(\Sigma \times \mathbb{R}^4)$ with inverse Fourier transform (at fixed σ) $\check{F}(\sigma, \cdot) \in L^1(\mathbb{R}^4)$. Via the formula $\mathcal{W}(F \star G) = \mathcal{W}(F)\mathcal{W}(G)$ it induces a product of symbols, the so-called twisted convolution product,

$$(F \star G)(\sigma, \cdot) = F(\sigma, \cdot) \star_\sigma G(\sigma, \cdot)\,.$$

[2] To be exact, it is mentioned here, that the states which are used to calculate the uncertainty relations must be in the domain of the commutators. For the definition cf. [27]: Let $\mathfrak{A}$ be a C^*-algebra to which A is affiliated (see page 9). A state $\omega \in \mathcal{S}(\mathfrak{A})$ is in the domain of A, if it is in the support of A and $\omega(A^2) \overset{\text{def}}{=} \sup\big(\omega(f(A)) \,|\, f \in C_0(\mathbb{R})_+, f(\lambda) \le \lambda^2, \lambda \in \mathbb{R}\big) < \infty$.

Then $\mathcal{E}$ is the enveloping C^*-algebra of the algebra with twisted convolution product. By definition,

$$
\begin{aligned}
F(\sigma,x)\star_\sigma G(\sigma,x) &= \int dkdp\, \check{F}(\sigma,k)\,\check{G}(\sigma,p)\,e^{-\frac{i}{2}k\sigma p}\,e^{i(k+p)x} \\
&= (4\pi)^{-4}\int dx_1dx_2\,F(\sigma,x_1)\,G(\sigma,x_2)\;e^{-2i\,(x_2-x)\sigma^{-1}(x-x_1)}\,.
\end{aligned}
\tag{1.6}
$$

Shorthand notations such as $kq = k_\mu q^\mu$ and $k^\mu\sigma_{\mu\nu}p^\nu = k\sigma p$ are employed throughout, and dk abbreviates d^4k. The exponential $\exp(-\frac{i}{2}k\sigma p)$ is referred to as the *twisting*.
Frequently, only $f(q)$ will be considered in calculations, and the dependence on Q will be suppressed. As $f(q)$ is independent of Q, it is an element of the C^*-algebra of continuous bounded (not vanishing at infinity) functions from Σ to $\mathcal{K}$. Products of such $f(q)$ depend on Q, and, the twisting being bounded, are again elements of this C^*-algebra, which is denoted $\tilde{\mathcal{E}}$ as in [27].
The symbol of an n-fold product $f_1(q)\cdots f_n(q)$ may be written as

$$
(f_1\dots f_n)(x) = \int dx_1\dots dx_n\; f_1(x_1)\dots f_n(x_n)\,C_n(\sigma;x_1-x,\dots,x_n-x)\,,
$$

with the nonlocal kernel C_n (cf. [27]),

$$
C_n(\sigma;x_1-x,\dots,x_n-x) = (2\pi)^{-4n}\int dk_1\dots dk_n\; e^{+i\sum_j k_j(x-x_j)}\,e^{-\frac{i}{2}\sum_{i<j}k_i\sigma k_j}\,. \tag{1.7}
$$

In the literature, (1.6) is often interpreted as the Weyl-Wigner-Moyal star-product

$$
f\star_M g\,(x) = e^{\frac{i}{2}\overleftarrow{\partial}_\mu\sigma^{\mu\nu}\overrightarrow{\partial}_\nu}f(x)g(x)\,.
$$

This requires the interchange of the momentum integration with the exponential series expansion. Hence, it is not surprising that for general smooth functions the star-product is a formal power series, and as such may have very different support properties from the twisted convolution product. However, apart from the models which employ θ-expanded actions, the calculations performed in the literature actually use the twisted convolution product, not the star-product. Therefore, the symbol $\star$ which is customary there is used also throughout this thesis.
The Poincaré group acts on the symbols in $\mathcal{E}$ by

$$
(\tau_{a,\Lambda}F)(\sigma,x) = (\det\Lambda)\;F(\Lambda^{-1}\sigma(\Lambda^{-1})^t,\Lambda^{-1}(x-a))\,,
$$

and, hence, acts as an automorphism on $\mathcal{E}$. Therefore, derivatives can be defined as infinitesimal generators of translations, and for $f(q)$ we obtain

$$
\partial_{q_\mu}f(q) = \partial_{a_\mu}f(q+a)\,|_{a=0}\,,\qquad \text{where } a\in\mathbb{R}^4\,.
$$

A $\mathcal{Z}$-valued trace on $\tilde{\mathcal{E}}$ was defined in [27] by

$$
Tr(f(q)) \stackrel{\text{def}}{=} \check{f}(0)\,,
$$

written symbolically as $Tr(f(q)) = \int d^4q\; f(q)$, and was shown to be indeed positive. It has the property

$$
Tr(f(q)g(q)) = \int d^4q\; f(q)\,g(q) = \int d^4x\; f(x)\star_\sigma g(x) = \int d^4x\; f(x)\,g(x)\,.
$$

Hence, in terms of the twisted convolution product we may say that "one star-product may be dropped under the trace". It was also shown in [27] that there is another positive $\mathcal{Z}$-valued map on $\tilde{\mathcal{E}}$, given by

$$\int_{q^0=t} d^3q\, f(q) \stackrel{\text{def}}{=} \int dk_0\, e^{ik_0 t}\, \check{f}(k_0, \mathbf{0})\,. \tag{1.8}$$

Here and in what follows, boldface letters stand for the spatial part of a 4-vector. In view of the uncertainty relations the map (1.8) can be understood as follows: one may consider an expression at a fixed time t at the cost of total ignorance regarding its position in space. Note, however, that evaluation in a point, $q^\mu \mapsto a$, $a \in \mathbb{R}^4$, fails to be a positive functional on $\mathcal{E}$.
States on $\mathcal{E}$ must be maps to $\mathbb{C}$, and therefore, one has to integrate out the dependence on $\mathcal{Z}$. Unfortunately, the Lorentz group is not compact and no Lorentz-invariant compact measure exists, and therefore, evaluation in a state will break Lorentz covariance. One may postpone this problem by first considering $\mathcal{Z}$-valued "states" and integrating over parts of Σ later. Both notations

$$\omega(A) \quad \text{and} \quad \langle \omega, A\rangle$$

for evaluation in a state ω will be used throughout. States on $\mathcal{E}$ with optimal localization both in space and in time can be defined. By definition, they minimize $(\Delta q_0)^2 + \cdots + (\Delta q_3)^2$, a characterization which is evidently invariant under rotations and translations, but not under Lorentz boosts. Explicitly, it was shown in [27] that for $F \in \mathcal{E}$ the optimally localized states are of the following form,

$$\omega_a(F) = \int_{\Sigma_1} d\mu_\sigma (\eta_a F)(\sigma)\,, \tag{1.9}$$

where $\eta_a : \mathcal{E} \to C_0(\Sigma_1)$ is the localization map with localization centre $a \in \mathbb{R}^4$,

$$(\eta_a F)(\sigma) \stackrel{\text{def}}{=} \int dk\, \check{F}(\sigma, k)\, e^{-\frac{1}{2}|k|^2}\, e^{ika}\,, \qquad |k|^2 = {k_0}^2 + \cdots + {k_3}^2\,, \tag{1.10}$$

and where μ is any probability measure on the distinguished subset Σ_1 of Σ,

$$\Sigma_1 \stackrel{\text{def}}{=} \{\sigma \in \Sigma \,|\, ||\sigma||^2 \stackrel{\text{def}}{=} \tfrac{1}{2}\textstyle\sum \sigma_{\mu\nu}^2 = \tfrac{1}{2}(\vec{e}^{\,2} + \vec{m}^2) = \vec{e}^{\,2} = \vec{m}^2 = 1\}\,, \tag{1.11}$$

whose definition is rotation- and translation-invariant but not invariant under boosts. In fact, Σ_1 is the orbit of $\sigma^{(0)}$ under the action of the orthogonal group.
For later purposes, it is convenient to introduce the C^*-algebra $\mathcal{E}_1$ generated by the symbols of $\mathcal{E}$ restricted to Σ_1 by the restriction map $\rho : \mathcal{E} \to \mathcal{E}_1$ with $\rho F = F\lceil_{\Sigma_1}$. A localization map η with localization $a = 0$ can then be written as the composition $\eta = \eta^{(1)} \circ \rho$, where $\eta^{(1)}$ is a positive map from $\mathcal{E}_1$ to $C_0(\Sigma_1)$ and where $\langle \eta, e^{ikq}\rangle = e^{-\frac{1}{2}|k|^2}$ is understood as a constant function of $\sigma \in \Sigma_1$.
We are now prepared for a first preliminary definition of a free quantum field on the noncommutative Minkowski space. It will be made precise in Definition 4.6 in chapter 4, but suffices for the following two chapters. By analogy with the definition of $f(q)$, a quantum field ϕ on the noncommutative Minkowski space is formally written as

$$\begin{aligned} \phi(q) &\stackrel{\text{def}}{=} \int dk\, \check{\phi}(k) \otimes e^{ikq} \\ &= (2\pi)^{-3/2} \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} \left(a(k) \otimes e^{-ikq} + a^\dagger(k) \otimes e^{ikq} \right)\big|_{k\in H_m^+} \end{aligned} \tag{1.12}$$

with ordinary annihilation and creation operators a and $a^\dagger$, $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$ and the ordinary positive mass-shell H_m^+. The field ϕ is to be interpreted as a linear map from states on $\mathcal{E}$ to smeared field operators,

$$\omega \mapsto \phi(\omega) = \langle I \otimes \omega,\ \phi(q)\rangle = \int dx\, \phi(x)\, \psi_\omega(x)\,,$$

where on the right-hand side, $\phi(x)$ is a quantum field on ordinary spacetime, smeared with a testfunction ψ_ω defined by $\check{\psi}_\omega(k) = \langle \omega,\ e^{ikq}\rangle$. If products of fields are evaluated in a state, the right-hand side will in general involve nonlocal expressions such as $\phi \star \phi\,(x)$. The tensor product sign will be dropped to save notation, if no confusion is to be expected. The field operators being unbounded are affiliated to the corresponding Weyl algebra, the ordinary field algebra denoted $\mathcal{F}$.
The smeared field operator is defined on the ordinary symmetric Fock space of the free scalar field,

$$\mathfrak{H} = \bigoplus_{n=0}^{\infty} \mathfrak{H}^{(n)} \quad \text{with scalar product} \quad \langle \boldsymbol{\psi}|\boldsymbol{\varphi}\rangle = \sum_{n=1}^{\infty} \langle \boldsymbol{\psi}^{(n)}|\boldsymbol{\varphi}^{(n)}\rangle\,, \quad \langle \boldsymbol{\psi}|\boldsymbol{\psi}\rangle < \infty\,, \tag{1.13}$$

where $\mathfrak{H}^{(n)}$ is the n-particle space with wavefunctions $\psi^{(n)} \in L^2(\mathbb{R}^{3n})$ and scalar product

$$\langle \boldsymbol{\psi}^{(n)}|\boldsymbol{\varphi}^{(n)}\rangle = \int \frac{d\,\mathbf{p}_1}{2\omega_{\mathbf{p}_1}} \cdots \frac{d\,\mathbf{p}_n}{2\omega_{\mathbf{p}_n}}\, \overline{\psi^{(n)}(\mathbf{p}_1, \ldots, \mathbf{p}_n)}\, \varphi^{(n)}(\mathbf{p}_1, \ldots, \mathbf{p}_n)\,.$$

Often, the wavefunction will be written as depending on 4-momenta, $\psi^{(n)}(p_1, \ldots, p_n)$, the measure restricting them to the positive mass-shell. The unique vacuum state is denoted $|\Omega\rangle$. Frequently, the invariant domain $\mathcal{D}$ will be employed, which consists of states $|\boldsymbol{\psi}\rangle$ with smooth wavefunctions,

$$\psi^{(n)}(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathcal{S}(\mathbb{R}^{3n})$$

Throughout the following chapter, the "improper eigenstates" of the momentum operator, are used,

$$|p_1, \ldots p_n\rangle \stackrel{\text{def}}{=} a^\dagger(p_1) \ldots a^\dagger(p_n)|\Omega\rangle\,.$$

The implicit assumption underlying the definition of the quantum field (1.12) is that the field and the noncommutative structure do not interact. A possibly very interesting modification would be to consider annihilation and creation operators which depend on $\sigma \in \Sigma$.
With (1.12) as it stands, large parts of the apparatus of ordinary quantum field theory can be used. In order to clarify the notation let us now recall some basic facts from ordinary quantum field theory. As a very first starting point, the ordinary definition of Wick ordering can be applied to products of fields on the noncommutative Minkowski space via the formula

$$:\!\phi^n(q)\!: \stackrel{\text{def}}{=} (2\pi)^{-4n} \int dk_N :\!\hat{\phi}(k_1) \ldots \hat{\phi}(k_n)\!: \prod_{l \in N} e^{-ik_l q}\,, \tag{1.14}$$

where the Wick product $:\!\hat{\phi}(k_1) \ldots \hat{\phi}(k_n)\!:$ is defined as usual by putting all annihilation operators to the right. A shorthand notation has been applied, where $N = \{1, \ldots, n\}$ and where dk_N denotes $\prod_{i \in N} dk_i$. This notation will be employed throughout. Note that by application of Wick ordering to the field-operator part of (1.12), the order of the noncommutative exponentials $e^{ik_l q}$ remains unaffected.
Equivalent to the ordering prescription is the following definition of Wick products in momentum space, useful in later calculations,

$$\begin{aligned} :\!\hat{\phi}(k_1) \ldots \hat{\phi}(k_n)\!: \;&=\; \hat{\phi}(k_1) \ldots \hat{\phi}(k_n) \;+ \\ &+ \sum_{\substack{J \subset N \\ J \neq \emptyset}} \;\sum_{\substack{\alpha : J \to N \setminus J \\ \text{injective} \\ \alpha(j) > j\ \forall j \in J}} \Big(\prod_{l \in J} (-i)\, (2\pi)^4\, \hat{\Delta}_+(k_l)\, \delta^{(4)}(k_l + k_{\alpha(l)}) \prod_{i \in (J \cup \alpha(J))^c} \hat{\phi}(k_i) \Big)\,, \end{aligned} \tag{1.15}$$

where ^{c} denotes the complement in N and where $\hat{\Delta}_+(k)$ is the Fourier transform of the 2-point function $\Delta_+(x-y)$,

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= i\Delta_+(x-y)\,,\\ \langle\Omega|\hat{\phi}(k)\hat{\phi}(p)|\Omega\rangle &= (2\pi)^4\, i\,\hat{\Delta}_+(k)\,\delta^{(4)}(k+p)\,.\end{aligned}$$

In what follows, the above vacuum expectation values both in position and momentum space are referred to as contractions of two fields, and any term consisting of a number of contractions of two fields and possibly further uncontracted fields is referred to as a *contraction*.
The lengthy proof of (1.15) is elementary. Instead of requiring that $\alpha(j) > j$ for all $j \in J$, we could alternatively introduce a factor $\frac{1}{2}$ for every contraction to absorb the effect of double counting. Note that $|J|$, the number of elements of J, must be smaller than or equal to $[\frac{n}{2}]$ (the integer part of $\frac{n}{2}$), as otherwise the condition that α be injective is empty.
It is well-known that in Minkowski space

$$:\phi(x_1)\dots\phi(x_n): = \int dk_N :\hat{\phi}(k_1)\dots\hat{\phi}(k_n): \prod_{i\in N} e^{-ik_ix_i}$$

with $:\hat{\phi}(k_1)\dots\hat{\phi}(k_n):$ as in (1.15) yields a well-defined distribution in coinciding points, i.e. it remains well-defined when evaluated in a sequence of compactly supported testfunctions $g_n \in C_c^\infty(\mathbb{R}^{4n})$ approaching

$$\delta(x-x_1)\cdots\delta(x-x_n)$$

with $\operatorname{supp} g_n \subset \operatorname{supp} g_{n+1}$ and $\bigcap \operatorname{supp} g_n = \{x\}$. The ordinary product of fields $\phi(x_1)\dots\phi(x_n)$ diverges in this limit. The question of how to generalize this notion of coinciding points as well as the investigation of the resulting product of fields on the noncommutative Minkowski space is the main subject of this thesis.
Another formula which will be needed later is the Wick theorem. On Minkowski space as well as on its noncommutative counterpart, the product of two Wick monomials of order n and m can be rewritten as a sum of Wick monomials of order $\leq n+m$ by application of Wick's theorem, which in momentum space is

$$\begin{aligned}&:\hat{\phi}(k_1)\dots\hat{\phi}(k_n)::\hat{\phi}(p_1)\dots\hat{\phi}(p_m): \qquad (1.16)\\ &= \sum_{I\subset N}\ \sum_{\substack{\alpha:I\to M\\ \text{injective}}}\ \prod_{i\in I} i\,(2\pi)^4\,\hat{\Delta}_+(k_i)\,\delta^{(4)}(k_i+p_{\alpha(i)}) :\prod_{i\in I^c}\hat{\phi}(k_i)\prod_{i\in\alpha(I)^c}\hat{\phi}(p_i):.\end{aligned}$$

For all further necessary conventions see appendix A. They mostly coincide with the ones in [12]. If not pointed out otherwise, natural units are employed.

Chapter 2

The Hamiltonian Approach

In [27], a definition of a field theory on noncommutative spacetime has been proposed which is based on the introduction of a Hamiltonian and yields a formally unitary S-matrix by means of the ordinary Dyson series. This approach was later also called the interaction point time-ordering approach [13]. The aim of this chapter is to discuss the modified expectation values emerging in this approach in a formal manner. We will see, in particular, why the arising perturbation theory is formally unitary, while a violation of unitarity [43] was found in the context of the modified Feynman rules [37, 38]. The chapter should moreover serve as an introduction to the kind of problems one encounters when defining a quantum field theory on a noncommutative spacetime with noncommutative time variable. Moreover, the ultraviolet behaviour of such an approach is discussed. It is shown, in particular, that the ultraviolet behaviour of a special Hamiltonian formalism is essentially improved compared to the modified Feynman rules, such that a theory with ϕ^3-self-interaction (and most likely, also ϕ^4) turns out to be finite.
Starting point is the observation that the free Hamiltonian on the ordinary Minkowski space can equivalently be understood as the positive linear functional $\int_{q^0=t} d^3q$ from (1.8) acting on a Hamiltonian density defined on the noncommutative Minkowski space,

$$
\begin{aligned}
H_0 &= \tfrac{1}{2}\int_{x^0=t} d^3x :(\,(\partial_0\phi(x))^2 + (\nabla\phi(x))^2 + m^2\phi(x)^2\,): \\
&= \tfrac{1}{2}\int_{x^0=t} d^3x :(\,(\partial_0\phi(x))^2 - (\partial_0^2\phi(x))\,\phi(x)\,): \\
&= \tfrac{1}{2}\int_{q^0=t} d^3q :(\,(\partial_0\phi(q))^2 - \phi(q)\,(\partial_0^2\phi(q))\,): \\
&= \tfrac{1}{2}\int_{q^0=t} d^3q :(\,(\partial_0\phi(q))^2 + (\nabla\phi(q))^2 + m^2\phi(q)^2\,):\,,
\end{aligned}
$$

where the equivalence of the second and the third line has been proved in [27] for free fields satisfying the Klein-Gordon equation. The double dots denote normal ordering in the sense of equation (1.14). We see that H_0 does not depend explicitly on Σ and that, as usual, H_0 is independent of the time t.
It is now natural to define the interaction Hamiltonian for a scalar self-interacting theory by analogy, simply replacing the ordinary interaction term $\phi^n(x)$ by a noncommutative counterpart $\phi^n(q)$ and employing the positive linear functional $\int_{q^0=t} d^3q$. As we will see in chapter 3, this is not the only possible generalization.
The twisting appearing in the product $\phi^n(q)$, the resulting operator depends on the possible values of the commutators $Q^{\mu\nu}$ for $n \geq 3$ (or, to be exact, it takes values in the centre $\mathcal{Z}$ of the multiplier algebra of $\mathcal{E}$). Somehow, one has to eliminate this dependence, since, finally, expectation values calculated from the theory must be numbers. It was proposed in [27] to get

rid of this dependence by integrating over Σ_1, the compact rotationally invariant subset of Σ, which arose in the discussion of best-localized states (1.11). Hence, the interaction Hamiltonian on the noncommutative Minkowski space proposed in [27] is

$$H_I(t) \;=\; \frac{g}{n!}\int_{\Sigma_1} d\mu_\sigma \int\limits_{q^0=t} d^3q \;{:}\phi^n(q){:}\,, \tag{2.1}$$

which can equivalently be written as

$$\begin{aligned} H_I(t) &= \tfrac{g}{n!}\int_{\Sigma_1} d\mu_\sigma \int\limits_{x^0=t} d^3\mathbf{x} \;{:}\phi \star_\sigma \cdots \star_\sigma \phi\,(x){:} \\ &= \tfrac{g}{n!}\tfrac{1}{(2\pi)^{4n}}\int_{\Sigma_1} d\mu_\sigma \int\limits_{x^0=t} d^3\mathbf{x}\int dk_N \;{:}\hat\phi(k_1)...\,\hat\phi(k_n){:}\; \exp\big(-\tfrac{i}{2}\sum_{i<j} k_i\sigma k_j\big)\, e^{-ix\sum k_i} \qquad (2.2)\\ &= \tfrac{g}{n!}\int_{\Sigma_1} d\mu_\sigma \int dx_N\; \mathcal{D}(\sigma,t;x_1,\ldots,x_n)\;{:}\phi(x_1)\cdots\phi(x_n){:}\,. \qquad (2.3) \end{aligned}$$

Here, the double dots indicate normal ordering of ordinary fields (in position or momentum space respectively), and where the position space kernel $\mathcal{D}$ is given by

$$\begin{aligned} \mathcal{D}(\sigma,t;x_1,\ldots,x_n) &= (2\pi)^{-4n}\int dk_N\; \exp\big(-\tfrac{i}{2}\sum_{i<j} k_i\sigma k_j\big)\,(2\pi)^3\,\delta^{(3)}(\textstyle\sum \mathbf{k}_i)\, e^{+i\sum k_i x_i}\, e^{-it\sum k_{i,0}} \\ &= \int\limits_{x^0=t} d\mathbf{x}\; C(\sigma; x-x_1,\ldots,x-x_n) \end{aligned}$$

with C as in equation (1.7). The rotation- and translation-invariant measure has the following explicit form,

$$d\mu_\sigma = (8\pi)^{-1}\, d\vec e\, d\vec m\, \delta(\vec e^{\,2}-1)\,\delta^{(3)}(\vec e \mp \vec m)\,. \tag{2.4}$$

There are more possibilities to rid H_I of the dependence on $\mathcal{Z}$ than by integrating over Σ_1, though unfortunately, no Lorentz-invariant yielding a finite result is available on Σ. Integrating over Σ_1 is not a Lorentz-invariant prescription, but at least rotation and translation invariance are kept in this approach.

Another approach, which is more commonly employed in the context of string-inspired noncommutative models, is to fix a particular value $\theta \in \Sigma$, or more frequently, to consider a fixed noncommutativity matrix $\theta \notin \Sigma$. For such a fixed noncommutativity matrix θ, the interacting Hamiltonian is

$$H_I^\theta(t) = \tfrac{g}{n!}\int\limits_{x^0=t} d^3\mathbf{x}\;{:}\phi \star_\theta \cdots \star_\theta \phi\,(x){:}\,. \tag{2.5}$$

This can be understood in terms of the Hamiltonian previously defined by using special point measures,

$$H_I^\theta(t) = \tfrac{g}{n!}\int d\tilde\mu_\sigma \int\limits_{q^0=t} d^3q\;{:}\phi(q)^n{:} = \tfrac{g}{n!}\int dx_N\;\mathcal{D}(\theta,t;x_1,\ldots,x_n)\;{:}\phi(x_1)\cdots\phi(x_n){:}\,,$$

where

$$d\tilde\mu_\sigma = d\vec e\, d\vec m\,\delta^{(3)}(e_i-\theta_{0i})\,\delta^{(3)}(m_i-\epsilon_{ilk}\theta_{lk})\,. \tag{2.6}$$

In particular, we can choose

$\theta \in \Sigma_1 \subset \Sigma$:	$d\tilde\mu_\sigma = d\vec e\, d\vec m\,\delta^{(3)}(\vec e-(1,0,0))\,\delta^{(3)}(\vec e\mp\vec m)\,,$
$\theta \notin \Sigma$ spacelike:	$d\tilde\mu_\sigma = d\vec e\, d\vec m\,\delta^{(3)}(\vec e)\,\delta^{(3)}(\vec m-(1,0,0))\,,$
$\theta \notin \Sigma$ dilation-covariant (lightlike):	$d\tilde\mu_\sigma = d\vec e\, d\vec m\,\delta^{(3)}(\vec e-(0,1,0))\,\delta^{(3)}(\vec m-(1,0,0))\,.$

Obviously, for $\theta \notin \Sigma$, the corresponding noncommutative algebra is not $\mathcal{E}_\sigma$, although similar in its algebraic properties. In particular, the uncertainty relations will differ when $\theta \notin \Sigma$, as discussed in chapter 1.

Of course, not only invariance under boosts but also rotation invariance is lost, if a fixed noncommutativity matrix is used, though translation invariance is kept as a consequence of the fact that the commutators are central. Using the symmetries which leave the property "spacelike" or "lightlike" invariant, we may however find measures which are not point measures. As is well-known from electrodynamics, in the lightlike case, such an extended measure will not be rotation-invariant, since rotations do not preserve the property "lightlike". One could choose, for instance,

$$d\mu_\sigma = d\vec{e}\, d\vec{m}\, \delta^{(3)}(\vec{e} - (0,0,1))\, \delta(m_3)\, \delta(\vec{m}^2 - 1)\,. \tag{2.7}$$

In what follows, different fixed noncommutativity matrices as well as the integration over Σ_1 will be considered. We will frequently work with a Hamiltonian which depends on σ,

$$\begin{aligned} H_I(\sigma;t) &= \tfrac{g}{n!}\tfrac{1}{(2\pi)^{4n}} \int_{x^0=t} d^3\mathbf{x} \int dk_N :\hat{\phi}(k_1)...\hat{\phi}(k_n): \exp\left(-\tfrac{i}{2}\sum_{i<j} k_i\sigma k_j\right) e^{-ix\sum k_i} \\ &= \tfrac{g}{n!}\int dx_1 \ldots dx_n\, \mathcal{D}(\sigma,t;x_1,\ldots,x_n) :\phi(x_1)\cdots\phi(x_n):\,, \end{aligned} \tag{2.8}$$

to perform calculations and bear in mind that we still need to integrate against a measure (2.4) or (2.6). Note that in the limit $\lambda_P \to 0$, where the twisting $\exp(-\frac{i}{2}\lambda_P^2 \sum k_i\sigma k_j)$ tends to 1, the kernels $\mathcal{D}$ tend to

$$\prod_{j=1}^{n} \delta(x_{j,0} - t) \prod_{j=2}^{n} \delta^{(3)}(\mathbf{x}_j - \mathbf{x}_1)\,,$$

and we recover the ordinary local Hamiltonian.

In [27] it was proposed to assume LSZ-asymptotic conditions [61] and to calculate expectation values from

$$\sum_{r=0}^{\infty} \tfrac{(-i)^r}{r!} \langle\Omega|\, T\left(\phi_0(x_1)\ldots\phi_0(x_n) \int dt_1 \ldots dt_r\, H_I(t_1)\ldots H_I(t_r)\right)|\Omega\rangle\,.$$

Before discussing the resulting graph theory, some general remarks on the properties of theories with nonlocal interaction in the Hamiltonian framework are to be made.

2.1 Dyson's series

Consider a Hamiltonian on Minkowski space, defined by some nonlocal kernel K,

$$H_I(t) = \tfrac{g}{n!} \int_{x^0=t} d\mathbf{x} \int dx_N\, K(x-x_1,...,x-x_n) :\phi(x_1)\cdots\phi(x_n): \tag{2.9}$$

$$= \tfrac{g}{n!} \int dx_N\, \mathcal{K}(t;x_1,\ldots,x_n) :\phi(x_1)\cdots\phi(x_n):\,, \tag{2.10}$$

where $\mathcal{K}$ is the spatial integral $\int d\mathbf{x}$ of K at fixed time t. Similarly to the usual case, we can then write

$$H_I(t) = e^{iH_0 t}\, H_{I,0}\, e^{-iH_0 t}\,,$$

where $H_{I,0}$ is defined by (2.10) with kernel $\mathcal{K}(0;x_1,\dots,x_n)$ at time 0. This is a consequence of the following identity

$$\int dx_N\ \mathcal{K}(0;x_1,\dots,x_n)\ e^{iH_0t} :\phi(x_{1,0},\mathbf{x}_1)\cdots\phi(x_{n,0},\mathbf{x}_n): e^{-iH_0t}$$
$$= \int dx_N \int\limits_{x_0=0} d\mathbf{x}\ K(x-x_1,\dots,x-x_n) :\phi(x_{1,0}+t,\mathbf{x}_1)\cdots\phi(x_{n,0}+t,\mathbf{x}_n):$$
$$= \int dx_N\ \mathcal{K}(t;x_1,\dots,x_n) :\phi(x_1)\cdots\phi(x_n): .$$

The kernels $K(...)=\int d\mu_\sigma\, C(\sigma;...)$ and $\mathcal{K}(t;...)=\int d\mu_\sigma\,\mathcal{D}(\sigma,t;...)$ with one of the measures $d\mu_\sigma$ from above provide special examples of such Hamiltonians. Pursuing this programme means to treat the theory on the noncommutative spacetime $\mathcal{E}$ as a theory on the ordinary Minkowski space with an effective nonlocal Hamiltonian density given by C. Compared to the ordinary case, it is important to note that the interaction Hamiltonian $H_{I,0}$ at time zero does not arise from a product of time-zero fields on the noncommutative spacetime $\mathcal{E}$. A direct definition of such time-zero fields on $\mathcal{E}$ does not seem to be available, as sharp localization is impossible (recall that – as was pointed out in chapter 1 – the δ-distribution is not a positive functional on $\mathcal{E}$).

Based on the above, we can formally follow the usual procedure and solve the Schrödinger equation for the time evolution operator $U(t,s)$ in the interaction picture,

$$i\,\frac{d}{dt}\,U(t,s)=H_I(t)\,U(t,s)\,,\qquad U(t,t)=1\,,$$

by iteration. This yields the ordinary Dyson series [31] with its time-ordered exponential. The interacting field in this framework then is (formally) defined as

$$\phi_{int}(x)=U^{-1}(x_0,-\infty)\,\phi(x)\,U(x_0,-\infty)\,,\quad x=(x^0,\mathbf{x})\,. \tag{2.11}$$

By Haag's theorem [50], we know that in ordinary field theory the interaction picture is inconsistent. Within the local approach to quantum field theory, this is usually circumvented, since the coupling constant is replaced here by a function $g\in C_0^\infty(\mathbb{R}^4)$ which is constant in some bounded region and vanishes completely outside some bounded region in spacetime. Translation invariance is broken and Haag's theorem does not apply anymore.

Let us now proceed in a similar way and introduce such a cutoff function g in the interaction, which renders all expressions well-defined in the infrared region (large distance). The interaction Hamiltonian then is

$$H_I^g(t)=e^{iH_0t}\,H_{I,t}^g\,e^{-iH_0t}$$

with

$$H_{I,t}^g=\tfrac{1}{n!}\int\limits_{x^0=0} d^3\mathbf{x}\ g(t,\mathbf{x})\int dx_N\ K(x-x_1,\dots,x-x_n) :\phi(x_1)\dots\phi(x_n):\,,$$

and the S-matrix can be defined as the limit of the corresponding time evolution operator $U(t,s)$, where $t\to\infty$ and $s\to-\infty$, such that

$$S[g]=1+\sum_{r=1}^{\infty}S_r[g]\,,$$

where (with π_s abbreviating $\pi(s)$ for an element π of the permutation group P_r)

$$\begin{aligned} S_r[g] &= \tfrac{(-i)^r}{r!}\sum_{\pi\in P_r}\int dt_1\dots dt_r\,\theta(t_{\pi_1}-t_{\pi_2})\dots\theta(t_{\pi_{r-1}}-t_{\pi_r})\,H_I^g(t_{\pi_1})\,\dots\,H_I^g(t_{\pi_r}) \\ &= (-i)^r\int dt_1\dots dt_r\,\theta(t_1-t_2)\dots\theta(t_{r-1}-t_r)\,H_I^g(t_1)\,\dots\,H_I^g(t_r) \end{aligned} \tag{2.12}$$

with θ denoting the Heaviside step function. Following common convention, the symbol θ is also used for the fixed noncommutativity matrix, but, since the Heaviside functions appear in a different context, no confusion is to be expected.
In this framework, the interacting field can be defined by Bogoliubov's formula as

$$\phi_{int}(x) = \frac{\delta}{i\delta h(x)}\, S(g,0)^{-1}\, S(g,h)\,\big|_{h=0}\,, \tag{2.13}$$

where $S(g,h)$ is the S-matrix with interaction term

$$h(x)\,\phi(x) \;+\; g(x)\int dx_N\, K(x-x_1,\dots,x-x_n)\,{:}\phi(x_1)\cdots\phi(x_n){:}$$

for g and h compactly supported testfunctions. This definition has the advantage over (2.11) that fields with definite time-component (time-zero-fields), which in higher order perturbation theory are ill-defined, do not appear explicitly[1].
The connection between the local formalism and the one without cutoff function is usually established by taking the adiabatic limit, where the cutoff function g tends to a constant. It can be performed either for the n-point functions ("weak adiabatic limit") or for the S-matrix itself ("strong adiabatic limit"). For massive local theories both limits were shown to exist [35, 36]. An alternative approach [55] was analysed recently in [14], where local observables in the sense of [51] are constructed from the S-matrix via Bogoliubov's formula. The construction depends only locally on the particular cutoff function g, such that if two such functions coincide on some bounded region of spacetime, the corresponding algebras of observables are unitarily equivalent. The construction relies on the fact that the S-matrix is causal.
Before investigating some properties of the above S-matrix for a nonlocal interaction, note that it can also be written using an effective "Lagrangian density",

$$S_r[g] = \tfrac{i^r}{r!}\int dx_1\dots dx_r\, S_r(x_1,\dots,x_r)\, g(x_1)\dots g(x_r)$$

with $S_r(x_1,\dots,x_r) =$

$$\begin{aligned} = \;& \left(\tfrac{-1}{n!}\right)^r \sum_{\pi\in P_r} \theta(x_{\pi_1,0}-x_{\pi_2,0})\cdots\theta(x_{\pi_{r-1},0}-x_{\pi_r,0})\;\cdot \\ & \cdot\int dx_1^1 ... \,dx_1^n \int\dots\int dx_r^1 ... \,dx_r^n \prod_{i=1}^{r}\Big(K(x_{\pi_i}-x^1_{\pi_i},\dots,x_{\pi_i}-x^n_{\pi_i})\;{:}\phi(x^1_{\pi_i}) ... \,\phi(x^n_{\pi_i}){:}\Big)\,. \end{aligned}$$

For example, if $K(\dots) = \int d\mu_\sigma C(\sigma;\dots)$, the second line in the above is

$$\int d\mu_{\sigma_1} ... \int d\mu_{\sigma_r}\;{:}\phi\star_{\sigma_1} ... \star_{\sigma_1}\phi\,(x_{\pi_1}){:}\;\dots\;{:}\phi\star_{\sigma_r} ... \star_{\sigma_r}\phi\,(x_{\pi_r}){:}\,.$$

Formally, the Hamiltonian H can indeed be derived from the "Lagrangian density". However, a precise definition of the energy-momentum tensor on the noncommutative Minkowski space has so far been derived in a satisfactory way only in [88] for classical fields, and the framework is not directly applicable here. Hence, the Hamiltonian approach is to be interpreted as an effective approach, enabling us to calculate expectation values of the S-matrix in analogy with the ordinary case. We will frequently encounter consequences of the fact that not all properties of ordinary local quantum field theory hold.

[1] Alternatively, we can still define operators U for the interaction with cutoff and formally define the interacting field as before.

Let us now recount some of the properties of the S-matrix defined by the Dyson series (2.12) with or without cutoff-function g.

Unitarity: First recall the following well-known argument which formally holds independently of whether we use the cutoff function g in the definition of the interaction Hamiltonian or not:

Remark 2.1 Any S-matrix defined by (2.12) is formally unitary (i.e. before renormalization),

$$S\,S^\dagger \;=\; 1 + S_1 + S_1^\dagger + (S_2 + S_1\,S_1^\dagger + S_2^\dagger) + \ldots \;=\; 1\,,$$

if the interaction Hamiltonian is symmetric, $H_I(t) = H_I(t)^\dagger$.

The well-known proof is repeated here only for the sake of demonstrating that the nonlocality of the interaction does not enter.

Proof: The claim is an immediate consequence of the way the time-ordering has been defined.

$$\begin{aligned}
&\sum_{N_1+N_2=N} S_{N_1}\,{S_{N_2}}^\dagger \quad = \\
&= \; i^N \sum_{N_1=0}^{N} (-1)^{N_1} \int dt_1 \ldots dt_N\, \theta(t_1-t_2)\ldots\theta(t_{N_1-1}-t_{N_1}) \;\cdot \\
&\qquad \cdot\, \theta(t_{N_1+1}-t_{N_1+2})\ldots\theta(t_{N-1}-t_N)\, H_I(t_1)\ldots H_I(t_{N_1})\, H_I(t_N)\ldots H_I(t_{N_1+1}) \\
&= \; i^N \int dt_1 \ldots dt_N\; H_I(t_1)\ldots H_I(t_N)\; \underbrace{\sum_{N_1=0}^{N} (-1)^{N_1} \prod_{i=1}^{N_1-1} \theta(t_i-t_{i+1}) \prod_{i=N_1+1}^{N-1} \theta(t_{i+1}-t_i)}_{=\,(*)}\,,
\end{aligned}$$

where the convention is such that Heaviside functions of arguments satisfying an empty condition such as $t_i - t_{i+1}$ with $i = N$, as well as empty products such as $\prod_{i=N}^{N-1} \ldots$, are set to 1. Obviously, $(*) = 1$ for $N = 0$, and using

$$\begin{aligned}
&\sum_{N_1=0}^{N+1} (-1)^{N_1} \prod_{i=1}^{N_1-1} \theta(t_i-t_{i+1}) \prod_{i=N_1+1}^{N} \theta(t_{i+1}-t_i) \\
&\quad = \sum_{N_1=0}^{N} (-1)^{N_1} \prod_{i=1}^{N_1-1} \theta(t_i-t_{i+1}) \prod_{i=N_1+1}^{N-1} \theta(t_{i+1}-t_i) \cdot \begin{cases} \theta(t_{N+1}-t_N) & N_1 < N \\ 1 & N_1 = N \end{cases} \\
&\qquad + (-1)^{N+1} \prod_{i=1}^{N} \theta(t_i-t_{i+1})
\end{aligned}$$

together with the fact that $1 = \theta(t_{N+1}-t_N) + \theta(t_N-t_{N+1})$ one proves by induction that $(*) = 0$ for $N \geq 1$. ■

Hence, if the time-ordering in the S-matrix is defined as in (2.12) with respect to the parameter times t appearing in the Hamiltonians $H_I(t)$, the theory will automatically be (formally) unitary. It follows in particular, that the S-matrix for any of the Hamiltonians introduced in this section is (formally) unitary, independent of whether we integrate over Σ_1 as in (2.1) or consider a Hamiltonian with fixed noncommutativity matrix as in (2.5).
It follows immediately that the outgoing field ϕ_{out} when defined as

$$\phi_{out} = S^\dagger \phi_{in} S\,,$$

satisfies the same commutation relations as the incoming field, i.e.

$$[\phi_{out}(x), \phi_{out}(y)] = [\phi_{in}(x), \phi_{in}(y)]\,,$$

where problems of renormalization have not yet been taken into account.
Thus the question arises whether this contradicts earlier results on nonlocal interactions [87, 60, 11] (cf. [57, 80] and the discussions at the "Conference of Theoretical Physics" in Kyoto and Tokyo in 1953, [56]). There, the above condition was shown to be violated (before renormalization), contrary to earlier claims [11], in theories which are nonlocal in time. As a consequence, a modification of the asymptotic condition was shown to be necessary in [53]. Starting point of these considerations is the field equation with a nonlocal interaction term, which is then solved recursively, a procedure called the Yang-Feldman approach [86, 59] in local quantum field theory. In chapter 4 of this thesis, this approach shall be analysed on the noncommutative Minkowski space. One of its virtues is that we are not compelled to treat the theory as an effective one on the ordinary Minkowski space with a nonlocal interaction. If however, we choose to do so, we will see in section 4.4 that the interacting field calculated by such methods differs from the one defined here. Hence, the two approaches are not immediately related and we may conclude that the counterexamples at fourth order of the perturbative expansion analysed in the publications cited above do not apply to the Hamiltonian formalism investigated here.
A point particularly worth mentioning in this context is the fact that the interacting field in the Hamiltonian formalism does not satisfy the ordinary equation of motion.

Remark 2.2 The interacting field defined by (2.13) for a nonlocal Hamiltonian does not satisfy the ordinary equation of motion,

$$(\Box + m^2)\phi_{int}(x) \neq U^{-1}(x_0, s)\, i[H_I(x_0), \partial^0\phi(x)]\, U(x_0, s)\,, \qquad s \to \infty\,, \qquad x = (x^0, \mathbf{x})\,,$$

unless the nonlocal kernel K is local with respect to the time variable,

$$K(x - x_1, \ldots, x - x_n) = \prod_{j=1}^{n} \delta(x_0 - x_{j,0})\, K_s(\mathbf{x} - \mathbf{x}_1, \ldots, \mathbf{x} - \mathbf{x}_n)\,,$$

where $K_s(\mathbf{x} - \mathbf{x}_1, \ldots, \mathbf{x} - \mathbf{x}_n)$ is nonlocal with respect to the spatial variables.

Proof: The calculation is straightforward, and already at first order we find a deviation. We employ a cutoff function g, which, however, is not crucial for the proof. The definition of the interacting field yields the following expression at first order,

$$\phi_{int,1}(x) = i \int dy\, g(y)\, \theta(x_0 - y_0)\, [\mathcal{H}(y), \phi(x)]\,,$$

where $\mathcal{H}(x) \stackrel{\text{def}}{=} \frac{1}{n!}\int dx_N\; K(x - x_1, \ldots, x - x_n)\, \phi(x_1) \ldots \phi(x_n)$ is the Hamiltonian density. Hence,

$$\partial^0\partial_0\, \phi_{int,1}(x) = i \int dy\, g(y)\, \partial_0 \left(\delta(x_0 - y_0)\, [\mathcal{H}(y), \phi(x)] + \theta(x_0 - y_0)\, [\mathcal{H}(y), \partial_0\phi(x)] \right).$$

In the local case, the first term is zero, since $[\mathcal{H}(y), \phi(x)]$ is proportional to the commutator function $\Delta(y - x)$ which is zero if $y_0 = x_0$. For a nonlocal interaction, the commutator yields

$$[\mathcal{H}(y), \phi(x)] = \frac{1}{n!} \int dy_N\; K(y - y_1, .., y - y_n) \sum_i \Delta(y_i - x) {:}\prod_{j \neq i} \phi(y_j){:}\,,$$

which will only vanish for $y_0 = x_0$ if $y_{i,0} = y_0\ \forall i$, i.e. if K is local with respect to the time variable. In the general nonlocal case we derive, using $[\mathcal{H}(y), \partial^0\partial_0\phi(x)] = (\nabla^2 - m^2)[\mathcal{H}(y), \phi(x)]$,

$$(\Box + m^2)\,\phi_{int,1}(x) \quad = \quad i\int dy\, g(y)\,\big(\,\partial^0(\,\delta(x_0-y_0)\,[\mathcal{H}(y),\phi(x)]\,) + \delta(x_0-y_0)\,[\mathcal{H}(y),\partial^0\phi(x)]\,\big)\,,$$

and, hence, taking into account that the fields are (operator-valued) distributions, we find

$$\int dx\,(\Box + m^2)\,\phi_{int,1}(x)\,h(x) \quad = \quad i\int dx\,[H(x_0),\partial^0\phi(x)]\,h(x) - i\int dx\,[H(x_0),\phi(x)]\,\partial^0 h(x)$$

for a testfunction h. Only the first term arises at first order in the right-hand side of the ordinary equation of motion, but the disturbance by the second term is absent only for interactions which are local with respect to the time variable. ■

Problems such as the well-posedness of the initial value problem and the derivation of the equations of motion from a Lagrangian density of classical fields may be investigated in the more rigorous framework elaborated in [88]. As emphasized before, however, the framework employed there is different from what is investigated here, and the results not immediately applicable.

Other attempts to give meaning to the Hamiltonian formalism in theories with nonlocal interaction or with higher derivatives have been investigated, for instance, in [42]. Here, the nonlocal theory is treated as a system with constraints, and a classical Hamiltonian is defined after an additional time-coordinate has been introduced.

Causality: In nonlocal theories, the S-matrix defined by Dyson's series (2.12) does not satisfy the causal factorization usually satisfied in local field theory, where for $\{x_1,\dots,x_l\}$ later than or spacelike to $\{x_{l+1},\dots,x_n\}$ (or equivalently for $\{x_{l+1},\dots,x_n\}$ not in the forward lightcones of $\{x_1,\dots,x_l\}$) we have

$$S_r^{loc}(x_{\pi_1},\dots,x_{\pi_r}) = S_l^{loc}(x_1,\dots,x_l)\,S_{r-l}^{loc}(x_{l+1},\dots,x_r)\,.$$

More particularly, we find in the local case the following locality condition for spacelike separated arguments $\{x_1,\dots,x_l\}$ and $\{x_{l+1},\dots,x_n\}$,

$$S_r^{loc}(x_{\pi_1},\dots,x_{\pi_r}) = S_l^{loc}(x_1,\dots,x_l)\,S_{r-l}^{loc}(x_{l+1},\dots,x_r) = S_{r-l}^{loc}(x_{l+1},\dots,x_r)\,S_l^{loc}(x_1,\dots,x_l)\,.$$

While for $\{x_1,\dots,x_l\}$ later than $\{x_{l+1},\dots,x_r\}$ the causality condition is satisfied for the S-matrix on the noncommutative spacetime, due to the time-ordering given by the Heaviside functions, the locality condition for spacelike separated arguments does not hold. To see this consider an example at second order. Let x be earlier than y, $(x-y)^2 > 0$, $x_0 < y_0$, then indeed we find

$$S_2(x,y) = \mathcal{H}(y)\,\mathcal{H}(x) = S_1(y)\,S_1(x)\,,$$

where $\mathcal{H}(x) = \frac{1}{n!}\int dx_N\ K(x-x_1,\dots,x-x_n)\,\phi(x_1)\dots\phi(x_n)$ is the effective Hamiltonian density, defined already in the proof of Remark 2.2, and where for now, normal ordering is not taken into account. Now let x be spacelike separated from y, $(x-y)^2 < 0$, then contrary to the above causality conditions we find

$$\begin{aligned} S_1(x)\,S_1(y) - S_1(y)\,S_1(x) \quad &= \quad \tfrac{1}{n!^2}\int dx_N dy_N\ K(x-x_1,\dots,x-x_n)\,\cdot \\ &\cdot K(y-y_1,\dots,y-y_n)\,\sum_{i=1}^{n}\,\Big(\prod_{k=1}^{i-1}\phi(x_k)\sum_{j=1}^{n}\big(i\Delta(x_i-y_j)\,\prod_{l\neq j}\phi(y_l)\big)\,\prod_{k=i+1}^{n}\,\phi(x_k)\Big) \\ &\neq\ 0\,. \end{aligned}$$

This commutator remains unequal to 0 even for equal time variables $x_0 = y_0$, unless the kernel K is local in time, such that $x_{i,0} = x_0$ and $y_{i,0} = y_0$ $\forall i$. In the next chapter a Hamilton operator on the noncommutative Minkowski space will be introduced where the kernels are such that this non-locality will decay like a Gaussian function.

