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# The Measurement of Length in Linear Quantum Gravity

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## Abstract

In the weak-field approximation of general relativity the Einstein equation is linear and upon introduction of the harmonic gauge condition becomes the gravitational wave equation. Canonical quantization and the Fock construction lead to the Krein space of gravitons, which by imposing a gauge condition of Gupta-Bleuler type becomes a physical Hilbert space. Within this framework of a linearized covariant approach to quantum gravity the length of a spacelike curve is defined as an operator-valued distribution and subsequently analyzed with special emphasis on gauge invariance. For certain length operators of spacelike curves smeared out in time the vacuum fluctuations are calculated. From this finally arises a restriction on the measurability of space and time intervals in the form of an uncertainty relation.

## Zusammenfassung

In der Näherung der allgemeinen Relativitätstheorie für schwache Gravitationsfelder ist die Einstein-Gleichung linear und wird durch die Einführung der harmonischen Eichbedingung zur Gravitationswellengleichung. Kanonische Quantisierung und die Fockkonstruktion führen auf den Kreinraum der Gravitonen, welcher durch eine Eichbedingung vom Gupta-Bleuler-Typ zu einem physikalischen Hilbertraum wird. Im Rahmen dieses linearisierten kovarianten Ansatzes zur Quantengravitation wird die Länge einer raumartigen Kurve als operatorwertige Distribution definiert und im Anschluß unter besonderer Berücksichtigung der Eichinvarianz untersucht. Es werden für bestimmte raumartige in der Zeit verschmierte Kurven die Vakuumfluktuationen berechnet. Daraus ergibt sich schließlich eine Einschränkung an die Meßbarkeit von Raum- und Zeitintervallen in Form einer Unschärferelation.

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# Introduction

**Quantum Gravity.** We are at the verge of leaving the glorious century at the beginning of which general relativity and quantum theory have been discovered. Yet the dream of a unification of the two fundamental theories is not anywhere near to being fulfilled. Still the term »quantum gravity« is used to describe a variety of more or less radical approaches towards a synthesis of general relativistic and quantum principles. In order to mention only the major programs, take the covariant[1, 2], the canonical[3, 4], the Euclidean[5] and the superstring[6] approach, including supergravity[7]. While current research interests focus on non-perturbative methods, the perturbative covariant approach has lost much of its appeal since it is known to be non-renormalizable[8, 9]. It is fair to say, though, that so far all other attempts at constructing a theory of quantum gravity are struggling as well with severe technical and conceptual difficulties. The confusion is made complete by the lack of experimental data, for which the smallness of the Planck length provides an explanation. Hence in the field of quantum gravity, in contrast to all other physical research areas, nature does not act as the usual selection criterion or even a guiding light. It should therefore be clear that as a general feature research in quantum gravity is wide open to speculation. Under these circumstances it might prove to be a good idea to approach the delicate problem »quantum gravity« in a rather conservative manner, on the technical as well as on the conceptual side.

**Linear Quantum Gravity.** In our opinion the covariant approach, as advocated by Bryce DeWitt for many years, is by far the most conservative approach to quantum gravity, as it is simply the well-known formalism of canonical quantization applied to the metric perturbation of a classical background spacetime. In a straightforward manner one is thus led to a quantum field theory of interacting spin-2 particles, the famous gravitons. As we have noted above, the covariant program unfortunately ran into a dead end with the discovery of its non-renormalizability. However, this is a problem that is solely connected to the interaction and may be separated by considering just the free field theory. At first sight ignoring the interaction might seem to be a great sacrifice, but actually it is not. After all the classical counterpart of the free field theory is the linearized approximation of general relativity, from which the majority of all observable general relativistic effects can be derived. In the quantum case the situation becomes even more extreme due to the smallness of the Planck length and we should therefore be safe in our ignorance towards next-to-leading-order effects of quantum gravity. With regard to its linear classical origin, we will simply denote the free quantum field theory of the graviton in the sequel as »linear quantum gravity«. Obviously, the linear approach to quantum gravity has two advantages. First, one has a technically well-posed theory, from which rigorous results may be derived, and second, one is able to give meaningful interpretations to the results obtained. In fact these two properties should be satisfied by any approach to quantum gravity in order to be taken seriously.

**Quantum Geometry.** It appears reasonable to analyse quantum gravity from a geometric point of view since general relativity is a geometric theory par excellence. Spacetime, being a differentiable manifold, leads directly to geometric objects such as curves, surfaces or hypersurfaces and via its metric to associated geometric quantities such as length, area, volume or curvature. Whereas a proper definition of these fundamental quantities even in the canonical approach poses a problem[10, 11], in linear quantum gravity one is simply given operator-valued distributions over the background spacetime. By choosing suitable test functions, one is able to ensure gauge invariance and obtains a perfectly well-defined set of observables, only waiting to be analysed.

**Quantum Spacetime.** It is appropriate to mention another recent development in connection with quantum gravity, which however aims at quantizing the underlying manifold of spacetime rather than its metric. We refer to the work of Doplicher, Fredenhagen