Covariance: Consider the kernels from the previous section. We have seen there that Lorentz covariance is problematic due to the fact that the product $\phi^n(q)$ depends on $\sigma \in \Sigma$ and that there is no Lorentz-invariant measure on Σ which would yield a finite integral. However, when the noncommutative spacetime $\mathcal{E}$ is considered (or the dilation covariant "lightlike" noncommutative spacetime from [27]), the question of covariance may be postponed – and in the meantime, we bear in mind that before integration over Σ_1 (or against some extended measure in the lightlike case), the Hamiltonian density (2.8) transforms covariantly under the full Poincaré group,

$$\begin{aligned} U(a,\Lambda)\,\phi(q)^n\,U(a,\Lambda)^{-1} &= \phi(\Lambda q + a)^n \\ &= \int dk_N\,\check{\phi}(k_1)\cdots\check{\phi}(k_n)\,e^{i(\sum_l k_l)\,(\Lambda q + a)}\,e^{-\frac{i}{2}\sum_{l<j} k_l\,(\Lambda\sigma\Lambda^\dagger)\,k_j} \end{aligned}$$

and by construction, $\Lambda\sigma\Lambda^\dagger$ is again an element of Σ. Once the integration over Σ_1 is performed, the resulting expression is at least invariant under rotations and translations.

Notwithstanding the problems sketched above, LSZ-asymptotics [61] are assumed to hold (for an investigation with commuting time-variable see [17]), such that expectation values of the S-matrix in multi-particle states can be calculated as usual by the following formula

$$\begin{aligned} \langle p_1 \ldots \mathrm{in} \mid S_r \mid q_1 \ldots \mathrm{in} \rangle &= \tfrac{(-i)^r}{r!} \sum_{\pi \in P_r} \int dt_1 \ldots dt_r\,\theta(t_{\pi_1} - t_{\pi_2}) \ldots \theta(t_{\pi_{r-1}} - t_{\pi_r}) \,\cdot \\ &\qquad \cdot \langle p_1 \ldots \mathrm{in} \mid H_I(t_{\pi_1}) \ldots H_I(t_{\pi_r}) \mid q_1 \ldots \mathrm{in} \rangle \end{aligned} \tag{2.14}$$

This was already proposed in [27] and was later also called the interaction point time-ordering approach [13]. In later chapters we will consider the proper Fock space (1.13) to calculate expectation values, but for the time being, no smearing in the momenta is employed for the time being. Instead, we consider the improper eigenstates $|p_1 \ldots p_s\rangle$, where

$$\langle p| :\phi(x)\phi(y): |q\rangle = \tfrac{1}{(2\pi)^3}\,(\,e^{+ipx-iqy} + e^{+ipy-iqx}\,)\,.$$

From the expressions derived in this manner, we can find the smeared expectation values by simply integrating over the external momenta on the mass-shell against appropriate functions $\psi^{(n)}$. Inserting the explicit form of the Hamiltonian, we see that the important step in the calculation is to evaluate expectation values in position or momentum space,

$$\langle p_1 \ldots \mathrm{in} \mid :\hat{\phi}(k_1^{\pi_1}) \ldots \hat{\phi}(k_n^{\pi_1}): \;\ldots\; :\hat{\phi}(k_1^{\pi_r}) \ldots \hat{\phi}(k_n^{\pi_r}): \mid q_1 \ldots \mathrm{in} \rangle \tag{2.15}$$

$$\langle p_1 \ldots \mathrm{in} \mid :\phi(x_1^{\pi_1}) \ldots \phi(x_n^{\pi_1}): \;\ldots\; :\phi(x_1^{\pi_r}) \ldots \phi(x_n^{\pi_r}): \mid q_1 \ldots \mathrm{in} \rangle\,, \tag{2.16}$$

employing Wick's theorem. It is convenient for some purposes to introduce graphical rules to treat the combinatorics of the perturbative expansion in the spirit of Feynman's rules, but it is emphasized that this is not necessary. To calculate an expectation value at r-th order perturbation theory (2.14) in an m_1- and an m_2-particle state, we simply consider all possibilities to contract $r{\cdot}n{-}(m_1{+}m_2)$ fields in (2.15) or (2.16), respectively. If the renormalization procedure in position space is employed, it is not necessary to sort the diagrams in terms of their loop number. Instead, the S-matrix is renormalized order by order in the coupling constant. As we have seen, a number of properties, such as formal unitarity of the S-matrix, can be derived without reference to graphs.

Nonetheless, in order to establish the connection with the literature and, in particular, with the modified Feynman rules, let us now derive graphical rules for the case where ordinary normal ordering is used. These rules will moreover turn out to be quite useful to investigate the renormalization in the Hamiltonian formalism sketched here. Also, they will serve us well in the next chapter where the adiabatic limit of a theory based on a regulated interaction term is discussed.
Since our interest in this chapter mainly lies in comparing the graphical rules employed in the literature, where g is a constant from the outset, with the graphs arising within the Hamiltonian formalism, the cutoff function is not considered anymore in the remainder of this chapter. Instead, the ordinary coupling *constant* is employed.

2.2 Graphs

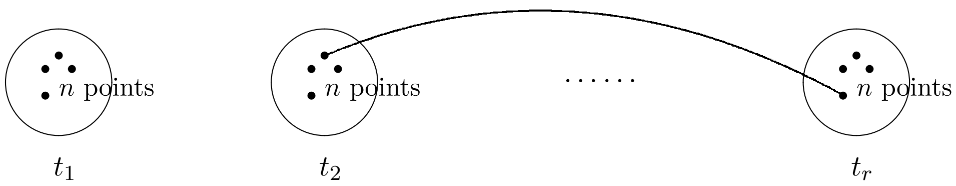
There are different ways to define graphical rules to simplify the calculation of the S-matrix' expectation values. The simplest one as far as its derivation is concerned is given in position space. Starting point is the Hamiltonian written in position space as in (2.3). To calculate the contribution to the S-matrix at the r-th order of the perturbative expansion, and with a number of l incoming and s outgoing external momenta, we have to calculate the following expression

$$\begin{aligned}
&(-i)^r \int dt_1 ... dt_r \prod_{i=1}^{r-1} \theta(t_i - t_{i+1}) \, \langle p_1 ... p_s \mid H_I(t_1) \ldots H_I(t_r) \mid q_1 ... q_l \rangle \\
&= \; (-i)^r \int dt_1 ... dt_r \prod_{i=1}^{r-1} \theta(t_i - t_{i+1}) \int \prod_{j=1}^{r} dx_j^1 ... dx_j^n \int d\mu_{\sigma_1} ... d\mu_{\sigma_r} \; \mathfrak{D}(\sigma_1, t_1; x_1^1, \ldots, x_1^n) \; \cdot \\
&\qquad \cdot \, \mathfrak{D}(\sigma_r, t_r; x_r^1, \ldots, x_r^n) \, \langle p_1 ... p_s \mid \prod_{j=1}^{r} {:}\phi(x_j^1) \cdots \phi(x_j^n){:} \mid q_1 ... q_l \rangle \, . \qquad (2.17)
\end{aligned}$$

Here, and throughout the rest of this chapter, the kernels C, and $\mathcal{D}$ are employed, but all results hold generally for any nonlocal kernel. In order to simplify the calculations, symmetrized kernels $\mathfrak{D}$,

$$\mathfrak{D}(\sigma, t; x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\pi \in S_n} \mathcal{D}(\sigma, t; x_{\pi_1}, \ldots, x_{\pi_n}) \, ,$$

are employed, which is legitimate since ${:}\phi(x_1) \cdots \phi(x_n){:} = {:}\phi(x_{\pi_1}) \cdots \phi(x_{\pi_n}){:}$ for an arbitrary permutation of the arguments.
By application of Wick's theorem, the products $\prod_{j=1}^{r} {:}\phi(x_1^j) \cdots \phi(x_n^j){:}$ are now to be brought into complete normal order, picking up the appropriate contractions, each of which yields a 2-point function (positive-frequency propagator) $i\Delta_+(x_{j_1}^{i_1} - x_{j_2}^{i_2})$, where $j_1 < j_2$. The non-trivial contributions to the expectation value $\langle p_1 ... p_s \mid \ \ldots \ \mid q_1 ... q_l \rangle$ are those where $n \cdot r - s - l$ fields are contracted. In order to keep track of the combinatorics, graphs may be helpful. The main difference to the graph theory on Minkowski space is that we have to employ vertices which consist of n points,



where $t_1 > \cdots > t_r$, and the points in the j-th vertex are labelled by x_j^i, $i = 1, \ldots, n$. Such vertices were called multi-vertices in [10]. The circles around the multi-vertices are merely a

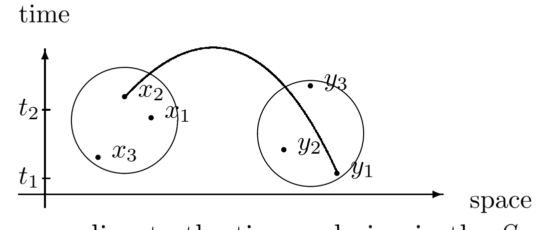
manner to keep track of which vertices belong to the same parameter time; since the nonlocal kernel's support is not compact, the multi-vertices are not confined to a particular region in spacetime.
Now, the different possibilities to contract a number of $n \cdot r - s - l$ fields are symbolized by the different possibilities to draw directed lines which pairwise connect points $x^{i_1}_{j_1}$ and $x^{i_2}_{j_2}$ with $j_1 < j_2$. Each line then symbolizes a contractions $i\Delta_+(x^{i_1}_{j_1} - x^{i_2}_{j_2})$, $j_1 < j_2$. No lines connecting points from the same vertex (tadpoles) appear since the interaction at every vertex is already normally ordered. Since the kernels $\mathfrak{D}$ are symmetric, it does not matter which of the n points in the vertices are used in a contraction. The fields which are not contracted are to be evaluated in improper states $\langle p_1 ... p_s \mid \ldots \mid q_1 ... q_l \rangle$. From these simple considerations we can already conclude that as in the ordinary case for ϕ^{2m}-theories, only graphs with an even number $s + l$ of external legs contribute, as the number of contracted fields $r \cdot 2m - s - l$ must be even.
If the graph theory is posed in this manner, problems of causality are apparent, cf. also [13]. It is a general feature of theories with a nonlocal interaction, and in [10] this problem was considered explicitly for the nonlocal Hamilton operator which will be investigated in the next chapter. The problem can briefly be described as follows: usually, in a local theory, the time-orderings in the S-matrix have the same argument as the fields appearing at each vertex, and they can be combined to yield causal commutators (Feynman propagators),

$$\theta(x_0 - y_0)\, \Delta_+(x - y) + \theta(-x_0 + y_0)\, \Delta_+(-x + y)\,.$$

As is well-known, this propagator is causal in the following sense: the matrix element $i\Delta_+(x - y) = \langle \Omega | \phi(x)\phi(y) | \Omega \rangle$ belongs to the process where a particle is created at y and annihilated at x. This process is causal if the creation takes place before the annihilation, i.e. if $x_0 > y_0$. Likewise, the process described by $\Delta_+(-x + y)$ is causal if $y_0 > x_0$. When a Hamilton operator is considered which is nonlocal in time, the time-ordering in the definition of the S-matrix does not refer to the arguments of the fields. Hence, a contraction between two fields from two different vertices which corresponds to the process of creating and annihilating a particle does not come together with an appropriate time-ordering and the creation may well take place after the annihilation.
Let us briefly discuss the situation for the particular Hamiltonian (2.3) in ϕ^3-theory, where the position space kernel is (see [27])

$$\mathcal{D}_3(\sigma, t; x_1, x_2, x_3) = c\, e^{2i(x_1 - x_2)\sigma^{-1}(x_2 - x_3)}\, \delta(-x_{1,0} + x_{2,0} - x_{3,0} + t)\,.$$

Consider the following simple graph at second order,



Here, it is obvious that, according to the time-ordering in the S-matrix, we find $t_1 < t_2$ for the parameter times $t_1 = x_{1,0} - x_{2,0} + x_{3,0}$ and $t_2 = y_{1,0} - y_{2,0} + y_{3,0}$, but $x_{2,0} > y_{1,0}$, such that the contraction between $\phi(x_2)$ and $\phi(y_1)$ is not causal.
This is a general feature of nonlocal theories and, unless the kernel employed is local in time, in general, processes will exhibit acausal behaviour.
If, for instance, a kernel $\mathcal{D}$ with fixed noncommutativity matrix θ is employed, many graphs are acausal in the above sense, although, as we will see in section 2.3, in some graphs (the so-called planar graphs) the twistings cancel and causal commutators appear. Considering only space-space-noncommutativity, where the kernel is local in time, will lead to causal behaviour.

The properties found here do not contradict the causal behaviour of the S-matrix we found in the preceding section. The causality problem as investigated refers to inner processes, while before we investigated the causal behaviour with respect to the overall parameter times t. In local interactions, these two considerations coincide.

Let us now consider another possibility to derive graphical rules which is more apt to establish the connection to the existing literature on the subject[2]. Though equivalent to the rules given above, they appear in a decisively different form, and, in particular, we will not need the multi-vertices introduced above. Before proceeding, let us first recall that on the ordinary Minkowski space, the graph theory is much simplified, if the asymptotic condition is used to show that

$$\begin{aligned} \langle p_1 \dots p_{l_1}, \mathrm{out} \,|\, q_1 \dots q_{l_2}, \mathrm{in} \rangle \;&=\; \langle p_1 \dots \mathrm{in} \,|\, S \,|\, q_1 \dots \mathrm{in} \rangle \\ = \;& \text{disconnected terms } + \\ & + (iZ^{-1/2})^{l_1+1} \int dy_1 \dots dy_{l_1} dx_1 \dots dx_{l_2}\, e^{i(\sum p_i y_i - \sum q_i x_i)} \;. \\ & \cdot (\Box_{y_1} + m^2) \dots (\Box_{x_{l_2}} + m^2) \, \langle \Omega | \, T(\phi(y_1) \dots \phi(x_{l_2})) \, |\Omega \rangle \end{aligned}$$

with a renormalization constant Z, such that all information is contained in the time-ordered vacuum expectation values of the interacting fields, the so-called n-point functions $G(x_1, \dots, x_n) = \langle \Omega | \, T(\phi(x_1) \dots \phi(x_n)) \, |\Omega \rangle$. For local interactions, the n-point functions can be calculated from the following formula,

$$\begin{aligned} & G(x_1, \dots, x_n) \, \langle \Omega | \, T \, \exp\Big(- i \int dt \, H_I(t) \Big) \, |\Omega \rangle = \\ & = \sum_{r=0}^{\infty} \tfrac{(-i)^r}{r!} \, \langle \Omega | \, T \, \Big(\phi_0(x_1) \dots \phi_0(x_n) \int dt_1 \dots dt_r \, H_I(t_1) \dots H_I(t_r) \Big) \, |\Omega \rangle \,, \end{aligned} \tag{2.18}$$

where all fields (also the ones in the Hamiltonians) are free incoming fields, and where the time-ordering T applies to the time variables $x_{1,0}, \dots, x_{n,0}, t_1, \dots, t_r$. The calculations are then greatly simplified by the fact that all connecting lines in such time-ordered vacuum expectation values are given by Feynman propagators. As we shall see below, this is no longer the case, if the interaction is nonlocal: while the formal apparatus can still be applied (under the assumption that LSZ-asymptotics hold), the conclusion that Feynman propagators alone serve as internal lines is no longer true. In fact, as was pointed out in [9], it is this assumption which leads to the violation of unitarity first observed in [43]. This point will be treated in more detail later.

In order to calculate connected expectation values for the nonlocal interaction one can indeed use (2.18) as a starting point as proposed in [27, (6.15)]. This has been done explicitly in [63]. It appears to be somewhat easier to directly calculate the on-shell matrix elements $\langle p_1 \dots \,|\, S \,|\, q_1 \dots \rangle$ and this will be done here. The rules thus derived indeed coincide with those given in [63], if the Hamiltonian (2.5) with fixed noncommutativity matrix θ is used.

The combinatorics are greatly simplified, if the customary symmetrized vertex factors

$$\mathfrak{S}(\sigma; k_1, \dots, k_n) \stackrel{\mathrm{def}}{=} \tfrac{1}{n!} \sum_{\pi \in P_n} \exp\big(- \tfrac{i}{2} \sum_{i<j} k_{\pi_i} \sigma k_{\pi_j} \big)$$

are introduced in order to make the momenta of one vertex indistinguishable. We thus use the following interaction Hamiltonian,

$$H_I(\sigma; t) = \tfrac{g}{n!} \tfrac{1}{(2\pi)^{4n}} \int dk_1 \dots dk_n : \hat{\phi}(k_1) \dots \hat{\phi}(k_n) : \mathfrak{S}(\sigma; k_1, \dots, k_n) \int d\mathbf{x} \; e^{i \, \mathbf{x} \sum_{j=1}^n \mathbf{k}_j \, - i \, t \, \sum_{j=1}^n k_{j,0}} \,.$$

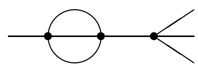
[2] In the meantime, a paper in which also the above position space graphical rules were spelled out has appeared [24].

Here, the nonlocality is hidden in the vertex factors $\mathfrak{S}$ and pointlike vertices can be used, each labelled by an ordinary 4-vector $(t, \mathbf{x})$. Note that the spatial integral $\int d\mathbf{x}$ yields 3-momentum conservation at the vertex. In the above definition, the dependence on σ has not yet been taken care of, so we still have the choice to either perform the integration $\int_{\Sigma_1} d\mu_\sigma$ at each vertex or to use one of the point measures (2.6) and, hence, to set all σ equal to a given θ.

To start with, consider the third order contribution to the S-matrix in ϕ^4-theory with one momentum flowing out and three momenta flowing in,

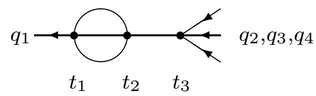
$$\begin{aligned}\langle q_1|S_3|q_2q_3q_4\rangle &= \tfrac{(-i)^3}{3!}\sum_{\pi\in P_3}\int dt_1dt_2dt_3\,\theta(t_{\pi_1}-t_{\pi_2})\theta(t_{\pi_2}-t_{\pi_3})\\ &\qquad\times\ \langle q_1|H_I(\sigma_{\pi_1};t_{\pi_1})\,H_I(\sigma_{\pi_2};t_{\pi_2})\,H_I(\sigma_{\pi_3};t_{\pi_3})|q_2q_3q_4\rangle\,.\end{aligned}$$

Here, S_3 is to be understood to depend on $\sigma_1, \sigma_2, \sigma_3$, and we still have to decide whether to integrate over Σ_1 or to set all σ_i equal to a given θ . The major complication compared to the ordinary case is that the different time orderings have to be treated separately. For instance, consider an ordinary graph at this order with four external momenta,



(2.19)

Assign the outgoing momentum q_1 to the vertex on the left and the incoming momenta q_2, q_3, q_4 to the vertex on the right, and then attribute times t_1, t_2, t_3 to the vertices from left to right,



(2.20)

Now, the contribution to S_3 corresponding to this graph will differ depending on the order of the t_i. For illustration, we will consider the explicit calculations for the cases $t_1 > t_2 > t_3$ (A) and $t_1 > t_3 > t_2$ (B).

First we pick the contribution corresponding to graph (2.19) which arises when the 3-fold product of normally ordered field monomials is brought into complete normal order by application of (1.16). This is a contraction where three fields from the first vertex are contracted with three fields from the second vertex, and the remaining field from this vertex is contracted with one field from the third vertex. Since we work with symmetrized twistings it does not matter which particular momenta we choose at each vertex.

Hence, for the time-ordering where $t_1 > t_2 > t_3$, the contribution to S_3 is given by

$$\begin{aligned} A &= \tfrac{i}{3!}\,\mathbf{C}\left(\tfrac{g}{(2\pi)^{16}\,4!}\right)^3\int dt_1dt_2dt_3\,\theta(t_1-t_2)\theta(t_2-t_3)\int dk_1\ldots dk_4\int dp_1\ldots dp_4\int dl_1\ldots dl_4\\ &\quad\cdot\int d\mathbf{x}_1\,e^{i\mathbf{x}_1(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)}\int d\mathbf{x}_2\,e^{i\mathbf{x}_2(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)}\int d\mathbf{x}_3\,e^{i\mathbf{x}_3(\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3+\mathbf{l}_4)}\\ &\quad\cdot\mathfrak{S}(\sigma_1;k_1,..,k_4)\,\mathfrak{S}(\sigma_2;p_1,..,p_4)\,\mathfrak{S}(\sigma_3;l_1,..,l_4)\,e^{-i\,t_1\sum k_{i,0}}\,e^{-i\,t_2\sum p_{i,0}}\,e^{-i\,t_3\sum l_{i,0}}\\ &\quad\cdot(2\pi)^{26}\,\delta^{(4)}(q_1+k_1)\,\delta^{(4)}(q_2-l_1)\,\delta^{(4)}(q_3-l_2)\,\delta^{(4)}(q_4-l_3)\,i\,\hat{\Delta}_+(k_2)\delta^{(4)}(k_2+p_1)\\ &\quad\cdot i\,\hat{\Delta}_+(k_3)\delta^{(4)}(k_3+p_2)\,i\,\hat{\Delta}_+(k_4)\delta^{(4)}(k_4+p_3)\,i\,\hat{\Delta}_+(p_4)\delta^{(4)}(p_4+l_4)\\ &= \tfrac{i}{3!}\left(\tfrac{g}{4!}\right)^3(2\pi)^{-18}\,\mathbf{C}\int dx_1dx_2dx_3\,\theta(x_{1,0}-x_{2,0})\theta(x_{2,0}-x_{3,0})\,e^{+iq_1x_1}\,e^{-ix_3(q_2+q_3+q_4)}\\ &\quad\cdot\int\frac{d\,\mathbf{k}_2}{2\omega_{\mathbf{k}_2}}\frac{d\,\mathbf{k}_3}{2\omega_{\mathbf{k}_3}}\frac{d\,\mathbf{k}_4}{2\omega_{\mathbf{k}_4}}\frac{d\,\mathbf{p}_4}{2\omega_{\mathbf{p}_4}}\,e^{-ik_2(x_1-x_2)}\,e^{-ik_3(x_1-x_2)}\,e^{-ik_4(x_1-x_2)}\,e^{-ip_4(x_2-x_3)}\\ &\qquad\cdot\mathfrak{S}(\sigma_1;-q_1,k_2,k_3,k_4)\,\mathfrak{S}(\sigma_2;-k_2,-k_3,-k_4,p_4)\mathfrak{S}(\sigma_3;-p_4,q_2,q_3,q_4)\,.\end{aligned}$$

The combinatorial factor $\mathbf{C}$ is calculated as usual in the following way. Consider the second vertex. There are $\binom{4}{3}\,3! = 4!$ possibilities to select an ordered tuple of 3 momenta out of four. These momenta are to be paired with 3 out of the 4 momenta of the first vertex. Here, we have to take an unordered sample to avoid double counting, so the symmetry factor is $\binom{4}{3} = 4$. The remaining momentum of the second vertex is then paired with one of the momenta of the third vertex, which gives an additional symmetry factor of $\binom{4}{1} = 4$. There is only one way to distribute the outflowing momentum q_1 to the one remaining momentum in the first vertex, but there are 3! possibilities to distribute the incoming momenta q_2, q_3, q_4 to the remaining three momenta of the third vertex. We conclude that $\mathbf{C} = 4!\,4^2\,3! = 2304$. Note that if we did not use symmetrized vertex factors, $4!\,4^2 = 386$ of these contributions would have to be treated separately.
The same graph with the distinct time-ordering $t_1 > t_3 > t_2$ is given by a very similar expression, the only difference being that the Hamiltonian $H_I(t_2)$ now stands right of $H_I(t_3)$. To make the resulting changes more lucid, the momenta belonging to $H_I(t_2)$ and $H_I(t_3)$ are denoted p_i and l_i as before. The only contraction that differs from the previous one then is $i\,\hat{\Delta}_+(l_4)\delta^{(4)}(l_4+p_4)$ which replaces $i\,\hat{\Delta}_+(p_4)\delta^{(4)}(p_4+l_4)$. Hence, the contribution to S_3 is

$$
\begin{aligned}
B \;=\; & \tfrac{i}{3!}\left(\tfrac{g}{4!}\right)^3 (2\pi)^{-18}\,\mathbf{C}\int d^4x_1 d^4x_2 d^4x_3\,\theta(x_{1,0}-x_{3,0})\theta(x_{3,0}-x_{2,0})\,e^{+iq_1x_1}\,e^{-ix_3(q_2+q_3+q_4)} \\
& \cdot\int \frac{d\,\mathbf{k}_2}{2\omega_{\mathbf{k}_2}}\frac{d\,\mathbf{k}_3}{2\omega_{\mathbf{k}_3}}\frac{d\,\mathbf{k}_4}{2\omega_{\mathbf{k}_4}}\frac{d\,\mathbf{l}_4}{2\omega_{\mathbf{l}_4}}\;e^{-ik_2(x_1-x_2)}\,e^{-ik_3(x_1-x_2)}\,e^{-ik_4(x_1-x_2)}\,e^{-il_4(x_3-x_2)} \\
& \quad\cdot\mathfrak{S}(\sigma_1;-q_1,k_2,k_3,k_4)\,\mathfrak{S}(\sigma_3;l_4,q_2,q_3,q_4)\,\mathfrak{S}(\sigma_2;-k_2,-k_3,-k_4,-l_4)\,.
\end{aligned}
$$

From these examples we can readily extract the general rules for a ϕ^n-theory in position space.

Remark 2.3 Graphical rules in position space, employing the $\mathfrak{S}$-kernels:

1. Draw all ordinary connected Feynman graphs of the process under consideration, characterized by the number of vertices (r vertices at r-th order of the perturbative expansion) and the number of external momenta. Since we consider a normally ordered interaction term, no tadpole graphs need to be considered, such that no internal line starts and ends at the same vertex. Consider all possibilities to distribute the external momenta to the vertices separately.
2. Pick one of the above graphs and assign vectors $x_1, \ldots, x_r$ with time components $x_{1,0}, \ldots, x_{r,0}$ to its vertices.
3. Choose one particular time order and write down the appropriate $r-1$ Heaviside functions
$$\int d^4x_1 \ldots d^4x_r\;\theta(\ldots)\cdots\theta(\ldots)\,.$$
4. For every internal line connecting x_i and x_j write down a mass-shell integral
$$\frac{1}{(2\pi)^3}\int \frac{d\,\mathbf{k}}{2\omega_{\mathbf{k}}} e^{-ik(x_i-x_j)}\,,$$
where $x_{i,0} > x_{j,0}$. The internal momentum k labels a directed line leading from x_j to x_i (earlier to later).
5. For an external momentum q leaving the vertex x_i multiply with $(2\pi)^{-3/2}\,e^{+iqx_i}$.
For an external momentum q entering the vertex x_i multiply with $(2\pi)^{-3/2}\,e^{-iqx_i}$.
6. At each vertex, the twisting $\mathfrak{S}$ is determined by the following rules:
   - an external momentum leaving the vertex enters with a $-$ sign;

- an external momentum flowing into the vertex enters with a + sign;
- an internal momentum enters with a + sign, if the vertex is the endpoint of the momentum's line, and it enters with a − sign if the vertex is the starting point of the momentum's line. In other words, if k labels the line from x_j to x_i (so $x_{i,0} > x_{j,0}$), k enters the twisting with a − sign at the vertex x_j and with a + sign at x_i.

As the twistings $\mathfrak{S}$ are symmetrized, the order of the momenta in $\mathfrak{S}$ is arbitrary.

7. For each of the r vertices multiply with a factor $\frac{g}{n!}$. Calculate the symmetry factor of the diagram as described on page 28 and multiply the expression with it. Multiply with a factor $\frac{(-i)^r}{r!}$.

The rules in momentum space follow immediately, we merely have to perform the time and space integrations, using the identity

$$\int d\tau\, \theta(\tau)\, e^{-i\tau k_0} = \frac{i}{2\pi} \frac{1}{-k_0 + i\epsilon}\,.$$

For illustration, let us again consider the above examples. Introducing relative time coordinates $\tau_i = t_i - t_{i+1}$, $i = 1, 2$, and performing the time and space integrations, we obtain[3] for A,

$$\begin{aligned} A \;=\; & \tfrac{i}{3!} \left(\tfrac{g}{4!}\right)^3 \left(\tfrac{i}{2\pi}\right)^2 (2\pi)^{-8}\, \mathbf{C} \int \frac{d\,\mathbf{k}_2}{2\omega_{\mathbf{k}_2}} \frac{d\,\mathbf{k}_3}{2\omega_{\mathbf{k}_3}} \frac{d\,\mathbf{k}_4}{2\omega_{\mathbf{k}_4}} \frac{d\,\mathbf{p}_4}{2\omega_{\mathbf{p}_4}} \frac{1}{q_{1,0} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_4} + i\epsilon} \cdot \\ & \cdot \frac{1}{q_{1,0} - \omega_{\mathbf{p}_4} + i\epsilon}\, \mathfrak{S}(\sigma_1; -q_1, k_2, k_3, k_4)\, \mathfrak{S}(\sigma_2; -k_2, -k_3, -k_4, p_4) \mathfrak{S}(\sigma_3; -p_4, q_2, q_3, q_4) \\ & \cdot \delta^{(3)}(-\mathbf{q}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)\, \delta^{(3)}(-\mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 + \mathbf{p}_4)\, \delta^{(3)}(\mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4 - \mathbf{p}_4) \\ & \cdot \delta(-q_{1,0} + q_{2,0} + q_{3,0} + q_{4,0})\,. \end{aligned}$$

The overall energy conservation is the result of the time integration dt_3. It yields the δ-distribution $\delta(k_{1,0} + \cdots + k_{4,0} + p_{1,0} + \cdots + p_{4,0} + l_{1,0} + \cdots + l_{4,0})$ in which only external momenta survive. Note that the spatial $\delta^{(3)}$-distributions can be rewritten in such a form that they also yield the distribution $\delta^{(3)}(-\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4)$. Therefore, energy and momentum of the external momenta are conserved.

More generally, consider a graph with times $t_1 > \cdots > t_r$. Introduce $r - 1$ relative time coordinates $\tau_i = t_i - t_{i+1}$, i.e. $t_i = \sum_{j=i}^{r-1} \tau_j + t_r$. The time integrations then yield

$$\frac{\left(\frac{i}{2\pi}\right)^{r-1} \delta\big(\sum_i (k_{i,0}^{(1)} + .. + k_{i,0}^{(r)})\big)}{(-\sum_i k_{i,0}^{(1)} + i\epsilon)\, (-\sum_i (k_{i,0}^{(1)} + k_{i,0}^{(2)}) + i\epsilon)\, \cdots\, (-\sum_i (k_{i,0}^{(1)} + .. + k_{i,0}^{(r-1)}) + i\epsilon)}\,,$$

where the 4-momenta $k_i^{(1)}$, $i = 1, \ldots, n$, belong to the latest vertex, and the momenta $k_i^{(r)}$ belong to the earliest one. These considerations enable us to state the general rules in momentum space:

Remark 2.4 Graphical rules in momentum space, employing the $\mathfrak{S}$-kernels:

1. As in Remark 2.3 item 1.
2. Pick one of the above graphs and assign times $t_1, \ldots, t_r$ to its vertices. Choose a particular time-ordering.
3. For every internal line write down a mass-shell integral

$$\frac{1}{(2\pi)^3} \int \frac{d\,\mathbf{k}}{2\omega_{\mathbf{k}}}$$

where $k = (\omega_{\mathbf{k}}, \mathbf{k})$ labels the directed line connecting the earlier vertex with the later one. k thus flows out of the earlier vertex into the later one.

[3] At the end of chapter 3, we will comment on the peculiar divergence which would arise, if q_1 were on the same mass-shell as the internal momentum p_4.

4. For each vertex j apart from the earliest one write down the following energy factors,

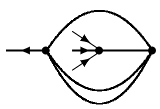
$$\frac{i}{2\pi}\frac{1}{-\sum_i k_{i,0}+i\epsilon},$$

where the sum runs over the 0-components of the following momenta:

   - the internal momenta flowing into the vertex (with a + sign), i.e. the ones labelling inner lines starting at earlier vertices and ending in the vertex under consideration;
   - the internal momenta flowing into any of the later vertices (with a + sign), provided they start at earlier vertices than the vertex under consideration;
   - the external momenta which flow into or out of the vertex under consideration or any of the later vertices of the graph. Here, outgoing momenta enter with a $-$ sign, while the ones flowing into a vertex enter with a + sign.

5. At each vertex impose 3-momentum conservation $(2\pi)^3\delta^{(3)}(...)$, where incoming momenta enter with a + sign, outgoing momenta with a $-$ sign. Impose overall energy conservation of the external momenta $2\pi\,\delta(...)$, where incoming momenta enter with a + sign, momenta flowing out of the graph with a $-$ sign. For every external momentum multiply with a factor $(2\pi)^{-3/2}$.
6. At each vertex, the twisting $\mathfrak{S}$ is determined by the same rules as stated in Remark 2.3 item 6.
7. As in Remark 2.3 item 7.

We thus conclude that these rules indeed coincide with the ones given in [63] when $\sigma_i = \theta\ \forall\, i$. Phenomenological consequences for noncommutative QED in this setting have been investigated in [62], where, due to the lack of covariance, measurements depend on the earth's movement through spacetime (measured relative to the absolute directions given by the fixed θ).
To calculate the S-matrix element at r-th order perturbation theory (2.14) using ordinary graphs by application of the rules given above, one has to sum over all possibilities to distribute the external momenta to the vertices as well as sum over all possible time-orderings.
Alternatively, one may start from the second line in (2.12) and treat only one time-ordering, as we have done in the derivation of the graphical rules employing the position space kernels $\mathfrak{D}$ and multi-vertices. The resulting graphs are calculated by the same rules as stated above, but more graphs have to be considered. For instance, in addition to (2.20) we would have to consider


,

$t_1 > t_2 > t_3$

which yields the same contribution as (2.20) with time-ordering $t_1 > t_3 > t_2$.
In view of later chapters, it is important that the explicit form of the kernel $\mathfrak{S}$ never entered in the derivation of the rules in momentum space as given in Remarks 2.3 and 2.4. It only mattered that $\mathfrak{S}$ was allowed to depend on the momenta's 0-components.
In the following two sections the rules given above will be compared to the modified Feynman rules. We will specialize to the nonlocal kernels arising in the Hamiltonian (2.8) on the noncommutative Minkowski space, mostly with fixed noncommutativity matrix θ.

2.3 Planar graphs and Feynman propagators

As first observed in [38], one of the crucial properties of the modified Feynman rules [37, 38] is the (partial) cancellation of twistings in particular graphs, which gives rise to two types of

graphs, planar and non-planar ones. The underlying mechanism is similar to the one known from the analysis of twisted Eguchi-Kawai matrix models [44].
Basis of the modified Feynman rules is the graph theory in momentum space as it arises in ordinary field theory where Feynman propagators serve as internal lines. The noncommutativity of spacetime modifies the ordinary rules only by additional twisting factors $\exp(\frac{i}{2}\sum k_i\theta k_j)$ which appear at every vertex. They depend on the momenta that flow into and out of the vertex. One of the major simplifications compared to the setup discussed in the previous section is that not only the 3-momentum but also the energy is conserved at each vertex, which in turn causes the expressions to be cyclically symmetric with respect to the momenta entering and flowing out of each vertex. To simplify the combinatorics, the twisting can be symmetrized as in the previous section, but this is not done here.
One way to justify the modified Feynman rules is to start from a theory with nonlocal interaction on Euclidean spacetime. On a Euclidean version of $\mathcal{E}$, where, compared to the spacetime discussed before, all Lorentz products are replaced by Euclidean scalar products, the Lagrangian from which one starts is

$$\int d^4q\ \left(\tfrac{1}{2}\partial_\mu\phi(q)\partial^\mu\phi(q) - \tfrac{1}{2}\,m^2\,\phi(q)^2 - \tfrac{1}{n!}\phi^n(q)\right),$$

where $\int d^4q$ is the trace on $\tilde{\mathcal{E}}$ (see chapter 1). As in the Hamiltonian formalism, the free part of the theory turns out to be the same as usual, since $\int d^4q\, f(q)g(q) = \int d^4x\, f(x)g(x)$, and only the interaction part $\phi^n(q)$ differs. Formally, the modified Feynman rules can then be derived by mimicking the ordinary Euclidean path integral with the ordinary Gaussian measure. Since the only propagator in ordinary field theory on a Euclidean spacetime is the Feynman propagator, these propagators will appear in the resulting perturbative expansion. Again, the theory is thus treated as an effective nonlocal theory on ordinary spacetime – a generalization of the path integral suitable for noncommutative vector spaces has not been found as yet. The question to start from would be to give meaning to the concept of a path on a noncommutative spacetime. In any case, starting from such a Euclidean theory, one has to extract results for the corresponding theory on Minkowski space. The problem here is to define what this corresponding theory on Minkowski space should be. First note that the ordinary analytic continuation fails if the time variable does not commute. For instance, by analytic continuation of two Euclidean Feynman propagators we would usually find

$$\frac{1}{-p_4^2-\mathbf{p}^2-m^2}\,\frac{1}{-k_4^2-\mathbf{k}^2-m^2} \longrightarrow \frac{1}{-(p_4+ip_0)^2-\mathbf{p}^2-m^2}\,\frac{1}{-(k_4+ik_0)^2-\mathbf{k}^2-m^2}$$

such that the product of two Minkowski Feynman propagators appears in the limit $p_4, k_4 \to 0$. However, if a twisting is present with $\theta^{0i} \neq 0$, this analytic continuation yields

$$\exp(-ik\theta p) \to \exp(-i(p_4+ip_0)\theta^{0i}k_i - ip_i\theta^{i0}(k_4+ik_0) - ip_i\theta^{ij}k_j)$$

which produces divergent terms. Unless the time variable commutes, one is compelled to consider the entries θ^{0i} of the noncommutativity matrix to be variables as well (which is usually not the case in string inspired models), and perform an analytic continuation in θ^{0i} which yields

$$\exp(-ik\theta p) \to \exp(-i(p_4+ip_0)(\tilde{\theta}^{0j}+i\theta^{0j})k_j - ip_j(\tilde{\theta}^{j0}+i\theta^{j0})(k_4+ik_0) - ip_i\theta^{ij}k_j)\,.$$

As can be checked explicitly, the result is indeed analytic also with respect to $\tilde{\theta}^{0j}+i\theta^{0j}$. The desired Minkowskian theory would then formally be the limit $p_4, k_4 \to 0$ and $\tilde{\theta}^{0j} \to 0$. There is, however, the serious problem that unless a generalization of the Osterwalder-Schrader positivity theorem has been proved, the relation between these two theories remains obscure. For a spacetime with noncommuting time variable such a generalization seems to be at least difficult

(while the Schwinger function itself is the same as usual and as such is positive, it is not clear whether and how this extends to higher orders).
The above ideas should not be confused with the prescription used in [43], where the replacement $\theta^{j0} \to i\,\theta^{j0}$ was needed to render an otherwise infinite integral over inner momenta finite. Unless an integration over θ is part of the expression, it is impossible to perform this change of variables. Nonetheless, the results of the paper remain valid (see, for instance [3]), and an alternative discussion was given in [9] which, in greater detail, is also subject of this chapter.
Let us now, though briefly, comment on field theories on noncommutative spacetimes which are derived as certain low-energy limits of open string theories on D-brane configurations in background magnetic fields [75, 73]. Here, the string tension approaches 0, while the magnetic field and the open string metric are kept fixed. The strings then do not collapse in the limit and the resulting theory is interpreted as a field theory on a space-space-noncommutative spacetime. To consider an electric field as well, would yield a theory with spacetime noncommutativity, but it seems that such a low-energy limit only yields a unitary field theory if one allows for tachyonic states with negative norm [3]. This has been taken as evidence that it should be impossible to define unitary perturbation theory on a noncommutative spacetime with time-space noncommutativity. As we have seen in the preceding section, this is not true, although it must be admitted that theories which are nonlocal in time exhibit strange properties, such as obeying the "wrong" equation of motion in the Hamiltonian formalism. However, if the motivation to consider noncommutative spacetimes is founded on considering uncertainty relations, there is no reason to assume only space-space noncommutativity.
This being said, let us turn to a comparison of the different sets of rules. As for the Hamiltonian formalism we will, unless otherwise stated, focus our attention on the Hamiltonian (2.5) with fixed noncommutativity matrix θ, where θ may be in Σ or not. We will see that the difference between the Hamiltonian framework and the one based on the modified Feynman rules is that in the latter, all internal lines are given by Feynman propagators, while in the former, Feynman propagators appear only in planar graphs (unless we impose space-space or lightlike noncommutativity). Moreover, it is shown that it is the assumption of Feynman propagators serving as internal lines also in nonplanar graphs which leads to a violation of the optical theorem in the context of the modified Feynman rules.
One of the complications compared to the modified Feynman rules is that in the Hamiltonian formalism only the overall energy is conserved, while at the vertices we merely find 3-momentum conservation. As a consequence, no cyclic symmetry is at hand and the question of which graphs are planar and which are not turns out to be quite complicated.

Definition 2.5 *A graph is planar if the cancellation of twistings from different vertices is such that the resulting twisting does not depend on the internal momenta.*

In the following analysis, instead of the symmetrized vertex factors, the original twisting

$$\mathfrak{t}_\theta(k_1,\ldots,k_n) = \exp\left(-\tfrac{i}{2}\sum_{i<j} k_i\theta k_j\right)$$

is employed, and all contributions which have previously been hidden in the combinatorial factor $\mathbf{C}$ are considered individually. The corresponding reasoning in terms of the symmetrized vertex factors, though leading to the same results, would be less transparent.
To start the analysis let us introduce symbols as a shorthand notation to label individual contractions which contribute to an expectation value. In the perturbation theory arising from the Hamiltonian (2.5),

$$H_I^\theta(t) = \frac{g}{n!}\frac{1}{(2\pi)^{4n}}\int dk_N \,:\hat\phi(k_1)\ldots\hat\phi(k_n):\,\mathfrak{t}_\theta(k_1,\ldots,k_n)\;\delta^{(3)}(\textstyle\sum_i \mathbf{k}_i\,)\,e^{-it\sum_i k_{i,0}}\,,$$

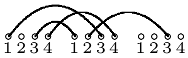
the Wick theorem in momentum space (1.16) can be put into a simple graphical language: For every vertex draw a group of n points in a horizontal line and for every contraction

$$i\,(2\pi)^4\,\hat{\Delta}_+(k_i^{(r)})\,\delta^{(4)}(k_i^{(r)}+k_{\alpha(i)}^{(m)})$$

draw a curve connecting the i-th point of the r-th group with the $\alpha(i)$-th point in the m-th group.

Remark 2.6 A graph which is planar in the context of the modified Feynman rules is not necessarily planar in the Hamiltonian setting.

To see this, consider once more the example graph (2.20). One of its particular contributions (which was previously hidden in the combinatorial factor **C**) is the contraction



(2.21)

Cancellations of twistings, if they occur, are independent of the time ordering. It therefore suffices to consider only the twistings together with the appropriate 3- and 4-momentum conservation,

$$\begin{aligned}&\mathfrak{t}_\theta\,(k_1,-q_1,k_2,k_3)\;\mathfrak{t}_\theta\,(-k_2,k_4,-k_1,-k_3)\;\mathfrak{t}_\theta\,(q_2,q_3,-k_4,q_4)\;\delta(-q_{1,0}+q_{2,0}+q_{3,0}+q_{4,0})\\&\quad\cdot\delta^{(3)}(\mathbf{k}_1-\mathbf{q}_1+\mathbf{k}_2+\mathbf{k}_3)\,\delta^{(3)}(-\mathbf{k}_2+\mathbf{k}_4-\mathbf{k}_1-\mathbf{k}_3)\,\delta^{(3)}(\mathbf{q}_2+\mathbf{q}_3-\mathbf{k}_4+\mathbf{q}_4)\,,\end{aligned}$$

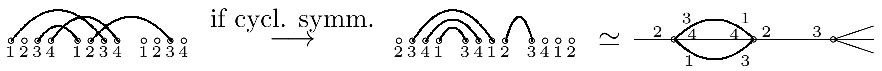
for external momenta q_1 (flowing out) and q_2, q_3, q_4 (entering the graph). A short calculation shows that this graph is not planar, as the evaluation of the first and the second $\delta^{(3)}$-distribution yields

$$\begin{aligned}\delta^{(4)}(q_2+q_3-q_1+q_4)\ \exp\Big(&-\tfrac{i}{2}\big(2(k_{1,i}+k_{2,i}-q_{1,i})\theta^{i0}(\omega_{\mathbf{q}_1-\mathbf{k}_1-\mathbf{k}_2}-\omega_{\mathbf{q}_1}+\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2})\\&+q_1\theta(q_2+q_3-q_4)+q_2\theta(q_3+q_4)+q_3\theta q_4\big)\Big)\,.\end{aligned}$$

We conclude that the twisting does depend on the internal momenta k_1 and k_2 – unless θ^{0i} is set to zero. Therefore, the contraction fails to be planar in the Hamiltonian setting for general θ. If, on the contrary, the 4-momentum is conserved at every vertex, then a simple calculation shows that the twisting

$$\mathfrak{t}_\theta\,(k_1,-q_1,k_2,k_3)\;\mathfrak{t}_\theta\,(-k_2,k_4,-k_1,-k_3)\;\mathfrak{t}_\theta\,(q_2,q_3,-k_4,q_4)$$

does not depend on internal momenta and, hence, that the contraction (2.21) is planar in the setting of the modified Feynman rules. In a graphical language, the mechanism can be understood as follows:



,

where the cyclic symmetry at the vertices is a consequence of the 4-momentum conservation. Nonetheless, planar graphs do appear also in the Hamiltonian formalism with fixed noncommutativity matrix θ even if $\theta^{0i} \neq 0$. As a short calculation shows, the contraction  with external momenta q_1, q_2 flowing out of the left vertex, and q_3, q_4 entering the vertex on the right, is indeed planar: its twisting is

$$\mathfrak{t}_\theta\,(k_1,k_2,-q_1,-q_2)\mathfrak{t}_\theta\,(q_3,q_4,-k_2,-k_1)=e^{-\frac{i}{2}\big((k_1+k_2)\theta(-q_1-q_2+q_3+q_4)+q_1\theta q_2+q_3\theta q_4\big)}\,,$$

and the δ-distributions to be considered are

$$\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{q}_1 - \mathbf{q}_2)\, \delta^{(3)}(\mathbf{q}_3 + \mathbf{q}_4 - \mathbf{k}_1 - \mathbf{k}_2)\, \delta(-q_{1,0} - q_{2,0} + q_{3,0} + q_{4,0})\,.$$

Hence, the first term in the twisting vanishes by energy-momentum conservation of the external momenta, and the twisting does not depend on the internal momenta k_1, k_2.
In the modified Feynman rules it is assumed from the outset that Feynman propagators serve as internal lines. Hence, in order to compare the Hamiltonian formalism with the modified Feynman rules, it is crucial to give a criterion when causal propagators appear in the Hamiltonian formalism.

Proposition 2.7 *Consider the perturbative expansion based on Dyson's series (2.12), where the Hamiltonian (2.5) with fixed noncommutativity matrix θ is employed. If an internal momentum in a particular graph does not appear in the twisting (planar contribution), the internal line can be written in terms of a Feynman propagator. If an internal line does appear in the twisting (nonplanar contribution), then the internal line can be written in terms of a Feynman propagator if and only if there is a timelike or lightlike 4-vector n such that $\theta^{\mu\nu} n_\nu = 0$.*

Proof: Since we sum over all different time-orderings as well as all possibilities to distribute the external momenta to the vertices, there are always two contributions to any graph in the Hamiltonian setting with everything but the time order coinciding. Hence, we find, according to the rules,

$$\ldots\ \theta(x_{i,0} - x_{j,0})\, \frac{1}{(2\pi)^3} \int \frac{d\,\mathbf{k}}{2\omega_{\mathbf{k}}}\, e^{-ik(x_i - x_j)}\, f(\omega_{\mathbf{k}}, \mathbf{k}, \ldots)\ \ldots$$

and

$$\ldots\ \theta(x_{j,0} - x_{i,0})\, \frac{1}{(2\pi)^3} \int \frac{d\,\mathbf{k}}{2\omega_{\mathbf{k}}}\, e^{-ik(x_j - x_i)}\, f(\omega_{\mathbf{k}}, \mathbf{k}, \ldots)\ \ldots\,,$$

where $f(\omega_{\mathbf{k}}, \mathbf{k}, \ldots)$ abbreviates the twistings at the vertices i and j, $\mathfrak{t}_\theta(\ldots, k, \ldots)\, \mathfrak{t}_\theta(\ldots, -k, \ldots)$, $k = (\omega_{\mathbf{k}}, \mathbf{k})$.
In a planar contribution, where k does not appear in the twisting, the integrations over $\int \frac{d\,\mathbf{k}}{2\omega_{\mathbf{k}}}$ can be performed and the two contributions can be combined to yield a Feynman propagator,

$$\theta(x_{i,0} - x_{j,0})\, i\,\Delta_+(x_i - x_j) \ldots\ +\ \theta(x_{j,0} - x_{i,0})\, i\,\Delta_+(x_j - x_i) \ldots\ =\ i\,\Delta_F(x_i - x_j) \ldots\,.$$

In general, this is not true in a nonplanar contribution, where f depends on k. Suppose that also in this case a Feynman propagator would serve as the internal line connecting vertex i and vertex j. Then the appropriate contribution would be (for better readability $x_i = x$, $x_j = y$)

$$\begin{aligned}
& (2\pi)^{-4} \int d^4k\, \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} f(k_0, \mathbf{k}, \ldots) \\
= \ & (2\pi)^{-4} \int d^4k\, \frac{1}{k^2 - m^2 + i\epsilon}\ e^{-ik_0(x_0 - y_0 + \theta_{0i} \sum_l \pm p_{l,i})}\ e^{+i\mathbf{k}(\mathbf{x} - \mathbf{y} + \sum_l \pm \tilde{\mathbf{p}}_l)}\ f_0(\ldots) \\
= \ & \theta(x_0 - y_0 + \theta_{0i} \sum_l \pm p_{l,i}) \tfrac{1}{(2\pi)^3\, i} \int \frac{d\,\mathbf{k}}{2\omega_{\mathbf{k}}}\ e^{-i\omega_{\mathbf{k}}(x_0 - y_0 + \theta_{0i} \sum_l \pm p_{l,i})}\ e^{+i\mathbf{k}(\mathbf{x} - \mathbf{y} + \sum_l \pm \tilde{\mathbf{p}}_l)}\ f_0(\ldots) \\
+ \ & \theta(y_0 - x_0 - \theta_{0i} \sum_l \pm p_{l,i}) \tfrac{1}{(2\pi)^3\, i} \int \frac{d\,\mathbf{k}}{2\omega_{\mathbf{k}}}\ e^{-i\omega_{\mathbf{k}}(y_0 - x_0 - \theta_{0i} \sum_l \pm p_{l,i})}\ e^{+i\mathbf{k}(-\mathbf{x} + \mathbf{y} - \sum_l \pm \tilde{\mathbf{p}}_l)}\ f_0(\ldots)\,,
\end{aligned}$$

where $\pm p_l$ are the momenta which have non-vanishing twisting with k and where $\tilde{\mathbf{p}}_l = (\theta_{1\mu} p_l^\mu, \theta_{2\mu} p_l^\mu, \theta_{3\mu} p_l^\mu)$. $f_0(\ldots)$ stands for that part of the twisting which is independent of k. By assumption, $\sum \pm p_l \neq 0$ (we consider a nonplanar contribution), and we conclude that the above expression is equal to the one derived before if $\theta^{0i} = 0$, or rather, since the expressions are covariant, if there is a timelike 4-vector n with $\theta^{\mu\nu}n_\nu = 0$. By continuity, this extends to the case of a lightlike n. There being no other possibilities for arbitrary p_l, the condition is not only sufficient but also necessary. ■

This is obviously consistent with the fact that without any twistings the rules in the Hamiltonian approach must reproduce the ordinary Feynman rules. Moreover, we conclude that the modified Feynman rules coincide with the ones based on the Hamiltonian approach with Hamiltonian (2.5) at fixed noncommutativity matrix θ, if and only if θ is such that we have space-space or lightlike noncommutativity. It follows that only in these cases we find the internal processes arising in the perturbation theory to be causal. Note that if it is possible to put the time-ordering and the contractions together such that only Feynman propagators appear as internal lines, we find 4-momentum conservation at this vertex.
It is noteworthy that if the Hamiltonian's dependence on σ is taken care of by integrating over Σ_1, cancellations of twistings from different vertices are impossible. There are, however, graphs in which the momenta entering and leaving a particular vertex are such that at this vertex itself, the twisting yields 1. Such graphs will be discussed at the end of this chapter.
It is now shown that the unitarity problem found in the context of the modified Feynman rules is related to the assumption that Feynman propagators serve as internal lines also in nonplanar graphs.

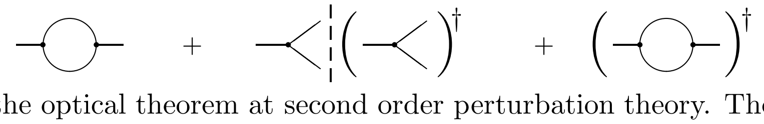
2.4 Unitarity

By re-considering the example first investigated in [43], we will see in this section how the modified Feynman rules lead to a non-unitary perturbation theory. As pointed out in [9], this failure can be linked to the assumption that all internal lines are given by Feynman propagators. Recall from Remark 2.1 that the Hamiltonian setup leads to a (formally) unitary S-matrix. Hence, there is no need to check the unitarity constraint graph by graph for particular expectation values. On the contrary, from the definition of the (formally) unitary S-matrix the optical theorem at second order perturbation theory can be *derived* as an identity given in terms of graphs. Consider, for instance, an expectation value with two external momenta at the second order of the perturbative expansion in ϕ^3-theory. Using the Dyson series (2.12), we immediately deduce that

$$\begin{aligned}\langle q_1|S_2^\dagger + S_1\,S_1^\dagger + S_2|q_2\rangle &= \langle q_1|\,2\,\mathrm{Re}\,S_2\,|q_2\rangle + \langle q_1|\,S_1\,S_1^\dagger\,|q_2\rangle \\ &= i^2\int dt_1 dt_2\,\big(\underbrace{\theta(t_2-t_1) - 1 + \theta(t_1-t_2)}_{=0}\big)\,\langle q_1|H_I(t_1)H_I(t_2)|q_2\rangle \;=\; 0\,. \end{aligned} \tag{2.22}$$

Remark 2.8 This identity can be understood in terms of graphs as

$$0 \overset{\text{Ham.form.}}{=}$$

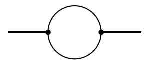


$,$ (2.23)

which is known as the optical theorem at second order perturbation theory. The corresponding explicit expression is given by

$$\begin{aligned} 0 \;=\; \tfrac{i^2}{2}\left(\tfrac{g}{3!}\right)^2 \tfrac{18}{(2\pi)^9} \int dx_1 dx_2 \int \frac{d\,\mathbf{k}_1}{2\omega_{\mathbf{k}_1}}\frac{d\,\mathbf{k}_2}{2\omega_{\mathbf{k}_2}} \Big(&- e^{-i(k_1+k_2)(x_1-x_2)} - e^{+i(k_1+k_2)(x_1-x_2)} \\ &+ \theta(x_{1,0}-x_{2,0})\,e^{-i(k_1+k_2)(x_1-x_2)} + \theta(x_{2,0}-x_{1,0})\,e^{+i(k_1+k_2)(x_1-x_2)} \\ &+ \theta(x_{2,0}-x_{1,0})\,e^{-i(k_1+k_2)(x_1-x_2)} + \theta(x_{1,0}-x_{2,0})\,e^{+i(k_1+k_2)(x_1-x_2)}\Big) \\ \cdot \Big(e^{+iq_1x_1}\,e^{-iq_2x_2}\,\mathfrak{S}(\sigma_1;-q_1,k_1,k_2)\,\mathfrak{S}(\sigma_2;-k_1,-k_2,q_2) + (q_1 \leftrightarrow -q_2)\Big)\,. \end{aligned}$$

Proof: First, we calculate $\langle q_1|S_2|q_2\rangle$, which according to the rules given in Remark 2.3 is equal to the following graph

 ,

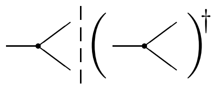
where the incoming momentum q_1 can either be assigned to the vertex on the left and the outgoing momentum q_2 to the vertex on the right, or the other way around. The times $x_{1,0}$ and $x_{2,0}$ are assigned to the vertices and the two possible time-orderings have to be considered. According to the rules, we thus have four contributions. Renaming the integration variables $x_1 \leftrightarrow x_2$ in two of these terms, and noting that σ_1 and σ_2 are treated on the same footing, such that we can switch $\sigma_1 \leftrightarrow \sigma_2$ as well (either they are both equal to θ or they are both integrated over the same measure $d\mu_\sigma$), we derive

$$\begin{aligned}\langle q_1|S_2|q_2\rangle \;=\; & i^2\left(\tfrac{g}{3!}\right)^2 (2\pi)^{-9}\mathbf{C}\int dx_1 dx_2\,\theta(x_{1,0}-x_{2,0})\int \frac{d\,\mathbf{k}_1}{2\omega_{\mathbf{k}_1}}\frac{d\,\mathbf{k}_2}{2\omega_{\mathbf{k}_2}}\, e^{-ik_1(x_1-x_2)}\, e^{-ik_2(x_1-x_2)} \\ & \cdot\Big(e^{+iq_1x_1}\, e^{-iq_2x_2}\,\mathfrak{S}(\sigma_1;-q_1,k_1,k_2)\,\mathfrak{S}(\sigma_2;-k_1,-k_2,q_2) + (q_1\leftrightarrow -q_2)\Big)\,.\end{aligned}$$

Here, $\mathbf{C} = \binom{3}{2}\,2!\,\binom{3}{2} = 18$ is the appropriate combinatorial factor[4]. Since $\langle q_1|S_2^\dagger|q_2\rangle = \overline{\langle q_2|S_2|q_1\rangle}$, where the bar symbolizes complex conjugation, it only remains to calculate the middle term,

$$\begin{aligned}\langle q_1|S_1\,S_1^\dagger|q_2\rangle \;&=\; -i^2\int dt_1dt_2\,\langle q_1|H_I(t_1)H_I(t_2)|q_2\rangle \\ &=\; \left(\tfrac{g}{3!}\right)^2 (2\pi)^{-9}\mathbf{C}'\int dx_1dx_2\int \frac{d\,\mathbf{k}_1}{2\omega_{\mathbf{k}_1}}\frac{d\,\mathbf{k}_2}{2\omega_{\mathbf{k}_2}}\, e^{-ik_1(x_1-x_2)}\, e^{-ik_2(x_1-x_2)} \\ &\qquad \cdot\Big(e^{+iq_1x_1}\, e^{-iq_2x_2}\,\mathfrak{S}(\sigma_1;-q_1,k_1,k_2)\,\mathfrak{S}(\sigma_2;-k_1,-k_2,q_2) + (q_1\leftrightarrow -q_2)\Big)\,,\end{aligned}$$

where $\mathbf{C}' = \binom{3}{2}\;2!\;\binom{3}{2} = 18$. In terms of graphs, this contribution can be written as

 ,

where the dashed line symbolizes the fact that two of the legs of each tree graph are contracted, such that the result has two external legs. Note that internal lines k_1 and k_2 remain on-shell. Adding all three terms, we thus find the explicit expression for the identity (2.22) as claimed, where we have to use that

$$\int dx_1dx_2 f(x_1,x_2) = \tfrac{1}{2}\int dx_1dx_2 f(x_1,x_2) + \tfrac{1}{2}\int dx_1dx_2 f(x_2,x_1)$$

as well as take advantage of the fact that the term involving the twistings and the external momenta is symmetric under the exchange of the integration variables x_1 and x_2. ■

[4]For $\sigma_1=\sigma_2=\theta$ this contribution can equivalently be written as

$$\begin{aligned}&\int dx_1dx_2\;\theta(x_{1,0}-x_{2,0})\cdot\big[\,\langle q_1|:\phi(x_1)\phi(x_2):|q_2\rangle \overset{sym}{\star_\theta}\;\Delta_+(x_1-x_2)\overset{sym}{\star_\theta}\;\Delta_+(x_1-x_2)\,\big] \\ +\;&\int dx_1dx_2\;\theta(x_{2,0}-x_{1,0})\cdot\big[\,\langle q_1|:\phi(x_1)\phi(x_2):|q_2\rangle \overset{sym}{\star_\theta}\;\Delta_+(x_2-x_1)\overset{sym}{\star_\theta}\;\Delta_+(x_2-x_1)\,\big]\,,\end{aligned}$$

where $\overset{sym}{\star_\sigma}$ stands for the symmetrized star-product with respect to both x_1 and x_2. This is the expression used in [9].

A more common convention in the derivation of the optical theorem is to use the transition matrix T instead of the S-matrix, $S = 1 - iT$. In this case, (2.22) can be rewritten as $0 = \langle q_1 | S_2^\dagger + S_1 S_1^\dagger + S_2 | q_2 \rangle = \langle q_1 | (-iT_2)^\dagger + T_1 T_1^\dagger - iT_2 | q_2 \rangle$ such that

$$0 = \langle q_1 | \, 2 \, \mathrm{Im} \, T_2 \, | q_2 \rangle + \langle q_1 | T_1 T_1^\dagger | q_2 \rangle \, .$$

If $\sigma_1 = \sigma_2 = \theta$ is fixed, the explicit form of the optical theorem is

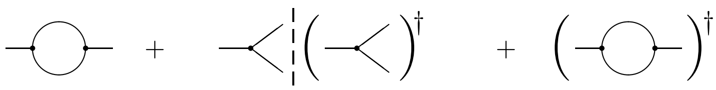
$$\begin{aligned} 0 \;=\; & (2\pi)^4 \, \delta^{(4)}(q_1 - q_2) \, \tfrac{i^2}{2} \int dz \int \frac{d\,\mathbf{k}_1}{2\omega_{\mathbf{k}_1}} \frac{d\,\mathbf{k}_2}{2\omega_{\mathbf{k}_2}} \Big(- e^{-i(k_1+k_2)z} - e^{+i(k_1+k_2)z} + \\ & + \theta(z_0) \, e^{-i(k_1+k_2)z} + \theta(-z_0) \, e^{+i(k_1+k_2)z} + \theta(-z_0) \, e^{-i(k_1+k_2)z} + \theta(z_0) \, e^{+i(k_1+k_2)z} \Big) \\ & \cdot \Big(e^{+iq_1 z} \, \mathfrak{S}(\theta; -q_1, k_1, k_2) \, \mathfrak{S}(\theta; -k_1, -k_2, q_2) + (q_1 \leftrightarrow -q_2) \Big) \, . \end{aligned}$$

From the preceding section we know that only the planar parts of the above, where the twisting becomes independent of k_1 and k_2, will yield Feynman propagators. In such planar contributions the optical theorem reads as usual,

$$0 = \int dx_1 dx_2 \, \langle q_1 | {:}\phi(x_1) \, \phi(x_2){:} | q_2 \rangle \, \Big(\underbrace{\Delta_F^2 + \overline{\Delta_F^2} - \Delta_+^2 - \Delta_-^2}_{=\;0} \Big)(x_1 - x_2) \, . \qquad (2.24)$$

In the nonplanar contributions the time-ordering cannot be absorbed into Feynman propagators alone, but, by construction, the unitarity constraint is satisfied.
Let us now consider the corresponding calculation in the context of the modified Feynman rules. It was shown in [43, 2] that in this setting the three terms which correspond to the graphs in (2.23) do not yield zero unless one assumes spacelike or lightlike noncommutativity. Let us now recall the argument given in [9] which puts this result in a slightly different language.

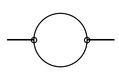
Remark 2.9 If the modified Feynman rules are employed, the right-hand side of (2.23) yields



$$\overset{\mathrm{mod\,F}}{\propto} \quad \int dx_1 dx_2 \, \langle q_1 | {:}\phi(x_1) \, \phi(x_2){:} | q_2 \rangle \, \Big(\Delta_{ret} \star \Delta_{av} + \Delta_{av} \star \Delta_{ret} \Big)(x_1 - x_2)$$

with the retarded and advanced propagators, $\Delta_{ret/av}(x) = \pm\theta(\pm\, x_0)\Delta(x)$.

Proof: The fish graph in the setting of the modified Feynman rules is given by the following expression,



$$\begin{aligned} \overset{\mathrm{mod\,F}}{=} \quad & \left(\tfrac{g}{3!}\right)^2 \tfrac{18}{(2\pi)^{11}} \int dx_1 dx_2 \int dk_1 dk_2 \frac{1}{k_1^2 - m^2 + i\epsilon} \, \frac{1}{k_2^2 - m^2 + i\epsilon} \, e^{-i(k_1+k_2)(x_1-x_2)} \\ & \cdot \Big(e^{+iq_1 x_1} \, e^{-iq_2 x_2} \, \mathfrak{S}(\theta; -q_1, k_1, k_2) \, \mathfrak{S}(\theta; -k_1, -k_2, q_2) + (q_1 \leftrightarrow -q_2) \Big) \, , \end{aligned}$$

which, by the fact that "one star-product can be dropped under the trace" $\int dx_i$ (see chapter 1), is equal to

$$\begin{aligned} = \;\; & \left(\tfrac{g}{3!}\right)^2 \tfrac{18}{(2\pi)^{11}} \int dx_1 dx_2 \int dk_1 dk_2 \frac{1}{k_1^2 - m^2 + i\epsilon} \, \frac{1}{k_2^2 - m^2 + i\epsilon} \, e^{-i(k_1+k_2)(x_1-x_2)} \\ & \cdot \Big(e^{+iq_1 x_1} \, e^{-iq_2 x_2} + e^{+iq_1 x_2} \, e^{-iq_2 x_1} \Big) \, \cos\left(\tfrac{1}{2} k_1 \theta k_2\right) \, \cos\left(\tfrac{1}{2} k_1 \theta k_2\right) \\ = \;\; & \tfrac{18}{2} \left(\tfrac{g}{3!}\right)^2 \int dx_1 dx_2 \Big(\Delta_F \star_{2\theta} \Delta_F \, (x_1 - x_2) + \Delta_F^2(x_1 - x_2) \Big) \, \langle q_1 | {:}\phi(x_1) \, \phi(x_2){:} | q_2 \rangle \, . \end{aligned}$$

Likewise, we calculate the term in the middle,

$$-\tfrac{18}{2}\,\left(\tfrac{g}{3!}\right)^2 \int dx_1 dx_2\, \langle q_1|:\phi(x_1)\,\phi(x_2):|q_2\rangle \Big(\Delta_+ \star_{2\theta}\, \Delta_+ + \Delta_+^2 + \Delta_- \star_{2\theta}\, \Delta_- + \Delta_-^2 \Big)\,(x_1 - x_2)\,.$$

Adding all three terms and dividing by the common prefactor, we thus obtain

$$\int dx_1 dx_2\, \langle q_1|:\phi(x_1)\,\phi(x_2):|q_2\rangle \Big(\Delta_F \star_{2\theta}\, \Delta_F + \Delta_F^2 + \overline{\Delta_F \star_{2\theta}\, \Delta_F} + \overline{\Delta_F^2}$$
$$- \Delta_+ \star_{2\theta}\, \Delta_+ - \Delta_+^2 - \Delta_- \star_{2\theta}\, \Delta_- - \Delta_-^2 \Big)\,(x_1 - x_2)\,.$$

Hence, the planar contributions which correspond to the expression (2.24) appearing in ordinary quantum field theory indeed add up to zero. On the contrary, the nonplanar part yields a star-product of retarded and advanced propagators, since (for better readability, $\star$ is used instead of $\star_{2\theta}$)

$$\begin{aligned}
\Delta_F \star \Delta_F + \overline{\Delta_F \star \Delta_F} &= (\theta\Delta - \Delta_-) \star (\theta\Delta - \Delta_-) + (-\theta\Delta + \Delta_+) \star (-\theta\Delta + \Delta_+) \\
&= \Delta_- \star \Delta_- + \Delta_+ \star \Delta_+ + \theta\Delta \star (\theta\Delta - \Delta) + (\theta\Delta - \Delta) \star \theta\Delta \\
&= \Delta_- \star \Delta_- + \Delta_+ \star \Delta_+ + \Delta_{ret} \star \Delta_{av} + \Delta_{av} \star \Delta_{ret}\,.
\end{aligned}$$

Here, Δ denotes the commutator function $\Delta_+ + \Delta_-$, and $\Delta_{ret/av}(x) = \pm\theta(\pm\, x_0)\Delta(x)$ stands for the retarded and the advanced propagator, respectively. This proves the claim. ■

Proposition 2.10 *The twisted convolution product of the retarded and the advanced propagator vanishes, if the time is assumed to commute with the space variables, i.e. when there is a timelike vector n with $\theta^{\mu\nu}n_\nu = 0$. This remains true if n is lightlike.*

Proof: The proof is essentially the same as the one of Proposition 2.7. Let g be a testfunction. Ignoring any problems regarding divergences, we find for $\theta^{0i} = 0$,

$$\int dx\, g(x) \int d\mathbf{y}_1 d\mathbf{y}_2\, (\Delta_{ret}(x_0,\mathbf{y}_1)\Delta_{av}(x_0,\mathbf{y}_2) + \Delta_{av}(x_0,\mathbf{y}_1)\Delta_{ret}(x_0,\mathbf{y}_2))$$
$$\times \int d\mathbf{p}\, \delta^{(3)}(y_{1,i} + \theta_{ij}p^j - x_i)\, e^{+i\mathbf{p}(\mathbf{y}_2 - \mathbf{x})} \;=\; 0\,,$$

as $\Delta_{av}(x_0,\mathbf{y}_1)\Delta_{ret}(x_0,\mathbf{y}_2) = 0$. In a covariant way this reads

$$\Delta_{ret/av}(x) = \pm\theta(\pm nx)\Delta(x) = \pm\theta(\pm nx) \star \Delta(x) = \pm\Delta(x) \star \theta(\pm nx)$$

for timelike n and hence $\Delta_{ret} \star \Delta_{av} = i^2 \Delta \star \theta \star (1-\theta) \star \Delta$. For $\theta^{\mu\nu}n_\nu = 0$ this vanishes as in this case $\theta \star (1-\theta) = \theta(1-\theta) = 0$. By continuity, this remains true when n approaches a lightlike vector. ■

Now let θ be a full noncommutativity matrix, $\theta \in \Sigma$, such that we have non-trivial commutation relations for all space and time variables and such that, in particular, there is no timelike or lightlike vector n such that a new time- or lightcone-coordinate $\tilde{q}^0 = n^\mu q_\mu$ commutes with all other coordinates. Then the twisted convolution product of the retarded and the advanced propagator does not vanish,

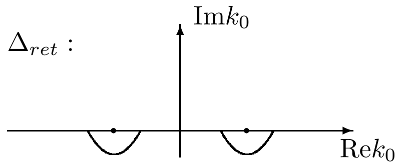
$$\Delta_{ret} \star \Delta_{av} + \Delta_{av} \star \Delta_{ret} \neq 0\,,$$

where, for now, all problems concerning divergences are ignored. This is a consequence of the fact that it was proved in [3] that the optical theorem is violated in the context of the modified Feynman rules. If the above product would vanish, the optical theorem would be complied with.

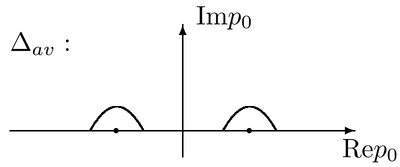
A direct proof of the fact that the product is non-zero is not available, but it may be understood as follows. First recall how it may be shown via Fourier transform that the pointwise product $\Delta_{ret}\Delta_{av}$ vanishes. The Fourier transforms of the retarded and the advanced propagator are

$$\frac{1}{m^2 - k^2 + 2i\epsilon k_0} \qquad \text{and} \qquad \frac{1}{m^2 - p^2 - 2i\epsilon p_0}\,,$$

respectively, where the term $2i\epsilon k_0$ encodes the prescription of how to integrate around the two poles which appear in each of the propagators,



and



$$(k^0)_{1/2} = \pm\omega_{\mathbf{k}} + i\epsilon \qquad\qquad (p^0)_{1/2} = \pm\omega_{\mathbf{p}} - i\epsilon$$

Taking the pointwise product of these propagators at x, the paths of integrations in k^0 and p^0 cannot be closed such that the singularities of both propagators are enclosed; for $x_0 > 0$ we have to close both paths in the upper half planes, thus excluding the singularities of Δ_{av}, and for $x_0 < 0$ we have to close both paths in the lower half planes, thus excluding the singularities of Δ_{ret}. Hence, by the residue integral theorem, the result is zero. If, on the contrary, a twisting is present in which p_0 and k_0 appear, the situation is changed. In $\Delta_{ret} \star \Delta_{av}$ we find the following dependence on the imaginary part of p_0 and k_0,

$$e^{-\mathrm{Im}\, k_0\, (x_0 - \theta_{0i}p^i)}\, e^{-\mathrm{Im}\, p_0\, (x_0 + \theta_{0i}k^i)}\,,$$

such that for fixed $\mathbf{k}$ and $\mathbf{p}$ we can close the path in the upper half for k_0 and the lower half for p_0, if $x_0 - \theta_{0i}p^i > 0$ and $x_0 + \theta_{0i}k^i < 0$. By the calculus of residue, the result is

$$\tfrac{i\pi}{\omega_{\mathbf{k}}}\,\tfrac{i\pi}{\omega_{\mathbf{p}}}\left(e^{i\omega_{\mathbf{k}}(x_0 - \theta_{0i}p^i)} - e^{-i\omega_{\mathbf{k}}(x_0 - \theta_{0i}p^i)}\right)\left(e^{i\omega_{\mathbf{p}}(x_0 + \theta_{0i}k^i)} - e^{-i\omega_{\mathbf{p}}(x_0 + \theta_{0i}k^i)}\right)$$

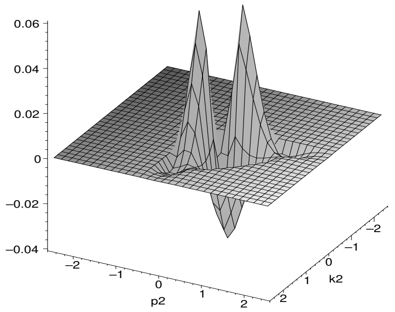
for $x_0 - \theta_{0i}p^i > 0$ and $x_0 + \theta_{0i}k^i < 0$. Now let $\theta = \sigma^{(0)}$, the standard symplectic form ($\lambda_p = 1$), then we find the following expression for $\Delta_{ret} \star \Delta_{av}$ (with $p_\perp = (p_1, p_3)$),

$$\int dp_\perp dk_\perp\; e^{-i(k_1p_3 - k_3p_1)}\, e^{-i(k_\perp + p_\perp)x_\perp}\;\cdot$$
$$\cdot\int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dk_2\; \underbrace{\theta(p_2 + x_0)\,\theta(k_2 - x_0)\,\frac{\sin(\omega_{\mathbf{k}}(x_0 - p_2))\sin(\omega_{\mathbf{p}}(x_0 + k_2))}{\omega_{\mathbf{k}}\,\omega_{\mathbf{p}}}\, e^{-i(k_2+p_2)x_2}}_{=:\, f(x_0, x_2; k_2, p_2; m^2 + p_\perp^2 + k_\perp^2)}\,.$$

This is not obviously zero by some symmetry in k and p, although the integrand f vanishes if $k_2 = -p_2$. A crude numerical analysis shows the following:

Consider the estimate where we "smear" f in x_0 with a double step function and with a Gauss function in x_2, regardless of the fact that the above product may be ill-defined on parts of the lightcone. Then, by a numerical analysis using `Maple`, we find that for fixed $k_\perp^2$ and $p_\perp^2$ with $k_\perp^2 + p_\perp^2 + m^2 = 4$, the smeared integrand

$$\int dx_0 dx_2\; (-\theta(x_0 - \tfrac{1}{2}) + \theta(x_0 + \tfrac{1}{2}))\, \frac{2\, e^{-\frac{1}{4}x_2^2}}{\sqrt{\pi}}\, f(x_0, x_2; k_2, p_2; 4)$$



as a function of p_2 and k_2 has the form shown in the figure above. Therefore, it is not to be expected that the integral over k_2 and p_2 to vanishes.

The situation is different if the formal series with the Moyal-star-product $\star_M$ is considered. By virtue of the fact that (at every order in θ) the star-product is local, the support of $\Delta_{ret} \star_M \Delta_{av}$ in this case consists only of 0. Since the product is singular in this point, an *infinite* number of renormalizations are needed to render the product well-defined (at each even order in θ; all odd orders vanishing). Renormalization conditions could then be used to set the product to 0, but it is not clear whether an infinite number of these makes sense. Treating a θ-expanded approach as an effective theory, however, one may truncate the formal power series at some order.
The observation that for lightlike and spacelike noncommutativity the obstruction in the optical theorem vanishes, has been taken as a motivation to consider only these special cases. While this may be natural when models are motivated by string theory, it is not if the motivation is based on spacetime uncertainty relations. In particular, Lorentz invariance should not easily be given up. While questions concerning covariance remain to be solved also in the context of [27], covariance is at least not lost from the start.
Therefore, the argument should be turned around. Starting point should be an approach which is formally unitary for a general noncommutative spacetime, such as the one based on Dyson's series (2.12). With the special choice of a fixed space-space or lightlike noncommutativity matrix θ, where obviously the theory is still unitary, the rules resulting from (2.12) simplify and *coincide* with the modified Feynman rules by Proposition 2.7.
We will see that the same is true in the Yang-Feldman approach which will be analysed in chapter 4.

2.5 Renormalizability

Let us conclude this chapter with some remarks on properties of the Hamiltonian formalism concerning renormalization and contrast them with those found in the framework of the modified Feynman rules.
In the framework of the modified Feynman rules, the most serious obstacle to applying ordinary renormalization procedures is a notorious mixing of ultraviolet and infrared divergences which was first discussed in [70]. Roughly speaking, the problem is that graphs which are ultraviolet finite may develop hard infrared divergences when inserted into larger graphs. To sketch the mechanism let us consider an example in the Euclidean regime, as discussed in [70]. Here, one of the nonplanar contributions to the fish graph in ϕ^3-theory is given by the following expression,

$$\delta^{(4)}(q-q') \int dkdp \, \frac{1}{k^2+m^2} \frac{1}{p^2+m^2} \, e^{-ik\theta p} \, \delta^{(4)}(k+p-q) \, ,$$

where q and q' are the momenta which enter and leave the diagram. Using Schwinger's parametrization,

$$\frac{1}{k^2+m^2} = \int_0^\infty d\alpha \, e^{-\alpha(k^2+m^2)} \, ,$$

a short calculation yields

$$\delta^{(4)}(q-q') \int_0^\infty d\alpha d\beta \, \tfrac{\pi^2}{(\alpha+\beta)^2} \, e^{-(\alpha+\beta)m^2} \, e^{-\frac{\alpha\beta}{\alpha+\beta}q^2 - \frac{1}{4(\alpha+\beta)}(\theta q)^2} \, .$$

Due to the effective ultraviolet cutoff $\exp(-\frac{1}{4(\alpha+\beta)}(\theta q)^2)$ arising from the twisting, this contribution to the fish graph is ultraviolet finite as long as $(\theta q)^2 \neq 0$. When this graph is inserted into a higher order diagram, it is no longer guaranteed that $\theta q \neq 0$ (for instance, q may be zero), and, hence, the ultraviolet divergence re-appears as an infrared divergence with leading order $\propto (\theta q)^{-2}$.

There are two different kinds of nonplanar graphs, where this effect manifests itself. A graph of the first kind is one which is ultraviolet finite due to the fact that an internal momentum has a nonvanishing twisting with an external momentum. Then, if the external momentum is 0 (infrared regime), the divergence reappears. A graph of the second kind is one which is ultraviolet finite due to the fact that an internal momentum has a nonvanishing twisting with another internal momentum in such a manner, that by using energy-momentum conservation at the vertex, the twisting can be rewritten as one between an internal and an external momentum, and, hence, can be reduced to a graph of the first kind. The example above is of this nature.
Let us now consider the two different kinds of graphs with ultraviolet-infrared mixing in the Hamiltonian framework. It is obvious, that a graph which is ultraviolet finite due to a twisting involving an internal and an external momentum alone will diverge, when the external momentum is set to zero. This is no longer true, when we take into account the fact that, in the Hamiltonian framework, all momenta are on the mass-shell. Consider the following contribution to the fish graph in ϕ^3-theory,

$$\delta^{(4)}(q-q')\int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}}\frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}\,\frac{1}{-(\omega_{\mathbf{k}}+\omega_{\mathbf{p}}-q_0)+i\epsilon}\,\delta^{(3)}(\mathbf{k}+\mathbf{p}-\mathbf{q})\,e^{-ik\theta q}\,.$$

As in section 2.3, we do not employ the symmetrized twisting here, since the results are more lucid, if we consider single contributions. Here, we have picked the contraction where the first and the third momentum of one vertex are contracted with the other vertex' second and first momentum, respectively. The choice of the time-ordering is irrelevant here. Setting $q=(m,\mathbf{0})$ and introducing spherical coordinates, where the 3-axis is given by $-\mathbf{e}$ with $e^i=\theta^{0i}$, we obtain

$$\int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}}\frac{1}{2\omega_{\mathbf{k}}}\frac{1}{-2\omega_{\mathbf{k}}}\,e^{-ik_i\,\theta^{i0}\,m}\propto\int dr\,\frac{r}{|\vec{e}|\,m\,(r^2+m^2)^{3/2}}\;\sin(r\,|\vec{e}|\,m)\,.$$

The integrand is well-defined for $r=0$, and for large r it decreases as fast as $1/r^2$, which is sufficient for the integral to remain well-defined. Similarly, we find for a contribution to the fish graph of the second kind,

$$\delta^{(4)}(q-q')\int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}}\frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}\,\frac{1}{-(\omega_{\mathbf{k}}+\omega_{\mathbf{p}}-q_0)+i\epsilon}\,\delta^{(3)}(\mathbf{k}+\mathbf{p}-\mathbf{q})\,e^{-ik\theta p}\,.$$

Even for $q=0$ the above remains well-defined. To see this, introduce spherical coordinates to perform the $\mathbf{k}$-integration, where the 3-axis is again given by $-\mathbf{e}$. Then we obtain

$$\int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}}\frac{1}{2\omega_{\mathbf{k}}}\frac{1}{-2\omega_{\mathbf{k}}}\,e^{-ik\theta(\omega_{\mathbf{k}},-\mathbf{k})}\propto\int dr\,\frac{r}{|\vec{e}|\,(r^2+m^2)^2}\;\sin(+2\,r\,\sqrt{r^2+m^2}\,|\vec{e}|)\,.$$

Again, the integrand is well-defined in $r=0$, and for large r it behaves like $\sin(r^2)/r^3$ and hence decreases fast enough for the integral to be well-defined. Due to the fact that the energy is not conserved at the vertex, the two graphs discussed differ from one another (which was not the case in the setting of the modified Feynman rules).
The calculations are consistent with the fact that for commuting time the modified Feynman rules and the Hamiltonian formalism coincide. For the first graph, we find that in the limit $|\vec{e}|\to 0$ (commuting time) the integrand becomes

$$\lim_{|\vec{e}|\to 0}\frac{r\,\sin(r\,m\,|\vec{e}|)}{|\vec{e}|\,(r^2+m^2)^{3/2}}=\frac{r^2\,m}{(r^2+m^2)^{3/2}}\;\overset{\mathrm{r}\gg\mathrm{m}^2}{\sim}\;1/r\,,$$

and similarly for the second graph,

$$\lim_{|\vec{e}|\to 0}\frac{r\,\sin(+2\,r\,\sqrt{r^2+m^2}\,|\vec{e}|)}{|\vec{e}|\,(r^2+m^2)^2}=\frac{2\,r^2}{(r^2+m^2)^{3/2}}\;\overset{\mathrm{r}\gg\mathrm{m}^2}{\sim}\;1/r\,,$$

such that the integrals over r diverge logarithmically as in the ordinary case and coincide with the result of the Euclidean calculation. The above argument illustrates the fact that the situation is more complicated in the Hamiltonian setting than in context of the modified Feynman rules; it is not claimed that this particular mixing of infrared and ultraviolet divergences is entirely absent in the Hamiltonian approach with fixed noncommutativity matrix θ and further investigations concerning these questions will follow elsewhere. Similar results, also at higher orders, have been obtained in [39], where it was furthermore claimed to be very plausible that the mixing of ultraviolet and infrared divergences should be entirely absent in this approach.

Last but not least, the ultraviolet behaviour in the Hamiltonian formalism is now shown to be quite different, if an integration over Σ_1 at each vertex is performed. As a matter of fact, it was already pointed out in [38] that differences were to be expected, but they have not as yet been analysed. First of all, as was mentioned already in section 2.3, no cancellations of twistings from different vertices can occur. And secondly, as we will see below, the integration itself supplies an additional factor which for large momenta $\mathbf{p}$ decreases fast as $1/|\mathbf{p}|^2$. This factor is actually sufficient to render both ϕ^3- and ϕ^4-theory ultraviolet finite (where in the latter case, the proof is unfortunately founded on an as yet unproved conjecture about the degeneracy of the twisting).

Let us perform the integration over Σ_1 explicitly in ϕ^3- and ϕ^4-theory, employing the measure (2.4) in the definition of the Hamiltonian. As already shown in [27], a direct calculation in spherical coordinates yields

$$\int_{\Sigma_1} d\mu_\sigma \; e^{-\frac{i}{2}a\sigma b} = \frac{\sin\gamma_+(a,b)}{\gamma_+(a,b)} + \frac{\sin\gamma_-(a,b)}{\gamma_-(a,b)} \,,$$

where $\gamma_\pm(a,b) = |-a_0\mathbf{b} + b_0\mathbf{a} \pm \mathbf{a}\times\mathbf{b}|$. Likewise, we find

$$\int_{\Sigma_1} d\mu_\sigma \; e^{-\frac{i}{2}a\sigma b - \frac{i}{2}c\sigma f} = \frac{\sin\beta_+(a,b,c,f)}{\beta_+(a,b,c,f)} + \frac{\sin\beta_-(a,b,c,f)}{\beta_-(a,b,c,f)} \,,$$

where $\beta_\pm(a,b,c,f) = |-a_0\mathbf{b} + b_0\mathbf{a} \pm \mathbf{a}\times\mathbf{b} - c_0\mathbf{f} + f_0\mathbf{c} \pm \mathbf{c}\times\mathbf{f}|$. The Hamiltonian in ϕ^3-theory thus takes the form

$$\begin{aligned} H_I(t) &= \tfrac{g}{3!}\int dk_1 dk_2 dk_3 \,\Big(\frac{\sin\gamma_+}{\gamma_+} + \frac{\sin\gamma_-}{\gamma_-}\Big)(k_1+k_2, k_2+k_3) \\ &\qquad :\check{\phi}(k_1)\check{\phi}(k_2)\check{\phi}(k_3):\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\, e^{+it\sum k_{i,0}} \,, \end{aligned}$$

while for a ϕ^4-self-interaction we find

$$\begin{aligned} H_I(t) &= \tfrac{g}{4!}\int dk_1 ... dk_4 \,\Big(\frac{\sin\beta_+}{\beta_+} + \frac{\sin\beta_-}{\beta_-}\Big)(k_1+k_2, k_2+k_3+k_4, k_3, k_4) \\ &\qquad :\check{\phi}(k_1)...\check{\phi}(k_4):\delta^{(3)}(\mathbf{k}_1+...+\mathbf{k}_4)\, e^{+it\sum k_{i,0}} \,. \end{aligned}$$

The factors $1/\gamma_\pm$ and $1/\beta_\pm$ arising from the integration over Σ_1 change the superficial degree of divergence. Both decrease like an inverse square for large momenta. Under the assumption (which will be weakened below) that the twisting is never trivial, we obtain the following naïve modified counting rules for a graph with r vertices (in 4 dimensions):

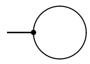
- for each internal line: propagator -1, integration $+3$;
- from momentum conservation: -3 at every vertex and once -1 for energy conservation;
- from the energy factors: -1 for $r-1$ of the vertices;
- from the integrated twisting: -2 at each one of the r vertices.

Therefore, for a theory with ϕ^n-self-interaction the superficial degree of divergence is

$$\omega(\Gamma) = 2b - 6r + 4 = n \cdot r - e - 6r + 4 = (n-6)r - e + 4 = \omega_{ord}(\Gamma) - 2r\,,$$

where e is the number of external lines, b the number of internal lines, i.e. $2b = n \cdot r - e$, and where ω_{ord} denotes the ordinary degree of divergence. As usual, $\omega < 0$ implies that a graph is superficially finite. Note that the oscillating sine-factors which may further improve the ultraviolet behaviour have not been taken into account here.

Remark 2.11 Application of the counting rules to ϕ^3-theory yields ultraviolet-finiteness for all graphs apart from the tadpole

 ,

since for $e = 1$, ω will be less than zero for any graph with $r > 1$, and for $e > 1$, ω is always less than zero (usually, $\omega < 0$ for $r > 4 - e$, $e \geq 1$).

Let us convince ourselves of the finiteness of the fish graph in an explicit calculation. To do so, we apply the rules from Remark 2.4 as they stand, with the only difference that the integration over Σ_1 is performed. If it is true that the twisting is never trivial, it suffices for our purposes to pick one particular contribution (instead of using the symmetrized kernels $\mathfrak{S}$) in the same manner as we have done it in section 2.3. Consider, for instance, the contribution where the second and the third field of the earlier vertex are contracted with the second and the third field of the later vertex, and q is an external momentum leaving the later vertex, while q' enters at the earlier one. Then the rules yield

$$\begin{aligned}
&\delta^{(4)}(q-q') \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} \frac{\delta^{(3)}(\mathbf{k}+\mathbf{p}-\mathbf{q})}{-(\omega_{\mathbf{k}}+\omega_{\mathbf{p}}-q_0)+i\epsilon} \Big(\Big(\frac{\sin\gamma_+}{\gamma_+} + \frac{\sin\gamma_-}{\gamma_-}\Big)(-q+k,k+p)\Big)^2 \\
&\quad = \delta^{(4)}(q-q') \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} \frac{1}{2\omega_{\mathbf{q-k}}} \frac{1}{-(\omega_{\mathbf{k}}+\omega_{\mathbf{q-k}}-q_0)+i\epsilon} \Big(2\,\frac{\sin\tilde{\gamma}_{\mathbf{k,q}}}{\tilde{\gamma}_{\mathbf{k,q}}}\Big)^2\,,
\end{aligned}$$

where

$$\tilde{\gamma}_{\mathbf{k,q}} = \gamma_\pm(-q+k, (\omega_{\mathbf{k}}+\omega_{\mathbf{q-k}}, \mathbf{0}) = (\omega_{\mathbf{k}}+\omega_{\mathbf{q-k}})\,|\mathbf{k}-\mathbf{q}|\,,$$

such that, after introducing spherical coordinates, we see that the integrand decreases as fast as $|\mathbf{k}|^{-5}$ (i.e. $\omega = -4$), rendering a well-defined integral. Note that this remains true for $\mathbf{q} = 0$. It is interesting to observe that the sine-factor is absent in part of the above expression due to the fact that it appears in a square.
In the same manner, application of the naïve counting rules yields finiteness of ϕ^4-theory. The main difference compared to the ordinary case is that ω is no longer independent of the number of vertices. As usual, there are no graphs with an odd number of external legs e (recall that the number of contractions in a graph arising in ϕ^n-theory is $\frac{1}{2}(r \cdot n - e)$ such that for $n = 4$, $4\,r - e$ must be even). Hence, the smallest (non-zero) possible number of external legs is 2.

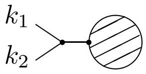
Remark 2.12 Provided that the twisting is never trivial, the superficial degree of divergence ω for graphs in ϕ^4-theory with at least two external legs, $e \geq 2$, (i.e. all non-vacuum graphs), is less than 0 for $r \geq 2$ (in the ordinary case $r \geq 4$ is necessary). By the same argument, graphs with $e \geq 4$ are also superficially finite, as in this case $\omega \leq -2r$ (usually, ω is independent of r, $\omega = -e + 4$ such that the number of external legs must be greater than 6 in order to find superficially finite graphs). Therefore, from the application of the naïve counting rules it follows that all graphs apart from vacuum graphs and graphs containing tadpoles (where $e = 2$, $r = 1$ is possible) are superficially finite.

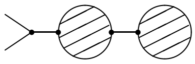
However, things are not quite so simple. The reason is that there are some graphs in which the twisting at a vertex may become trivial due to particular relations between the momenta

entering and leaving it. Before investigating this question, we recall that Σ_1 is the orbit of $\sigma^{(0)}$ (which is non-degenerate) under the action of the orthogonal group, and, hence, that the space where the twisting becomes trivial is the same as the one where $\gamma_\pm = 0$ or $\beta_\pm = 0$, respectively.
In ϕ^3-theory the situation is still fairly simple. At the vertex, we make use of 3-momentum conservation to replace $\mathbf{k}_3$, which for the twisting yields ($\mu_i = \pm\,\omega_{\mathbf{k}_i}$, $\mu_{i+j} = \pm\,\omega_{\mathbf{k}_i+\mathbf{k}_j}$),

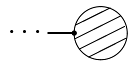
$$-(\mu_1 + 2\mu_2 + \mu_{1+2})\,\sigma^{0i}\,k_{1,i} + (\mu_2 + \mu_{1+2})\,\sigma^{0i}\,k_{2,i} + k_{1,i}\,\sigma^{ij}\,k_{2,j}\,.$$

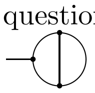
It follows that the only dependence between k_1 and k_2 which can arise from the graph theory and renders the twisting trivial is that $k_1 = -k_2$, including the energy-component (it follows that $\mathbf{k}_3 = 0$). This is only possible if either: k_1 and k_2 are the only external momenta of the graph, and k_3 connects the vertex under consideration with a bubble graph (or a number of bubble graphs),



,  etc. ,

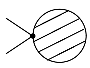
or: k_1 and k_2 are internal momenta of a bubble, one of which enters and one of which leaves the only vertex of the bubble where some (external or internal) momentum (k_3) enters,

 .

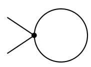
We conclude in particular, that in the fish graph the twisting is never trivial. We know already that the tadpole graph has to be subtracted, which may be taken care of by using a normally ordered interaction term. Therefore, the important question is, whether the only other graph which has to be renormalized in ordinary ϕ^3-theory,  , is finite or not. If it is, ϕ^3-theory is ultraviolet finite in this approach.
Taking into account that one of the twistings may become trivial in this graph, the counting rules still yield the desired result: since the graph ordinarily diverges logarithmically, the remaining two vertices where the twisting stays non-trivial suffice to render a well-defined integration ($\omega = -4$). This can be confirmed by a tedious explicit calculation.
In ϕ^4-theory the situation is more complicated. Here, 3-momentum at the vertex yields (with $\mu_i = \pm\,\omega_{\mathbf{k}_i}$, $\mu_{i+j+l} = \pm\,\omega_{\mathbf{k}_i+\mathbf{k}_j+\mathbf{k}_l}$),

$$\begin{aligned}-(\mu_1 + 2\mu_2 + 2\mu_3 + \mu_{1+2+3})\,\sigma^{0i}\,k_{1,i} - (\mu_2 + 2\mu_3 + \mu_{1+2+3})\,\sigma^{0i}\,k_{2,i}\\ -(\mu_3 + \mu_{1+2+3})\,\sigma^{0i}\,k_{3,i} + k_{1,i}\,\sigma^{ij}\,(k_{2,j} + k_{3,j}) + k_{2,i}\,\sigma^{ij}\,k_{3,j}\,.\end{aligned}$$

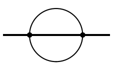
Conjecture: The only dependence between the momenta which can arise in the graph theory and which results in the above being 0, is: $k_1 = -k_2$ and the energy-components of k_3 and k_4 have opposite sign ($\mathbf{k}_3 = -\mathbf{k}_4$ follows from $\mathbf{k}_1 = -\mathbf{k}_2$).
The only non-vacuum graph in which a vertex with such a dependence arises is the following:

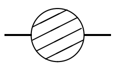
 ,

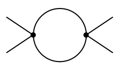
where one of the internal momenta of the bubble enters, the other leaves the vertex, and where 4-momentum conservation of the external momenta ensures that their energy-components have opposite sign. Again, by application of the counting rules, we conclude that, even taking into account that one of the twistings may become trivial, only the tadpole graph

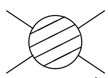


diverges, since, otherwise, ω is at least equal to -2. Therefore, all possible graphs apart from vacuum graphs (and graphs containing tadpoles) are superficially finite, since


$\omega = -2$


$\omega \leq -2$


$\omega = -4$

 .
$\omega < -4$

Although the theories turn out to be ultraviolet finite in this approach, we will see in section 3.3 that a mass renormalization becomes necessary, as, otherwise, infrared divergences may occur in graphs which are not one-particle-irreducible. Since the theory is not Lorentz-invariant, such a mass renormalization will in general not be Lorentz-invariant. Therefore, the dispersion relation may be modified when such a scheme is applied. The mechanism of the occurrence of such infrared divergences, which may be understood as a subtle form of the ultraviolet-infrared mixing problem, will be studied in section 3.3. In section 5.3 we will encounter a similar effect within the Yang-Feldman approach.

Chapter 3

Regularized Wick monomials

In the previous chapter, ordinary normal ordering was applied to products of fields $\phi^n(q)$ to define an interaction term. In this chapter, we explore one of the possibilities to generalize the ordinary construction in a way more suitable for fields on the noncommutative spacetime $\mathcal{E}$.
Let us first recall the situation in ordinary quantum field theory. As emphasized in chapter 1, normal ordering is necessary because products of fields are in general not well-defined when one or more of their arguments coincide. This is due to the fact that fields are (operator-valued) distributions. A normally ordered product arises from the naïve product by subtraction of suitable counterterms ($:1: \stackrel{\text{def}}{=} 1$),

$$\begin{aligned} &:\phi(x_1)\dots\phi(x_n): \qquad (3.1)\\ &= \phi(x_1) :\phi(x_2)\dots\phi(x_n): - \sum_{i=2}^{n} i\Delta_+(x_1-x_i) :\phi(x_2)\dots\phi(x_{i-1})\phi(x_{i+1})\dots\phi(x_n):, \end{aligned}$$

yielding a well-defined distribution in the limit of coinciding points (while the individual terms on the right-hand side diverge). This definition of Wick products coincides with the rule to put all creation operators to the left in momentum space (normal order). As it turns out, this is not the case on the noncommutative Minkowski space: defining a well-defined product at different points and then taking a suitable limit of coinciding points whose definition is adapted to the fact that the theory is defined on a noncommutative spacetime, yields an interaction term other than $:\phi^n(q):$. In particular, a natural definition of this limit of coinciding points makes it unnecessary to subtract any counterterms – the limit is well-defined in itself. Starting point of the construction are n mutually commuting sets of quantum coordinates q_i^μ, $i = 1, \dots, n$, defined as the n-fold tensor product

$$q_i^\mu \stackrel{\text{def}}{=} I \otimes \dots \otimes I \otimes q^\mu \otimes I \otimes \dots \otimes I \qquad (3.2)$$

with q^μ in the i-th tensor factor. As in the ordinary case, the product of fields at different points,

$$\phi(q_1)\dots\phi(q_n) = (2\pi)^{-4n} \int dk_N\, \check{\phi}(k_1)\dots\check{\phi}(k_n)\, e^{ik_1q_1}\dots e^{ik_nq_n},$$

is well-defined, mapping states to operators affiliated to the ordinary field algebra $\mathcal{F}$ as in chapter 1.
Contrary to the ordinary case, however, it is not possible to let the (Euclidean) distance between two sets of coordinates tend to zero, since the relative coordinates $q_i - q_j$ fail to commute with each other such that the map $f(q_i - q_j) \mapsto f(0)$ fails to be positive. Hence, the limit of coinciding points cannot be assumed as in the ordinary case, and the best we can do is to minimize the relative coordinates using the states with minimal uncertainty. This can be seen

as an "approximate limit" of coinciding points [41], the construction of which is the content of the following section (see also [69]). For a related discussion see also [19].
The interaction term arising from taking the approximate limit of coinciding points is different from the term $:\phi^n(q):$ used previously. In particular, we will see that the resulting S-matrix is ultraviolet-finite at every order of the perturbative expansion. Most of the statements made in this chapter have been published in a joint paper [10], but the method employed there to prove ultraviolet-finiteness differs from the one used in this chapter. Moreover, in addition to the results published in [10], the adiabatic limit is investigated.

3.1 An approximate limit of coinciding points

The first step in [10] to define the approximate limit of coinciding points was to take the tensor product in (3.2) not over the complex numbers $\mathbb{C}$ but over $\mathcal{Z}$, the centre of the multiplier algebra of $\mathcal{E}$. This simplification is natural in the following sense. Let $n = 2$. We want to minimize the Euclidean distance of the two sets of mutually commuting variables $q_1^\mu = q^\mu \otimes I$ and $q_2^\mu = I \otimes q^\mu$. Although it is not possible to set the difference $q_1 - q_2$ of the variables themselves equal to 0, we may be able to set the difference of certain functions of q_1 and q_2 equal to zero. In particular, it is possible to set the difference of their commutators to zero,

$$[q_1^\mu, q_1^\nu] - [q_2^\mu, q_2^\nu] = [q^\mu, q^\nu] \otimes I - I \otimes [q^\mu, q^\nu] = 0\,.$$

To implement this, the tensor product in (3.2) is taken over $\mathcal{Z}$ such that

$$q_i^\mu \stackrel{\text{def}}{=} I \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} I \otimes_{\mathcal{Z}} q^\mu \otimes_{\mathcal{Z}} I \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} I \tag{3.3}$$

with q^μ in the i-th entry. The quantum coordinates q_j then satisfy the canonical commutation relation[1],

$$[q_j^\mu, q_j^\nu] = iQ^{\mu\nu}\,,$$

where the right-hand side does not depend on j. Here, Q is subject to the quantum conditions (1.2) and (1.3), and we use units such that $\lambda_P = 1$. Note that employing the tensor product over $\mathcal{Z}$ also implies that the mean coordinate $\frac{1}{n}(q_1 + \cdots + q_n)$ commutes with the relative coordinates

$$q_{ij}^\mu \stackrel{\text{def}}{=} q_i^\mu - q_j^\mu\,,$$

$$\tfrac{1}{n}[\, q_1^\mu + \cdots + q_n^\mu\,,\, q_{ij}^\nu\,] = \tfrac{1}{n}[\, q_1^\mu + \cdots + q_n^\mu\,,\, q_i^\nu - q_j^\nu\,] = \tfrac{1}{n}[\, q_i^\mu\,,\, q_i^\nu\,] - \tfrac{1}{n}[\, q_j^\mu\,,\, q_j^\nu\,] = 0$$

and that the mean coordinate itself behaves like a quantum coordinate *of characteristic length* $1/\sqrt{n}$, i.e.

$$\tfrac{1}{n^2}[q_1^\mu + \cdots + q_n^\mu, q_1^\nu + \cdots + q_n^\nu] = i\,\tfrac{1}{n}Q^{\mu\nu}\,.$$

Now consider $\mathcal{E} \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} \mathcal{E}$, the $(n+1)$-fold tensor product of $\mathcal{E}$ over $\mathcal{Z}$. Since the mean coordinates and the relative coordinates satisfy the canonical commutation relations (the former up to a factor), it is possible to identify the algebra to which the mean coordinates are affiliated with the first tensor factor of $\mathcal{E} \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} \mathcal{E}$, while the algebra to which the relative coordinates are affiliated can be identified with the factors 2 to $n+1$. As first shown in [69] and is demonstrated below, this construction enables us to apply an n-fold tensor product $\eta^{n\otimes_{\mathcal{Z}}}$ of localization maps (1.10) to minimize all the distance variables simultaneously. To see this, we will need some C^*-algebraic properties. First note that since the coordinates q_i^μ satisfy the canonical commutation relations, the correspondence

$$\mathcal{W}(g \otimes f) = g(Q)f(q), \quad g \in C_0(\Sigma), f \in C_0(\mathbb{R}^4), \check{f} \in L^1(\mathbb{R}^4),$$

[1] Strictly speaking, the tensor product over $\mathcal{Z}$ yields the desired commutation relations in Weyl form.

as in (1.5) naturally extends to generalized symbols $F = F(\sigma; x_1, \dots, x_n)$, called n-symbols, by

$$\mathcal{W}^{(n)}(g \otimes f) = g(Q) f(q_1, \dots, q_n), \quad g \in C_0(\Sigma), f \in C_0(\mathbb{R}^{4n}), \check{f} \in L^1(\mathbb{R}^{4n}),$$

with

$$f(q_1, \dots, q_n) = \int dk_N \; \check{f}(k_1, \dots, k_n) \, e^{i(k_1 q_1 + \dots + k_n q_n)} \,.$$

It induces a product and an involution on the generalized n-symbols, and the enveloping C^*-algebra of the resulting algebra is $\mathcal{E}^{(n)} = \mathcal{E} \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} \mathcal{E}$.
Now consider coordinates $\mathfrak{q}^\mu$ of characteristic length $1/\sqrt{n}$, and define the following coordinates with $n+1$ tensor factors,

$$\bar{\mathbf{q}}^\mu \stackrel{\text{def}}{=} \mathfrak{q}^\mu \otimes_{\mathcal{Z}} \underbrace{I \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} I}_{n \text{ factors}} \qquad \text{and} \qquad \mathbf{q}^\mu_{ij} \stackrel{\text{def}}{=} I \otimes_{\mathcal{Z}} q^\mu_{ij} \,,$$

where q^μ_{ij} is the relative coordinate $q^\mu_i - q^\mu_j$. Then the n coordinates

$$\mathbf{q}^\mu_i \;\; \stackrel{\text{def}}{=} \;\; \bar{\mathbf{q}}^\mu + \tfrac{1}{n} \sum_{j=1}^{n} \mathbf{q}^\mu_{ij} \,,$$

which by construction are elements of $M(\mathcal{E}^{(n+1)})$, the multiplier algebra of $\mathcal{E}^{(n+1)}$, satisfy canonical commutation relations,

$$[\mathbf{q}^\mu_j, \mathbf{q}^\nu_j] = i\, Q^{\mu\nu} \otimes_{\mathcal{Z}} I^{n \otimes_{\mathcal{Z}}} \;\; \stackrel{\text{def}}{=} \;\; i\, \mathbf{Q}^{\mu\nu} \qquad \text{and} \qquad [\mathbf{q}^\mu_i, \mathbf{q}^\nu_j] = 0 \quad \text{for } i \neq j \,.$$

This is due to the fact that, by definition,

$$[\bar{\mathbf{q}}^\mu, \bar{\mathbf{q}}^\nu] = i\, \tfrac{1}{n} \mathbf{Q}^{\mu\nu} \,, \qquad\qquad [\bar{\mathbf{q}}^\mu, \mathbf{q}^\nu_{ij}] = 0 \,,$$

and that for the sum of the relative coordinates we find

$$[\sum_l q^\mu_{jl}, \sum_k q^\nu_{jk}] = [\tfrac{n-1}{n}\, q^\mu_j - \tfrac{1}{n} \sum_{l \neq j} q^\mu_l, \tfrac{n-1}{n}\, q^\nu_j - \tfrac{1}{n} \sum_{k \neq j} q^\nu_k] = i\, \tfrac{n-1}{n}\, \mathbf{Q}^{\mu\nu} \,,$$
$$[\sum_l q^\mu_{il}, \sum_k q^\nu_{jk}] = [\tfrac{n-1}{n}\, q^\mu_i - \tfrac{1}{n} \sum_{l \neq i} q^\mu_l, \tfrac{n-1}{n}\, q^\nu_j - \tfrac{1}{n} \sum_{k \neq j} q^\nu_k] = -i\, \tfrac{1}{n}\, \mathbf{Q}^{\mu\nu} \quad \text{for } i \neq j \,.$$

Remark 3.1 By von Neumann uniqueness (at each fixed $\sigma \in \Sigma$, as elaborated in [27]), there exists a faithful *-homomorphism which maps the coordinates defined by (3.3) to the coordinates defined above,

$$\beta^{(n)} : \mathcal{E}^{(n)} \mapsto M(\mathcal{E}^{(n+1)}) \qquad \text{with} \qquad \beta^{(n)}(q_i) = \mathbf{q}_i$$

where $M(\mathcal{E}^{(n+1)})$ denotes the multiplier algebra of $\mathcal{E}^{(n+1)} = \mathcal{E} \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} \mathcal{E}$.

Proof: See [69, 10]: The map $q^\mu_i \mapsto \mathbf{q}^\mu_i$ determines a *-homomorphism $\beta_i : \mathcal{E} \to M(\mathcal{E}^{(n+1)})$ (whose canonical extension to $M(\mathcal{E})$ is also denoted by β_i). The ranges of β_i and β_j commute for $i \neq j$ as q_i and q_j commute for $i \neq j$. Moreover, β_i restricted to $\mathcal{Z}$ is an isomorphism independent of i. By the universal properties of the tensor product and its uniqueness for nuclear C^*-algebras, there is a *-homomorphism $\beta^{(n)}$ of $\mathcal{E}^{(n)}$ to $M(\mathcal{E}^{(n+1)})$, s.t. $\beta(A_1 \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} A_n) = \beta_1(A_1) \dots \beta_n(A_n)$ for $A_i \in \mathcal{E}$. By construction, $\beta^{(n)}$ is faithful on $\mathcal{Z}$. This is sufficient for $\beta^{(n)}$ to be faithful on $\mathcal{E}^{(n)}$: since $\mathcal{K} \otimes \mathcal{K} \sim \mathcal{K}$ as C^*-algebras ($\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space), we conclude that $\mathcal{E} \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} \mathcal{E} \sim \mathcal{E} \sim C_0(\Sigma, \mathcal{K})$, and hence that the closed 2-sided ideals in $\mathcal{E} \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} \mathcal{E}$ are in a 1-1 correspondence with the closed subsets of Σ. ∎

Explicitly, we have $\beta^{(n)}\big(g(Q)f(q_1,\dots,q_n)\big) = g(\mathbf{Q})f(\mathbf{q}_1,\dots,\mathbf{q}_n)$, where

$$f(\mathbf{q}_1,\dots,\mathbf{q}_n) = \int dk_1\dots dk_n\ \check{f}(k_1,\dots,k_n)e^{ik_1\mathbf{q}_1}\dots e^{ik_n\mathbf{q}_n}\,.$$

This construction enables us to define the approximate limit of coinciding points, where the Euclidean differences $|q_{ij}|^2$ are minimized. Main ingredient is the observation that the action of an n-fold tensor product of best-localized states on (an exponential of) a difference variable $e^{ik_\mu q^\mu_{ij}}$ is equal to the action of one best-localized state on (an exponential of) the ordinary noncommutative coordinate $e^{ik_\mu q^\mu}$, as

$$\langle\, \eta \otimes_{\mathcal{Z}} \dots \otimes_{\mathcal{Z}} \eta\,,\, e^{\frac{i}{\sqrt{2}}k_\mu q^\mu_{ij}}\,\rangle = e^{-\frac{1}{2}|k|^2} = \langle\, \eta, e^{i\,k_\mu q^\mu}\,\rangle\,.$$

Hence, the above is a constant function of $\sigma \in \Sigma_1$, since the use of best-localized states restricts the spectrum of the commutators to Σ_1, such that η may be written as $\eta = \eta^{(1)} \circ \rho$, where ρ is the restriction map from $\mathcal{E}$ to $\mathcal{E}_1$ (cf. the remarks following equation (1.11)). We are now able to define the map which replaces the notion of coinciding points by a notion of approximate coincidence adapted to the noncommutative setting.

Definition 3.2 *Let $E^{(n)} : \mathcal{E}^{(n)} \to \mathcal{E}_1$ be defined by*

$$E^{(n)} = (\rho \otimes_{\mathcal{Z}} \underbrace{\eta \otimes_{\mathcal{Z}} \dots \otimes_{\mathcal{Z}} \eta}_{n\,\text{times}}) \circ \beta^{(n)}\,,$$

where $\beta^{(n)} : \mathcal{E}^{(n)} \mapsto M(\mathcal{E}^{(n+1)})$ is defined in Remark 3.1, $\eta : \mathcal{E} \to C_0(\Sigma_1)$ is the localization map (1.10), and where $\rho : \mathcal{E} \to \mathcal{E}_1$ is the restriction map. Then $E^{(n)}$ is called the quantum diagonal map and replaces the ordinary limit of coinciding points.

Note that the generators of the algebra in which $E^{(n)}$ takes values have characteristic length $1/\sqrt{n}$. In what follows, the use of the letter $\mathfrak{q}$ instead of q will always indicate that the coordinate is of characteristic length $1/\sqrt{n}$.
We close this section by giving the explicit form of the quantum diagonal map acting on a generalized n-symbol.

Proposition 3.3 *Let $f \in C_0(\mathbb{R}^{4n})$, $\check{f} \in L^1(\mathbb{R}^{4n})$. Then the explicit form of the quantum diagonal map acting on f is given by*

$$E^{(n)}(f(q_1,\dots,q_n)) = \int dk_N\ \check{f}(k_N)\, r_n(k_N)\, e^{i(\sum_i k_i)\mathfrak{q}}\,,$$

where $\mathfrak{q}$ is a quantum coordinate with characteristic length $1/\sqrt{n}$ and where the kernel r_n is given by

$$r_n(k_1,\dots,k_n) = \exp\big(-\tfrac{1}{2}\sum_{i=1}^{n}\big|k_i - \tfrac{1}{n}\sum_{l=1}^{n}k_l\big|^2\,\big)\,. \tag{3.4}$$

Equivalently, the quantum diagonal map can be written as $E^{(n)}\big(f(q_1,\dots,q_n)\big) = h_f(\mathfrak{q})$ where h_f is the symbol of the function $h_f(x) = c_n\int da_N\ f(x+a_1,\dots,x+a_n)\,\tilde{r}_n(a_N)$ with $c_n = \frac{n^2}{4}(2\pi)^{-2n+4}$ and with the position space kernel $\tilde{r}_n$ given by

$$\tilde{r}_n(a_1,\dots,a_n) = \exp\big(-\tfrac{1}{2}|a_1|^2 - \dots - \tfrac{1}{2}|a_n|^2\,\big)\,\delta^{(4)}\big(\,a_1 + \dots + a_n\,\big)\,. \tag{3.5}$$

In particular, $E^{(n)}\big(f(q_1,\dots,q_n)\big)$ is a constant function of $\sigma \in \Sigma_1$ such that there is no explicit dependence on σ.

Proof: By definition, $\beta^{(n)}\big(f(q_1,\dots,q_n)\big) = \int dk_1\cdots dk_n\, \check f(k_1,\dots,k_n)\, e^{ik_1\mathbf{q}_1+\cdots+ik_n\mathbf{q}_n}$, and with

$$\exp\big(i\sum_i k_i\mathbf{q}_i\big) = \exp\big(i\sum_i k_i\bar{\mathbf{q}}\big) \otimes_{\mathcal Z} \exp\big(i\sum_{i=1}^n (k_i - \tfrac1n \sum_{j=1}^n k_j)q_i\big)$$

we find

$$\begin{aligned}
&(\rho\otimes_{\mathcal Z}\eta^{n\otimes_{\mathcal Z}})\circ\beta^{(n)}\big(f(q_1,\dots,q_n)\big)\\
&= \int dk_1\cdots dk_n\, \check f(k_1,\dots,k_n)\, e^{i\sum_i k_i\mathbf{q}} \prod_{i=1}^n \langle\,\eta\,,\,\exp\big(i(k_i-\tfrac1n\sum_{j=1}^n k_j)q_i\big)\rangle
\end{aligned}$$

which proves (3.4). Let us recall that, by construction, the above is a constant function of $\sigma\in\Sigma_1$: the best-localized states restrict the spectrum of the $Q^{\mu\nu}$ to Σ_1, but, since the $\mathbf{q}_i$ commute among themselves, no twistings appear explicitly.
The configuration space kernel (3.5) follows by direct calculation from

$$\begin{aligned}
h_f(x) &= (2\pi)^{-4n}\int dx_N\, f(x_N)\int dk_N\, e^{-i\sum k_i(x_i-x)}\, r_n(k_N)\\
&= \int dy_N\, f(x+y_1,\dots,x+y_n)\,\underbrace{(2\pi)^{-4n}\int dk_N\, e^{-i\sum k_iy_i}\, r_n(k_N)}_{=\; c_n\,\tilde r_n(y_N)}\;.
\end{aligned}$$

To calculate $\tilde r_n$, we first perform a reparametrization which makes the Gaussian functions independent of one of the integration variables (here, I_4 is the 4×4-identity matrix),

$$\begin{pmatrix} k_1\\ k_2\\ \vdots\\ k_{n-1}\\ k_n\end{pmatrix} \mapsto \begin{pmatrix} p_1\\ p_2\\ \vdots\\ p_{n-1}\\ p_n\end{pmatrix} = -\frac1n \begin{pmatrix} (1-n)\,I_4 & I_4 & \dots & I_4 & I_4\\ I_4 & (1-n)\,I_4 & \dots & I_4 & I_4\\ \vdots & & & & \\ I_4 & I_4 & \dots & (1-n)\,I_4 & I_4\\ 0 & 0 & \dots & 0 & -n\,I_4\end{pmatrix} \begin{pmatrix} k_1\\ k_2\\ \vdots\\ k_{n-1}\\ k_n\end{pmatrix},$$

such that

$$\begin{aligned}
c_n\,\tilde r_n(y_N) &= \frac{n^4}{(2\pi)^{4n}}\int dp_N\,\exp\big(-i\sum_{i=1}^{n-1}(p_1+\cdots+2p_i+\cdots+p_n)\,y_i - ip_ny_n\big)\;\cdot\\
&\qquad\cdot\exp\big(-\tfrac12\,|p_1|^2-\cdots-\tfrac12\,|p_{n-1}|^2-\tfrac12\,|p_1+\cdots+p_{n-1}|^2\big)\\
&= (2\pi)^{-4(n-1)}\, n^4\,\delta^{(4)}(y_1+\cdots+y_n)\;\cdot\\
&\qquad\cdot\int dp_{N-1}\, e^{-i\sum_{i=1}^{n-1}p_i(y_i-y_n)}\, e^{-\frac12|p_1|^2-\cdots-\frac12|p_{n-1}|^2-\frac12|p_1+\cdots+p_{n-1}|^2}\\
&= (2\pi)^{-4(n-1)}\, n^4\delta^{(4)}\big(\textstyle\sum y_i\big)\left(\pi^{\frac n2}\displaystyle\prod_{i=2}^n\sqrt{\tfrac{2\,(i-1)}{i}}\right)^4 e^{-\frac12|y_1|^2-\cdots-\frac12|y_{n-1}|^2-\frac12|y_n|^2}\;.
\end{aligned}$$

Since $\prod_{i=2}^n\sqrt{\frac{2\,(i-1)}{i}} = \frac{2^{\frac{n-1}{2}}}{\sqrt n}$, it follows that $c_n = \frac{n^2}{4}(2\pi)^{-2n+2}$. ■

3.2 Regularized field monomials

In this section, the quantum diagonal map $E^{(n)}$ is used to define regularized monomials of quantum fields. Employing such monomials as interaction terms, we can show that the resulting S-matrix is finite at every order of the perturbative expansion if an adiabatic switching is applied.

Definition 3.4 *Let $\phi(q)$ be a field as defined in (1.12), and let $E^{(n)} : \mathcal{E}^{(n)} \to \mathcal{E}_1$ be the quantum diagonal map of Definition 3.2. Then*

$$\begin{aligned}
\phi^n_R(\mathfrak{q}) &= E^{(n)}(\phi(q_1)\dots\phi(q_n)) \\
&= \int dk_N\; r_n(k_1,\dots,k_n)\,\check\phi(k_1)\cdots\check\phi(k_n)\,e^{i(k_1+\dots+k_n)\mathfrak{q}}
\end{aligned}$$

is called the regularized field monomial.

The regularized field monomials are now used to define an effective theory on the ordinary Minkowski space, by the same principles as applied in the previous chapter. Our investigation is based on the symbol $\phi^n_R(x)$ of $\phi^n_R(\mathfrak{q})$,

$$\begin{aligned}
\phi^n_R(x) &= \int dk_N\; r_n(k_1,\dots,k_n)\,\check\phi(k_1)\cdots\check\phi(k_n)\,e^{i(k_1+\dots+k_n)x} \\
&= c_n\int da_N\; \tilde r_n(x-a_1,\dots,x-a_n)\,\phi(a_1)\cdots\phi(a_n)\,.
\end{aligned}$$

Due to the kernel r_n, the "regularized field monomials" are indeed regular:

Proposition 3.5 *The regularized field monomial $\phi^n_R(x)$ is a well-defined distribution, mapping Schwartz functions to operators acting on the invariant domain $\mathcal{D}$ of smooth wavefunctions (as defined in chapter 1).*

Proof: Let us first introduce the notation

$$\underline{a} = (a_1,\dots,a_n)\in\mathbb{R}^{4n}\,,\qquad d\underline{a} = d^4a_1\dots d^4a_n$$

which for later purposes replaces the notation a_N, da_N in this chapter. Evaluating $\phi^n_R(x)$ in a function $g\in\mathcal{S}(\mathbb{R}^4)$, we obtain

$$\int dx\, g(x)\,\phi^n_R(x) = c\int d\underline{a}\; g(\kappa(\underline{a}))\,\prod_{i=1}^n \exp\left(-\tfrac12|a_i-\kappa(\underline{a})|^2\right)\,\phi(a_1)\dots\phi(a_n)\,,$$

where $\kappa(\underline{a})\in\mathbb{R}^4$ is the mean of $\underline{a}$,

$$\kappa(\underline{a}) = \tfrac1n\sum_{i=1}^n a_i\,.$$

Since the n-fold tensor product of fields $\phi(a_1)\dots\phi(a_n)$ is a well-defined distribution which after evaluation in a testfunction on $\mathbb{R}^{4n}$ yields a well-defined operator acting on elements of $\mathcal{D}$, the invariant domain, it suffices to show that $g(\kappa(\underline{a}))\,\prod_{i=1}^n\exp\left(-\frac12|a_i-\kappa(\underline{a})|^2\right)$ is a Schwartz function on $\mathbb{R}^{4n}$. Now, the Gaussian functions are translation invariant, and they merely yield a Schwartz function on $\mathbb{R}^{4(n-1)}$, when each a_i is understood as the vector $(0,\dots,0,a_i,0,\dots,0)\in\mathbb{R}^{4n}$, such that all a_i are linearly independent and such that $\kappa(\underline{a})$ is equal to $1/n\,(a_1,\dots,a_n)\in\mathbb{R}^{4n}$. Only $n-1$ of the arguments $a_i-\kappa(\underline{a})$, $i=1,\dots,n$, turn out to be linearly independent, while the sum of all n arguments vanishes,

$$\sum_{i=1}^n (a_i-\kappa(\underline{a})) = 0\,.$$

We have seen this effect already in the proof of Proposition 3.3, where a coordinate transformation (in that case on the momenta) was employed which rendered the Gaussian functions independent of one of the new coordinates. However, it can be seen easily that $\kappa(\underline{a})$ and $a_i-\kappa(\underline{a})$, $i=1,\dots,n-1$, (understood again as vectors in $\mathbb{R}^{4n}$) are linearly independent: for $a_i\in\mathbb{R}^{4n}$,

$i = 1, \dots, n$ linearly independent, the $4n \times 4n$-matrix with rows given by $a_i \in \mathbb{R}^{4n}$, $i = 1, \dots, n-1$, and $\kappa(\underline{a}) \in \mathbb{R}^{4n}$, has non-zero determinant, since

$$\det\Bigg(-\frac{1}{n} \begin{pmatrix} (1-n)\, I_4 & I_4 & \dots & I_4 & I_4 \\ I_4 & (1-n)\, I_4 & \dots & I_4 & I_4 \\ \vdots & & & & \\ I_4 & I_4 & \dots & (1-n)\, I_4 & I_4 \\ -I_4 & -I_4 & \dots & -I_4 & -I_4 \end{pmatrix} \Bigg) \neq 0 \, .$$

We may thus conclude that $g(\kappa(\underline{a})) \prod_i \exp\left(-\frac{1}{2}|a_i - \kappa(\underline{a})|^2\right)$ is a Schwartz function on $\mathbb{R}^{4n}$. ■

In other words, there is no need to bring the annihilation and creation operators in the regularized field monomial $\phi_R^n(x)$ into normal order as in the ordinary case. In particular, all subtraction terms (the tadpoles) are well-defined due to the regularizing kernels $\tilde{r}_n$.
For reasons to be explained below, we nevertheless define an effective Hamiltonian with a normally ordered interaction,

$$H_I^g(t) = \tfrac{1}{n!} \int dx\, \delta(x_0 - t)\, g(x) :\!\phi_R^n(x)\!: , \tag{3.6}$$

where the regularized Wick monomial $:\!\phi_R^n(x)\!:$ is defined as

$$:\!\phi_R^n(x)\!: = c_n \int da_N\; \tilde{r}_n(x - a_1, \dots, x - a_n) :\!\phi(a_1) \cdots \phi(a_n)\!: .$$

For the time being, an adiabatic switching $g \in \mathcal{S}(\mathbb{R}^4)$ is applied to postpone all questions concerning the infrared-behaviour of the theory, and Haag's theorem as mentioned in section 2.1 is circumvented. Note that the adiabatic switching has no counterpart on the noncommutative side, but is part only of the effective theory. As in chapter 2, the effective Hamiltonian does not arise from taking the product of time zero fields, but from the following expression instead,

$$e^{iH_0 t} :\!\phi_R^n(0, \mathbf{x})\!: e^{-iH_0 t} = c_n \int da_N\; \tilde{r}_n(x - a_1, \dots, x - a_n)\big|_{x_0 = 0}\, e^{iH_0 t} :\!\phi(a_1) \cdots \phi(a_n)\!: e^{-iH_0 t} \, .$$

Here, H_0 is the free Hamiltonian already applied in the preceding chapter. Hence, the free Hamiltonian is treated on a different footing compared to the effective interaction Hamiltonian (3.6). This is necessary as the use of the regularized field monomials in the free Hamiltonian density would result in a modified Hamiltonian that is no longer the zero component of a Lorentz vector. This is due to the fact that, in order to define best-localized states which in turn yield the regularized interaction term, one has to choose a fixed Lorentz frame. It is emphasized again, however, that translation and rotation invariance are preserved.
Contrary to the Hamiltonian arising from the interaction $\phi^n(q)$ employed in the previous chapter, $H_I^g(t)$ is a constant function of $\sigma \in \Sigma_1$ by definition. The situation is thus improved in comparison to the one discussed in the previous chapter, where a particular measure had to be chosen to rid the Hamiltonian of its dependence on Σ.
We will now investigate the perturbation theory following the principles already explained in section 2.1. A normally ordered interaction is chosen since normal ordering is required in the free Hamiltonian. Also it does not seem natural to use an interaction with non-vanishing vacuum expectation value, which, as we shall see in the next section, moreover diverges in the adiabatic limit where the cutoff is removed.

Remark 3.6 Contrary to the ordinary case, $H_I^g(t)$ is a well-defined operator on the ordinary invariant domain $\mathcal{D}$.

Proof: Let us first consider the ordinary case. Here, the normally ordered product of fields $:\phi^n(x):$ at a fixed time t when acting, for instance, on the vacuum, yields the following expression,

$$\int \prod_{i=1}^{n} \frac{d\mathbf{k}_i}{2\omega_{\mathbf{k}_i}}\, \tilde{g}_t(\sum \mathbf{k}_i)\ e^{it\sum \omega_{\mathbf{k}_i}}\ a^\dagger(k_1)\dots a^\dagger(k_n)\,\Omega\,,$$

where $\tilde{g}_t(\sum \mathbf{k}_i) = \int d\mathbf{x}\ g(t,\mathbf{x}) e^{-i\mathbf{x}\sum \mathbf{k}_i}$. This is ill-defined, as the function $\tilde{g}_t(\sum \mathbf{k}_i)\, e^{it\sum \omega_{\mathbf{k}_i}}$ does not decrease fast enough on $\mathbb{R}^{3n}$. If, on the contrary, we employ regularized Wick monomials $:\phi^n_R(x):$, we derive the following function of the momenta $\mathbf{k}_i$,

$$\int d\mathbf{x}\ g(t,\mathbf{x}) \int da_1 ...\, da_{n-1}\, e^{-\frac{1}{2}\sum_{i=1}^{n-1}|a_i|^2}\ e^{-\frac{1}{2}|\sum_{i=1}^{n-1} a_i|^2}\ e^{i\sum_{i=1}^{n-1} k_i((t,\mathbf{x})-a_i)}\ e^{ik_n((t,\mathbf{x})+\sum_{i=1}^{n-1} a_i)}$$
$$= \tilde{g}_t(\sum \mathbf{k}_i)\ e^{it\sum \omega_{\mathbf{k}_i}} f(k_n - k_1, \dots, k_n - k_{n-1})\,\big|_{k_i\in H_m^+} = h(\mathbf{k}_1, \dots, \mathbf{k}_n)\,,$$

where f is the Fourier transform of a Schwartz function on $\mathbb{R}^{4(n-1)}$,

$$f(k_n - k_1, \dots, k_n - k_{n-1}) = \int da_1 ...\, da_{n-1}\ e^{i\sum_{i=1}^{n-1} a_i(k_n - k_i)}\ e^{-\frac{1}{2}|\sum_{i=1}^{n-1} a_i|^2}\ e^{-\frac{1}{2}\sum_{i=1}^{n-1}|a_i|^2}\,.$$

Hence, h is square-integrable on $\mathbb{R}^{3n}$. In the same manner it can be shown that the Hamiltonian H_I^g is well-defined when acting on a particle state $|\psi^{(s)}\rangle$ with smooth wavefunction $\psi_s(\mathbf{p}_1, .., \mathbf{p}_s) \in \mathcal{S}(\mathbb{R}^{3s})$. ■

This being settled, let us follow the procedure of the preceding chapter and define an S-matrix via the Dyson series.

Remark 3.7 The effective Hamiltonian is symmetric, $H_I^g(t)^\dagger = H_I^g(t)$, and it follows, by Remark 2.1, that the S-matrix is (formally) unitary when defined by the formal series

$$S[g] = I + \sum_{r=1}^{\infty} S_r[g]\,,$$

where

$$\begin{aligned} S_r[g] &= \left(\tfrac{-i}{n!}\right)^r \int dt_1 \dots dt_r\, \theta(t_1 - t_2)\dots\theta(t_{r-1} - t_r)\ \cdot \\ &\quad \cdot \int d\mathbf{x}_1 \dots d\mathbf{x}_r\ g(t_1,\mathbf{x}_1) :\phi^n_R(t_1,\mathbf{x}_1): \dots g(t_r,\mathbf{x}_r) :\phi^n_R(t_r,\mathbf{x}_r):\,. \end{aligned} \tag{3.7}$$

Moreover, due to the regularizing kernel, the S-matrix $S[g]$ is well-defined in every order r, as the following Proposition shows.

Proposition 3.8 *For any Schwartz function $g \in \mathcal{S}(\mathbb{R}^4)$ the r-th order contribution to the S-matrix $S[g]$, as given by equation (3.7), can be written as a finite sum of closable operators on the invariant domain $\mathcal{D}$. No ultraviolet divergences appear.*

In [10] a proof of the above claim was given in momentum space for factorized Schwartz functions $g(x) = g_t(x_0) g_s(\mathbf{x})$. To complement this result a proof in position space and for general g is provided here (see also [8]). Let us first consider ordinary quantum field theory, where the S-matrix at r-th order is given by

$$S_r[g] = c \int dx^1 \cdots dx^r\ g(x^1)\dots g(x^r)\ \prod_{j=1}^{r-1} \theta\big(x_0^j - x_0^{j+1}\big)\ \prod_{j=1}^{r} :\phi(x^j)^n:\,.$$

Expectation values in multi-particle states (without smearing in the momenta) are typically of the following form,

$$\prod_{j<j'} \Delta_+(x^j - x^{j'})^{n(j,j')} \, \langle p_{(l)} | :\phi(x^1)^{m_1} \dots \phi(x^r)^{m_r}: | q_{(s)} \rangle \,,$$

for some choice of indices $j, j' \in R = \{1, \dots, r\}$. Here, $n(j,j') = n(j',j) \in \mathbb{N}_0$ and $m_j = n - \sum_{i \in R} n(j,i)$. While the multiplication of a translation-invariant distribution with a Wick product of fields is well-defined (Theorem 0 in [35]), ultraviolet divergences arise, since the product of a Heaviside function θ with contractions Δ_+^n is ill-defined in 0 for $n \geq 2$. Such divergences do not appear, if regularized Wick monomials are employed.

Proof of Proposition 3.8: In addition to the notation used in the proof of Proposition 3.5, let us furthermore introduce

$$\underline{a} - x = (a_1 - x, a_2 - x, \cdots, a_n - x), \quad x, a_j \in \mathbb{R}^4 \,, \underline{a} \in \mathbb{R}^{4n} \,.$$

Now, write down the regularized Wick monomials $:\phi_R^n(x_j):$ which appear in the S-matrix at r-th order using the position space kernels $\tilde{r}_n$ given by formula (3.5). After evaluation of the δ-distributions, we find the following expression for $S_r[g]$,

$$\begin{aligned} S_r[g] \;\; &= \;\; c \int d\underline{a}^1 \cdots d\underline{a}^r \; g(\kappa(\underline{a}^1)) \dots g(\kappa(\underline{a}^r)) \prod_{j=1}^{r-1} \theta\big(\kappa_0(\underline{a}^j) - \kappa_0(\underline{a}^{j+1})\big) \\ &\quad \cdot \prod_{j=1}^{r} \Big(\; \exp\big(-\tfrac{1}{2} |\underline{a}^j - \kappa(\underline{a}^j)|^2\big) :\phi(a_1^j) \dots \phi(a_n^j): \Big) \,, \end{aligned} \tag{3.8}$$

where, as before, $\kappa(\underline{a}) \in \mathbb{R}^4$ is the mean of $\underline{a}$, and has the time component $\kappa_0(\underline{a})$,

$$\kappa(\underline{a}) = \tfrac{1}{n} \sum_{i=1}^{n} a_i \,, \qquad \kappa_0(\underline{a}) = \tfrac{1}{n} \sum_{i=1}^{n} a_{i,0} \,.$$

Taking expectation values of $S_r[g]$ in multi-particle states (without smearing in the momenta) then yields terms of the form

$$\Big(\prod_{\substack{j<j' \in J \\ (i,i') \in I}} \Delta_+(a_i^j - a_{i'}^{j'}) \Big) \, \langle p_{(l)} | \underbrace{:\phi(a_1^1) \dots\dots\dots \phi(a_n^r):}_{\phi(a_i^j)\,,\, \phi(a_{i'}^{j'}) \text{ missing}} | q_{(s)} \rangle$$

for some choice of index sets $J \subset R = \{1, .., r\}$, $I \subset N \times N$, $N = \{1, .., n\}$. Here, the total number of contractions Δ_+ is equal to $\frac{1}{2}(r \cdot n - s - l)$. Contrary to the ordinary case, every contraction appears only once. Hence, multiplication with the Heaviside functions does not pose a problem. Moreover, as the product of Heaviside functions in (3.8) is a translation-invariant distribution,

$$u(\underline{a}^1, \dots, \underline{a}^r) = \prod_{j=1}^{r-1} \theta\big(\kappa_0(\underline{a}^j) - \kappa_0(\underline{a}^{j+1})\big) = u(\underline{a}^1 - \underline{x}, \dots, \underline{a}^r - \underline{x}) \qquad \text{for } \underline{x} \in \mathbb{R}^{4n} \,,$$

its multiplication with the Wick product of fields is not problematic. And finally, as the arguments of the contractions differ from those appearing in the fields, the product of contractions and fields is actually a tensor product and as such automatically well-defined. It remains to be shown that the Gaussians together with the adiabatic switching functions g are sufficient as testfunctions. This is indeed the case by the same argument as employed in Proposition 3.5, as for each $j \in \{1, \dots, r\}$, the vectors $\kappa(\underline{a}^j)$ and $a_i^j - \kappa(\underline{a}^j)$, $i = 1, \dots, n-1$, understood as elements of $\mathbb{R}^{4n}$, are linearly independent. Hence, the Gaussian functions together with the r adiabatic switchings at r-th order of the perturbative expansion yield a testfunction on $\mathbb{R}^{4nr}$. ■

It is emphasized that the effect of the above regularization is similar to other attempts to derive a regularized field theory, but in our case the regularizing kernels are intrinsically motivated by the underlying noncommutative spacetime $\mathcal{E}$. Note that if we were to reintroduce units where $\lambda_P \neq 1$, the Planck length would appear as the width of the Gaussians in the position space kernels and as the inverse width of the ones in the momentum space kernels.
If no normal ordering is employed in the definition of the interaction Hamiltonian, the S-matrix is still finite at every order of the perturbative expansion as long as an adiabatic switching g is employed. To see this, rewrite the regularized monomial ϕ_R^n as a normally ordered product plus appropriate contractions (the tadpoles) by repeated application of formula (1.15), and consider one of the tadpole contributions to $S_r[g]$, for instance a contribution with one tadpole contraction in the first vertex. Then, instead of (3.8), we have to consider

$$\int d\underline{a}^1 \cdots d\underline{a}^r \; g(\kappa(\underline{a}^1)) \dots g(\kappa(\underline{a}^r)) \prod_{j=1}^{r-1} \theta\big(\kappa_0(\underline{a}^j) - \kappa_0(\underline{a}^{j+1})\big) \cdot$$
$$\cdot \Delta_+(a_1^1 - a_2^1) :\phi(a_3^1) \dots \phi(a_n^1): \prod_{j=1}^{r} \exp\left(-\tfrac{1}{2}|\underline{a}^j - \kappa(\underline{a}^j)|^2\right) :\phi(a_1^j) \dots \phi(a_n^j):.$$

Multi-particle expectation values of this expression are well defined, since, as in Proposition 3.5 and Theorem 3.8, it is sufficient that the adiabatic switchings $g(\kappa(\underline{a}^j))$ in combination with the Gaussian functions $\exp\left(-\frac{1}{2}|\underline{a}^j - \kappa(\underline{a}^j)|^2\right)$ yield a Schwartz function on $\mathbb{R}^{4nr}$.

3.3 Graphs and the adiabatic limit

So far, an adiabatic switching g was employed to cut off the interaction. As discussed in section 2.1, there are at least two possibilities to treat the dependence on g. One is to calculate the so-called adiabatic limit [35, 36], where g tends to a constant, and another is to show that the algebra of local observables is independent of the chosen cutoff function (in the sense that if two such functions coincide on some bounded region of spacetime, the corresponding algebras of observables are unitarily equivalent), cf. [55, 14]. This latter construction relies on a causal behaviour of the (relative) S-matrix. However, by the same principles as explained in section 2.1, the S-matrix defined in Remark 3.8 is not causal. Instead, we find, with notation as before and without taking normal ordering into account,

$$\begin{aligned} 0 \;&\neq\; S_1(x)\, S_1(y) - S_1(y)\, S_1(x) \\ &=\; c \int d\underline{x} d\underline{y} \; \tilde{r}_n(\underline{x} - x)\, \tilde{r}_n(\underline{y} - y) \sum_{i=1}^{n} \Big(\prod_{k=1}^{i-1} \phi(x_k) \sum_{j=1}^{n} \big(i\Delta(x_i - y_j) \prod_{l \neq j} \phi(y_l)\big) \prod_{k=i+1}^{n} \phi(x_k)\Big) \end{aligned}$$

for spacelike separated x, y. Although at least compared to the case investigated in section 2.1 (see p. 22) the nonlocality decreases fast like a Gaussian function, the local approach will not be investigated here.
Instead, it is shown that if regularized Wick monomials are used as interaction terms, all expectation values are well-defined as long as a cutoff in time is kept. Furthermore, it is shown that if this cutoff in time is also replaced by a constant, the one-particle irreducible (1PI) expectation values of the resulting S-matrix remain well-defined, while graphs which are not one-particle irreducible, may develop a peculiar kind of divergence. It is emphasized, however, that no mixing

of ultraviolet and infrared divergences appears by insertion of lower order graphs into higher order diagramms. Let us start the analysis by considering the vacuum expectation values, which behave like the vacuum graphs in the ordinary theory and diverge when g tends to a constant.

Remark 3.9 In the adiabatic limit, where the function g is a constant, the vacuum expectation values of the S-matrix (3.7) diverge.

Proof: The mechanism is similar to the one in the ordinary case. It is analysed in position space using formula (3.8). In vacuum expectation values all fields are contracted, such that at r-th order of the perturbative expansion the maximal possible number (i.e. $\frac{r \cdot n}{2}$) of contractions $\Delta_+(a_i^j - a_{i'}^{j'})$ appears. Therefore, the distributions θ and Δ_+ in the integrand of $\langle\Omega|\, S_r\, |\Omega\rangle$ only depend on relative coordinates between different vertices. Now perform a coordinate transformation at every vertex $j = 1, .., r$, where

$$a_i^j \mapsto v_1 + \cdots + v_{i-1} + 2\, v_i + v_{i+1} \cdots + v_n \qquad \text{for } i = 1, .., n-1 \qquad\qquad \text{and} \qquad a_n^j \mapsto v_n$$

such that for $i = 1, .., n-1$, $a_i^j - \kappa(\underline{a}^j) = v_i^j$ and $a_n^j - \kappa(\underline{a}^j) = -\sum_{i=1}^{n-1} v_i^j$. (This transformation is the inverse of the transformation used in the proof of Proposition 3.3.) Now the Gaussian functions do not depend on v_n, and for relative coordinates between different vertices $a_i^j - a_{i'}^{j'}$ we immediately find that v_n^j and $v_n^{j'}$ enter only as the differences $v_n^j - v_n^{j'}$. Hence, there is a transformation which renders the integrand independent of one of the coordinates v_n^j, $j = 1, .., r$. As a result, the integration over this variable diverges. ■

Special vacuum contributions are the tadpole graphs arising in ϕ^{2m}-theories at first order perturbation theory. As we have seen above, they diverge in the adiabatic limit. Similarly, we find that in ϕ^{2m+1}-theories, some (non-vacuum) tadpole contributions at first order perturbation theory diverge in this limit. Replacing the adiabatic switching function g by a constant, we find from (3.7) the following explicit expression for such contributions:

$$\begin{aligned} &\int dx \int ad_1 ... da_n\ \phi(a_1) \prod_{i=2}^{2m} \Delta_+(a_i - a_{i+1}) \tilde{r}_n(x - a_1, ..., x - a_n) \\ &= \int dx\ \phi(x) \int da_1 ... da_n \prod_{i=2}^{2m} \Delta_+(a_i - a_{i+1}) \prod_{i=1}^{n} e^{-\frac{1}{2}|a_i|^2}\, \delta(\textstyle\sum a_i)\,. \end{aligned}$$

The kernel $\tilde{r}_n$ being symmetric, it is irrelevant which field remains uncontracted. While the integrals $\int da_1 ... da_n$ are well-defined, the integration over x is not, since for a one-particle state $|\boldsymbol{\psi}^{(1)}\rangle$, the expectation value

$$\langle\Omega| \int dx\, \phi(x)\, |\boldsymbol{\psi}^{(1)}\rangle = \int \frac{d\,\mathbf{p}}{2\omega_{\mathbf{p}}} \psi^{(1)}(\mathbf{p})\, \delta^{(4)}(p) = \psi^{(1)}(\mathbf{0})\, \delta(m)$$

is ill-defined. This provides further motivation to use Wick ordered regularized field monomials as interaction terms, since otherwise some tadpole contributions to the Hamiltonian diverge in the adiabatic limit.

Let us now consider diagrams with more than one external leg. The graphical rules as given in Remarks 2.3 and 2.4 can be employed, since their derivation did not rely on the particular form of the nonlocal kernels which appear in the Hamiltonian. Let us briefly consider the position space graphs which were sketched at the beginning of section 2.2. In the approach investigated here, according to formula (3.8), the time-ordering of the multi-vertices in position space graphs will be defined with respect to the average times $\kappa_0(\underline{a}^j)$. By the same mechanism as discussed

in section 2.2 for the kernels $\mathcal{D}$, problems concerning causality become apparent which have already been mentioned in [10].
For our purposes, it is more convenient to work in momentum space, employing the rules from Remarks 2.3 and 2.4. Let us first consider again the fish graph in ϕ^3-theory with one of the two possible time-orderings, where q and q' are the external momenta leaving and entering the graph,

$$\delta^{(4)}(q-q')\int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}}\frac{1}{2\omega_{\mathbf{q}-\mathbf{k}}}\,\frac{1}{-(\omega_{\mathbf{k}}+\omega_{\mathbf{q}-\mathbf{k}}-q_0)+i\epsilon}\,r_3(-q,k,(\omega_{\mathbf{q}-\mathbf{k}},\mathbf{q}-\mathbf{k}))^2\,.$$

Since the square of $r_3(-q,k,(\omega_{\mathbf{q}-\mathbf{k}},\mathbf{q}-\mathbf{k}))$ is equal to

$$\exp\big(-\tfrac{1}{2}\,|\mathbf{q}|^2-\tfrac{1}{2}|\mathbf{k}|^2-\tfrac{1}{2}|\mathbf{q}-\mathbf{k}|^2-\tfrac{1}{9}\,(2q_0+\omega_{\mathbf{k}}+\omega_{\mathbf{q}-\mathbf{k}})^2-\tfrac{1}{9}\,(2\omega_{\mathbf{k}}-\omega_{\mathbf{q}-\mathbf{k}}+q_0)^2-\tfrac{1}{9}\,(2\omega_{\mathbf{q}-\mathbf{k}}-\omega_{\mathbf{k}}+q_0)^2\big)\,.$$

and as the Fourier transform of the Heaviside function never reaches its singularity in 0, the expression is well-defined. Even if $q=0$, the regularizing effect of the kernel remains, yielding

$$\exp\big(-\tfrac{2}{3}\,|\omega_{\mathbf{k}}|^2-2\,|\mathbf{k}|^2\big)\,.$$

More generally, the special planar graphs from the previous chapter with vanishing twisting are not problematic in the approach pursued here. This is due to the fact that the kernel r_n is 1 if and only if all of its arguments are zero. In particular,

$$r_3(0,k,-k)=e^{-|k|^2}\,,\qquad r_4(q,-q,k,-k)=e^{-|q|^2-|k|^2}\,,\qquad r_n(k,0,\ldots,0)=e^{-\frac{n-1}{2n}\,|k|^2}\,.$$

This is consistent with the fact that the kernels $\mathfrak{S}_n$ as well as $\sin\gamma_\pm/\gamma_\pm$ and $\sin\beta_\pm/\beta_\pm$, respectively, arose directly from the twisting in the product $\phi^n(q)$, whereas the kernel r_n is constructed in a very different way.

Proposition 3.10 *The expectation values for the S-matrix remain finite when the adiabatic switching function is of the form $g(x)=g_0(x_0)\cdot g_s$, where g_s is a constant.*

Proof: From the rules given in Remark 2.3 we derive the general form of an expectation value at r-th order perturbation theory with an arbitrary fixed time-ordering without spatial cutoff. The only difference is that we now have to consider $\prod\theta(x_{i,0}-x_{i+1,0})\prod g_0(x_{i,0})$ instead of only Heaviside functions as before. Performing the space-integrations we again find 3-momentum conservation at all vertices. Hence, the rules from Remark 2.4 remain valid as far as points 3., 6. and 7. are concerned. Point 5. remains valid apart from the overall energy conservation. And instead of the energy factors described in point 4., the time-integrations yield a bounded function of the internal and external momenta. This is due to the fact that the integrand is

$$\prod\theta(x_{i,0}-x_{i+1,0})\prod g_0(x_{i,0})\,\exp\big(\textstyle\sum x_{i,0}(\sum\limits_{j\in T_{(i)}}\pm k_j)\big)\,,$$

where $T_{(i)}$ labels the momenta entering (+ sign) and leaving (− sign) the i-th vertex. Since $\prod\theta(x_{i,0}-x_{i+1,0})\prod g_0(x_{i,0})$ is certainly L^1 (which it is not if the cutoff is removed), its Fourier transform is bounded (and if some k does not appear in the exponential, the integration yields a constant (hence bounded) function of k). It remains to be shown that the integrations over the internal momenta remain well-defined.
Let b be the number of internal lines, and let $l=n\cdot r-2b$ be the number of external momenta, then an expectation value is of the form

$$\int\prod_{i=1}^{b}\frac{d\mathbf{k}_i}{2\omega_{\mathbf{k}_i}}\,t(k_B,q_L)\,\prod_{s=1}^{r}\delta^{(3)}\big(\sum_{i\in I_s}\mathbf{k}_i-\sum_{j\in J_s}\mathbf{k}_j+\sum_{l\in L_s}\pm q_l\big)\,r_n\big(k_{I_s},-k_{J_s},\pm q_{L_s}\big)$$

for some sets I_s, J_s and L_s with $|I_s|+|J_s|+|L_s| = n$ to be specified below. Here, $t(k_B, q_L)$ stands for the bounded function of internal and external momenta discussed above ($B = \{1, ..., , b\}$, $L = \{1, ..., , l\}$), and in the δ-distributions and the kernels r_n every internal momentum appears twice, once with a positive sign, once with a negative sign, but never twice in the same kernel r_n (no tadpoles). In other words,

$$\bigcup_{s=1}^{r} I_s = \bigcup_{s=1}^{r} J_s = \{1, ..., b\}\,, \quad I_s \cap J_s = \emptyset\,, \quad I_s \cap I_{s'} = \emptyset = J_s \cap J_{s'}\,, \quad \bigcup_{s=1}^{r} L_s = \{1, ..., l\}\,.$$

As in the ordinary case, the arguments of the δ-distributions are linearly independent, provided that there are external momenta (no vacuum graphs). Hence, their product is well-defined, and evaluating them by absorbing a number of mass-shell integrations will yield conservation of the external 3-momenta. The important difference compared to the approaches analysed in the previous chapter now is that the kernels r_n yield quickly decreasing testfunctions in the remaining internal momenta, since at every vertex the evaluation of the δ-distribution yields

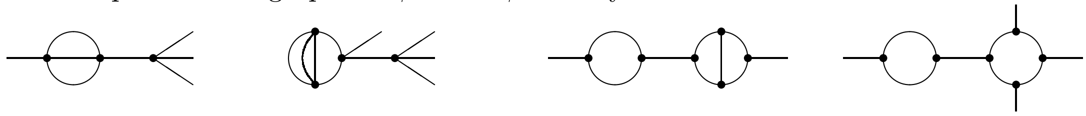
$$\begin{aligned}
&\delta^{(3)}\big(\sum_{i\in I_s}\mathbf{k}_i - \sum_{j\in J_s}\mathbf{k}_j + \sum_{l\in L_s}\pm q_l\big) \prod_{r\in I_s\cup J_s} \exp\big(-\tfrac{1}{2}|k_r - \tfrac{1}{n}(\sum_{i\in I_s}k_i - \sum_{j\in J_s}k_j + \sum_{l\in L_s}\pm q_l)|^2\big) \\
&\qquad\cdot \prod_{r\in L_s} \exp\big(-\tfrac{1}{2}|\pm q_r - \tfrac{1}{n}(\sum_{i\in I_s}k_i - \sum_{j\in J_s}k_j + \sum_{l\in L_s}\pm q_l)|^2\big) \\
&\longrightarrow \prod_{i\in I_s\cup J_s} \exp\big(-\tfrac{1}{2}|\mathbf{k}_i|^2\big) \prod_{j\in J_s} \exp\big(-\tfrac{1}{2}|\mathbf{k}_j|^2\big) \prod_{l\in L_s} \exp\big(-\tfrac{1}{2}|\pm \mathbf{q}_l|^2\big)\ \exp\big(\text{zero components}\big)\,,
\end{aligned}$$

where one of the internal momenta is replaced according to the δ-distribution and where the function depending on the 0-components is a Gaussian (i.e. certainly bounded) function depending on the spatial parts of the momenta. Since the Gaussian functions from different vertices cannot cancel each other, we conclude that the expectation values are well-defined. ■

As far as the smoothing effect of the kernels at high momenta is concerned, the removal of the cutoff function g_0 does not do any harm. However, if g_0 is replaced by a constant, the time integrations may well yield unbounded functions. In fact, as the following remark shows, graphs which are not one-particle-irreducible (i.e. those which may fall into two pieces when an internal line is cut) may develop a divergence of a peculiar kind.

Remark 3.11 Let G be a graph which is not one-particle-irreducible. Call every line a single line which, if cut, renders two separate graphs. Let l be a single line. Then the removal of the time cutoff renders an infrared divergence if one (ore both) of the vertices which are connected by l is connected by (possibly many) internal lines with a vertex into which only one external momentum enters, provided that between this latter vertex and the former one no other external momenta enter.

Typical examples of such graphs in ϕ^4- and ϕ^3-theory are

 .

To understand the problem in principle, it is sufficient to consider only the first example already investigated in section 2.2. Pick the time-ordering where the vertex on the left is the latest one and the one on the right is the earliest, then according to the rules we find

$$\delta^{(4)}(q - \sum q_i') \int \prod_{i=1}^{4} \frac{d\,\mathbf{k}_i}{2\omega_{\mathbf{k}_i}} \frac{\delta^{(3)}(\mathbf{k}_1 - \mathbf{q})}{-(\omega_{\mathbf{k}_1} - q_0) + i\epsilon} \frac{\delta^{(3)}(\sum_{i=2}^{4}\mathbf{k}_i - \mathbf{q})}{-(\omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3} + \omega_{\mathbf{k}_4} - q_0) + i\epsilon}\, r_4(...)\, r_4(...)$$

and the first term is obviously ill-defined for q on the mass-shell. This is not an effect particular to the noncommutative case, but rather a consequence of the fact that in our approach the bare and the physical mass coincide. If the physical mass were equal to m, such divergences would also appear in ordinary field theory: applying the ordinary rules to the first graph, one finds the Fourier transform of a Feynman propagator evaluated in an external momentum $\hat{\Delta}_F(q) = (q^2 - m^2 + i\epsilon)^{-1}$, which is ill-defined if $q^2 = m^2$. Note, however, that in the n-point functions (2.18), where the external lines correspond to Feynman propagators (in the ordinary case), this divergence remains unnoticed until the appropriate Klein-Gordon operator is applied and the integration performed.

The reason why these graphs are not problematic in the ordinary case is that the mass in the internal lines is not the physical mass. Following the ordinary renormalization procedure, one first sums up all self-energy contributions and then uses dressed propagators to calculate skeleton graphs only. In particular, the graph discussed above would be absorbed in the vertex of a tree graph with four legs (one vertex). The application of this programme to the interaction investigated here will not be pursued in this thesis, but it is mentioned that the mass renormalization resulting from this programme should not be Lorentz-invariant, since the whole setup uses a fixed Lorentz frame. This might result in a distorted dispersion relation, which may be of great interest as a means to find experimental bounds for the noncommutative parameter λ_P. Therefore, it is desirable to formulate a realistic theory such as quantum electrodynamics on the noncommutative Minkowski space within the above framework and pursue the programme of mass renormalization as proposed above. A similar mass renormalization is necessary in the framework of the finite theories discussed in section 2.5, since, apart from the different explicit form of the twisting, the graph theory is the same. One should therefore pursue the same programme there and compare the results. In section 5.3, a similar effect which occurs in the framework of the Yang-Feldman equation and leads to a modified dispersion relation will be discussed.

Chapter 4

The Yang-Feldman equation

As shown in the previous chapters, the Hamiltonian formalism provides an acceptable perturbative setup on a noncommutative spacetime. In the remainder of this thesis, yet another possibility to define quantum field theory on the noncommutative Minkowski space is introduced. It is based on what is called the Yang-Feldman equation [86, 59] and, contrary to field theory on ordinary Minkowski space, turns out to be inequivalent to the Hamiltonian approach. The field equation is used as a starting point and the interacting field is constructed iteratively. The formalism is Lorentz-covariant and permits to study the asymptotic behaviour of the interacting field directly. The interaction picture is avoided, as the formalism works exclusively in the Heisenberg picture; initial conditions are given not at a fixed instant in time, but asymptotically at infinite times. The earlier investigations in nonlocal field theory [87, 60, 11] mentioned in section 2.1 were based on this approach.

4.1 The classical perturbative setup

Due to the success of Feynman's rules, the Yang-Feldman approach seems to have lost some of its early popularity, and it is worthwhile to briefly mention how to set up perturbation theory for a classical field on the noncommutative Minkowski space in this approach. Consider the field equation of a scalar classical field on the noncommutative Minkowski space with a self-interaction given by $\phi^{n-1}(q)$,

$$(\Box_q - m^2)\,\phi(q) = -g\,\phi^{n-1}(q)\,, \qquad g \in \mathbb{R}\,, \tag{4.1}$$

where, according to chapter 1, the Klein-Gordon operator is defined by

$$(\Box_q - m^2)\,\phi(q) = (\Box_a - m^2)\,\phi(q + aI)\,|_{a=0}\,,$$

with a 4-vector $a \in \mathbb{M}^4$ and identity I. In what follows, $\phi(q + a)$ will be used as a shorthand notation for $\phi(q + aI)$. For the time being, $g > 0$ is a real coupling constant.

Remark 4.1 The field equation (4.1) can be solved recursively by the ansatz

$$\phi(q) = \sum_{\kappa=0}^{\infty} g^{\kappa}\,\phi_{\kappa}\,(q)\,,$$

where

$$\begin{aligned}
\phi_{\kappa}(q) &= \sum_{\kappa_1+\cdots+\kappa_{n-1}=\kappa-1} (G \times \phi_{\kappa_1}\ldots\phi_{\kappa_{n-1}})(q) \\
&= \sum_{\kappa_1+\cdots+\kappa_{n-1}=\kappa-1} \int dx\; G(x)\,\phi_{\kappa_1}(q-x)\ldots\phi_{\kappa_{n-1}}(q-x)
\end{aligned} \tag{4.2}$$

with some *ordinary* Green function G of the Klein-Gordon equation. G is fixed by initial conditions given at infinite times, in the sense that $\phi(q+t\,e_0)$ with a timelike 4-vector e_0 is given for t approaching $\pm\infty$.

Proof: The field equation at order κ reads

$$g^\kappa\,(\Box_a - m^2)\,\phi_\kappa(q+a)\,|_{a=0} = -g^\kappa \sum_{\sum\kappa_i=\kappa-1} \phi_{\kappa_1}\dots\phi_{\kappa_{n-1}}(q)\,,$$

and is solved by (4.2), since

$$\begin{aligned}(\Box_a - m^2) &\sum_{\sum\kappa_i=\kappa-1}\int dx\; G(x)\,\phi_{\kappa_1}(q+a-x)\dots\phi_{\kappa_{n-1}}(q+a-x)\,|_{a=0} \\ = (\Box_a - m^2) &\sum_{\sum\kappa_i=\kappa-1}\int dx\; G(x+a)\,\phi_{\kappa_1}(q-x)\dots\phi_{\kappa_{n-1}}(q-x)\,|_{a=0} = -\sum_{\sum\kappa_i=\kappa-1}\phi_{\kappa_1}\dots\phi_{\kappa_{n-1}}(q)\,.\end{aligned}$$

Initial conditions at infinity apply in the following sense:

$$\lim_{t\to-\infty}\phi(q+te_0) = \lim_{t\to-\infty}\sum_\kappa g^\kappa \sum_{\sum\kappa_i=\kappa-1}\int dx\; G(x+te_0)\,\phi_{\kappa_1}(q-x)\dots\phi_{\kappa_{n-1}}(q-x)\,.$$

■

The integrals above make sense if we choose the symbols of the fields and their Fourier transforms such that they decrease fast enough and such that the convolution of G and ϕ is well-defined. For instance, at first order we find

$$\phi_1(q) = \int dk\; e^{ikq}\underbrace{\int dk_N\,\delta(k-\textstyle\sum_i k_i)\,e^{-\frac{i}{2}\sum k_iQk_j}\,\check\phi(k_1)\dots\check\phi(k_{n-1})\,\check G(\sum_i k_i)}_{=\check h_\phi(k)}$$

where ϕ is chosen such that $h_\phi(x)$ decreases fast and is sufficiently smooth. Hence, the expression (4.2) for $\kappa=2$ makes sense, and likewise one proceeds to higher orders κ.
For the initial condition that ϕ_0, the field at zero-th order, is the incoming field, the interacting field at κ-th order is

$$\phi_\kappa(q) = \sum_{\sum\kappa_i=\kappa-1}\int dx\;\Delta_{ret}(x)\,\phi_{\kappa_1}(q-x)\dots\phi_{\kappa_{n-1}}(q-x)$$

with the retarded propagator $\Delta_{ret}=\theta(x_0)\,\Delta(x)$, since for $G=\Delta_{ret}$ we find

$$\lim_{t\to-\infty}\phi_\kappa(q+te_0) = \lim_{t\to-\infty}\int dx\;\Delta(x+te_0)\sum_{\sum\kappa_i=\kappa-1}\phi_{\kappa_1}\dots\phi_{\kappa_{n-1}}(q-x) = 0 \qquad \text{for } \kappa\neq 0\,.$$

Remark 4.2 By construction, the interacting field is Hermitean,

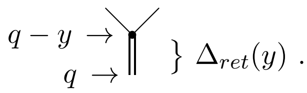
$$\phi_\kappa(q)^\dagger = \sum_{\sum\kappa_i=\kappa-1}\int dx\;\Delta_{ret}(x)\,\phi^\dagger_{\kappa_{n-1}}(q-x)\dots\phi^\dagger_{\kappa_1}(q-x) = \phi_\kappa(q)\,, \tag{4.3}$$

if the zero-th order field is Hermitean, $\phi_0^\dagger=\phi_0$. The standard proof follows immediately from the fact that the retarded propagator Δ_{ret} is real.

The graph theory of the Yang-Feldman equation is given by rooted trees, where the end branches symbolize free fields ϕ_0 and where the connecting lines between inner vertices symbolize the appropriate Green function G. Consider as an example a theory with ϕ^3-self-interaction and initial conditions as above. Then the interacting field at first order is given by

$$\phi_1(q) \quad = \quad \int dy \;\; \Delta_{ret}(y)\,\phi_0(q-y)\phi_0(q-y)\,,$$

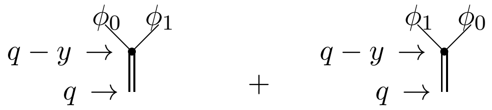
which can be symbolized by the following graph,



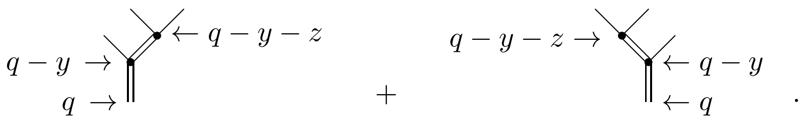
At second order of the perturbative expansion, we find

$$\begin{aligned}
\phi_2(q) &= \int dy \;\; \Delta_{ret}(y)\,(\,\phi_0(q-y)\phi_1(q-y)+\phi_1(q-y)\phi_0(q-y)\,)\\
&= \int dy \;\; \Delta_{ret}(y)\int dz \;\; \Delta_{ret}(z)\,\big(\,\phi_0(q-y)\,\phi_0^2(q-y-z)\;+\;\phi_0^2(q-y-z)\,\phi_0(q-y)\,\big)\,,
\end{aligned}$$

which can be cast into a graphical language as follows: start from



and append the first order graph to obtain



More generally, the classical graph theory (tree level) of the Yang-Feldman approach is constructed by appending trees of lower order to one another. Each tree starts with a root, has a number of inner vertices connected by lines, as well as a number of endpoints. The lines connecting inner vertices correspond to the Green functions, and the end-branches symbolize free fields. The following rule applies:

Remark 4.3 For a self-interacting ϕ^n-theory, every vertex which is neither the root at q nor an endpoint has $n-1$ successors. Such vertices are referred to as inner vertices. At order κ in the perturbative expansion, κ inner vertices occur in each tree, and each tree at order κ has $\kappa\,(n-2)+1$ endpoints and one root.

Proof: By induction: At first order, we have indeed $n-1 = 1\,(n-2)+1$ endpoints. Appending one more vertex to the endpoint of a graph of order $\kappa-1$, which, by induction hypothesis, has $(\kappa-1)\,(n-2)+1$ endpoints, swallows one endpoint while adding $n-1$ new ones, leaving us with $(\kappa-1)\,(n-2)+1-1+(n-1) = \kappa\,(n-2)+1$ endpoints. ■

The number of graphs at κ-th order perturbation theory is therefore given by a well-known combinatorial formula:

Remark 4.4 For a ϕ^n-self-interacting theory, the number of tree graphs at the κ-th order of the perturbative expansion in the Yang-Feldman approach is

$$\frac{(\kappa\,(n-1))\,!}{(\kappa\,(n-2)+1)\,!\,\kappa\,!}\,. \tag{4.4}$$

Proof: The number $P(\vec{r})$ of plane forests of type $\vec{r} = (r_0, \ldots, r_m)$, where r_i counts the number of vertices (including the endpoints, but not counting the root) which have i successors, is given by (cf. [77])

$$P(\vec{r}) = \frac{s}{l}\,\frac{l\,!}{r_0\,!\,r_1\,!\,\ldots\,r_m\,!}\,,$$

where l is the total number of vertices, $l = \sum r_i$, and where s is the number of components, $s = \sum_i (1-i)\, r_i > 0$. In a Yang-Feldman tree graph, the $\kappa\,(n-2)+1$ endpoints have 0 successors, while each of the κ inner vertices is succeeded by $n-1$ vertices. Not counting the root, the total number of vertices (inner vertices plus endpoints) is $l = \kappa\,(n-2)+1+\kappa = \kappa\,(n-1)+1$. Hence, the number of trees is given by $P(\vec{r})$ with $\vec{r} = (\kappa\,(n-2)+1, 0, \ldots, 0, \kappa) \in \mathbb{N}^n$. The number of components indeed is $s = \kappa\,(n-2)+1+(1-(n-1))\,\kappa = 1$ (the trees are connected). ■

As we have seen in the explicit calculation for a ϕ^3-self-interaction, there is one tree graph at first order, and two at second order. At third order, one has to consider 5 different graphs, while in fourth order we have to take 14 graphs into account. In a theory with ϕ^4-self-interaction, in second order perturbation theory 3 graphs are to be investigated, in third and fourth order 12 and 55 graphs, respectively.
Of course, the above analysis does not yet take into account quantum properties of the fields, which by application of Wick's theorem will result in the appearance of loop graphs. Before proceeding along these lines, let us consider the support properties of the interacting field's integrands, the so-called retarded products $R_\kappa(q; x_1, \ldots, x_\kappa)$,

$$\phi_\kappa(q) = \int dx_1 \ldots dx_\kappa\, R_\kappa(q; x_1, \ldots, x_\kappa)\,.$$

By construction, supp $R_\kappa(q;\,\cdot\,) \subset \{(x_1, \ldots, x_\kappa)\,|\,x_i \in \overline{V_+}\,,\ i = 1, \ldots, \kappa\}$, since at order κ a product of κ retarded propagators $\Delta_{ret}(x_1)\ldots\Delta_{ret}(x_\kappa)$ appears. Redefining the integration variables, we moreover find

$$\begin{aligned}
R_1(q; x_1) &= \Delta_{ret}(x_1)\,\phi_0(q-x_1)^{n-1}\,,\\
R_2(q; x_1, x_2-x_1) &= \Delta_{ret}(x_1)\,\Delta_{ret}(x_2-x_1)\sum_{l=1}^{n-1}\phi_0(q-x_1)^{l-1}\,\phi_0(q-x_2)^{n-1}\,\phi_0(q-x_1)^{n-1-l}\,,
\end{aligned}$$

$$\begin{aligned}
&R_3(q; x_1, x_2-x_1, x_3-x_2) =\\
&= \Delta_{ret}(x_1)\,\Delta_{ret}(x_2-x_1)\,\Delta_{ret}(x_3-x_2)\,\cdot\\
&\quad\cdot\sum_{l_1=1}^{n-1}\phi_0(q-x_1)^{l_1-1}\Big(\sum_{l_2=1}^{n-1}\phi_0(q-x_2)^{l_2-1}\,\phi_0(q-x_3)^{n-1}\,\phi_0(q-x_2)^{n-1-l_2}\Big)\,\phi_0(q-x_1)^{n-1-l_1}\\
&+ \Delta_{ret}(x_1)\,\Delta_{ret}(x_2-x_1)\,\Delta_{ret}(x_3-x_1)\,\cdot\\
&\quad\cdot\sum_{l_1=1}^{n-2}\phi_0(q-x_1)^{l_1-1}\,\phi_0(q-x_2)^{n-1}\sum_{l_2=1}^{n-l_1-1}\phi_0(q-x_1)^{l_2-1}\,\phi_0(q-x_3)^{n-1}\,\phi_0(q-x_1)^{n-1-l_1-l_2}\,.
\end{aligned}$$

More generally, $R_\kappa(q;\cdot)$ can always be written as a function of $x_1, x_2-x_1, \ldots, x_{i+1}-x_i, \ldots, x_\kappa - x_{\kappa-1}$.

Remark 4.5 The support of the retarded products with respect to the variables $x_1, \ldots, x_\kappa$, , supp $R_\kappa(q;\,\cdot\,)$, is contained in the set

$$\{(x_1, \ldots, x_\kappa)\,|\,x_1 \in \overline{V_+}\,,\ (x_i - x_{i-1}) \in \overline{V_+}\,,\ i = 2, \ldots, \kappa\}\,.$$

This is a consequence of the fact that, with notation $x_i - x_{i-1}\,|_{i=1} \stackrel{\text{def}}{=} x_1$, from $x_i - x_{i-1} \in \overline{V_+}$ for $i = 1, \ldots, \kappa$ it can be inferred that the 4-vector $x_{i+l} - x_i$, for any $0 < l \leq \kappa - i$, is again inside the future lightcone, since

$$x_{i+l} - x_i = (x_{i+l} - x_{i+l-1}) + (x_{i+l-1} - x_{i+l-2}) + \cdots + (x_{i+1} - x_i) \;\in\; \overline{V_+}\,.$$

Contrary to the ordinary case, however, q is not included in this causality structure. If q were a 4-vector $x \in \mathbb{M}^4$, we could perform the coordinate transformation $x_i \mapsto y_i = x - x_i$ and, from the fact that $x - y_i \in \overline{V_+}$, $y_i - y_j \in \overline{V_+}$ implies that $x - y_j = (x - y_i) + (y_i - y_j) \in \overline{V_+}$, we derive the well-known result that on ordinary Minkowski space,

$$\text{supp}\;R_\kappa(x;\,\cdot\,) \subset \{(y_1, \ldots, y_\kappa)\,|\,x - y_i \in \overline{V_+}\,,\; i = 1, \ldots, \kappa - 1\,\}\,.$$

4.2 Wick products and unitarity

Let us now turn to the definition of quantum fields on the noncommutative Minkowski space which will give rise to loop graphs in the perturbative expansion (4.2). Starting point is the definition of the free quantum field (1.12) on the noncommutative Minkowski space first given in [27],

$$\phi(q+x) \;\;=\;\; (2\pi)^{-3/2} \int \frac{d\,\mathbf{k}}{2\omega_{\mathbf{k}}} \Big(a(k) \,\otimes\, e^{-ik(q+x)} + a^\dagger(k) \,\otimes\, e^{+ik(q+x)} \Big)\Big|_{k\in H_m^+}\,,$$

where H_m^+ denotes the positive mass-shell, and where a and $a^\dagger$ are the ordinary annihilation and creation operators on the symmetric Fock space $\mathfrak{H}$. In the setting which is investigated here, this formal equation is to be understood in terms of the following definition.

Definition 4.6 *Let $\phi(x)$ be an ordinary Wightman field [78] on $\mathbb{M}^4$ with invariant domain $\mathcal{D}$, and let $\mathcal{F}$ be the corresponding field algebra. A q-field ϕ_q on the noncommutative spacetime $\mathcal{E}$ associated to ϕ is a linear map, formally denoted*

$$g \mapsto \phi_q(g) = \int dx\;\phi(q + xI)\,g(x)\,,$$

from testfunctions $g \in \mathcal{S}(\mathbb{R}^4, \mathcal{Z})$ to closable operators affiliated to $\mathcal{F} \otimes \mathcal{E}$, such that, for each state ω on $\mathcal{E}$ with $\omega(e^{ikq}) \in C_b^\infty(\mathbb{R}^4, \mathcal{Z})$ and for each unit vector $\Psi \in \mathcal{D}$ with $\mathbf{\Psi}(\cdot) = \langle\Psi|\,\cdot\,|\Psi\rangle$, the following two conditions are satisfied:

1. *$\mathbf{\Psi} \otimes \omega$ is in the domain[1] of $\phi_q(g)$, $g \in \mathcal{S}(\mathbb{R}^4, \mathcal{Z})$, i.e., after the field is smeared with a testfunction, evaluating in a state on $\mathcal{E}$ and taking an expectation value are well-defined operations.*
2. *$(\mathbf{\Psi} \otimes \omega)\,(\phi_q(\cdot)) \in \mathcal{S}(\mathbb{R}^4, \mathcal{Z})'$, i.e. evaluation in a state on $\mathcal{E}$ as well as taking an expectation value yields a continuous linear functional on the testfunction space.*

Here, $\mathcal{Z}$ denotes the centre of the multiplier algebra of $\mathcal{E}$ and $'$ denotes the dual space.

Again, as usual, the shorthand notation $\phi(q + x)$ will replace $\phi(q + xI)$. The above is still a preliminary definition. In section 5.2, we shall see that the testfunction space as well as the invariant domain and the admissible states on $\mathcal{E}$ have to be modified. A certain property

[1] cf. [27]: let $\mathfrak{A}$ be a C^*-algebra to which A is affiliated (see page 9). A state ω on $\mathfrak{A}$ is in the domain of A, if it is in the support of A and $\omega(A^2) \stackrel{\text{def}}{=} \sup\big\{\omega(f(A))\,|\,f \in C_0(\mathbb{R})_+, f(\lambda) \leq \lambda^2, \lambda \in \mathbb{R}\big\} < \infty$.

regarding analyticity will be required. The testfunctions may take values in $\mathcal{Z}$, and at times it will be suitable to write $g(Q;x)$ to emphasize this. Since $I \in \mathcal{Z}$, it is possible to identify $g(x)$ with $g(x)I$ such that $\mathcal{S}(\mathbb{R}^4) \subset \mathcal{S}(\mathbb{R}^4, \mathcal{Z})$ and, hence, ordinary testfunctions taking values in $\mathbb{C}$ are included in the definition. If a fixed noncommutativity matrix θ is employed, g is simply an element of $\mathcal{S}(\mathbb{R}^{4n})$ and $(\boldsymbol{\Psi} \otimes \omega)\,(\phi_q(\cdot)) \in \mathcal{S}(\mathbb{R}^4)'$. Note that the admissible states have to be in the domain of any monomial of the q_μ in order to ensure that $\omega(e^{ikq})$ be smooth, since taking derivatives of the exponential will result in the appearance of polynomials.
Again, the fields are merely affiliated to the field algebra for the same reason and in the same sense as usual: the field operators are unbounded and the field algebra $\mathcal{F}$ is the corresponding Weyl algebra. This is merely a fact to be kept in mind; in all calculations which follow, the unbounded field operators are used.
Explicitly, after smearing in a testfunction f and a state ω, the free q-field acts on state $|\boldsymbol{\varphi}\rangle \in \mathfrak{H}$ as

$$\begin{aligned}(\omega(\phi_q(f))\,|\boldsymbol{\varphi}\rangle)^{(n)}(p_1,\dots,p_n) &= c\sqrt{n+1}\int\frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}\;\psi_\omega(p)\,\hat f(p)\,\varphi^{(n+1)}(p,p_1,..,p_n)\\ &\quad+\frac{c}{\sqrt{n}}\sum_{i=1}^{n}\psi_\omega(-p_i)\,\hat f(-p_i)\,\varphi^{(n-1)}(p_1,..,\widehat{p}_k,..,p_n)\,, \end{aligned} \tag{4.5}$$

where $\widehat{\ }$ indicates omission of the argument, and where $\psi_\omega(k) = \omega(e^{ikq})$ and $c > 0$.
The n-fold tensor product of q-fields associated to ϕ can easily be defined in this framework. It is a linear map, formally denoted

$$\phi_q^{\otimes n}(g) = \int dx_1 \dots dx_n\; \phi(q+x_1)\cdots\phi(q+x_n)\,g(x_1,\dots,x_n)\,, \tag{4.6}$$

which maps testfunctions $g \in \mathcal{S}(\mathbb{R}^{4n}, \mathcal{Z})$ to closable operators affiliated to $\mathcal{F}\otimes\mathcal{E}$ such that the two conditions from Definition 4.6 are satisfied (with $\mathbb{R}^4$ replaced by $\mathbb{R}^{4n}$). The integral kernel $\phi^{\otimes n}(q;x_N)$ of $\phi_q^{\otimes n}$ is given by

$$\phi(q+x_1)\cdots\phi(q+x_n) = \phi^{\otimes n}(q;x_N) = \int dk_N\;\hat\phi(k_1)\dots\hat\phi(k_n)\,\otimes\prod_{l\in N} e^{-i\,k_l\,(q+x_l)} \tag{4.7}$$

with the ordinary field's Fourier transform $\hat\phi$. More general distributions on the noncommutative Minkowski space can be defined as follows.

Definition 4.7 *A q-distribution u_q associated to a Wightman field ϕ on $\mathbb{M}^4$ is a linear map, formally denoted*

$$u_q(g) = \int dx_1 \dots dx_n\; u(q;x_1,\dots,x_n)\,g(x_1,\dots,x_n)\,,$$

from testfunctions in $\mathcal{S}(\mathbb{R}^{4n}, \mathcal{Z})$ to closable operators affiliated to $\mathcal{F}\otimes\mathcal{E}$ such that the conditions from Definition 4.6 are satisfied (with $\mathbb{R}^4$ replaced by $\mathbb{R}^{4n}$).
A Q-distribution u_Q associated to ϕ is a q-distribution which takes values in closable operators affiliated to $\mathcal{F}\otimes\mathcal{Z}$, the tensor product of the field algebra and the centre $\mathcal{Z}$. It is formally denoted

$$u_Q(g) = \int dx_1 \dots dx_n\; u(Q;x_1,\dots,x_n)\,g(x_1,\dots,x_n)\,,$$

where $g \in \mathcal{S}(\mathbb{R}^{4n}, \mathcal{Z})$.

Notation: From now on, all q-distributions considered are associated to the *free* field, and will simply be called "q-distributions".

From the classical setup discussed in the previous section we may conclude that the quantum perturbation theory in the Yang-Feldman approach on the noncommutative Minkowski space gives rise to q-distributions. One of the tasks in setting up the perturbation theory will be to give meaning to products of fields whose arguments may coincide, for example,

$$\phi(q+x_1)\phi(q+x_2)\dots\phi(q+x_1)\dots\phi(q+x_n)\,.$$

This situation differs from the one analysed in chapter 3, where coinciding points were defined with respect to the noncommuting coordinates q^μ.

Definition 4.8 *For a q-distribution u_q on $\mathcal{S}(\mathbb{R}^{4n},\mathcal{Z})$, the limit of coinciding points is defined by evaluating u_q in a sequence of compactly supported testfunctions $g_r \in C_0(\mathbb{R}^{4n})$ which approaches*

$$\delta^{(4)}(x-x_1)\cdots\delta^{(4)}(x-x_n)$$

with $\operatorname{supp} g_r \subset \operatorname{supp} g_{r+1}$ *and* $\bigcap \operatorname{supp} g_r = \{(x,\dots,x)\} \in \mathbb{R}^{4n}$*. If it is well-defined, it renders a q-distribution u_q on $\mathcal{S}(\mathbb{R}^4,\mathcal{Z})$.*

A field monomial is not well-defined in this limit. This follows after evaluation in a testfunction $g \in \mathcal{S}(\mathbb{R}^4,\mathcal{Z})$ and a state ω by the same argument as usual: for $k_i \in H_m^\pm$, the function $g(\sum k_i)\,\psi_\omega(\sum k_i)$, where $\psi_\omega(\sum k_i) = \omega(e^{i\sum k_i\, q})$, does not decrease fast in all directions[2].
Now re-consider the products of fields, where all annihilation and creation operators are normally ordered, in view of the above definitions. Such products arise from (4.7) by applying the ordinary definition of normal-ordering in momentum space (1.15) to the product $\hat{\phi}(k_1)\dots\hat{\phi}(k_n)$,

$$:\phi_q^{\otimes n}(x_1,\dots,x_n): \stackrel{\text{def}}{=} (2\pi)^{-4n}\int dk_N\, :\hat{\phi}(k_1)\dots\hat{\phi}(k_n): \otimes \prod_{l\in N} e^{-ik_l(q-x_l)}\,. \tag{4.8}$$

From ordinary quantum field theory we may conclude that the above yields a well-defined q-distribution $:\phi_q^n:$ on $\mathcal{S}(\mathbb{R}^4,\mathcal{Z})$ when evaluated in the limit of coinciding points. These normally ordered products will serve as a preliminary definition; in chapter 5, a thorough investigation of an appropriate definition of Wick products will follow and the so-called quasiplanar Wick products will be introduced. In what follows, the tensor product sign is dropped to simplify the notation. It is emphasized that in equation (4.8) the order of the exponentials is *not* changed when the fields are brought into normal order. Any manipulation concerning the ordering of the fields, such as the application of Wick's theorem (1.16), will leave the noncommutative exponentials alone.
Let us now consider the quantum interacting field in the Yang-Feldman approach on $\mathcal{E}$ and find the general rules for the perturbation theory resulting from the use of the above normally ordered products defined above. We start by considering the first two orders arising in ϕ^3-theory as an example. Throughout, initial conditions are applied such that ϕ_0 is the incoming field.

Notation: In what follows, the zero order field $\phi_0 = \phi_{in}$ being a free field is simply labelled by ϕ (as in the definition of q-fields). The full solution of the Yang-Feldman equation (the interacting field) is labelled by ϕ_{int}.

Since the fields are now operator-valued distributions, the convolutions with Green functions are no longer a priori well-defined. Hence, an infrared cutoff is introduced via an adiabatic cutoff

[2] See [30] for an investigation of distributions and the twisted convolution product, where it was shown, for instance, that the twisted convolution product of speedily decreasing distributions is again a speedily decreasing distribution; a result which is unfortunately not applicable here, since the distributions under consideration are not of this type.

function g such that

$$\phi_1^g(q) \overset{\text{def}}{=} \int dy\ g(y)\,\Delta_{ret}(y)\,{:}\phi(q-y)\phi(q-y){:} = \big((g\Delta_{ret}) \times {:}\phi^2{:}\big)(q)\,,$$

and for $\kappa \geq 2$

$$\phi_\kappa^g(q) = \sum_{\kappa_1+\cdots+\kappa_{n-1}=\kappa-1} \int dx\ g(x)\,\Delta_{ret}(x)\,\phi_{\kappa_1}(q-x)\ldots\phi_{\kappa_{n-1}}(q-x)\,. \tag{4.9}$$

In ordinary quantum field theory, the (temporary) introduction of such a cutoff is not problematic; one simply replaces the original interaction term $\phi^{n-1}(x)$ by $\phi^{n-1}(x)g(x)$ which in (4.2) yields

$$\phi_\kappa^g(y) = \sum_{\kappa_1+\cdots+\kappa_{n-1}=\kappa-1} \int dx\ \Delta_{ret}(x)\,g(y-x)\,\phi_{\kappa_1}(y-x)\ldots\phi_{\kappa_{n-1}}(y-x)\,.$$

Contrary to that, (4.9) does not follow from such a modified interaction term. Bearing in mind, however, that in the end one will pass to the adiabatic limit of expectation values in a similar manner as discussed in the framework of the Hamiltonian approach, this is not a major drawback, but should instead be seen as an effective intermediate step in defining the theory[3].
In the following chapters, emphasis is put on the analysis of ultraviolet divergences.
To save notation, the superscript ^{g} is now dropped again. For the interacting field at second order we then find

$$\begin{aligned}\phi_2(q) \quad = \quad & \int dy\ g(y)\,\Delta_{ret}(y) \int dz\ g(z)\,\Delta_{ret}(z)\ \cdot \\ & \qquad \cdot \big(\,\phi(q-y)\,{:}\phi^2(q-y-z){:}\ +\ {:}\phi^2(q-y-z){:}\,\phi(q-y)\,\big)\,.\end{aligned}$$

Applying Wick's theorem (1.16), the products of Wick monomials are now rewritten as a sum of normally ordered expressions,

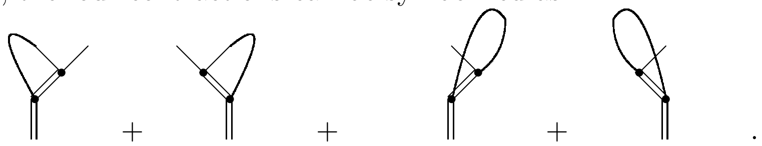
$$\begin{aligned}\phi(q-y)\,{:}\phi^2(q-y-z){:} \;=\; & {:}\phi(q-y)\,\phi^2(q-y-z){:}\ + \\ & + i\,(2\pi)^{-8}\int dp\,dk_1dk_2\ \hat{\Delta}_+(p)\,\Big(\delta^{(4)}(p+k_1)\,{:}\hat{\phi}(k_2){:}\ +\ \delta^{(4)}(p+k_2)\,{:}\hat{\phi}(k_1){:}\Big) \\ & \qquad\qquad \cdot\ e^{-ip(q-y)}\,e^{-ik_1(q-y-z)}\,e^{-ik_2(q-y-z)} \\ =\; & {:}\phi(q-y)\,\phi^2(q-y-z){:}\ +\ i\,\Delta_+(z)\,\phi(q-y-z) \\ & + i\,(2\pi)^{-8}\int dp\,\hat{\Delta}_+(p)\,e^{-ipz}\int dk\ \hat{\phi}(k)\,e^{-ipQk}\,e^{-ik(q-y-z)}\,.\end{aligned}$$

In the same manner ${:}\phi^2(q-y-z){:}\,\phi(q-y)$ is normally ordered, and we obtain

$$\begin{aligned}\phi_2(q) \;=\; & \int dy\,g(y)\,\Delta_{ret}(y)\int dz\,g(z)\,\Delta_{ret}(z)\,\big(\,{:}\phi(q-y)\,\phi^2(q-y-z){:}\ +{:}\phi^2(q-y-z)\,\phi(q-y){:}\big) \\ & + \int dy\,g(y)\,\Delta_{ret}(y)\int dz\,g(z)\,(\,\Delta_{ret}\cdot\,i\,\Delta_+(z) - \Delta_{ret}\cdot\,i\,\Delta_-(z)\,)\,\phi(q-y-z) \\ & + (2\pi)^{-4}\int dy\,g(y)\,\Delta_{ret}(y)\int dz\,g(z)\,\Delta_{ret}(z)\int dk\,\big(\,i\,\Delta_+(z+Qk)\,e^{-ik(q-y-z)}\,\hat{\phi}(k) \\ & \qquad\qquad\qquad\qquad - i\,\Delta_-(z-Qk)\,e^{-ik(q-y-z)}\,\hat{\phi}(k)\,\big)\,.\end{aligned}$$

[3] Again it should be mentioned that a very different possibility to treat the infrared problem may be based on the approach pursued in [88].

In terms of graphs, the four contractions can be symbolized as



To see this, consider the second and the fourth graph. They correspond to the contractions which arise from bringing $\phi(q-y):\phi^2(q-y-z):$ into normal order. In doing so, the field $\phi(q-y)$ is contracted once with the first field at the vertex $q-y-z$, and once with the second field at the vertex $q-y-z$ (counted from left to right).

Similar to what was found within the Hamiltonian framework in chapter 2, we have thus arrived at two qualitatively different contributions; one is a product of a field and an ordinary distribution, while the other depends on the twisting (or rather on the operator Q). As should be obvious from the above calculation, the mechanism leading to the different contributions is the same as in the Hamiltonian approach or in the modified Feynman rules, and again the contributions are referred to as planar and nonplanar graphs, respectively. The following section will be concerned with a systematic analysis of the graph theory in this context.

Before proceeding, let us consider the above nonplanar graphs in more detail, and reconsider the unitarity problem in the Yang-Feldman approach. In order to make the connection with the modified Feynman rules, the cutoff is dropped for the rest of this section. Using the fact that in every fibre $\mathcal{E}_\sigma$, the following equation holds, $\int dz f(z)\cdot(g\star h)(z)=\int dz(f\star g)(z)\cdot h(z)$, the second order term may be rewritten as

$$\begin{aligned}
c\int dy\,\Delta_{ret}(y)\int dz\int dl\;\hat{\Delta}_{ret}(l)\int dp\int dk\;\; &\big(\hat{\Delta}_+(p)\,e^{-ipz-ilz}\,e^{+ilQp}\,e^{-ik(q-y-z)}\,\hat{\phi}(k)\\
&-\;\hat{\Delta}_-(p)\,e^{-ipz-ilz}\,e^{-ilQp}\,e^{-ik(q-y-z)}\,\hat{\phi}(k)\big)\\
=\;\int dy\,\Delta_{ret}(y)\int dz\,\big(\,i\,\Delta_+\star_{2Q}\Delta_{ret}(z)&-\Delta_{ret}\star_{2Q}\;i\,\Delta_-(z)\,\big)\,\phi(q-y-z)\,.
\end{aligned}$$

By abuse of notation, $\star_{2Q}$ stands for the twisted convolution product with respect to $2Q$. This notation is used to emphasize that the fibrewise product with 2σ instead of σ is used, but that the expression is understood as being defined on the full bundle. In more precise terms, the expression above is a q-distribution on $\mathcal{S}(\mathbb{R}^8,\mathcal{Z})$ which may be written as a tensor product of a Q-distribution and a field ϕ_q.

Let us now turn again to the question of the connection between time-ordering and unitarity. As in the Hamiltonian formalism, we find that the time-ordering, hidden here in the retarded propagator, can be absorbed into a Feynman propagator in the planar contribution, since

$$\Delta_{ret}\,(i\Delta_+-i\Delta_-)=i\,\Delta_F^2-i\Delta_-^2\,.$$

This is not the case for the nonplanar contribution. Again, an additional product of retarded and advanced propagators appears,

$$i\,\Delta_+\star_{2Q}\Delta_{ret}-\Delta_{ret}\star_{2Q}\;i\,\Delta_-\;\;=\;\;i\,\Delta_F\star_{2Q}\Delta_F-\,i\,\Delta_-\star_{2Q}\Delta_--\Delta_{av}\star_{2Q}\;i\,\Delta_{ret}\,.$$

As discussed in section 2.4, this contribution does not vanish for general $Q_{\mu\nu}$.

Not having assumed from the outset that Feynman propagators serve as internal lines, no problems with formal unitarity arise. Explicitly, from the equations

$$\begin{aligned}
\overline{i\,\Delta_+\,\Delta_{ret}(z)-\Delta_{ret}\,i\,\Delta_-(z)}\;\;&=\;\;-\Delta_{ret}\,i\,\Delta_-(z)+\,i\,\Delta_+\,\Delta_{ret}(z)\\
\overline{i\,\Delta_+\star_{2Q}\Delta_{ret}(z)-\Delta_{ret}\star_{2Q}\;i\,\Delta_-(z)}\;\;&=\;\;-\Delta_{ret}\star_{2Q}\;i\,\Delta_-(z)+\,i\,\Delta_+\star_{2Q}\Delta_{ret}(z)
\end{aligned}$$

we obtain $\phi_2^\dagger=\phi_2$. In fact, this is true at any order perturbation theory.

Remark 4.9 If the incoming field ϕ is Hermitean, the interacting field $\phi_\kappa(q)$ is Hermitean at any order κ.

Proof: By induction as in the classical case (4.3). The important fact to note is that normal ordering does not spoil Hermiticity, which in turn is a consequence of the fact that normal ordering is defined by subtracting Hermitean terms (namely vacuum expectation values). ■

As mentioned in section 2.1, it was recognized early that for interactions on the ordinary Minkowski space which are nonlocal in time, the Yang-Feldman approach yields out-going fields which will in general not satisfy the canonical commutation relations. It was then deduced that there is no unitary operator connecting the asymptotic in- and the out-fields. Concerning the asymptotic behaviour of the theory considered here, we will see in section 5.3, that it is to be seriously modified, and, therefore, that the question of the unitarity of the S-matrix has to be reconsidered on a more fundamental level. On the level of the interacting field, however, unitarity means that the field is Hermitean, which is satisfied in the above framework.

4.3 Contractions and microcausality

In this section the prerequisites are provided for stating the general rules to calculate the interacting field at any given order, and general properties of the resulting n-point functions are discussed. From the example considered in the preceding section, we expect planar as well as nonplanar graphs to appear, and the combinatorics will turn out to be quite complicated compared to the one used in field theory on the ordinary Minkowski space. In order to treat such graphs systematically, some notations and combinatorial prerequisites are needed.
Let J be an ordered subset of $\mathbb{N}$ with $2a$ elements, $J = (j_1, \ldots, j_{2a})$ with $j_l < j_k$ if $l < k$. Then an *ordered pairing of* J is a partition of J into a ordered pairs, i.e. into pairwise disjoint ordered subsets of two elements $(i_1, j_1), \ldots, (i_a, j_a)$ with $i_1 < i_2 < \cdots < i_a$ and $i_k < j_k$.
Let N be an ordered set of n elements, $N = (1, \ldots, n)$. An *ordered pairing in* N is an ordered pairing of a subset $J \subset N$. Such a pairing will be labelled by a tuple (A, α) where $A \subset J$ is an ordered subset of $J \subset N$ and α is an injective map $\alpha : A \to N$ with $i < \alpha(i)$ for $i \in A$, such that the ordered pairs are given by $(i, \alpha(i))$, $i \in A$, and $J = A \cup \alpha(A)$. The first element of the pair $(i, \alpha(i))$ is also used as a label for the whole pair. Small letters denote the number of elements of sets, i.e. $a = |A|$.

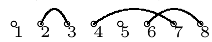
Definition 4.10 "Intersection and Enclosure Matrix" *Let (A, α) be an ordered pairing in N as above, and let U be the ordered subset $U = N \backslash (A \cup \alpha(A))$. The pair $(i, \alpha(i))$ encloses the index $l \in U$ if $i < l < \alpha(i)$. Two pairs intersect if $i < j < \alpha(i) < \alpha(j)$.*
The enclosure matrix E of the pairing is the $a \times u$-matrix with $E_{il} = 1$, if the pair $(i, \alpha(i))$ encloses the index l, and 0 otherwise ($i \in A, l \in U$). The intersection matrix I is defined as the upper diagonal $a \times a$-matrix with $I_{ij} = 1$ if the pairs $(i, \alpha(i))$ and $(j, \alpha(j))$ intersect and 0 otherwise ($i < j \in A$).
A pairing is called planar if both its intersection and enclosure matrix are trivial, i.e. if $I = E = 0$.

With the sign function $\epsilon(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$, the intersection and the enclosure matrix of a pairing (A, α) can be given explicitly as follows: $I_{ij} = 0$ for $i > j$, $E_{il} = 0$ for $i > l$, and

$$\begin{aligned} I_{ij} &= \tfrac{1}{2} \left(\epsilon(j-i) - \epsilon(\alpha(j)-i) - \epsilon(j-\alpha(i)) + \epsilon(\alpha(j)-\alpha(i)) \right) \qquad \text{for } i < j\,, \\ E_{il} &= \tfrac{1}{2} \left(\epsilon(l-i) - \epsilon(l-\alpha(i)) \right) \qquad \text{for } i < l\,. \end{aligned}$$

In terms of graphs, these definitions can be understood more easily as follows:

Remark 4.11 For the ordered set $N = (1, \dots, n)$ draw a number of n points in a horizontal line. Then, for fixed ordered pairing (A, α), connect each point $i \in A$ with its respective partner $\alpha(i)$ by a curve. Then two pairs intersect if and only if their connecting curves intersect, and a pair encloses an index $l \in N \backslash (A \cup \alpha(A))$ if and only if their connecting curve encloses it.

Example: Consider the pairing (A, α) in $N = (1, \dots, 8)$ where $A = (2, 4, 6)$ and $\alpha(A) = (3, 7, 8)$. The corresponding graph then is 

, and it allows to read off intersection and enclosure matrix: $I_{46} = 1$, $E_{45} = 1$, all others 0.

Using this notation, a Wick-ordered product of q-fields as given by equation (4.8) may be rewritten in terms of products of q-fields in the following convenient way.

Proposition 4.12 *Let $\phi_q^{\otimes n}$, $:\!\phi_q^{\otimes n}\!:$ denote the q-distributions as in (4.6) and (4.8). Then*

$$:\!\phi_q^{\otimes n}\!: \;=\; \phi_q^{\otimes n} + \sum_{(A,\alpha)} (-1)^{|A|} K_n^{(A,\alpha)}, \tag{4.10}$$

where the sum runs over all ordered pairings in N with $A \neq \emptyset$, and where $K_n^{(A,\alpha)}$ is a q-distribution with kernel

$$K_n^{(A,\alpha)}(q; x_N) = c \int dk_A dk_U \,\mathcal{I}(k_A)\,\mathcal{E}(k_A, k_U) \prod_{j \in A} i\,\hat{\Delta}_+(k_j)\, e^{-ik_j(x_j - x_{\alpha(j)})} \prod_{l \in U} \hat{\phi}(k_l)\, e^{-ik_l(q + x_l)},$$

where $c = (2\pi)^{-4(n-|A|)}$, $U = N \backslash (A \cup \alpha(A))$,

$$\mathcal{I}(k_A) \;=\; \exp\big(-i \sum_{\substack{i<j \\ i,j \in A}} I_{ij} k_i Q k_j\big) \quad \textit{and} \quad \mathcal{E}(k_A, k_U) \;=\; \exp\big(-i \sum_{\substack{i<l \\ i \in A, l \in U}} E_{il} k_i Q k_l\big)$$

with intersection matrix I and enclosure matrix E as in Definition 4.10.

Proof: Apply (1.15) to rewrite the normally ordered product $:\!\hat{\phi}(k_1) \dots \hat{\phi}(k_n)\!:$ which appears in (4.8) in terms of un-ordered products of fields. With the notations introduced in Definition 4.10, we find (4.10) with

$$\begin{aligned}(-1)^{|A|} K_n^{(A,\alpha)}(g) \;&=\; (2\pi)^{-4n} \int dx_N\, g(x_N) \int dk_N \,\exp\big(-i \sum_{j \in N} k_j(q + x_j)\big) \\ &\quad \cdot \exp\big(-\tfrac{i}{2} \sum_{\substack{i<j \\ i,j \in N}} k_i Q k_j\big)\, (2\pi)^{4|A|} \prod_{i \in A} (-i)\,\hat{\Delta}_+(k_i)\, \delta^{(4)}(k_i + k_{\alpha(i)}) \prod_{l \in U} \hat{\phi}(k_l).\end{aligned}$$

Let $\epsilon(r)$ again denote the sign of $r \in \mathbb{Z}$, and rewrite the twisting as

$$\sum_{\substack{i<j \\ i,j \in N}} k_i Q k_j \;=\; \tfrac{1}{2} \sum_{i,j \in N} \epsilon(j - i)\, k_i Q k_j\,.$$

Evaluating the δ-distributions, i.e. replacing $k_{\alpha(i)}$ by $-k_i$, we then obtain for the twisting

$$\begin{aligned}
&\sum_{\substack{i<j\\ i,j\in U}} k_iQk_j + \tfrac{1}{2}\sum_{i,j\in A}\big(\epsilon(j-i)-\epsilon(\alpha(j)-i)-\epsilon(j-\alpha(i))+\epsilon(\alpha(j)-\alpha(i))\big)\,k_iQk_j\\
&\quad+\tfrac{1}{2}\sum_{\substack{i\in U\\ j\in A}}(\epsilon(j-i)-\epsilon(\alpha(j)-i))\,k_iQk_j+\tfrac{1}{2}\sum_{\substack{j\in U\\ i\in A}}(\epsilon(j-i)-\epsilon(j-\alpha(i)))\,k_iQk_j\\
&\qquad=\sum_{\substack{i<j\\ i,j\in U}} k_iQk_j + 2\sum_{i<j\in A} I_{ij}\,k_iQk_j + 2\sum_{\substack{i<l\\ i\in A,l\in U}} E_{il}\,k_iQk_l\,,
\end{aligned}$$

and the claim follows. ■

The q-distribution $K_n^{(A,\alpha)}$ is referred to as a general contraction. It is uniquely determined by the ordered pairing (A,α) in N. Hence, a unique graph may be assigned to any contraction according to Remark 4.11, and the corresponding twisting can be read off directly. For example, the twisting of the contraction $K_8^{(A,\alpha)}$ where $A=(2,4,6)$ and $\alpha(A)=(3,7,8)$ is $\exp(-ik_4Qk_6-ik_4Qk_5)$. The momenta which are not contracted are referred to as external momenta, while the contracted fields' momenta are called internal momenta. A contraction is called planar if its pairing is planar (i.e. if $I=E=0$).

The graphs can be used to find all contractions which appear in (4.10). To that end, simply draw all possibilities to pairwise connect a number of $2a$ points out of n points drawn in a horizontal line, where a runs from 1 to the Gauss bracket $[n/2]=\max_{m\in\mathbb{N}}(m\leq n/2)$. Note that while these graphs bear some similarity with the ones introduced in section 2.3, the calculation of the resulting twisting is different. The reason that such graphs are convenient in both approaches is that they enable us to keep track of the position of the contracted fields in (tensor) products of fields.

A general contraction with $a=n/2$, where n is even, is a Q-distribution. This follows clearly from the proof of Proposition 4.12: if all fields are contracted, $U=\emptyset$, and there is no longer any dependence on q, while the twisting in such contractions may well be nontrivial.

Remark 4.13 The contractions with $a=n/2$, n even, constitute the vacuum expectation values of free fields,

$$W(Q;x_1,\dots,x_n)\stackrel{\text{def}}{=}\langle\phi(q+x_1)\cdots\phi(q+x_n)\rangle_0=\begin{cases}\sum\limits_{\substack{(A,\alpha)\\ a=n/2}}K_n^{(A,\alpha)}(Q;x_1,\dots,x_n) & \text{for } n \text{ even}\\[2ex] 0 & \text{for } n \text{ odd}\,,\end{cases}$$

where with the notation of Definition 4.6, $\langle\,\cdot\,\rangle_0=\mathbf{\Omega}\otimes\mathrm{id}$ with $\mathbf{\Omega}(\cdot)=\langle\Omega|\,\cdot\,|\Omega\rangle$, Ω being the ordinary Fock vacuum of the scalar free field. As in the ordinary case, vacuum expectation values of an odd number of fields vanish. Moreover, W is a translation-invariant Q-distribution which only depends on the differences of the coordinates x_i. In analogy with Wightman theory we will speak of n-point functions.

Proof: The claims are proved by the same method as used in the proof of Proposition 4.12. ■

Consider as an example the 4-point function

$$W(Q;x_1,\dots,x_4)=\langle\phi(q+x_1)\cdots\phi(q+x_4)\rangle_0\,,$$

which in terms of the graphs introduced in Remark 4.11 may be symbolized as the sum of all different possibilities to pairwise connect four points,

$$\langle \circ \ \circ \ \circ \ \circ \rangle_0 \quad = $$

From the above discussion it follows that the 4-point function has one non-planar and two planar contributions,

$$\begin{aligned} W(Q; x_1, \ldots, x_4) \;&=\; i^2 \Delta_+(x_1 - x_2)\, \Delta_+(x_3 - x_4) \,+\, i^2\, \Delta_+(x_1 - x_4)\, \Delta_+(x_2 - x_3) \\ &\quad + (2\pi)^{-6} \int \frac{d\,\mathbf{k}_1}{2\omega_{\mathbf{k}_1}} \frac{d\,\mathbf{k}_2}{2\omega_{\mathbf{k}_2}}\, e^{-ik_1\,(x_1 - x_3) - ik_2\,(x_2 - x_4)}\, e^{-ik_1 Q k_2}\,. \end{aligned}$$

In fact, the only n-point function (for n even) which is completely independent of Q is the 2-point function,

$$W(Q; x_1, x_2) = i\Delta_+(x_1 - x_2)\,.$$

All other n-point functions have some nonplanar contributions, since we sum over all possible pairings (A, α) of length $n/2$.
Let us collect some properties of the n-point functions which follow directly from the definition together with the classic results for fields on the ordinary Minkowski space (cf., for instance, [78]). The n-point functions behave as ordinary Wightman functions under complex conjugation,

$$\overline{W(Q; x_1, \ldots, x_n)} = W(Q; x_n, \ldots, x_1)\,,$$

and they are covariant under Poincaré transformations (with $\sigma \to \Lambda\sigma\Lambda^t$), since

$$U(\Lambda, a)\, \phi(q + x)\, U(\Lambda, a)^{-1} = \phi(\Lambda(q + x) + a)\,.$$

Moreover, the n-point functions are translation-invariant with respect to the commutative arguments x_i, and hence their Fourier transforms have support only in $\sum_{i=1}^n p_i = 0$. Furthermore, the spectral property holds by definition, since we treat only free fields.
After evaluation in a state ω, the n-point functions are moreover positive definite in the usual sense ($\phi^* = \phi$)

$$\begin{aligned} &\sum_{j,k} \int dx_1 \ldots dx_j\, dy_1 \ldots dy_k\; \overline{f_j(x_1, \ldots, x_j)}\, \cdot \\ &\qquad \cdot\, \omega\big(\langle \phi(q + x_j) \cdots \phi(q + x_1)\, \phi(q + y_1) \cdots \phi(q + y_k) \rangle_0 \,\big)\; f_k(y_1, \ldots, y_k) \geq 0\,, \end{aligned}$$

for any sequence of testfunctions $\{f_j\}$, $f_j \in \mathcal{S}(\mathbb{R}^{4j}, \mathcal{Z})$ with $f_j = 0$ except for a finite number of j. This is proved as in the ordinary case (see [78]) by using the fact that the inequality above is another way of saying that evaluation of the norm of the state

$$f_0\, |\Omega\rangle + \int dx_1\; \phi(q + x_1)\, f_1(x_1)\, |\Omega\rangle + \textstyle\int dx_1 dx_2\; \phi(q + x_1)\, \phi(q + x_2)\, f_2(x_1, x_2)\, |\Omega\rangle + \ldots$$

in ω yields a non-negative result.
But, apart from the lowest possible order, the n-point functions (for n even) are not local in the sense of Wightman theory. While for the 2-point function we find

$$W(Q; x_1, x_2) = i\Delta_+(x_1 - x_2) = i\Delta_+(x_2 - x_1) = W(Q; x_2, x_1) \quad \text{for } (x_1 - x_2)^2 < 0\,,$$

this ceases to be the case when more fields are involved,

$$W(Q; x_1, \ldots, x_i, x_j, \ldots, x_n) \neq W(Q; x_1, \ldots, x_j, x_i, \ldots, x_n) \quad \text{for } (x_i - x_j)^2 < 0\,,$$

as a short calculation for the 4-point function shows. It follows that the vacuum expectation value of the commutator of two q-fields at spacelike separated points vanishes,

$$\langle [\phi(q+x_1), \phi(q+x_2)] \rangle_0 = 0 \quad \text{for } (x_1-x_2)^2 < 0\,.$$

But contrary to the ordinary case, this is no longer true, if the vacuum expectation value is not taken: for the commutator itself we find

$$\begin{aligned}
[\phi(q+x_1), \phi(q+x_2)] \; &= \\
&= \int dk_1 dk_2 \, \big(\check{\phi}(k_1)\check{\phi}(k_2)\, e^{-\frac{i}{2}k_1 Q k_2} - \check{\phi}(k_2)\check{\phi}(k_1) e^{-\frac{i}{2}k_2 Q k_1} \,\big)\, e^{i(k_1+k_2)q + ik_1x_1 + ik_2x_2} \\
&= \int dk_1 dk_2 \, \check{\phi}(k_1)\check{\phi}(k_2) \,(e^{-\frac{i}{2}k_1 Q k_2} - e^{+\frac{i}{2}k_1 Q k_2} \,)\, e^{i(k_1+k_2)q + ik_1x_1 + ik_2x_2} \\
&\quad + \int dk_1 dk_2 \, [\check{\phi}(k_1), \check{\phi}(k_2)]\, e^{+\frac{i}{2}k_1 Q k_2}\, e^{i(k_1+k_2)q + ik_1x_1 + ik_2x_2}
\end{aligned}$$

which is in general not 0 even for $(x_1-x_2)^2 < 0$. Hence, microcausality in the ordinary sense is lost.

As an aside, it is mentioned here, that, since the two noncommutative structures (the fields on one side and the noncommutative spacetime on the other) are completely unrelated, one might think that the ordinary commutator of tensor products is not the correct notion to encode locality. For instance, considering $q_1^\mu = q^\mu \otimes I$ and $q_2^\mu = I \otimes q^\mu$ such that $[q_1, q_2] = 0$ as in chapter 3, we find that the commutator $[\phi(q_1+x_1), \phi(q_2+x_2)]$ indeed vanishes for $(x_1-x_2) < 0$. This may provide sufficient motivation for an attempt to produce local commutation relations by introducing a new commutator for q-fields. For example, one could consider

$$[\phi(q+x_1), \phi(q+x_2)]_1 \stackrel{\text{def}}{=} \int dk_1 dk_2 \, [\check{\phi}(k_1), \check{\phi}(k_2)]\, e^{i(k_1+k_2)q + ik_1x_1 + ik_2x_2}\, e^{-\frac{i}{2}k_1 Q k_2}$$

or

$$\begin{aligned}
[\phi(q+x_1), \phi(q+x_2)]_2 \;&\stackrel{\text{def}}{=}\; \int dk_1 dk_2 \, [\check{\phi}(k_1), \check{\phi}(k_2)]\, (e^{ik_1(q+x_1)} e^{ik_2(q+x_2)})_{sym} \\
&= \; \int dk_1 dk_2 \, [\check{\phi}(k_1), \check{\phi}(k_2)]\, e^{i(k_1+k_2)q + ik_1x_1 + ik_2x_2}\, (e^{-\frac{i}{2}k_1 Q k_2} + e^{-\frac{i}{2}k_1 Q k_2})
\end{aligned}$$

which both lead to local commutation relations for two free fields. Obviously, such a construction must be made consistent with the way the Wick ordering is defined, and possibly interesting results could be produced along these lines. The disadvantage is that both commutators violate the Jacobi identity, while $[\phi(q+x_1), \phi(q+x_2)]_1$ is not even antisymmetric. In any case, these prescriptions fail to produce local commutation relations when products of fields are considered.

Quite surprisingly the cluster decomposition property is still valid for free fields. While the proof for general Wightman fields uses locality (cf. [78], and references therein), it is not required when the vacuum expectation values of *free* fields are considered. This is a consequence of the fact that, in the case of free fields, the two-point functions are known to decrease exponentially for spacelike arguments.

Proposition 4.14 "Cluster decomposition property" *The cluster decomposition property holds for the vacuum expectation values of free q-fields: Let a be a spacelike vector, and $\lambda > 0$. Then*

$$W(Q; x_1, \ldots, x_j, x_{j+1} + \lambda a, \ldots, x_n + \lambda a) \to W(Q; x_1, \ldots, x_j)\, W(Q; x_{j+1}, \ldots, x_n)$$

for $\lambda \to \infty$ as distributions in $\mathcal{S}'(\mathbb{R}^{4n}, \mathcal{Z})$.

Proof: By definition, the n-point function (for n even) is a sum over all pairings in N, each labelled by a tuple (A, α) such that $N = A \cup \alpha(A)$. All pairings, where for all $i \in A$ both i and $\alpha(i)$ are either in $\{1, \ldots, j\}$ or in $\{j+1, \ldots, n\}$, are independent of λa. They constitute the right-hand side of the above equation. All other pairings, i.e. those with at least one pair $(i, \alpha(i))$ such that $i \in \{1, \ldots, j\}$ and $\alpha(i) \in \{j+1, \ldots, n\}$, yield a dependence on λa such that these terms vanish for $\lambda \to \infty$. To see this, consider such a contraction and evaluate it in a testfunction g which is assumed to be symmetric in its arguments. The contraction then is of the form

$$\int dx_1 \ldots dx_n \int \frac{d\,\mathbf{k}_A}{2\omega_{\mathbf{k}_A}} \prod e^{-ik_i(x_i - x_{\alpha(i)} + \lambda_i a)} \; e^{-\frac{i}{2}\sum I_{ij} k_i Q k_j} \, g(x_1, \ldots, x_n) \,,$$

where $\lambda_i = 0$, if both i and $\alpha(i)$ are either in $\{1, \ldots, j\}$ or in $\{j+1, \ldots, n\}$, and $\lambda_i = \lambda$ otherwise (and where, by assumption, $\lambda_i = \lambda$ for at least one $i \in A$). Performing the integrations over $x_1, \ldots, x_n$ yields

$$\int \frac{d\,\mathbf{k}_A}{2\omega_{\mathbf{k}_A}} \prod e^{-ik_i \lambda_i a} \; e^{-\frac{i}{2}\sum I_{ij} k_i Q k_j} \, \check{g}(k_1, \ldots, k_n) \,.$$

Now define $\check{h}(k_1, \ldots, k_n) = e^{-\frac{i}{2}\sum I_{ij} k_i Q k_j} \, \check{g}(k_1, \ldots, k_n)$, which is again a testfunction, then the claim follows since the 2-point functions decrease exponentially for spacelike argument. ■

The physical significance of the cluster decomposition property for general fields (not necessarily free fields) is that, if two systems are separated by a very large spacelike distance, the interaction between them decreases quickly (in fact, it decreases exponentially, $e^{-\lambda m}$, in the ordinary massive case and at least as λ^{-2} in massless theories). The locality condition being violated, the cluster decomposition property probably ceases to be true for vacuum expectation values of interacting fields. In a way, the above proposition may be seen as a consequence of the fact that the free field remains the same, and only in the interaction the effects of the noncommutative structure of spacetime become apparent.
A different approach to the problem of locality which works for theories with *fixed* space-space-noncommutativity matrix θ has been pursued in [4]. Lorentz covariance being broken in such a setup, one takes the largest symmetry group which preserves the noncommutative structure, $O(1,1) \times SO(2)$, and defines microcausality with respect to the group $O(1,1)$ (light-wedge causality).

4.4 General rules

In this section, the general rules to calculate the interacting field at any given order are presented. As a prerequisite we need Wick's theorem for ordinary normally ordered products on the noncommutative Minkowski space.

Remark 4.15 Proposition 4.12 allows to rewrite Wick's theorem (1.16) for n- and m-fold tensor products of q-fields in the following compact notation:

$$:\phi^{\otimes n}::\phi^{\otimes m}: \quad = \quad :\phi^{\otimes(n+m)}: \quad + \sum_{\substack{(A,\alpha) \text{ with} \\ \text{property } (N|M)}} :K_{n+m}^{(A,\alpha)}: \,, \tag{4.11}$$

where property $(N|M)$ indicates that the sum runs over all ordered pairings in $N \cup M$ such that $\emptyset \neq A \subset N$, $\alpha(A) \subset M$. In $:K_{n+m}^{(A,\alpha)}:$ all uncontracted fields are normally ordered with respect to each other.

For fixed $a = |A|$, there are $\frac{n!\,m!}{(n-a)!\,(m-a)!\,a!}$ ordered pairings satisfying property $(N|M)$, since this is the number of possibilities to connect a elements of the ordered set N with a elements of the ordered set M. Hence, the total number of terms in the sum in (4.11) is

$$\sum_{a=1}^{\min(m,n)} \frac{n!\,m!}{(n-a)!\,(m-a)!a!}\,.$$

The contractions $K_{n+m}^{(A,\alpha)}$ with (A,α) having property $(N|M)$ can be symbolized by the graphs introduced in Remark 4.11 in the following way: draw $n+m$ points in a horizontal line, separated by a small dash, then every contraction K with property $(N|M)$ corresponds to one of the possibilities to connect a points on the left-hand side with a points on the right-hand side. The twisting can then be read off as in Remark 4.11.
The combinatorics of Wick's theorem on the noncommutative Minkowski space as given by equation (4.11) can also be applied, if some of the arguments in the Wick products coincide, although, strictly speaking, the Wick product with coinciding arguments,

$$:\phi(q+x_1)\dots\phi(q+x_i)\dots\phi(q+x_i)\dots\phi(q+x_n):\,,$$

is no longer a tensor product $\phi_q^{\otimes n}$. Hence, the interacting field at order κ is calculated from formula (4.9) as follows. Start at the first order. Here, the fields are normally ordered by definition of the interaction term, $\phi_1 = (g\Delta_{ret}) \times :\phi^{n-1}:$. Proceed to second order by inserting ϕ_1 at the appropriate position in

$$\phi_2(q) = \int dy\; g(y)\,G(y) \sum_{l=1}^{n-1} (\phi\cdots \overset{l-th}{\phi_1} \cdots\phi)(q-y)\,.$$

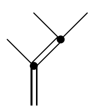
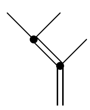
Bring the result into normal order by application of (4.11). Proceed to third order ϕ_3 by inserting ϕ_1 as well as the expression found for ϕ_2 (which is now normally ordered) at the appropriate positions according to formula (4.9). Proceed in this way to order κ.
To write down directly all expressions which appear at a given order κ one may also apply the graphical rules given below. For illustration, every step is followed by an example.

1. At order κ of the perturbation theory draw all

$$\frac{(\kappa\,(n-1))\,!}{(\kappa\,(n-2)+1)\,!\,\kappa\,!}$$

rooted tree graphs which occur according to Remark 4.4. The lines in these graphs which connect the vertices are called connecting lines and are symbolized by double lines. According to Remark 4.3, each tree graph possesses $\kappa(n-2)+1$ endpoints, each of which symbolizes a field ϕ.

Example:

 +  .

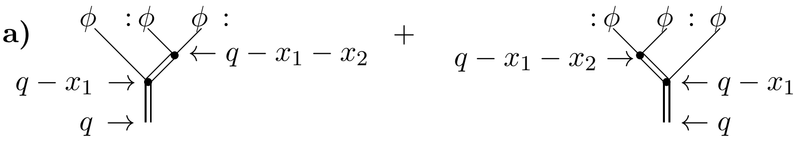
2. Starting from the root, label the first inner vertex as $q - x_1$. The inner vertices which are connected to the vertex $q-x_1$ are then labelled $q-x_1-x_2$, $q-x_1-x_3$ through $q-x_1-x_m$. Now consider one of these vertices, say $q - x_1 - x_3$ and again label all vertices which are

connected to it $q - x_1 - x_3 - x_{m+1}$, $q - x_1 - x_3 - x_{m+2}$ through $q - x_1 - x_3 - x_l$. Proceed in this way until all inner vertices are labelled.

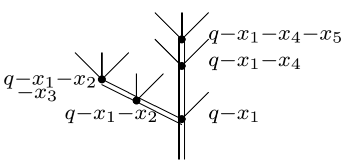
The argument of each field ϕ is then given by the label of the inner vertex from which the field-line starts. The order in which they appear is read off from left to right. Specifically, one can proceed as follows: draw a horizontal line above the graph and elongate the field-lines in such a manner that they do not intersect until they reach this line. Then read off the arguments from left to right.

If a normally ordered interaction is employed, where $\phi_1 = (g\,G) \times :\phi^{n-1}:$, fields which start at the same inner vertex are normally ordered with respect to each other, if the inner vertex is an extremal one, i.e., if the vertex is connected to the graph by only *one* connecting line. None of the other fields are normally ordered with respect to each other.

Example: **a)**



b)

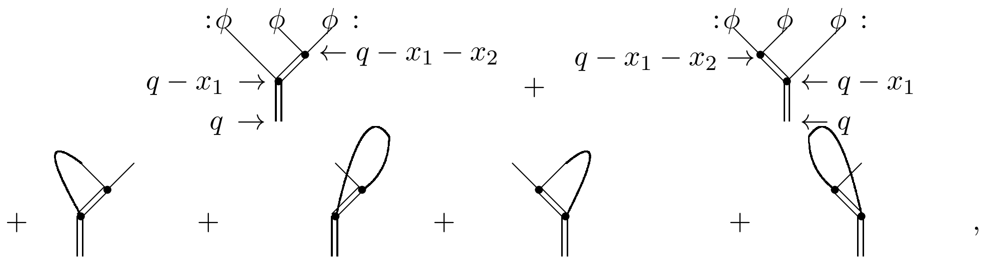


$$\rightarrow \; :\phi(q - x_1 - x_2 - x_3)^3 : \phi(q - x_1 - x_2)^2 \, \phi(q - x_1 - x_4)$$
$$:\phi(q - x_1 - x_4 - x_5)^3 : \phi(q - x_1 - x_4)\, \phi(q - x_1)$$

Here, only the fields starting from the vertices $q - x_1 - x_2 - x_3$ and $q - x_1 - x_4 - x_5$ are normally ordered with respect to each other.

3. In each tree graph a number of contractions $:K^{(A,\alpha)}_{\kappa(n-2)+1}:$ arises when the product of the $\kappa(n-2)+1$ fields is brought into normal order by application of Wick's theorem (4.11). Each of the contractions is symbolized by one of the possibilities to pairwise connect a certain number of the fields with each other such that no connections between fields starting at the same extremal vertex occur.

Example: From **a)** above we derive



where the second line corresponds to the contractions

$$\sum_{i=1}^{2} K^{(A_i,\alpha_i)}(q; -x_1, -x_1 - x_2, -x_1 - x_2) + \sum_{i=3}^{4} K^{(A_i,\alpha_i)}(q; -x_1 - x_2, -x_1 - x_2, -x_1)$$

with $A_1 = (1), \alpha_1(1) = 2\,, \quad A_2 = (1), \alpha_2(1) = 3\,, \quad A_3 = (2), \alpha_3(2) = 3\,, \quad A_4 = (1), \alpha_4(1) = 2\,.$

4. The analytic expression for each of the graphs thus arising, is then given as follows:

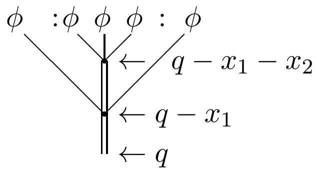
First write down the integrals over the inner vertices as well as the retarded propagators symbolized by the connecting lines

$$\int dx_1 \dots dx_\kappa \, g(x_1)\, \Delta_{ret}(x_1) \dots \, g(x_\kappa)\, \Delta_{ret}(x_\kappa)\,.$$

Then for the graph under consideration write down the appropriate contraction term $:K^{(A,\alpha)}_{\kappa(n-2)+1}:$ or the Wick-ordered product $:\phi(q-\sum_{I_1} x_i)\dots\phi(q-\sum_{I_l} x_i):$, respectively, whose arguments are read off from the graph from left to right according to item 2. in this list. The contraction's intersection and enclosure matrix may then be read off from the graph according to Remark 4.11.

Example: Applycation of the rule above to the graphs in the second line of the preceding example reproduces the analytic expressions explicitly calculated on page 68.

As a more complicated example consider the following second order contribution in ϕ^4-theory,



Here, only the three fields starting at $q - x_1 - x_2$ are normally ordered with respect to each other and the contribution to ϕ_2 before applying Wick's theorem is

$$\int dx_1 dx_2 \, g(x_1)\, g(x_2)\, \Delta_{ret}(x_1)\, \Delta_{ret}(x_2)\, \phi(q-x_1) :\phi(q-x_1-x_2)^3: \phi(q-x_1)\,.$$

The contractions which arise from normal ordering are

$$\sum_{\substack{(A,\alpha)' \\ A\neq\emptyset}} :K_5^{(A,\alpha)}(q; -x_1, -x_1-x_2, -x_1-x_2, -x_1-x_2, -x_1):\,,$$

where $'$ indicates that (A,α) must be chosen according to the fact that the fields in second, third and fourth position are already Wick ordered with respect to each other such that no contractions among the second, the third and the fourth index occur. Therefore, the sum runs over the following pairings (A,α):

A	$\alpha(A)$	
(1)	$(2),(3),(4),(5)$	
(2)	(5)	
(3)	(5)	
(4)	(5)	
$(1,2)$	$(3,5),(4,5)$	
$(1,3)$	$(2,5),(4,5)$	
$(1,4)$	$(3,5),(2,5)$	

The graph theory according to Remark 4.11 allows to read off the intersection and enclosure matrices of the above contractions. Summing all terms under the integral

$$\int dx_1 dx_2 \, g(x_1)\, g(x_2)\, \Delta_{ret}(x_1)\, \Delta_{ret}(x_2)$$

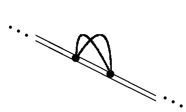
gives the full contribution.
The contraction in the above example with $A = (1)$ and $\alpha(1) = 5$ is a tadpole contribution. Matters such as whether it has to be subtracted or not will occupy us in the next chapter. In ϕ^3-theory such tadpole contributions which enclose fields starting at another vertex do not occur. Only two lines start at every vertex, and, hence, either both of them are fields, in which case the vertex is an extremal one and the fields are normally ordered with respect to each other, or one or both of the lines is a connecting line Δ_{ret}, in which case normal ordering is not necessary. Note that if one starts from an interaction which is not normally ordered, the application of Wick's theorem (4.11) also yields contraction terms with tadpole contractions between fields starting from the same extremal inner vertex.
Let us now collect some properties of the graphs which appear in the perturbative expansion. First we note that after application of Wick's theorem any graph will be of the following general form: it has at least one external leg (the root). A tree (indicated by double lines) is embedded in the graph in such a manner that every inner vertex is reached by the branches of the tree, without branches forming loops. By construction, any graph always appears together with its mirror image at a given order perturbation theory (unless it is mirror-symmetric itself). Moreover, the following claim holds:

Remark 4.16 Let

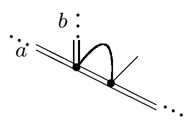
$$K^{(A,\alpha)}_{\kappa(n-2)+1}(q;\ \underbrace{-\textstyle\sum_{I_1} x_i}_{=:\, y_1},\ \ldots\ ,\ \underbrace{-\textstyle\sum_{I_{\kappa(n-2)+1}} x_i}_{=:\, y_{\kappa(n-2)+1}})$$

be one of the contractions resulting from the application of Wick's theorem (4.11) at order κ in ϕ^4-theory. Let $A_1 \subset A$ be the set which labels those contractions of two fields within K connecting two neighbouring vertices (i.e. vertices which are connected by one connecting line Δ_{ret}) and intersecting only other contractions from $(A_1, \alpha\restriction_{A_1})$. Then there also is a contraction arising from Wick's theorem which differs from K only in that for $i \in A_1$ the exponentials $e^{-ik_i(y_i - y_{\alpha(i)})}$ appear with the opposite sign, $e^{-ik_i(-y_i + y_{\alpha(i)})}$.

Proof: Consider ϕ^4-theory. Two generic contractions satisfying the assumptions are



and



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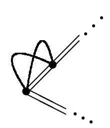
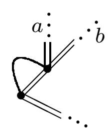
Since all possible trees appear in the perturbative expansion, there will also be tree graphs which have the following contractions in place of the ones above, but otherwise coincide with them:



and



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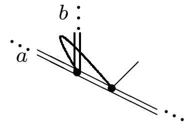
Their analytic expressions are identical apart from the sign in the exponential. ∎

The claim made in Remark 4.16 is not valid in general for contractions of neighbouring fields which intersect others or enclose other fields. This point is illustrated by the following example



for which we cannot find a tree graph with a contraction as in Remark 4.16.

Consider now a contraction K with contractions of two fields which satisfy the condition of the above remark. If they do not intersect each other (by definition, they do not intersect any other lines), they yield a product of propagators $\Delta_+(y_i - y_{\alpha(i)})$, $i \in A_1$. The sum of this contraction K and the corresponding contraction K' with $\Delta_+(-y_i + y_{\alpha(i)}) = -\Delta_-(y_i - y_{\alpha(i)})$ (and everything else coinciding) thus yields the propagator

$$\Delta^{(1)} = \Delta_+ - \Delta_- \, .$$

The planar graphs at second order discussed in section 4.2 satisfy the conditions of Remark 4.16, and, hence, as was calculated there explicitly, this propagator did appear. On the ordinary Minkowski space, where no intersections occur at all, all graphs can be rewritten such that only the two propagators Δ_{ret} and $i\,\Delta^{(1)}$ appear. For this reason, the graphs in the Yang-Feldman approach have been called "Dyson's double graphs" [32]. In this case, the identity $\Delta_{ret}\,(i\Delta_+ - i\Delta_-) = i\,\Delta_F^2 - i\Delta_-^2$ already employed in section 4.2, thus allows one to absorb the time-ordering appearing by means of the retarded propagators in any graph into Feynman propagators.

Last but not least, it is now shown that if the theory analysed in this chapter is interpreted as a theory on the ordinary Minkowski space with a nonlocal interaction, the interacting field as well as all expectation values differ from the ones calculated in the Hamiltonian formalism in chapter 2.

There are different ways to define the time-ordered vacuum expectation values of the interacting field (4.2). One possibility is to consider

$$\langle\Omega|\, T\big(\phi_{int}(q + x_1) \dots \phi_{int}(q + x_n)\big)\, |\Omega\rangle$$

and to evaluate this in a state ω. Here, T is the time-ordering with respect to the variables $x_{1,0}, \dots, x_{n,0}$. Another possibility is to consider

$$\langle\Omega|\, T\big(\omega_{x_1}(\phi_{int}) \dots \omega_{x_n}(\phi_{int})\big)\, |\Omega\rangle$$

with best-localized states ω_{x_i} centered at x_i, $\omega_x(e^{ikq}) = e^{-\frac{1}{2}|k|^2}\, e^{-ikx}$, where the time ordering is defined with respect to the variables $x_{1,0}, \dots, x_{n,0}$. One further possibility is given by

$$\langle\Omega|\, T\big(\phi_{int}(q_1) \dots \phi_{int}(q_n)\big)\, |\Omega\rangle \, , \tag{4.12}$$

where the q_i are mutually commuting quantum variables (i.e. we consider n-fold tensor products as in (3.3)) such that the time ordering T with respect to $q_1, \dots, q_n$ is well-defined. Then one still has to evaluate the time-ordered expectation values in suitable states. In all these proposals, the dependence on Σ could be treated by integrating over Σ_1 (keeping only rotation and translation invariance). It will be interesting to investigate consequences of the different definitions and to fully understand their physical interpretation.

For fixed noncommutativity matrix, the last of the above proposals yields a perturbative expansion in the spirit of the programme pursued in the context of theories with nonlocal interaction on the ordinary Minkowski space [60]. Understood as an effective theory on the ordinary Minkowski space, the time-ordered expectation values of this approach are

$$\langle\Omega|\, T\big(\phi_{int}(x_1) \dots \phi_{int}(x_n)\big)\, |\Omega\rangle \tag{4.13}$$

with

$$\phi_{int}(x) = \phi_{in}(x) + \sum_{\kappa=1}^{\infty} g^{\kappa} \sum_{\sum \kappa_i = \kappa - 1} \Delta_{ret} \times (\phi_{\kappa_1} \star \dots \star \phi_{\kappa_{n-1}})\,(x) \, . \tag{4.14}$$

Here, the infrared-cutoff is removed and the switching function g is replaced by a constant.

Remark 4.17 The interacting field $\phi^H_{int}(x)$ defined in the Hamiltonian approach by (2.11) does not coincide with the interacting field defined by (4.14) unless $\theta^{0i} = 0$ (or rather unless there is a timelike 4-vector n_μ with $n_\mu \theta^{\mu\nu} = 0$). Moreover, their time-ordered expectation values are not the same unless there is a timelike 4-vector n_μ with $n_\mu \theta^{\mu\nu} = 0$.

Proof: Consider ϕ^3-theory as an example. To match the conventions, absorb the factorial 1/3! into the Hamiltonian's coupling constant. Already at first order in the coupling constant, the two approaches yield different results:

$$\begin{aligned}
\phi^H_1(x) &= -i\,(2\pi)^{-11} \int dy\, \theta(x_0 - y_0) \int dk_1 dk_2 \int \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} :\hat{\phi}(k_1)\hat{\phi}(k_2): e^{-i(k_1+k_2)y} \\
&\qquad \cdot \left(e^{-ip(x-y)}\, \mathfrak{S}(\theta; k_1, k_2, -p) - e^{+ip(x-y)}\, \mathfrak{S}(\theta; k_1, k_2, +p)\right) \\
&\neq \int dy\, \theta(x_0 - y_0) \left(\Delta_+(x-y) - \Delta_+(y-x)\right) :\phi \star \phi(y):
\end{aligned}$$

unless $\theta^{0i} = 0$ (or rather unless there is a timelike 4-vector n_μ with $n_\mu \theta^{\mu\nu} = 0$), in which case $\int d\mathbf{y}$ is a (partial) trace with the property

$$\int d\mathbf{y}\, (\Delta(x-y) \star \phi(y) \star \phi(y))_{sym} = \int d\mathbf{y}\, \Delta(x-y) \cdot (\phi \star \phi)(y)\,.$$

To see that neither the time-ordered vacuum expectation values calculated from the two approaches coincide, note that

$$\begin{aligned}
&\langle\Omega|\, T(\phi^H_1(x)\phi^H_1(y))\, |\Omega\rangle = \\
&= (-i)^2\, \theta(x_0 - y_0) \int dx_1\, \theta(x_0 - x_{1,0}) \int dy_1\, \theta(y_0 - y_{1,0}) \,\cdot \\
&\quad \cdot \langle\Omega| :\left(\Delta(x - x_1) \star \phi(x_1) \star \phi(x_1)\right)_{sym} :: \left(\Delta(y - y_1) \star \phi(y_1) \star \phi(y_1)\right)_{sym} : |\Omega\rangle \\
&\quad + (x \leftrightarrow y)
\end{aligned}$$

is not the same as

$$\begin{aligned}
&\langle\Omega|\, T(\phi_1(x)\phi_1(y))\, |\Omega\rangle = \\
&= (-i)^2\, \theta(x_0 - y_0) \int dx_1\, \theta(x_0 - x_{1,0})\, \Delta(x - x_1) \int dy_1\, \theta(y_0 - y_{1,0})\, \Delta(y - y_1) \,\cdot \\
&\quad \cdot \langle\Omega| :\phi(x_1) \star \phi(x_1) :: \phi(y_1) \star \phi(y_1) : |\Omega\rangle \\
&\quad + (x \leftrightarrow y)
\end{aligned}$$

unless $\theta^{0i} = 0$. ■

This result should not come as a surprise, since it was already shown in section 2.1 that the Hamiltonian interacting field does not satisfy the ordinary equation of motion for general time-space-noncommutativity.
From Remark 4.17 it follows in particular, that the Yang-Feldman approach and the modified Feynman rules coincide in the case of space-space noncommutativity.

Chapter 5

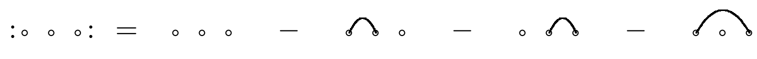
Quasiplanar Wick products

In the preceding chapter, ordinary normal ordering was used in defining products of fields. In other words, *all* tadpoles were subtracted from products as counterterms. The question which remains yet to be answered is whether there is a notion of locality of counterterms suitable in the Yang-Feldman approach on the noncommutative Minkowski space, and if so, whether subtracting only such local counterterms renders well-defined products of q-fields. As this chapter's analysis will show, this is indeed feasible; the crucial property being a notion of so-called q-locality. This is a weaker version of the locality property ordinarily demanded of counterterms, which does comply with one of the crucial requests usually made regarding support properties. The q-local subtractions then turn out to be those whose enclosure matrix vanishes, a property which is called quasiplanarity. Here, the prefix "quasi" indicates that the intersection matrix of q-local subtractions is in general nontrivial.[1] After proving that the subtractions which are not q-local remain finite in the limit of coinciding points, a definition of the so-called quasiplanar Wick products is given, where only quasiplanar subtractions are admitted. Furthermore, the corresponding Wick theorem is formulated.

Let us first consider some examples in order to become familiar with the kind of subtractions and terms we will be dealing with. Consider a three-fold product of q-fields. Then from (4.10) it follows that the normally ordered product is equal to

$$
\begin{aligned}
:\phi(q+x_1)\phi(q+x_2)\phi(q+x_3): \;=\;& \phi(q+x_1)\phi(q+x_2)\phi(q+x_3) \\
&- i\,\Delta_+(x_1-x_2)\,\phi(q+x_3) - i\,\Delta_+(x_2-x_3)\,\phi(q+x_1) \qquad (5.1)\\
&- i(2\pi)^{-8}\int dk_1 dk_2\,\hat{\Delta}_+(k_1)\,e^{-ik_1Qk_2}\,e^{ik_1(x_1-x_3)}\,\hat{\phi}(k_2)\,e^{-ik_2(q+x_2)}\,, \qquad (5.2)
\end{aligned}
$$

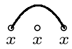
which in terms of the graphs introduced in the preceding section can be symbolized as



In the limit of coinciding points defined as in Definition 4.8, where the above q-distribution is evaluated in a sequence of compactly supported testfunctions approaching $\delta(x-x_1)\,\delta(x-x_2)\,\delta(x-x_3)$, the monomial of q-fields as well as the two subtraction terms in (5.1) yield ill-defined expressions as in the ordinary case (for the subtractions this is obvious, for the field monomial see page 67). In contrast to this, the term (5.2), which has a non-vanishing enclosure

[1]It would be preferable to say also "quasilocal" instead of q-local in order to avoid the inflationary use of the prefix q in contemporary physics, but quasilocality is already used as a technical term to denote a different property. Now, the term q-locality is used here to indicate that the definition is suitable for q-distributions, a term which serves as an abbreviation for "distribution taking values in operators affiliated to $\mathcal{F}\otimes\mathcal{E}$".

matrix, yields the following expression in the limit of coinciding points


$$= (2\pi)^{-4}\, i \int dk\, \Delta_+(Qk)\, \hat{\phi}(k)\, e^{-ik(q+x)}\,,$$

which turns out to be a well-defined q-distribution. To see this, recall that for the (free) q-field the momentum integration only runs over the (positive and negative) mass-shell, such that Qk is spacelike. Hence, $\Delta_+(Qk)$ is a smooth bounded function of k. Smearing with a testfunction g,

$$\int dx\, g(x) \int dk\, \Delta_+(Qk)\, \hat{\phi}(k)\, e^{-ik(q+x)} = \int dk\, \Delta_+(Qk)\, \hat{g}(-k)\, \hat{\phi}(k)\, e^{-ikq}\,,$$

and evaluating in a state on $\mathcal{E}$ then yields a well-defined operator, since the evaluation in a state ω on $\tilde{\mathcal{E}}$ yields the following expression

$$\Delta_+(Qk)\, \hat{g}(-k)\, \omega(e^{ikq}) \;=\; \Delta_+(Qk)\, \hat{g}(-k)\, \psi_\omega(k)\,,$$

which is square-integrable on the mass-shell (and even smooth if $\hat{g}$ and ψ_ω are smooth). Note that $\Delta_+(Qk)$ alone would not suffice to render the momentum integration well-defined, since it does not decrease quickly in all directions. To see this, consider the frame of reference where $Q = \sigma^{(0)}$. Then for k on the mass-shell, we find $(Qk)^2 = -m^2 - 2k_\perp$, where $k_\perp = (k_1, k_3)$. Since Δ_+ depends only of the Lorentz square of its argument (and the sign of the zero-component, unless the argument is spacelike, as is the case here), it follows that $\Delta_+(Qk)$ is constant as a function of k_2 and decreases exponentially in the perpendicular directions $k_\perp$. Hence, we may conclude that the tadpole (5.2), if subtracted, merely produces a finite mass renormalization.

As a more complicated example consider now the product of five q-fields. One of the subtractions which appear by application of Wick's theorem (4.10) can be symbolized by . In the limit of coinciding points, we find explicitly (up to numerical factors)


$$\propto \int dk \int \frac{d\mathbf{p}}{2\omega_\mathbf{p}}\, \Delta_+(Qp)\, e^{-ipQk}\, \hat{\phi}(k)\, e^{-ik(q+x)}\,.$$

The integration over $\mathbf{p}$ is well-defined by the same argument as employed above, but it is not immediately clear that the resulting function is a bounded function of $k \in H_m^\pm$ (or at least that it does not increase faster than a polynomial). To see that all is well consider again the frame of reference where $Q = \sigma^{(0)}$, such that, for p on the mass-shell, $\Delta_+(Qp) = h(p_\perp)$ is a constant function of p_2 which decreases exponentially in the perpendicular directions $p_\perp$. Then in the term under consideration,

$$\int dp_\perp\, h(p_\perp)\, e^{+ip_\perp \tilde{k}_\perp} \int dp_2\, \frac{1}{\sqrt{m^2+p_\perp^2+p_2^2}}\, e^{-i\sqrt{m^2+p_\perp^2+p_2^2}\,\tilde{k}_0 + ip_2 \tilde{k}_2} \qquad \text{with } \tilde{k} = Qk\,,$$

the integration over the second component of p yields the two-dimensional $\Delta_+^{2\mathrm{D}}$-propagator of mass $m^2 + p_\perp^2$. However, since k is on the mass-shell, $(\tilde{k}_0, \tilde{k}_2)^2 = -m^2$ is spacelike and therefore $\Delta_+^{2\mathrm{D}}$ is a bounded function of k. The integration over the perpendicular directions $p_\perp$ then yields a bounded function of k, since the product of h and $\Delta_+^{2\mathrm{D}}$ is integrable with respect to $p_\perp$. As before, boundedness in k suffices, since the q-field is still to be evaluated in a testfunction g and a suitable state ω.

The last example to be discussed here will illustrate that the structure of divergences in the Euclidean approach and the one on Minkowski space are not compatible and that a non-vanishing intersection matrix alone is not enough to render a tadpole finite. Consider the contraction  which also appears when Wick's theorem (4.10) is applied to a product of five q-fields and whose explicit form in the limit of coinciding points is

$$\int \frac{d\mathbf{p}}{2\omega_\mathbf{p}}\, \Delta_+(Qp)\, \phi(q+x)\,.$$

Contrary to the terms discussed above, there is no enclosure matrix. Now, by the same argument as employed before, we conclude that $\Delta_+(Qp)$ is a bounded function of $p \in H_m^+$, but that it is not integrable. Instead, in the frame of reference where $Q = \sigma^{(0)}$, the integration over p_2 diverges logarithmically. This means that results on renormalization gained in a Euclidean theory (see for instance [18]) cannot be applied to the Minkowskian regime: in a Euclidean regime, the above tadpole would take the following form,

$$\int dp \int dk \, \frac{1}{k^2+m^2} \, \frac{1}{p^2+m^2} \, e^{-ipQk} = c \int_0^\infty d\alpha d\beta \, e^{-(\alpha+\beta)m^2} \, \frac{1}{(\alpha\beta + \frac{\theta^2}{4})^2} \,,$$

where Q is assumed to have maximal rank and, without loss of generality, $(Qp)^2 = \theta^2(p_2^2 + p_1^2 + p_0^2 + p_3^2)$. As usual, the integrands fall off quickly at infinity, but contrary to the ordinary case, they are well-defined also for $\alpha, \beta = 0$, and the integration yields a finite result.

5.1 q-Locality

In chapter 3, a concept of locality was applied, where the ordinary limit of coinciding points is replaced by the approximate limit of coinciding points. This provided an appropriate generalization of the ordinary local interaction term $:\phi^n(x):$ suitable for the noncommutative Minkowski space $\mathcal{E}$. Here, a different line of thought is pursued: a notion of so-called q-local q-distributions is introduced, which will enable us to discriminate between admissible and non-admissible counterterms. This notion will furthermore allow to *a posteriori* justify the choice of an interaction term of the form $\phi^n(q)$.

In a very general sense, we may say that a map acting on a function is to be considered local, if it does not increase the function's support. It is not difficult to make this definition precise also for functions which take values in $\mathcal{Z}$. As mentioned before, such a definition comprises the case of ordinary complex-valued functions, since $I \in \mathcal{Z}$.

Definition 5.1 *A map $\gamma : \mathcal{S}(\mathbb{R}^{4n}, \mathcal{Z}) \to \mathcal{S}(\mathbb{R}^{4u}, \mathcal{Z})$, $0 \le u \le n$, is local, if it is linear and continuous and if the support of $\gamma(g)$ is canonically embedded in that of g,*

$$\operatorname{supp} \gamma(g) \subset \bigcup_{\substack{U \subset N \\ |U|=u}} P_U \operatorname{supp} g \,, \tag{5.3}$$

where $P_U : \mathbb{R}^{4n} \to \mathbb{R}^{4u}$ is the projection map $P_U(x_N) = x_U$, $N = \{1, \dots, n\}$.

Consider again the recursive definition of Wick products on the ordinary Minkowski space (3.1) and recall that any term v appearing on the right-hand side of the equation is a distribution of the following form, when evaluated in a testfunction g,

$$\langle v, g \rangle = \int dx_N \, g(x_{\pi_1}, \dots, x_{\pi_n}) \, K(x_{u+1}, \dots, x_n) \, \phi(x_1) \dots \phi(x_u) \tag{5.4}$$

for some permutation $\pi \in S_n$. The translation-invariant distributions K are also called the coefficient functions. Obviously, the map γ, where

$$\gamma(g)(x_1, \dots, x_u) = \int dx_{u+1} \dots dx_n \, g(x_{\pi_1}, \dots, x_{\pi_n}) \, K(x_{u+1}, \dots, x_n) \,,$$

is local in the above sense. For this reason, we may call v a local counterterm, since

$$\langle v, g \rangle = \langle \phi^{\otimes u} \circ \gamma, g \rangle = \langle \phi^{\otimes u}, \gamma(g) \rangle \,.$$

Note that a local map γ respects the limit of coinciding points: for a sequence of testfunctions $g_r(x_N)$ approaching $\delta(x-x_1)\ldots\delta(x-x_n)$ with $\operatorname{supp} g_{r+1} \subset \operatorname{supp} g_r$ and $\cap \operatorname{supp} g_r = \{(x,\ldots,x)\}$, we find $\operatorname{supp}\gamma(g_{r+1}) \subset \operatorname{supp}\gamma(g_r)$ and $\cap\operatorname{supp}\gamma(g_r) = \{(x,\ldots,x)\} \subset \mathbb{R}^{4u}$. It turns out that this concept of locality can be directly applied to q-distributions.

Definition 5.2 "q-Locality" *A q-distribution $v(q;x_1,\ldots,x_n)$ associated to ϕ is q-local of order $u \in \mathbb{N}$, if there is a local map $\gamma : \mathcal{S}(\mathbb{R}^{4n},\mathcal{Z}) \to \mathcal{S}(\mathbb{R}^{4u},\mathcal{Z})$ such that $v = \phi^{\otimes u}\circ\gamma$,*

$$\langle v, g\rangle = \langle \phi^{\otimes u}, \gamma(g)\rangle\,.$$

More generally, a linear combination of q-local q-distributions (possibly of different orders) is called q-local. For $0 < u \leq n$ we will also speak of a q-local product of u q-fields.

The slightly unusual index u is employed above as it labels the uncontracted fields.
To put Definition 5.2 in other words, a q-local distribution v is a product of $\mathcal{Z}$-valued coefficient functions H and q-fields,

$$\int dx_N\, g(x_N)\, v(q;x_1,\ldots,x_n) = \int dx_N\, g(x_N)\, H(Q; x_{N\setminus U}) \prod_{l\in U} \phi(q+x_l)\,,$$

where

$$\int dx_{N\setminus U}\, g(x_N)\, H(Q;x_{N\setminus U}) = (\gamma(g))(x_U)\,.$$

Note that the monomial of q-fields and the contractions containing no q-fields are not excluded in the definition, as they correspond to $\gamma = \mathrm{id}$ and to $\gamma(\cdot) = \langle v,\cdot\rangle$. By definition, q-local distributions bear much similarity with the local counterterms (5.4) ordinarily employed in quantum field theory, although the product of q-fields is non-local when re-expressed in terms of the twisted convolution product on the ordinary Minkowski space. It is important to realize that this notion of locality was already employed implicitly in chapter 2 as well as in the previous chapter, where $\phi^n(q)$ was claimed to be a natural generalization of the local interaction term $\phi^n(x)$.
The important observation now is, that not all contractions which appear when ordinary Wick ordering is applied to a product of q-fields (see Proposition 4.12), are q-local, and that typical non-q-local contractions do not approach the limit of coinciding points in the correct manner. In fact, we will see below that if a typical non-q-local distribution is evaluated in a sequence of testfunctions which approaches a product of δ-distributions $\prod_i \delta(x-x_i)$ as in Definition 4.8, there is a finite neighbourhood of x not in the support of the resulting testfunction $\gamma(g)$. Hence, such non-q-local distributions should not be admitted as counterterms. The scope of this chapter is to show that it is possible to define products of q-fields in the spirit of ordinary Wick products where only q-local counterterms are subtracted. The resulting products, called quasiplanar Wick products and denoted by triple points, $\vdots\,\phi^n(q+x)\,\vdots$, turn out to be well-defined q-local q-distributions in coinciding points.
The term q-locality is used to distinguish the definition from properties concerning causality (as in "local commutation relations"). Contrary to the situation in ordinary field theory, q-locality does not imply causal behaviour, as the q-fields themselves do not satisfy the proper commutation relations. However, we will see below that certain causality properties are preserved for particular q-local distributions, while they are violated for those which are not q-local. Before proceeding, it is shown that γ is not uniquely determined by the q-distribution v.

Remark 5.3 Let v be a q-distribution with $v = \phi^{\otimes u}\circ\gamma$ for some linear and continuous map $\gamma : \mathcal{S}(\mathbb{R}^{4n},\mathcal{Z}) \to \mathcal{S}(\mathbb{R}^{4u},\mathcal{Z})$. Then γ is uniquely determined up to the range of the Klein-Gordon operator, $v = \phi^{\otimes u}\circ\gamma = \phi^{\otimes u}\circ\gamma'$, with

$$\gamma' = \gamma + \sum_{j=1}^{u} (\Box_j + m^2)\beta^{(j)}\,,$$

where $\beta^{(j)} : \mathcal{S}(\mathbb{R}^{4n}, \mathcal{Z}) \to \mathcal{S}(\mathbb{R}^{4u}, \mathcal{Z})$ are linear and continuous.

Proof: By the properties of a tensor product of distributions, we have

$$\int dx_1 dx_2 \, (v_1 \otimes v_2)(x_1, x_2) \, g(x_1, x_2) = \int dx_1 \, v_1(x_1) \, f(x_1) \,,$$

where f is determined up to the kernel of v_1. In our case (with $u = 1$), this means that

$$v(g) = \langle v, g \rangle = \langle \phi \circ \gamma, g \rangle = \langle \phi, \gamma(g) \rangle = \langle \phi, \gamma(g) \rangle + \langle \phi, (\Box + m^2) h \rangle \,,$$

as the kernel of ϕ consists of functions of the form $(\Box + m^2)h$. As for any $h \in \mathcal{S}(\mathbb{R}^4, \mathcal{Z})$ there exists a linear continuous map $\beta : \mathcal{S}(\mathbb{R}^{4n}, \mathcal{Z}) \to \mathcal{S}(\mathbb{R}^4, \mathcal{Z})$ s.t. $h = \beta(g)$, the claim follows. To prove this for $u > 1$, it suffices to note that

$$\ker(\phi^{\otimes u}) = \sum_{i=1}^{u} V \otimes \dots V \otimes \overset{i-\mathrm{th}}{K} \otimes V \cdots \otimes V$$

where $V = \mathcal{S}(\mathbb{R}^4, \mathcal{Z})$ and K is the free field's kernel. ■

Since q-locality will serve as a means to rule out certain subtraction terms as unphysical, we have to bear in mind that γ is not uniquely determined. Let us first analyse which of the contractions arising when ordinary Wick ordering is applied to a product of q-fields as in (4.10), are q-local. Let us first introduce the notion of quasiplanarity, which turns out to be the crucial property in the following discussion.

Definition 5.4 "Quasiplanarity" *A contraction $K_n^{(A,\alpha)}(q; x_1, \dots, x_n)$ is called quasiplanar if its enclosure matrix is trivial, $E_{il} = 0$ for all $i \in A$, $l \in U = N \backslash (A \cup \alpha(A))$.*

The intersection matrix of a quasiplanar contraction is in general non-trivial, and therefore, a quasiplanar contraction is in general not completely planar.

Proposition 5.5 *A contraction $K_n^{(A,\alpha)}(q; x_1, \dots, x_n)$ is q-local if it is quasiplanar. If it is q-local, it is automatically q-local of order $u = n - 2|A|$.*

Proof: First, let $2|A| = n$. As $U = N \backslash (A \cup \alpha(A)) = \emptyset$, K has a trivial enclosure matrix. Since K is a Q-distribution, it is q-local of order 0. Likewise, the n-fold tensor product of q-fields (i.e. the contraction K_n with $A = \emptyset$, $|A| = 0$, E trivial) is q-local of order n.
Now let $0 < 2|A| < n$. Assume $K_n^{(A,\alpha)}$ to have a trivial enclosure matrix, $E_{il} = 0$, for all $i \in A, l \in U$. By Proposition 4.12, $K_n^{(A,\alpha)} = \phi^{\otimes(n-2|A|)} \circ \gamma$, where

$$(\gamma(g))(x_U) = (2\pi)^{-4|A|} \int dx_A dx_{\alpha(A)} \, g(x_N) \int dk_A \, \mathcal{I}(k_A) \prod_{i \in A} i \, \hat{\Delta}_+(k_i) \, e^{-ik_i(x_i - x_{\alpha(i)})} \,.$$

Now, γ obviously satisfies the locality condition (5.3) if g is a tensor product of two functions, $g(x_N) = f(x_A, x_{\alpha(A)}) \, h(x_U)$, as in this case $(\gamma(g))(x_U) = c \cdot h(x_U)$ with

$$c = (2\pi)^{-4|A|} \int dk_A \, \hat{f}(-k_A, k_A) \prod_{i \in A} i \, \hat{\Delta}_+(k_i) \, \mathcal{I}(k_A)$$

such that supp $\gamma(g) =$ supp h. Recall now that a general function $g \in \mathcal{S}(\mathbb{R}^{4n}, \mathcal{Z})$ with compact support can be written in terms of tensor products as $\sum_j f_j(x_A, x_{\alpha(A)}) \, h_j(x_U)$, and hence, by the continuity of γ, (5.3) holds for g. We thus conclude that K is q-local of order $u = n - 2|A|$.
■

Proposition 5.6 *A contraction K which has at least one non-vanishing entry in the enclosure matrix is equal to $\phi^{\otimes u} \circ \gamma$ with a linear, continuous map $\gamma : \mathcal{S}(\mathbb{R}^{4n}, \mathcal{Z}) \to \mathcal{S}(\mathbb{R}^{4u}, \mathcal{Z})$ violating the locality condition (5.3) given in Definition 5.1 for some testfunction $g(x_N) = \prod_{i \in N} g_i(x_i)$.*

Should it be impossible to use the freedom in the definition of γ as per Remark 5.3 to render the map γ appearing in the above proposition local, it would follow that q-locality implies triviality of the enclosure matrix, such that the *"if"* in Proposition 5.5 could be replaced by an *"if and only if"*.

Proof of Proposition 5.6: The simplest contraction with nonvanishing enclosure matrix is given by the pairing (A, α), where $1 \in A$, $\alpha(1) = 3$, and $2 \notin A$,

$$K_n^{(A,\alpha)} = \quad K_{n-3}^{(B,\beta)}$$

with $B = A\backslash(1)$, $\beta = \alpha|_B$. By Proposition 4.12,

$$\begin{aligned}
\langle K_n^{(A,\alpha)}, g\rangle &= \tfrac{1}{(2\pi)^3} \int dx_N \prod_{i \in N} g_i(x_i) \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}}\, e^{-ik(x_1 - x_3)}\, \phi(q + x_2 - Qk)\, K_{n-3}^{(B,\beta)}(q; x_4, \ldots, x_n) \\
&= \int dx_4 \ldots dx_n \int dx_2\, (\gamma(g))(x_2, x_4, \ldots, x_n)\, \phi(q + x_2)\, K_{n-3}^{(B,\beta)}(q; x_4, \ldots, x_n)
\end{aligned},$$

where we have performed the fibrewise-defined coordinate transformation $x_2 \to x_2 + \sigma k$ such that

$$(\gamma(g))(x_2, x_4, \ldots, x_n) = \tfrac{1}{(2\pi)^3} \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} \int dx_1 dx_3\; g_1(x_1) g_2(x_2 + Qk) g_3(x_3) \prod_{i \geq 4} g_i(x_i)\, e^{-ik(x_1 - x_3)}$$

We immediately infer that $\gamma(g)$ does *not* have compact support, as long as supp $g_2 \neq \emptyset$. Therefore, its support cannot be embedded in supp g, and we deduce that γ is not local.
This procedure can be directly applied to the general case. Let $L \subset U = N\backslash(A \cup \alpha(A))$ denote the index set labelling the q-fields which are enclosed by some contraction (i.e. $\forall\, l \in L\ \exists\, i \in A$: $E_{il} \neq 0$). Then the (fibrewise) change of integration variables $x_l \to x_l + \sigma \sum_{i \in A} E_{il} k_i$ for all $l \in L$ yields

$$\gamma(g)(x_U) = (2\pi)^{-3|A|} \int \frac{d\mathbf{k}_A}{2\omega_{\mathbf{k}_A}} \int dx_{N\backslash U} \prod_{s \in N\backslash L} g_s(x_s) \prod_{l \in L} g_l(x_l + Q \sum_{j \in A} E_{jl} k_j) \prod_{i \in A} e^{-ik_i(x_i - x_{\alpha(i)})}\, \mathcal{I}(k_A)$$

and $\gamma(g)$ will in general not have compact support with respect to the variables x_L. ■

It is now shown that the freedom in the definition of γ cannot be used to render the nonlocal contraction $K =$ local by adding a correction from the range of the Klein-Gordon operator, which provides evidence for the general case. In the proof, K is considered in the limit of coinciding points, $\langle K, g_c\rangle$, with $g_c(x_1, x_2, x_3) = \int dx\, g(x)\, \delta(x - x_1)\, \delta(x - x_2)\, \delta(x - x_3)$, which by the remarks at the beginning of this chapter is well-defined,

$$\langle K, g_c\rangle = \int dx \quad g(x) = (2\pi)^{-3} \int dx \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}}\, g(x + Qk)\, \phi(q + x) = \langle \phi, \gamma(g_c)\rangle\,. \tag{5.5}$$

Proposition 5.7 *The freedom in the definition of γ as per Remark 5.3 cannot be used to render the nonlocal map γ,*

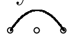
$$(\gamma(g_c))(x) = (2\pi)^{-3} \int \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}\, g(x+Qp)\,,$$

local.

Proof: By contradiction. Suppose that $\langle \phi, \gamma(g_c)\rangle = \langle \phi, f\rangle$, where f has compact support. Then $\gamma(g_c) - f = (\Box + m^2)h$ for some h and thus, in particular, $\gamma(g_c) \times \Delta = f \times \Delta$ where Δ is the commutator function. Go to the frame of reference where $Q = \sigma^{(0)}$. Let g be symmetric with respect to $x_0 = 0$. Then the Cauchy data of $\gamma(g_c) \times \Delta$ are

$$\begin{aligned} \gamma(g_c) \times \Delta\,(0,\mathbf{x}) &= c \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}}\, e^{i\mathbf{k}\mathbf{x}}\, (\hat{g}(\omega_{\mathbf{k}},\mathbf{k}) - \hat{g}(-\omega_{\mathbf{k}},\mathbf{k}))\, \Delta_+(\sigma^{(0)}k)\,|_{k\in H_m^+} = 0 \\ \partial_{x_0}(\,\gamma(g_c)\times\Delta\,)(x)\,|_{x_0=0} &= \tilde{c}\int d\mathbf{k}\, e^{i\mathbf{k}\mathbf{x}}\, \hat{g}(\omega_{\mathbf{k}},\mathbf{k})\, \underbrace{\frac{1}{\sqrt{m^2+2k_\perp^2}}\, K_1\big(\, m\sqrt{m^2+2k_\perp^2}\,\big)}_{\stackrel{\text{def}}{=}\, h(k_\perp)} \end{aligned}$$

with $k_\perp = (k_1, k_3)$ as before and with irrelevant constants $c, \tilde{c}$. Now, $\partial_{x_0}(f \times \Delta)(0,\mathbf{x})$ is a function with compact support. In order for $\partial_{x_0}(\gamma(g_c) \times \Delta\,)(0,\mathbf{x})$ to have compact support as well, $\hat{g}(\omega_{\mathbf{k}},\mathbf{k})\, h(k_\perp)$ must be an analytic function for $\mathbf{k} \to \boldsymbol{\xi} + i\boldsymbol{\eta}$. But $h(\xi_\perp + i\eta_\perp)$ is ill-defined if $2(\xi_\perp + i\eta_\perp)^2 = -m^2$, which is satisfied, for instance, if $\xi_\perp\eta_\perp = 0$ and $\xi_\perp^2 = \eta_\perp^2 - m^2/2 \geq 0$. The attempt to fix this by allowing only for testfunctions g whose Fourier transforms are not supported in the region where h is not analytic will only result in $\hat{g}(\omega_{\mathbf{k}},\mathbf{k})\, h(k_\perp) \equiv 0$, as $\hat{g}$ is analytic, and, being zero in an open subset in $\mathbb{C}^4$, it will be zero everywhere.
Hence, $\hat{g}(\omega_{\mathbf{k}},\mathbf{k})\, h(k_\perp)$ is not analytic for all η, ξ, so $\gamma(g_c)$ does not have compact Cauchy data. Therefore, the initial assumption must be wrong. ■

The nonlocal maps γ which appear in non-quasiplanar contractions do not in general behave properly when the limit of coinciding points is taken. Consider again the nonlocal contraction $K =$  . This being a typical non-q-local contraction the claim is expected to hold in a similar manner also for general non-q-local contractions.

Remark 5.8 Let $g_r^{(x)}(w,y,z) = h_r(w)h_r(y)h_r(z)$ approach $\delta(x-w)\,\delta(x-y)\,\delta(x-z)$. Evaluate $K =$  in $g_r^{(x)}$ which yields a q-distribution $u_r(q;x) = \int dy\, \gamma(g_r^{(x)})(y)\, \phi(q+y)$, where

$$\begin{aligned} \gamma(g_r^{(x)})(y) &= c\int dwdz\, h_r(w)\, h_r(z) \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}}\, h_r(y+Qk)\, e^{-ik(w-z)} \\ &= c\int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}}\, h_r(y+Qk)\, \hat{h}_r(-k)\, \hat{h}_r(k)\,. \end{aligned}$$

Then $\gamma(g_r^{(x)})(y)$ does not approach $\delta(x-y)$. In fact, there is a finite neighbourhood of x which is not necessarily in the support of $\gamma(g_r^{(x)})$, even as $r \to \infty$.

Proof: Consider the frame where $Q = \lambda_P\, \sigma^{(0)}$. Let the support of h_r be contained in the box $[-a_0 + x_0, x_0 + a_0] \times \cdots \times [-a_3 + x_3, x_3 + a_3] \subset \mathbb{R}^4$, where $0 < a_\mu < \frac{1}{2}\lambda_P\, m$. Note that $H_m^+ \cap \operatorname{supp} \hat{h}_r \neq \emptyset$ since h_r has compact support such that $\hat{h}_r$ is analytic and hence, cannot vanish in an open subset without being identically 0. Then x is not in the support of $\gamma(g_r^{(x)})$. To see this, consider the second spatial component y_2. Clearly, $\operatorname{supp}\gamma(g_r^{(x)}) \subset \operatorname{supp}\,(h_r(\,\cdot\,+Qk)\,|_{k\in H_m^+})$ and in order to have $h_r(y+Qk)\,|_{k\in H_m^+} \neq 0$, the second spatial component y_2 must be such that

$x_2 - a_2 \le y_2 + \lambda_P \sqrt{m^2 + \mathbf{k}^2} \le x_2 + a_2$. However, since by assumption $a_2 < \frac{1}{2}\lambda_P\, m$, a finite neighbourhood of $y = (y_0, y_1, x_2, y_3)$ is not in the support of $\gamma(g_r^{(x)})$. And since this remains true for increasing r, we may conclude that $\gamma(g_r^{(x)})(y)$ does not approach $\delta(x-y)$. ■

It is interesting to note that the nonlocal map arising in (5.5),

$$\gamma(g_c)(x) = (2\pi)^{-3} \int \frac{d\,\mathbf{k}}{2\omega_{\mathbf{k}}}\, g(x + Qk)\,,$$

does not take values in $\mathcal{S}(\mathbb{R}^4, \mathcal{Z})$. To see this, consider the Fourier transform

$$\widehat{\gamma(g_c)}(p) = (2\pi)^{-3} \int dx\, e^{ipx} \int \frac{d\,\mathbf{k}}{2\omega_{\mathbf{k}}}\, g(x+Qk) \;\propto\; \Delta_+(-Qp)\,\hat{g}(p)\,.$$

As pointed out in the beginning of this chapter, for p on the mass-shell, $\Delta_+(-Qp)$ is a bounded function of p. However, allowing for arbitrary p, the singular support of $\Delta_+(-Qp)$ is not empty such that $\Delta_+(-Qp)\,\hat{g}(p)$ is not a Schwartz function on $\mathbb{R}^4$ and, therefore, neither is $\gamma(g_c)$.
Let us give one further motivation as to why q-locality should be considered a natural physical property of q-distributions. In [27], it was shown that the commutator of two massless free q-fields, evaluated in best-localized states $\omega(e^{ipq}) = e^{-\lambda_P^2\,|p^2|/2}$, decreases as a Gaussian function in spacelike directions,

$$\begin{aligned}[][\omega(\phi(q+x)), \omega(\phi(q+y))] \;&=\; \frac{-i\,\lambda_P}{4\pi\sqrt{8\pi}\,|\mathbf{x}-\mathbf{y}|}\,\big(e^{-\frac{1}{8\,\lambda_P^2}((x_0-y_0)-|\mathbf{x}-\mathbf{y}|)^2} - e^{-\frac{1}{8\,\lambda_P^2}((x_0-y_0)+|\mathbf{x}-\mathbf{y}|)^2}\big) \\ &\overset{\text{def}}{=}\; C_\omega(x-y)\,. \end{aligned} \tag{5.6}$$

In the limit $\lambda_P \to 0$, C converges to the commutator function D of the massless field.

Remark 5.9 Let $K_n^{(A,\alpha)}$ and $K_m^{(B,\beta)}$ be two quasiplanar contractions, each with one uncontracted q-field, $N\backslash(A \cup \alpha(A)) = (l)$ and $M\backslash(B \cup \beta(B)) = (k)$. Then, in the massless case, their commutator is proportional to the product of $C_\omega(x_l - x_k)$ with a Q-distribution depending on $x_{N\backslash(l)}$ and $x_{M\backslash(k)}$. It therefore falls off like a Gaussian function if $x_l - x_k$ is spacelike. If one of the contractions is not quasiplanar, the commutator does not have this property.

Proof: A q-local contraction with one uncontracted q-field can be written as

$$K_n^{(A,\alpha)}(q; x_N) = K_{n-1}^{(A,\alpha)}(Q; x_{N\backslash(l)})\,\phi(q + x_l)\,.$$

Since $l \notin A \cup \alpha(A)$, the same symbols (A,α) are used on the right-hand side and on the left-hand side of the equation, although strictly speaking, on the right-hand side $A \subset N\backslash(l)$, $\alpha : A \to N\backslash(l)$. As the q-fields commute with the Q-distributions, it follows immediately that

$$\big[\omega\big(K_n^{(A,\alpha)}(q;x_N)\big), \omega\big(\big(K_m^{(B,\beta)}(q;x_M)\big)\big] \;=\; K_{n-1}^{(A,\alpha)}(Q; x_{N\backslash(l)})\; K_{m-1}^{(B,\beta)}(Q; x_{M\backslash(k)})\; C_\omega(x_l - x_k)\,.$$

Now consider a contraction which is not q-local, for instance $K = K_3^{(A,\alpha)}$ with $A = (1)$, $\alpha(1) = 3$, where

$$\omega\big(K(q; x_1, x_2, x_3)\big) = (2\pi)^{-3} \int \frac{d\,\mathbf{k}}{2|\mathbf{k}|}\, e^{-ik(x_1 - x_3)}\, \omega(\phi(q + x_2 - Qk))\,.$$

A direct calculation shows that the commutator with $\omega(\phi(q+y))$ is proportional to

$$\int \frac{d\,\mathbf{k}}{2|\mathbf{k}|}\, e^{-ik(x_1-x_3)}\, C_\omega(x_2 - Qk - y)$$

Even in the limit of coinciding points, where $x_i = x$, this commutator does not fall off for spacelike $x-y$ as in (5.6). This behaviour is shown by any contraction with $|A| > 1$, $N\backslash(A\cup\alpha(A)) = (l)$ and non-vanishing enclosure matrix, whose commutator with $\omega(\phi(q+y))$ is proportional to

$$\int \frac{d\mathbf{k}_A}{2|\mathbf{k}_A|} \prod_{i\in A} e^{-ik_i(x_i - x_{\alpha(i)})}\, \mathcal{I}(k_A)\, C_\omega(x_l - Q\sum_{i\in A} E_{il}k_i - y)$$

and, again, does not decrease like a Gaussian for spacelike $x_l - y$. ■

Note, however, that the argument only applies to contractions with only one uncontracted field, $|U| = 1$. A commutator of a product of q-fields (with or without contractions), $[\omega(\phi(q+x_1)\ldots\phi(q+x_n))\,,\,\omega(\phi(y))]$, does not satisfy (5.6).

5.2 Quasiplanar Wick products

In this section, products of q-fields are defined, where only quasiplanar subtraction terms are admitted. Again, some combinatorial prerequisites are necessary, and the notion of a connected contraction is introduced. In terms of index sets, the characterization of this property turns out to be quite technical, but, fortunately, an alternative simple characterization in terms of graphs can be given.

Definition 5.10 "Connectedness" *Let (A,α) be a pairing in N, and let I^s be the symmetrized intersection matrix of (A,α),*

$$I^s_{ij} = \begin{cases} I_{ij} & \text{for } i<j \\ 0 & \text{for } i=j \\ I_{ji} & \text{for } i>j \end{cases}$$

Then two pairs $(i,\alpha(i))$ and $(j,\alpha(j))$, $i\neq j\in A$, are connected by contractions in A if either they intersect or if there exist pairs $(j_l,\alpha(j_l))$, $j_l\in A$, $l=1,\ldots,\kappa$, such that

$$I^s_{i\,j_1}\Big(\prod_{l=1}^{\kappa-1} I^s_{j_l\,j_{l+1}}\Big) I^s_{j_\kappa\,j} \neq 0\,.$$

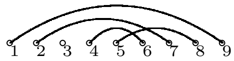
Here, j_k is not necessarily smaller than j_{k+1}.
The connected pairings in (A,α) are given by all (B,β) with $\beta=\alpha|_B$, where $B\neq\emptyset$ is an ordered subset of A such that for all $i,j\in B$ $(i,\beta(i))$ and $(j,\beta(j))$ are connected by contractions in B and $\cup B = A$.

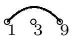
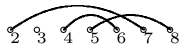
A connected component of a contraction $K_n^{(A,\alpha)}$ is defined to be a contraction $C_m^{(B,\beta)}$, where (B,β) is a connected pairing in (A,α). The connected component contains all uncontracted q-fields which are enclosed, i.e. all q-fields labelled by the index set $U'\subset U = N\backslash(A\cup\alpha(A))$,

$$U' = \{l\in U\,|\,\exists\, i\in B \text{ s.t. } E_{il}\neq 0\} = \{l\in U\,|\,\exists\, i\in B \text{ with } i<l<\beta(i)\}\,,$$

such that, by construction, a connected component is a q-distribution on $\mathcal{S}(\mathbb{R}^{4m},\mathcal{Z})$ with $m = 2|B|+|U'|$.
In terms of the graphs introduced before, the connected components of a contraction can be characterized in a much simpler way: a connected component is given by all connecting curves which can be drawn "without lifting the pencil" (drawing backwards along a curve is allowed) together with all enclosed points.
Example: The contraction $K_9^{(A,\alpha)}$ where $A=(1,2,4,5)$, and $\alpha(A)=(9,7,6,8)$ has two connected components, labelled by $B_1=(1)$ and $B_2=(2,4,5)$. Indeed, $(2,7)$ is connected to $(4,6)$

by $(5,8)$ as $I^s_{25}\, I^s_{54} \neq 0$. In terms of graphs, this can be understood more easily: the connected components of



are $K_3^{(B_1,\beta_1)}(x_1,x_3,x_9) =$  and $K_7^{(B_2,\beta_2)}(q;x_2,\ldots,x_8) =$  .

Let us now prove the main theorems of this chapter on which the definition of q-local Wick products will rely. Roughly speaking, it is shown that a contraction $:K_n^{(B,\beta)}(q;x_1,\ldots,x_n):$, whose connected components are not quasiplanar and whose uncontracted q-fields are normally ordered, is well-defined in the limit of coinciding points (in the sense of Definition 4.8).
Before proving this, we need to put the contractions' explicit expression as given in Proposition 4.12 in a slightly different form. First replace the Fourier transforms of the fields $\hat{\phi}(k_l)$, $l \in V = N\backslash(B \cup \beta(B))$ by the inverse Fourier transforms $(2\pi)^4\, \check{\phi}(-k_l)$ and perform a change of integration variables in the external momenta, $k_V \to -k_V$. Write out the uncontracted fields $\check{\phi}(k_l)$, $l \in V$, in positive and negative mass-shell components,

$$\check{\phi}(k) = (2\pi)^{-3/2} \int \frac{d\,\mathbf{p}}{2\omega_{\mathbf{p}}} \big(a(\omega_{\mathbf{p}},\mathbf{p})\,\delta(p+k) + a^\dagger(\omega_{\mathbf{p}},\mathbf{p})\,\delta(p-k) \big) = \check{\phi}_-(k) + \check{\phi}_+(k)\,,$$

and write the product of uncontracted fields in $:K:$ as the sum,

$$:\prod_{s\in V} \check{\phi}(k_s): = \sum_{\substack{V_1,V_2\\ V_1\cup V_2 = V}} \prod_{l\in V_1} \check{\phi}_+(k_l) \prod_{r\in V_2} \check{\phi}_-(k_r)\,. \tag{5.7}$$

Now rename the indices labelling the external and internal momenta such that the smallest indices are assigned to the external momenta on the positive mass shell, the largest ones to the external momenta on the negative mass-shell, and the ones in between to the internal momenta:

Remark 5.11 Let again small letters abbreviate the length of a set. Let ρ be a map from the ordered sets V_1, B, V_2, where $v_1+b+v_2 = n$, to an ordered set $U_1\times A\times U_2 \subset \mathbb{N}^n$, $U_1 = (1,\ldots,v_1)$, $A = (v_1+1,\ldots,v_1+b)$, $U_2 = (v_1+b+1,\ldots,n)$, $U = U_1 \cup U_2$, with

$$\rho(l_s) = \begin{cases} s & \text{for } l_s \text{ the } s\text{-th element of } V_1 \\ v_1+s & \text{for } l_s \text{ the } s\text{-th element of } B \\ v_1+b+s & \text{for } l_s \text{ the } s\text{-th element of } V_2 \end{cases}$$

Let π be the inverse of ρ. Then the redefinition of integration variables $k_i \to k_{\rho(i)}$ yields

$$\begin{aligned} &:K_n^{(B,\beta)}(q;x_1,\ldots,x_n): \;= \\ &= c_{n,b} \sum_{U_1,U_2} \int dk_{U_1} dk_{U_2} \prod_{s\in U_1} \check{\phi}_+(k_s) \prod_{r\in U_2} \check{\phi}_-(k_r)\, \exp\big(+i \textstyle\sum_{l\in U} k_l\,(q+x_{\pi(l)}) - \frac{i}{2} \sum_{\substack{l,l'\in U\\ \pi(l)<\pi(l')}} k_l Q k_{l'}\big)\,\cdot \\ &\qquad \cdot \underbrace{\int dk_A\, \mathcal{J}(k_{U_1},k_A,k_{U_2}) \prod_{j\in A} i\,\hat{\Delta}_+(k_j)\, e^{-ik_j(x_{\pi(j)}-x_{\beta(\pi(j))})}}_{= F(x_{(\pi(A)\cup\beta(\pi(A)))}\,;\;k_{U_1},k_{U_2})}\,. \end{aligned} \tag{5.8}$$

Here, the twisting $\mathcal{J}$ is of the form

$$\begin{aligned} \mathcal{J}(k_{U_1},k_A,k_{U_2}) &= \exp\big(-i \textstyle\sum_{\substack{r<i\\ r\in U_1, i\in A}} E_{ri}\, k_r Q k_i\big) \exp\big(-i \sum_{\substack{i<j\\ i,j\in A}} I_{ij} k_i Q k_j\big) \exp\big(+i \sum_{\substack{i<r\\ i\in A, r\in U_2}} E_{ir}\, k_i Q k_r\big) \\ &\stackrel{\text{def}}{=} \exp\big(-i \textstyle\sum_{\substack{s<t\\ s,t\in U_1\cup A\cup U_2}} J_{st}\, k_s Q k_t\big)\,, \end{aligned}$$

where the intersection matrix I and the enclosure matrix E are calculated from the intersection matrix $\tilde{I}$ and the symmetrized enclosure matrix $\tilde{E}^s$ of $K_n^{(B,\beta)}$ as follows: $I_{ij} = \tilde{I}_{\pi(i)\pi(j)}$ for $i,j \in A$, $E_{ri} = \tilde{E}^s_{\pi(r)\pi(i)}$ for $r \in U_1$, $i \in A$, $r < i$, and $E_{ir} = \tilde{E}^s_{\pi(i)\pi(r)}$ for $r \in U_2$, $i \in A$, $i < r$. The symmetrized enclosure matrix $\tilde{E}^s$ is defined analogously to I^s in Definition 5.10.

Proof: The proof is straightforward. The twisting between internal momenta follows immediately, since ρ respects the order of B. As for the calculation of the enclosure matrix, one has to be careful, since ρ does not respect the order of the external momenta. For the same reason, one has to take care not to change the order of the noncommuting exponentials e^{ikq}. ■

In order to write out the explicit expression in a clearer way, the order of integration has been changed. This is justified by the fact that the integrals are well-defined when smeared with a testfunction g and a state ω.
Let us now analyse a contraction $:K^{(B,\beta)}:$, whose connected components are not quasiplanar, in the limit of coinciding points. We will explicitly investigate the limit of coinciding points in the variables x_i, $i \in (\pi(A) \cup \beta(\pi(A)))$. Since the external fields are normally ordered, it then follows that the limit of coinciding points may also be performed with respect to the arguments of the uncontracted fields, x_l, $l \in U_1 \cup U_2$.

Proposition 5.12 *Consider a contribution to a normally ordered contraction $:K_n:$ which does not possess any quasiplanar connected components and where all external momenta are on the positive mass-shell. Let F be defined as in (5.8) and let g_m be a sequence of compactly supported testfunctions which converges to $\delta(x-x_1)\dots\delta(x-x_{2a})$ such that $\operatorname{supp} g_m \subset \operatorname{supp} g_{m+1}$ and $\bigcap \operatorname{supp} g_m = \{(x,\dots,x)\} \subset \mathbb{R}^{8a}$. Then in*

$$F(g_m; k_{U_1}) \overset{\text{def}}{=} \int dx_1 \dots dx_{2a}\, F(x_1,\dots,x_{2a}\,;\, k_{U_1})\, g_m(x_1,\dots,x_{2a})$$

the integrations over the internal momenta dk_A in F and the integrations over dx_i may be interchanged and, in the limit $m \to \infty$ the integrations over the internal momenta remain finite,

$$\int dk_A\, \mathcal{J}(k_{U_1}, k_A) \prod_{j \in A} i\, \hat{\Delta}_+(k_j) < \infty\,,$$

for fixed external momenta k_{U_1}.

Proof: Assume K (and hence F) to be connected. This may be done without loss of generality; the general case only involves more tedious notation. Let, moreover, g_m be symmetric in its arguments in order to further simplify the notation. By assumption, $U_2 = \emptyset$.
Introduce hyperbolic coordinates on the mass-shell, $k^\mu = (w\cosh\theta, v_1, w\sinh\theta, v_2)$, with $w = \sqrt{v^2+m^2}$, $v = (v_1, v_2) \in \mathbb{R}^2$, and $\theta \in \mathbb{R}$, such that $\int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} f(\omega_{\mathbf{k}}, \mathbf{k}) = \frac{1}{2}\int d^2v\, d\theta\, f(k(v,\theta))$. Let $\underline{x}$ abbreviate $(x_1,\dots,x_{2a})$. Go to a Lorentz frame such that $Q = \sigma^{(0)}$. Then, up to numerical constants, $F(g_m; k_U)$ is equal to

$$\int d\underline{x}\, g_m(\underline{x}) \int d^2v_A\, d\theta_A\, \exp\big(-i \sum_{s<t} J_{st}(w_s w_t \sinh(\theta_s - \theta_t) + v_s \wedge v_t)\big) \prod_{j \in A} e^{-i(x_j - x_{j+1})\, k_j(v_j, \theta_j)}\,, \tag{5.9}$$

where $v_s \wedge v_t = v_{s,1}v_{t,2} - v_{s,2}v_{t,1}$ and where the indices s,t are elements of the index set $U_1 \cup A$. The arguments of the exponentials which originally appear in the expression,

$$\prod_{j \in A} \exp\big(-ik_j(x_{\pi(j)} - x_{\beta(\pi(j))})\big)\,,$$

have been renamed which is possible since the g_m are symmetric in their arguments.
Now consider the integrations over θ_r in (5.9) to be cut off at finite values $\pm R_r$. Shift the integration over the rapidity variables θ_r, $r \in A$, into the complex plane, $\theta_r + i\eta_r$, where

$$0 < \eta_{u_1+1} < \cdots < \eta_{u_1+a} < \pi\,.$$

By the residue calculus, the integrations around the rectangles of lengths $2R_r$ and heights η_r will be zero. This is now used to replace the original integrations over θ_r by integrations along the curves given by $\alpha(\theta_r) = \theta_r + i\eta_r$. Set $\eta_{s\in U_1} = 0$. Then, using the formulas

$$\sinh(\theta + i\eta) = \sinh\theta\cos\eta + i\cosh\theta\sin\eta \quad \text{and} \quad \cosh(\theta + i\eta) = \cosh\theta\cos\eta + i\sinh\theta\sin\eta\,,$$

we replace the integral (5.9) with cutoffs R_r by the following integrals,

$$\begin{aligned}\int\! d\underline{x}\; g_m(\underline{x})\!\int\! d^2v_A\!\int'\! d\theta_A\; &\exp\big(-i\sum_{s<t} J_{st}\,(w_sw_t\sinh\theta_{st}\cos\eta_{st} + v_s\wedge v_t) + \sum_{s<t} J_{st}\,w_sw_t\cosh\theta_{st}\sin\eta_{st}\big)\;\cdot\\ &\cdot\prod_{j\in A}\exp\big(-i\,w_j\,\cos\eta_j\,(\cosh\theta_j\,\xi_{j,0} - \sinh\theta_j\,\xi_{j,2}) + i\,(v_{j,1},v_{j,2})(\xi_{j,1},\xi_{j,3})\big)\cdot\\ &\qquad\cdot\prod_{j\in A}\exp\big(w_j\,\sin\eta_j\,(\sinh\theta_j\xi_{j,0} - \cosh\theta_j\xi_{j,2})\big)\\ &+ \text{boundary terms } (R_A)\,,\end{aligned}$$

where ξ_j abbreviates $x_j - x_{j+1}$ and where $\int'$ indicates that the integrations over $\theta_{r\in A}$ are cut off by $\pm R_r$. The boundary terms are the integrals along the edges of the rectangle (from 0 to the respective η_r), whose integrands are given by the same expression as above with $\theta_r = \pm R_r$. With increasing m, the support of g_m becomes smaller and smaller. Hence, for any $\delta > 0$, there is an $M \in \mathbb{N}$, such that for all $m > M$, the support of g_m is contained in the set $\{|x_\mu - x_{j,\mu}| < \frac{\delta}{2}\}$ for all $j \in A$ and all $\mu = 0, \ldots, 3$, where $\{(x, \ldots, x)\} = \bigcap \operatorname{supp} g_m$. In that case, $|x_{j,\mu} - x_{j+1,\mu}| < \delta$, and for $\delta > 0$ sufficiently small, the above integrand is bounded by

$$\exp\big(\sum J_{st}w_sw_t\cosh\theta_{st}\sin\eta_{st}\big)\,|g_m(x_1, \ldots, x_{2a})|\,\exp\big(-\sum_{j\in A} w_j\,\sin\eta_j\,e^{-\theta_j}\delta\big)\,. \tag{5.10}$$

This decreases fast in $\theta_1, \ldots, \theta_a$, since by construction,

$$J_{st}w_sw_t\sin\eta_{st} < 0 \qquad \text{for all } s < t \in U_1 \cup A\,.$$

By the same mechanism, the boundary terms vanish as the side lengths R_r tend to infinity. To see this, we first perform the integration over η_{u_1+a} with all other η_j fixed at 0, then the one over η_{u_1+a-1} and so on. Then the integrals vanish for $R_r \to \infty$.
Hence, $F(g_m; k_U)$ can be replaced by the integral along the shifted integration variables, and the result is independent of the particular values of η_r. Due to the boundedness of the integrand, the integration over the momenta and the one over dx_i commute, and we can pass to the limit $m \to \infty$ before performing the momentum integrations.
It thus remains to be shown that in this limit, where $x_j = x$ for all $j \in A$, and, hence, $x_j - x_{j+1} = 0$, the integral

$$\int\! d^2v_A\!\int\! d\theta_A\;\exp\big(-i\sum_{s<t} J_{st}\,(w_sw_t\sinh\theta_{st}\cos\eta_{st} + v_s\wedge v_t)\big)\,\exp\big(\sum_{s<t} J_{st}\,w_sw_t\cosh\theta_{st}\sin\eta_{st}\big)$$

indeed remains well-defined. That this is true relies on the fact that the contraction K is connected and not quasiplanar. For one thing, connectedness ensures that all θ_j, $j \in A$, appear in the exponential $\exp\left(\sum J_{st}\, w_s w_t \cosh\theta_{st}\sin\eta_{st}\right) = \exp\left(-\sum |J_{st}\, w_s w_t \cosh\theta_{st}\sin\eta_{st}|\right)$ by which the integrand is bounded. This, however, is not sufficient for the integrals over k_A to exist. If K were quasiplanar, the exponential would only depend on differences of the internal rapidity variables. Hence, it would be possible to eliminate one of the θ_r by introducing relative rapidity variables $\theta_{r_1} - \theta_{r_2}$, such that the integration over one of the rapidity variables would diverge. This is prevented in a contraction which is not quasiplanar. In this case, the enclosure matrix E has at least one nontrivial entry, $E^s_{rt} \neq 0$ for some $r \in A, t \in U_1$, and hence, the exponential does not depend on the differences of the rapidity variables alone. This proves the claim. ■

Trying to prove the same claim for a normally ordered contraction in which also uncontracted fields on the *negative* mass-shell appear ($U_2 \neq \emptyset$), one encounters the following difficulty. In the above proof it was crucial that $J_{st} w_s w_t \sin\eta_{st} < 0$ for all $s < t \in U_1 \cup A$. Now, if $U_2 \neq \emptyset$, we find the following twisting between internal momenta and external negative mass-shell momenta

$$\exp\Big(+i \sum_{\substack{i<r\\ i\in A, r\in U_2}} E^s_{ir}\, k_i Q k_r\Big)\,.$$

Since for $t \in U_2$, $k_t^\mu = (w_t\cosh\theta_t, v_{t,1}, w_t\sinh\theta_t, v_{t,2})$, $w_t = -\sqrt{v^2+m^2}$, the analytic continuation in θ_r, $r \in A$, as in the proof above yields $+E^s_{rt} w_r |w_t| \cosh\theta_{rs}\sin(\eta_r - \eta_t)$ which is greater than zero if $\eta_t = 0$, and $0 < \eta_r < \pi$.
If *all* external momenta are on the negative mass-shell, this can be fixed by choosing a sequence $-\pi < \eta_r < \eta_{r+1} < 0$, for $r \in A$, and $\eta_t = 0$ for $t \in U_2$, and proceed as in the proof above. However, if both negative and positive mass-shell momenta appear, this is impossible, and it turns out that we have to perform an analytic continuation also in the rapidity variables θ_l, $l \in U_2$.

Proposition 5.13 *Consider a contribution to a normally ordered contraction* $:K_n:$ *which does not possess any quasiplanar connected components and where both* $U_1 \neq \emptyset$ *and* $U_2 \neq \emptyset$. *Let* F *be defined as in (5.8), and let, as in Proposition 5.12,* g_m *be a sequence of compactly supported testfunctions which converges to* $\delta(x-x_1)\dots\delta(x-x_{2a})$ *such that* $\operatorname{supp} g_m \subset \operatorname{supp} g_{m+1}$ *and* $\bigcap \operatorname{supp} g_m = \{(x,\dots,x)\} \subset \mathbb{R}^{8a}$. *Then in*

$$\int dk_{U_2} \prod_{t\in U_2} \check{\phi}_-(k_t)\ \exp\Big(+i \sum_{l\in U} k_l\,(q + x_{\pi(l)}) - \tfrac{i}{2} \sum_{\substack{l,l'\in U\\ \pi(l)<\pi(l')}} k_l Q k_{l'}\Big)\cdot$$
$$\cdot \int dx_1 \dots dx_{2a}\ F(x_1,\dots,x_{2a}\,;\ k_{U_1}, k_{U_2})\, g_m(x_1,\dots,x_{2a})$$

the integrations over the internal momenta dk_A *in* F, *as in (5.8), and the integrations over* dx_i *may be interchanged, if, after evaluation in a suitable state* ω *and a testfunction* h, *it is possible to analytically continue the above integrand with respect to the rapidity variables* θ_t, $t \in U_2$, *with* $0 \leq \operatorname{Im}\theta_t \leq \pi$, *where* $k_t^\mu = (w_t\cosh\theta_t, v_{t,1}, w_t\sinh\theta_t, v_{t,2})$, $w_t = -\sqrt{v^2+m^2}$. *In the limit* $m \to \infty$, *the integrations over the internal momenta then remain finite for fixed external momenta* k_{U_1}.

Proof: Regarding the integration over the internal momenta, the proof is similar to the one of Proposition 5.12, the only difference being that we now consider a continuation in the rapidity variables θ_l, $l \in A \cup U_2$, with

$$0 < \eta_{u_1+1} < \cdots < \eta_{u_1+a} < \eta_{u_1+a+1} < \cdots < \eta_{u_1+a+u_2} < \pi\,,$$

and consider the integral

$$\int d\underline{x}\; g_m(\underline{x}) \int d^2 v_A \int' d\theta_A \;\exp\big(-i \sum_{s<t} J_{st}\,(w_s w_t \sinh\theta_{st}\cos\eta_{st} + v_s \wedge v_t) + \sum_{s<t} J_{st}\, w_s w_t \cosh\theta_{st}\sin\eta_{st}\big) \cdot$$

$$\cdot \prod_{j\in A} \exp\big(-i\, w_j\; \cos\eta_j\, (\cosh\theta_j\, \xi_{j,0} - \sinh\theta_j\, \xi_{j,2}) + i\,(v_{j,1}, v_{j,2})(\xi_{j,1},\xi_{j,3})\big)\cdot$$

$$\cdot \prod_{j\in A} \exp\big(w_j\; \sin\eta_j\, (\sinh\theta_j \xi_{j,0} - \cosh\theta_j \xi_{j,2})\big)$$

$$+ \text{ boundary terms } (R_A, R_{U_2})\,,$$

where the twisting now includes an enclosure matrix between positive and negative mass-shell momenta.
It remains to be shown that the analytic continuation in the rapidity variables θ_t, $t\in U_2$, makes sense. This is not trivial, since, for instance, the twisting between the external momenta may contain parts which diverge exponentially,

$$\exp(-\tfrac{1}{2}\, w_s |w_t| \cosh\theta_{st} \sin(-\eta_t)\,, \qquad s\in U_1\,, t\in U_2\,.$$

That the integrations remain well-defined nonetheless and that the boundary terms vanish, follows, if the testfunction h and the state ω are chosen such that the integrands are still analytic and decrease fast enough.
The above integral may then be replaced by the integral along the shifted integration variables and the claim follows in the same manner as in the proof of Proposition 5.12. In the resulting integral, we may set $\eta_l = \pi$ for all $l \in U_2$, such that the twisting between external momenta will remain unchanged. ■

We have thus seen that the infinite momentum integrations over internal momenta $k_{j\in A}$ in a contraction remain finite in the limit of coinciding points, if they are part of a non-quasiplanar connected component. It was essential in the proof that the external momenta are on-shell and, hence, that they always have a non-vanishing energy-component. Setting one of the q_r, $r\in U$, equal to zero in such an ultraviolet-finite term would produce an *infrared singularity.* This effect, which was encountered in the context of the modified Feynman rules, is *absent* as long as the external momenta are on the (positive or negative) mass-shell.

Remark 5.14 From the proof of Proposition 5.13 we conclude that the domain on which the q-fields are defined has to be modified. It is no longer sufficient that the wavefunctions be Schwartz (or L^2), but, furthermore, their Fourier transforms must be analytic in the rapidity variable and decrease fast enough such that in the above proof the integrations remain finite and the boundary terms vanish. The resulting space of wavefunctions is denoted D_a. It is expected to be a subset of $\mathcal{S}$ (or L^2), since the request made on analyticity and fast decrease in the rapidity variable is an additional restriction, which does not contradict the properties of Schwartz functions. Therefore, D_a still is a space of testfunctions for q-distributions.

It remains to be proved that the resulting domain D_a is invariant under application of q-fields. To see that this is most likely true, consider a state $|\boldsymbol{\varphi}\rangle \in \mathfrak{H}$ with wavefunctions $\in D_a$ and apply a field operator which is smeared in a testfunction f and a state ω with $\psi_\omega(k) = \omega(e^{ikq}) \in D_a$ and $\hat{f}\in D_a$ to $|\boldsymbol{\varphi}\rangle$ as in (4.5). Since taking products and integrating out one of the momentum variables should not change the defining properties of D_a, no problems are anticipated here.
If the limit of coinciding points is also performed with respect to the arguments of the normally ordered fields, then it must also be required that $\hat{g}_m \in D_a$ for the compactly supported testfunctions g_m which define this limit.

Note that the best-localized states are not admissible states, if an analytic continuation in θ_t as in the above proof is to be performed. This follows by a direct calculation which shows that the analytic continuation in the rapidity variable of a Gaussian function results in a function having parts which increase too rapidly.
From Proposition 5.13 together with the above remarks we can immediately infer the following result:

Proposition 5.15 *Consider the normally ordered q-distribution* $:\phi^n(q; x_1, \dots, x_n):$. *Then the q-distribution which arises by revoking the subtraction of non-quasiplanar counterterms,*

$$:\phi^n(q; x_1, \dots, x_n): \quad + \sum_{k=1}^{[\frac{n-1}{2}]} \sum_{\substack{(A,\alpha),\, |A|=k \\ \text{non-quasiplanar}}} :K_n^{(A,\alpha)}(q; x_1, \dots, x_n):, \tag{5.11}$$

is a well-defined q-distribution on D_a *in coinciding points. Here, the second sum runs over all non-quasiplanar pairings, i.e. over all pairings in which* none *of the connected components are quasiplanar.*

Proof: The claim follows from the fact that the non-quasiplanar contractions are finite. Therefore, we change the q-distribution only by finite mass renormalizations, if we refrain from subtracting them. ■

The sum runs from one to $\left[\frac{n-1}{2}\right]$, since at least one field in K must remain uncontracted, such that for n even, the sum runs to $\frac{n}{2} - 1$ and for n odd, it runs to $\frac{n-1}{2}$. Proposition 5.15 allows for the following definition of quasiplanar Wick products.

Proposition 5.16 "Quasiplanar Wick products" *Let a quasiplanar Wick monomial, denoted* $\vdots\phi^n(q, x_1, \dots, x_n)\vdots$ *or* $\vdots\phi(q + x_1) \dots \phi(q + x_n)\vdots$, *be defined recursively as the following q-distribution,*

$$\begin{aligned} \vdots\phi^n(q; x_1, \dots, x_n)\vdots \quad &\stackrel{\text{def}}{=} \quad \phi(q + x_1)\, \vdots\phi^{n-1}(q; x_2, \dots, x_n)\vdots \\ &- \sum_{k=1}^{[\frac{n}{2}]} \sum_{\substack{(A,\alpha),\, |A|=k \\ \text{connected}}} K_{2k}^{(A,\alpha)}(Q; x_1, ..., x_{2k})\, \vdots\phi^{n-2k}(q; x_{2k+1}, ..., x_n)\vdots, \end{aligned} \tag{5.12}$$

in shorthand notation,

$$\vdots\phi^{\otimes n}\vdots \quad \stackrel{\text{def}}{=} \quad \phi \otimes \vdots\phi^{\otimes(n-1)}\vdots \quad - \sum_{k=1}^{[\frac{n}{2}]} \sum_{\substack{(A,\alpha),\, |A|=k \\ \text{connected}}} K_{\otimes(2k)}^{(A,\alpha)} \otimes \vdots\phi^{\otimes(n-2k)}\vdots.$$

Then $\vdots\phi^n(q; x_1, \dots, x_n)\vdots$ *is a quasiplanar (thus q-local) q-distribution.*

Furthermore, a quasiplanar Wick product can be written in terms of ordinary Wick products, where the subtraction of non-quasiplanar counterterms has been revoked as in (5.11),

$$\vdots\phi^{\otimes n}\vdots \quad = \quad :\phi^{\otimes n}: \quad + \sum_{k=1}^{[\frac{n-1}{2}]} \sum_{\substack{(A,\alpha),\, |A|=k \\ \text{non-quasiplanar}}} :K_{\otimes n}^{(A,\alpha)}:, \tag{5.13}$$

and in the limit of coinciding points it is a well-defined q-distribution on D_a.

Proof: Observing that only quasiplanar subtractions are admitted in the definition, we conclude that the quasiplanar Wick product is indeed a quasiplanar q-distribution. q-locality then follows from Proposition 5.5.
That the quasiplanar Wick products are indeed q-distributions of the form (5.11), is demonstrated in appendix B.1. Well-definedness in coinciding points then follows from Proposition 5.15. ■

Note that, contrary to the behaviour of an ordinary Wick product, the quasiplanar Wick product is not symmetric in its arguments,

$$\vdots\,\phi^n(q;x_1,\ldots,x_n)\,\vdots \quad \neq \quad \vdots\,\phi^n(q;x_{\pi(1)},\ldots,x_{\pi(n)})\,\vdots\,, \qquad \pi\in S_n\,,$$

unless it were defined with such a complete symmetrization. As in the ordinary case, however, vacuum expectation values of quasiplanar Wick products vanish,

$$\langle\Omega|\,\vdots\,\phi^n(q;x_1,\ldots,x_n)\,\vdots\,|\Omega\rangle = 0\,.$$

Observing that in (5.13) at least one uncontracted field appears in every term, this follows immediately from the fact that the vacuum expectation values of ordinary Wick products vanish.

Proposition 5.17 "Quasiplanar Wick theorem": *The product of two quasiplanar Wick monomials of order n and m is a quasiplanar (thus q-local) q-distribution which may be written as a sum of Wick monomials of order $\leq n+m$, multiplied with Q-distributions Δ_{kl},*

$$\begin{aligned}
&\vdots\,\phi^n(q;x_1,\ldots,x_n)\,\vdots\,\vdots\,\phi^m(q;y_{n+1},\ldots,y_{n+m})\,\vdots\\
&= \;\vdots\,\phi^{n+m}(q;x_1,\ldots,x_n,y_{n+1},\ldots,y_{n+m})\,\vdots\\
&\qquad + \sum_{k=1}^{n}\sum_{l=1}^{m}\Delta_{kl}(Q;x_{n-k+1},\ldots,x_n,y_{n+1},\ldots,y_{n+l})\\
&\qquad\qquad \times\,\vdots\,\phi^{n+m-l-k}(q;x_1,\ldots,x_{n-k},y_{n+l+1},\ldots,y_{n+m})\,\vdots\,,
\end{aligned}$$

where $\Delta_{kl}=0$ for $k+l$ odd, and for $k+l$ even,

$$\Delta_{kl}(Q;x_{n-k+1},\ldots,y_{n+l}) = \sum_{\substack{(A,\alpha)\text{ with}\\ \text{property }(*)}} K_{k+l}^{(A,\alpha)}(Q;x_{n-k+1},\ldots,y_{n+l})\,,$$

with property $(*)$ denoting: A and $\alpha(A)\subset\{n-k+1,\ldots,n+l\}$ and in every connected component (C,γ) of (A,α), $\exists\; i\in C$ with $i\in\{n-k+1,\ldots,n\}$ and $\gamma(i)\in\{n+1,\ldots,n+l\}$.

Proof: The right-hand side of the above is obviously quasiplanar, and, hence, by Proposition 5.5, it is q-local. The proof that the right-hand side is indeed equal to the left-hand side may be found in appendix B.2. ■

Before proceeding, note that the Q-distributions which appear above, $K_{k+l}^{(A,\alpha)}$ with $|A|=k+l$, are translation-invariant. This follows directly from the explicit form of a general contraction as given in Proposition 4.12, and we have seen this already in the discussion of the n-point functions in the previous section.

Conjecture 5.18 "Quasiplanar Wick Theorem for coinciding points": *Let $g_r\in D_a(\mathbb{R}^{4(n+m)},\mathcal{Z})$ be a sequence of compactly supported testfunctions which converges to*

$\delta(x-x_1)\dots\delta(x-x_n)\delta(y-x_{n+1})\dots\delta(y-x_{n+m})$ such that $\operatorname{supp} g_r \subset \operatorname{supp} g_{r+1}$ *and* $\bigcap \operatorname{supp} g_r = \{(x,\dots,x,y,\dots,y)\} \subset \mathbb{R}^{4(n+m)}$. *Then, in the limit $r \to \infty$ the right-hand side of the quasiplanar Wick theorem yields a well-defined q-distribution which maps testfunctions in $D_a(\mathbb{R}^8, \mathcal{Z})$ to closable operators affiliated to $\mathcal{F} \otimes \mathcal{E}$. Analogous results hold for the evaluation on the partial diagonal, where g_r converges to $\prod_{i\in I}\delta(x-x_i)\prod_{j\in J}\delta(y-x_j)$ for some selection of indices $I \subset \{1,\dots,n\}$ and $J \subset \{n+1,\dots n+m\}$.*

Steps towards a Proof:

It has to be shown that Δ_{kl} is a well-defined Q-distribution in the limit $r \to \infty$, and that it may be multiplied with the field monomials $\vdots\,\phi^{n+m-l-s}\,\vdots$.

- The Q-distributions Δ_{kl} are well-defined in the limit of coinciding points. The proof goes analogously to the one of Proposition 5.12:

  Consider one of the connected components of Δ_{kl}, labelled by (A, α). First we note that only positive mass-shell integrations appear and that, by Proposition 5.17, for at least one of the momenta k_r, $r \in A$, an exponential $e^{ik_r(x-y)}$ is part of the integrand.

  The proof then is the same as the one of Proposition 5.12 with only one minor difference. In Proposition 5.12, it was necessary that at least one external field was enclosed by one of the contractions in each of the contraction's connected components. This prevented the integrand from depending only on the differences of the internal rapidity variables which would have rendered one of the rapidity integrations infinite.

  Here, this role is played by the contraction of two fields which appears in each connected component connecting one of the first k indices with one of the last l indices, i.e. the contraction of two fields labelled by the pair $(j, \gamma(j))$ with $j \in \{n-k+1,\dots,n\}$ and $\gamma(j) \in \{n+1,\dots,n+l\}$, which yields an exponential $e^{-k_j(x-y)}$ (instead of the twisting e^{-ik_jQp} with an external momentum p used in the proof of Proposition 5.12).

- Now recall that the quasiplanar Wick products $\vdots\,\phi^{n+m-l-s}(q;x,\dots,x,y,\dots,y)\,\vdots$ can be rewritten in terms of ordinary Wick products. As a result, twisted convolution products of fields and 2-point functions at $x-y$ appear (and additionally, some integrations over internal momenta which are connected to the 2-point functions by means of a twisting but have 0 as their arguments).

- The Q-distributions Δ_{kl} may indeed be multiplied with these products, since the twistings, being bounded and smooth, do not enlarge the set of directions in which the Fourier transform of a distribution does not decrease fast. As Δ_+ may be multiplied with itself, we conclude that this is also true for Δ_{kl} and the twisted convolution products of Δ_+ which appear in the quasiplanar Wick product $\vdots\,\phi^{n+m-l-s}\,\vdots$. Moreover, the Q-distributions Δ_{kl} are translation-invariant with respect to the commuting variables and by Theorem 0 in [35], translation-invariant distributions may be multiplied with ordinary Wick products.

But, strictly speaking, the theorem used above has only been proved for numerical (or at best for vector-valued) distributions, and does not necessarily hold for $\mathcal{Z}$-valued distributions. Moreover, as was pointed out in Remark 5.14, it has not yet been proved that the domain D_a is invariant, although as pointed out, no problems are anticipated here. However, it has to be emphasized that the above yields very strong evidence as to the well-definedness of the quasiplanar Wick theorem for coinciding points.

Remark 5.19 The application of quasiplanar Wick products in the framework of the Yang-Feldman equation is straightforward. In the rules given in section 4.4 the change amounts to replacing the ordinary Wick products by quasiplanar Wick products, and the application of the ordinary Wick theorem by the application of the quasiplanar Wick theorem.

5.3 The dispersion relation

The requirement that only quasiplanar counterterms should be subtracted has important consequences for the asymptotic behaviour of the theory, resulting in a distorted dispersion relation. In a more realistic theory than self-interacting bosonic fields such as quantum electrodynamics, this is a measurable quantity. A first analysis of this point is to be the last line of thought pursued in this thesis.
Assume that the interacting field may indeed be renormalized by subtracting quasiplanar counterterms. Then we derive the renormalized field equation for a ϕ^n-self-interacting theory, involving, in particular, a finite mass m. As discussed at the end of chapter 4, we may calculate the vacuum expectation value of two interacting fields at different, mutually commuting "points",

$$\langle \phi(q_1)\,\phi(q_2) - \phi(q_2)\,\phi(q_1)\rangle_0\,, \tag{5.14}$$

the result of which is the dressed commutator function at $q_1 - q_2$. Taking the Fourier transform of the above in ordinary quantum field theory one would obtain $\epsilon(p_0)\delta(p^2 - m^2) = (p_0/|p_0|)\,\delta(p^2 - m^2)$ plus the Källén-Lehmann integral, if the mass renormalization was done with respect to the *physical* mass. Suppose now that we had chosen to perform the mass renormalization with respect to a finite but unphysical mass m^2, then we would allow for additional finite mass counterterms δm^2 in the field equation. The Fourier transform of the dressed commutator would then yield the following expansion,

$$\epsilon(p_0)\delta(p^2 - m^2) + \epsilon(p_0)\delta'(p^2 - m^2)\,\delta m^2 + \frac{1}{2}\,\epsilon(p_0)\delta''(p^2 - m^2)\,(\delta m^2)^2 + \dots\,, \tag{5.15}$$

which as usual can be summed up to give $\epsilon(p_0)\delta(p^2 - m^2 - \delta m^2)$. If, however, quasiplanar Wick products are employed, the calculation of (5.14) is expected to be different, since we will find unusual contractions in the non-q-local tadpoles which have not been subtracted. For instance, in ϕ^4-theory, where the effect is more easily calculated than in ϕ^3-theory, the finite tadpole at first order is

$$= i \int dk\, \Delta_+(Qk)\, \check{\phi}(k)\, e^{ikq}\,.$$

It leads to the mass renormalization $\epsilon(p_0)\delta'(p^2 - m^2)\,\Delta_+(Qp)$. It follows, in particular, that $\Delta_+(Qp)$ results in a modification of the mass which is not Lorentz-invariant. Hence, the dispersion relation will be modified. Similar discussions in the context of space-space-commutativity, which are not founded on the general construction of quasiplanar Wick products, may be found in [64, 4, 17].
Taking the concept of q-locality serious, this contraction can indeed not be subtracted to recover the ordinary dispersion relation, since it was proved in Proposition 5.7 that this contraction is not q-local.
Assuming that higher orders in g can be added up consistently as in (5.15),

$$\epsilon(p_0)\delta(p^2 - m^2 - g\, i\, \Delta_+(Qp) - \dots)\,,$$

we deduce that $p_0^2 = \mathbf{p}^2 + m^2 + g\, i\, \Delta_+(Qp) + \dots$. Allowing for additional counterterms, α and βp^2, the dispersion relation at this order is found to be

$$p^2 - m^2 - g\,(i\,\Delta_+(Qp; m) + \alpha_1 + \beta_1 p^2) = 0\,.$$

Now choose the frame of reference where $Q = \sigma^{(0)}$ and $(Qp)^2 = \lambda_P^4\,(-p_0^2 + p_2^2 - p_\perp^2) = \lambda_P^4\,(-p^2 - 2\,p_\perp^2)$ with $p_\perp = (p_1, p_3)$, such that for timelike p^2 we find

$$i\,\Delta_+(Qp; m) = \frac{m}{4\pi^2\,\lambda_P^2\,\sqrt{p^2 + 2\,p_\perp^2}}\,K_1\Big(\lambda_P^2\,m\,\sqrt{p^2 + 2\,p_\perp^2}\,\Big)\,.$$

Hence, the transversal velocity is given by

$$\begin{aligned}\frac{dp_0}{dp_\perp} &= \frac{p_\perp}{p_0}\,\frac{1 + \frac{g}{2\,p_\perp\,(1-g\beta_1)}\frac{\partial}{\partial p_\perp}\Big(\frac{m}{4\pi^2\,\sqrt{-(Qp)^2}}\,K_1\big(\,m\sqrt{-(Qp)^2}\,\big)\Big)}{1 - \frac{g}{2\,p_0\,(1-g\beta_1)}\frac{\partial}{\partial p_0}\Big(\frac{m}{4\pi^2\,\sqrt{-(Qp)^2}}\,K_1\big(\,m\sqrt{-(Qp)^2}\,\big)\Big)}\\ &= \frac{p_\perp}{p_0}\,\frac{1 + \frac{g}{1-g\beta_1}\,\eta((Qp)^2)}{1 - \frac{g}{1-g\beta_1}\,\eta((Qp)^2)}\,,\end{aligned}$$

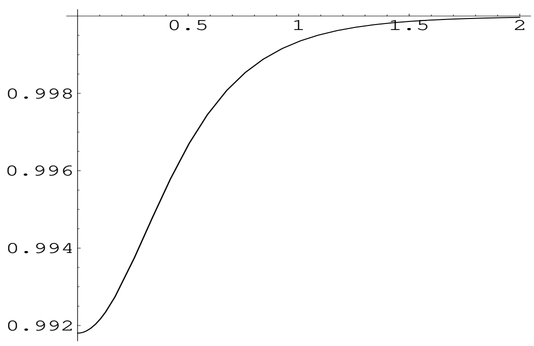
where

$$\eta((Qp)^2) = -\frac{m^2\,\lambda_P^4\,K_2(\lambda_P^2 m\,\sqrt{-(Qp)^2})}{-8\,\pi^2\,(Qp)^2} = -\frac{m^2\,K_2(\lambda_P^2 m\,\sqrt{{p_0}^2 - {p_2}^2 + {p_\perp}^2})}{8\,\pi^2\,({p_0}^2 - {p_2}^2 + {p_\perp}^2)}\,.$$

Now let us set the momentum p on the physical mass-shell, $p^2 = M^2$, which may be done consistently by fixing the constant α_1 above accordingly. Then

$$\eta((Qp)^2)\,\big|_{p^2=M^2} = -\frac{m^2\,K_2(\lambda_P^2 m\,M\,\sqrt{1 + \frac{2\,p_\perp^2}{M^2}})}{8\,\pi^2\,M^2\,\Big(1 + \frac{2\,p_\perp^2}{M^2}\Big)}\,.$$

If the mass m and the physical mass are of the same order of magnitude, the resulting deviation from the usual dispersion relation is much too large. Even assuming that M and m are of the order of the Planck mass, much larger than that of any known theory's particle, the resulting deviation $\frac{1+\frac{g}{1-g\beta_1}\,\eta((Qp)^2)}{1-\frac{g}{1-g\beta_1}\,\eta((Qp)^2)}$ of the group velocity in the perpendicular directions $p_\perp$ is



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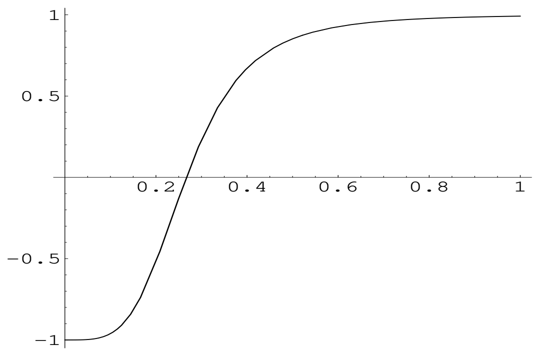
as a function of $p_\perp$ (one-dimensional), where $\beta_1 = 0$, $g = 1/5$, $m = M = 1$, and $\lambda_P = 1$ such that all quantities are measured in Planck units. Surprisingly, the maximal deviation occurs not at high momenta but at $p_\perp = 0$. In the above numerical setting, this point of maximal deviation is very large,

$$(1 + g\eta_0)/(1 - g\eta_0) \simeq 0.991802\,, \qquad \eta_0 = \eta((Qp)^2)\,\big|_{\substack{p_\perp^2=0\\ p^2=m^2}} = \eta(-m^2)\,.$$

Using a smaller mass m, the deviation becomes even larger, as we can see in the following plot, where the minimum of the above function, $\frac{1+g\,\eta(-m^2)}{1-g\,\eta(-m^2)}$, (i.e. the maximal deviation from 1) is plotted against the mass m, ranging from 0 to 1, (with $M = m$, $\beta_1 = 0$, $g = 1/5$, $\lambda_P = 1$),



plotted with `Mathematica`.

We see that for small masses, the group velocity may even become negative. Integrating over, say, Σ_1 would not improve the situation, since the scale λ_P remains fixed.
Taking into account however, that m and M may differ from one another, it is possible to allow for small physical masses M, while taking m to be very large. To see this, recall that at $p_\perp = 0$ and $p^2 = M^2$,

$$\eta((Qp)^2) = -\frac{m^2\,K_2(\lambda_P^2 m\,M)}{8\,\pi^2\,M^2} \propto -\frac{m^2}{M^2}\,K_2(\lambda_P^2\,M^2\,\frac{m}{M})$$

and since (cf. for instance [47]),

$$\alpha^2 K_2(\beta\,\alpha) \longrightarrow 0 \qquad \text{for } \alpha \text{ large enough}\,,$$

it is possible to make the deviation arbitrarily small even for small masses by choosing m large enough. It remains to be investigated whether this scheme can be applied consistently to all orders.
However small, the modification of the dispersion relation has serious consequences. In ordinary local quantum field theory, the Hilbert space of the asymptotic fields is the Fock space of the free fields with fixed (constant) mass. The above analysis shows that this cannot be true for the asymptotic fields in the framework considered here, since their mass will in general depend on the momentum.
The distorted dispersion relation could provide predictions which are very sensitive to experimental data and put serious restrictions on the scale of noncommutativity, which was taken to be of the order of the Planck scale in the above. Depending on how questions concerning the consistency of the renormalization procedure as sketched above can be solved, it is not impossible that in a realistic model such as quantum electrodynamics, where phenomenological calculations so far have provided lower bounds for the energy scale of noncommutativity, an *upper* bound for the energy scale could be derived in this way.

Outlook

Future work certainly should focus on the application of quasiplanar Wick products to the Yang-Feldman perturbative approach. It is reasonable to hope that q-local counterterms suffice to treat all ultraviolet divergences consistently. Furthermore, the quasiplanar Wick products could also be applied within the Hamiltonian framework.
Regarding the modified dispersion relation, different lines of thought are possible. One is that the dispersion relation should be calculated for a realistic theory such as quantum electrodynamics on the noncommutative Minkowski space to get predictions which might eventually be experimentally tested. Another possibility is to consider supersymmetric extensions, where quadratic divergences may cancel each other, and, possibly, the ordinary dispersion relation could be restored.
In more general terms, the modified dispersion relation means that the asymptotic behaviour of the theory is considerably changed compared to the ordinary case. In particular, the ordinary Fock space, whose definition relies on a mass which is independent of the momentum, cannot serve as the Fock space of incoming and outgoing fields, and a modification of the asymptotic conditions becomes necessary. The effect clearly pertains to the theory's infrared behaviour, and the distorted dispersion relation should be seen as a subtle form of the infrared-ultraviolet mixing property.
A similar effect was hinted at in the framework of the regularized Hamiltonian approach. Here, the necessity to use dressed propagators in order to be able to remove the adiabatic cutoff in the time-variable, forces one to allow for a mass renormalization which is not Lorentz-invariant. This in turn is due to the fact that a Lorentz frame had to be chosen in order to define the best-localized states. The renormalized theory then is expected to possess a distorted dispersion relation. The same is anticipated for the other Hamiltonian approaches, since in these cases, it is not clear whether the mass renormalization will lead to an ordinary dispersion relation. These questions certainly deserve further study.
On a more fundamental level, one of the most challenging problems is to gain a better understanding of the spectrum of the commutators, which is deeply connected with the quest for Lorentz invariance.

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Appendix A

Conventions and useful formulas

The Fourier transform:

$$
\begin{aligned}
\check{f}(k) &= (2\pi)^{-n} \int d^n x \, e^{-ikx} \, f(x) \\
\hat{f}(k) &= \int d^n x \, e^{ikx} \, f(x) \\
\delta^{(n)}(x-y) &= (2\pi)^{-n} \int d^n k \, e^{+ik(x-y)}
\end{aligned}
$$

The free field:

$$
\begin{aligned}
\phi(q) &= \int dk \, e^{ikq} \, \check{\phi}(k) \;=\; \int dk \, e^{ikq} \, (2\pi)^{-4} \int dx \, e^{-ikx} \, \phi(x) \\
&= \int dk \, e^{ikq} \, (2\pi)^{-3/2} \int \frac{d\,\mathbf{p}}{2\omega_{\mathbf{p}}} \left(a(p) \, \delta^{(4)}(k+p) + a^\dagger(p) \, \delta^{(4)}(k-p) \right) \\
&= (2\pi)^{-3/2} \int \frac{d\,\mathbf{p}}{2\omega_{\mathbf{p}}} \left(a(p) \, e^{-ipq} + a^\dagger(p) \, e^{ipq} \right) \\
\text{with} \quad [a(p), a^\dagger(q)] &= 2\,\omega_{\mathbf{p}} \, \delta^{(3)}(\mathbf{p}-\mathbf{q})
\end{aligned}
$$

The 2-point function:

$$
\langle \Omega | \phi(x) \, \phi(y) | \Omega \rangle = i \, \Delta_+(x-y) = (2\pi)^{-3} \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} \, e^{-ik(x-y)}
$$

$$
\begin{aligned}
\langle \Omega | \phi(x) \, \phi(y) | \Omega \rangle - \langle \Omega | \phi(y) \, \phi(x) | \Omega \rangle &= i \, \Delta(x-y) \;=:\; i \left(\Delta_+(x-y) + \Delta_-(x-y) \right) \\
&= i \, \Delta_+(x-y) - i \, \Delta_+(y-x)
\end{aligned}
$$

such that

$$
i\Delta_-(x-y) = -i\Delta_+(y-x) = -(2\pi)^{-3} \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} \, e^{+ik(x-y)}
$$

$$
\hat{\Delta}_+(p) = \int dx \, e^{ipx} \frac{1}{i(2\pi)^3} \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} \, e^{-ikx} = (2\pi)^4 \frac{1}{i(2\pi)^3} \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} \, \delta^{(4)}(p-k)
$$

for spacelike argument:

$$
i \, \Delta_+(x) = \frac{m}{4\,\pi^2 \sqrt{-x^2}} \, K_1(m \sqrt{-x^2}\,)
$$

with the Bessel function K of the second kind of order 1.

The commutator function:

$$i\,\Delta(x) \;=\; \frac{i}{2\pi}\,\epsilon(x_0)\,\delta(x^2) - \frac{i\,m}{4\pi\sqrt{x^2}}\,\theta(x^2)\,\epsilon(x_0)\,J_1(m\sqrt{x^2})$$

The causal (Feynman) propagator

$$\Delta_F(x) = \theta(x_0)\,\Delta_+(x) + \theta(-x_0)\,\Delta_+(-x) = \theta(x_0)\,\Delta_+(x) - (1-\theta(x_0))\,\Delta_-(x)$$

Green's functions:

$$\begin{aligned}\partial_t^2\,T(\phi(x)\phi(x')) \;&=\; \partial_t\Big(\underbrace{\delta(t-t')\left(\phi(x)\phi(x')-\phi(x')\phi(x)\right)}_{=0}\Big)\\ &\quad + \partial_t\Big(\theta(t-t')\,\partial_t\phi(x)\phi(x') + \theta(t'-t)\,\phi(x')\partial_t\phi(x)\Big)\end{aligned}$$

Replace $\partial_t^2\phi(x)$ by $(\Delta_x - m^2)\phi(x)$:

$$\begin{aligned}(\Box_x + m^2)\,T(\phi(x)\phi(x')) \;&=\; \delta(t-t')\left(\partial_t\phi(x)\phi(x') - \phi(x')\partial_t\phi(x)\right)\\ &=\; \delta(t-t')\,i\,\partial_t\,\Delta(x-x')\\ &=\; -i\,\delta^{(4)}(x-x')\,.\end{aligned}$$

Similarly for $\Delta_{ret}(x) = \theta(x_0)\,\Delta(x)$:

$$\begin{aligned}\partial_t^2\,\Delta_{ret}(x-x') \;&=\; \partial_t\Big(\underbrace{\delta(t-t')\,\Delta(x-x')}_{=0}\Big) + \partial_t\Big(\theta(t-t')\,\partial_t\Delta(x-x')\Big)\\ &=\; -\delta^{(4)}(x-x') + \theta(t-t')\,\partial_t^2\Delta(x-x')\end{aligned}$$

and thus

$$(\Box_x + m^2)\,\Delta_{ret}(x-x') \;=\; -\,\delta^{(4)}(x-x')$$

Appendix B

Formal proofs

B.1 Proof of Proposition 5.16

It is shown that the recursive definition of the quasiplanar Wick product (5.12) coincides with the prescription (5.11), where the subtraction of non-quasiplanar counterterms from the ordinary Wick product is revoked.

Proof by induction: For $n = 2$, there is no non-quasiplanar pairing, and the quasiplanar Wick product coincides with the ordinary one,

$$\vdots\phi^2(q; x_1, x_2)\vdots = \phi(q; x_1)\phi(q; x_2) - K_2(Q; x_1, x_2) = {:}\phi(q; x_1, x_2){:}$$

where K_2 is the connected contraction $K_2^{(A,\alpha)} = \frown$ with $A = (1)$, $\alpha(1) = 2$, $K_2(Q; x_1, x_2) = i\Delta_+(x_1 - x_2)$. Here, $\phi(q; x_1)$ stands for $\phi(q + x_1)$. For $n = 3$ we find that the recursive definition (5.12) coincides with (5.11), since

$$\begin{aligned}
\vdots\phi^3(q; x_1, x_2, x_3)\vdots &= \phi(q; x_1)\vdots\phi^2(q; x_2, x_3)\vdots - K_2(Q; x_1, x_2)\,\phi(q; x_3) \\
&= \phi(q; x_1){:}\phi^2(q; x_2, x_3){:} - K_2(Q; x_1, x_2)\,\phi(q; x_3) \\
&= {:}\phi^3(q; x_1, x_2, x_3){:} + {:}K_3^{(A,\alpha)}(q; x_1, x_2, x_3){:}
\end{aligned}$$

where $A = (1)$, $\alpha(1) = 3$, which yields the only non-quasiplanar contraction at this order. The last line follows by application of the ordinary Wick theorem. Assume now that for $l \le n$ the claim is true. At order $n + 1$, the recursive definition yields

$$\vdots\phi^{\otimes(n+1)}\vdots \;=\; \phi \otimes \vdots\phi^{\otimes n}\vdots - \sum_{k=1}^{[\frac{n+1}{2}]} \sum_{\substack{(A,\alpha),\,|A|=k \\ \text{connected}}} K^{(A,\alpha)}_{\otimes(2k)} \otimes \vdots\phi^{\otimes(n+1-2k)}\vdots\,,$$

where $K^{(A,\alpha)}_{\otimes 2k}$ with $k = |A|$ is a shorthand notation for the the Q-distribution $K^{(A,\alpha)}_{2\,|A|}(Q; x_1, \dots, x_{2\,|A|})$. Note that it is not necessary to indicate by some additional symbols that all fields in a contraction $K_{\otimes 2k}$ are contracted, if the respective sum runs only over pairings with $|A| = k$.

Now replace $\vdots\,\phi^{\otimes n}\,\vdots$ and $\vdots\,\phi^{\otimes(n+1-2k)}\,\vdots$ according to the induction hypothesis to obtain

$$\begin{aligned}
\vdots\,\phi^{\otimes(n+1)}\,\vdots \;&=\; \phi\otimes :\phi^{\otimes n}: \,+\, \phi\otimes \sum_{m=1}^{[\frac{n-1}{2}]} \sum_{\substack{(A,\alpha),\,|A|=m\\ \text{non-quasiplanar}}} :K_{\otimes n}^{(A,\alpha)}: \\
&\quad - \sum_{k=1}^{[\frac{n+1}{2}]} \sum_{\substack{(A,\alpha),\,|A|=k\\ \text{connected}}} K_{\otimes(2k)}^{(A,\alpha)} \otimes :\phi^{\otimes(n+1-2k)}: \\
&\quad - \sum_{k=1}^{[\frac{n+1}{2}]} \sum_{\substack{(A,\alpha),\,|A|=k\\ \text{connected}}} K_{\otimes(2k)}^{(A,\alpha)} \otimes \sum_{m=1}^{[\frac{n-2k}{2}]} \sum_{\substack{(A,\alpha),\,|A|=m\\ \text{non-quasiplanar}}} :K_{\otimes(n+1-2k)}^{(A,\alpha)}:
\end{aligned}$$

Note that for n odd, the second summation in the last line is empty for $k = \left[\frac{n+1}{2}\right] = \frac{n+1}{2}$ and $k = \left[\frac{n+1}{2}\right] - 1 = \frac{n-1}{2}$. Similarly, for n even, the second summation in the last line is empty for $k = \left[\frac{n+1}{2}\right] = \frac{n}{2}$. In these cases, the sum is assumed to have the value 0. This notation is a shorthand which, in particular, makes it unnecessary to distinguish between the cases n even and n odd.
By application of the ordinary Wick theorem, we obtain for the first term in the above formula:

$$\phi\otimes :\phi^{\otimes n}: = :\phi^{\otimes(n+1)}: + K_{\otimes 2}\otimes :\phi^{\otimes(n-1)}: + \sum_{j=3}^{n+1} :K_{\otimes(n+1)}^{(1,\alpha(1)=j)}: .$$

As for the second term, three different kinds of contractions arise, when the field on the left is contracted with the uncontracted fields in $:K_{\otimes n}^{(A,\alpha)}:$. The first possibility is that the first field on the left is contracted such that the resulting contraction is still non-quasiplanar. The second possibility is that the resulting contraction is a product of a connected contraction with no more uncontracted fields and a field monomial (or merely a connected contraction with no more uncontracted fields), and the last possibility is that the resulting contraction is a product of a connected contraction with no more uncontracted fields and a non-quasiplanar contraction. In terms of the shorthand notation from above,

$$\begin{aligned}
\phi\otimes \sum_{m=1}^{[\frac{n-1}{2}]} \sum_{\substack{(A,\alpha),\,|A|=m\\ \text{non-quasiplanar}}} :K_{\otimes n}^{(A,\alpha)}: \;&=\; :\phi\otimes \sum_{m=1}^{[\frac{n-1}{2}]} \sum_{\substack{(A,\alpha),\,|A|=m\\ \text{non-quasiplanar}}} K_{\otimes n}^{(A,\alpha)}: \\
&\quad + \sum_{m=2}^{[\frac{n}{2}]} \sum_{\substack{(A,\alpha)\\ |A|=m,\,1\in A\\ \text{non-quasiplanar}}} :K_{\otimes(n+1)}^{(A,\alpha)}: \,+\, \sum_{k=2}^{[\frac{n+1}{2}]} \sum_{\substack{(A,\alpha),\,|A|=k\\ \text{connected}}} K_{\otimes(2k)}^{(A,\alpha)} \otimes :\phi^{\otimes(n+1-2k)}: \\
&\quad + \sum_{k=1}^{[\frac{n+1}{2}]} \sum_{\substack{(A,\alpha),\,|A|=k\\ \text{connected}}} K_{\otimes(2k)}^{(A,\alpha)} \otimes \sum_{m=1}^{[\frac{n-2k}{2}]} \sum_{\substack{(A,\alpha),\,|A|=m\\ \text{non-quasiplanar}}} :K_{\otimes(n+1-2k)}^{(A,\alpha)}:
\end{aligned}$$

and we obtain

$$\vdots\,\phi^{\otimes(n+1)}\,\vdots \;=\; :\phi^{\otimes(n+1)}: + \sum_{k=1}^{[\frac{n}{2}]} \sum_{\substack{(A,\alpha),\,|A|=k\\ \text{non-quasiplanar}}} :K_{\otimes(n+1)}^{(A,\alpha)}:$$

$\blacksquare$

B.2 Formal proof of the quasiplanar Wick Theorem

Let $C_{\otimes(2k)}$ be the Q-distribution $\sum_{\substack{(A,\alpha),\,|A|=k \\ \text{connected}}} K^{(A,\alpha)}_{\otimes(2k)}$.
First we prove the following formula for the product of two quasiplanar Wick monomials:

$$\vdots\phi^{\otimes m}\vdots\vdots\phi^{\otimes n}\vdots \;=\; \vdots\phi^{\otimes(m+n)}\vdots + \sum_{l=1}^{m}\sum_{k=[\frac{l}{2}]+1}^{[\frac{n+l}{2}]} \vdots\phi^{\otimes(m-l)}\vdots \otimes C_{\otimes(2k)} \otimes \vdots\phi^{\otimes(n-2k+l)}\vdots \tag{B.1}$$

Proof: Application of the recursive definition (5.12) yields

$$\begin{aligned}\vdots\phi^{\otimes m}\vdots\vdots\phi^{\otimes n}\vdots \;=\; & \vdots\phi^{\otimes(m+n)}\vdots + \sum_{l=1}^{m}\sum_{k=1}^{[\frac{n+l}{2}]} \vdots\phi^{\otimes(m-l)}\vdots \otimes C_{\otimes(2k)} \otimes \vdots\phi^{\otimes(n+l-2k)}\vdots \\ & - \sum_{l=1}^{m}\sum_{k=1}^{[\frac{m-l+1}{2}]} \vdots\phi^{\otimes(m-l-2k+1)}\vdots \otimes C_{\otimes(2k)} \otimes \vdots\phi^{\otimes(n+l-1)}\vdots\end{aligned}$$

Since the last sum can be rewritten equivalently as

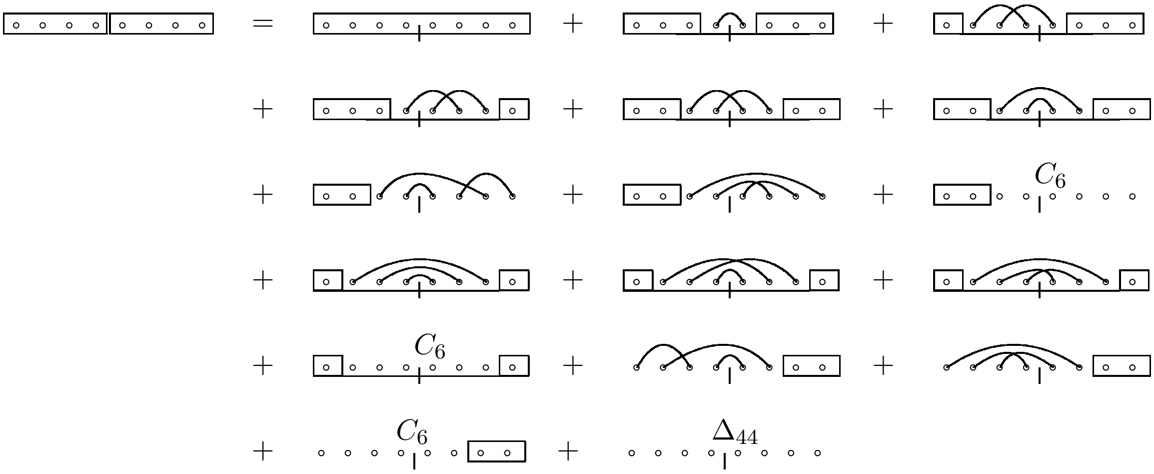
$$- \sum_{l=2}^{m}\sum_{k=1}^{[\frac{l}{2}]} \vdots\phi^{\otimes(m-l)}\vdots \otimes C_{\otimes(2k)} \otimes \vdots\phi^{\otimes(n+l-2k)}\vdots$$

the claim follows. □

Applying (B.1) recursively, until all products in (B.1) are quasiplanar Wick products, we derive the quasiplanar Wick theorem. ■

In the two examples which follow, the graphs introduced in chapter 4 are employed. Quasiplanar Wick products are symbolized by boxes, and the underscore symbolizes quasiplanar Wick ordering of fields which are not neighbours in the tensor product.

Example:



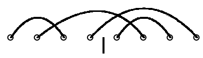
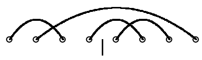
where C_6 is the Q-distribution

$$C_6 = \sum_{\substack{(A,\alpha),\,|A|=3 \\ \text{connected}}} K_6^{(A,\alpha)} =$$

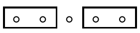
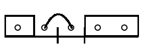
and where Δ_{44} is the Q-distribution

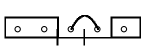
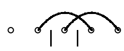
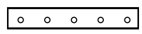
$$
\begin{aligned}
\Delta_{44} = \sum_{\substack{(A,\alpha)\text{ with}\\ \text{property}(*)}} K_{2|A|}^{(A,\alpha)} &= \sum_{A=(1,2,3,4)} K_{2|A|}^{(A,\alpha)} \\
&+ \sum^{*}_{\substack{A=(1,2,4,5)\\ \alpha(1)=3,\alpha(5)=7}} K_{2|A|}^{(A,\alpha)} + \sum^{*}_{\substack{A=(1,2,4,5)\\ \alpha(1)=3,\alpha(5)=8}} K_{2|A|}^{(A,\alpha)} + \sum^{*}_{\substack{A=(1,2,4,6)\\ \alpha(1)=3,\alpha(6)=8}} K_{2|A|}^{(A,\alpha)} \\
&+ \sum^{*}_{\substack{A=(1,2,3,5)\\ \alpha(2)=4,\alpha(5)=7}} K_{2|A|}^{(A,\alpha)} + \sum^{*}_{\substack{A=(1,2,3,5)\\ \alpha(2)=4,\alpha(5)=8}} K_{2|A|}^{(A,\alpha)} + \sum^{*}_{\substack{A=(1,2,3,6)\\ \alpha(2)=4,\alpha(6)=8}} K_{2|A|}^{(A,\alpha)} \\
&+ \sum^{*}_{\substack{A=(1,2,3,5)\\ \alpha(1)=4,\alpha(5)=7}} K_{2|A|}^{(A,\alpha)} + \sum^{*}_{\substack{A=(1,2,3,5)\\ \alpha(1)=4,\alpha(5)=8}} K_{2|A|}^{(A,\alpha)} + \sum^{*}_{\substack{A=(1,2,3,6)\\ \alpha(1)=4,\alpha(6)=8}} K_{2|A|}^{(A,\alpha)}
\end{aligned}
$$

where, for instance,

$$\sum^{*}_{\substack{A=(1,2,4,5)\\ \alpha(1)=3,\alpha(5)=7}} K_{2|A|}^{(A,\alpha)} =$$  $+$ 

Example:

 $=$  $+$  $+$  $+$

$+$  $+$  $+$  $+$ 

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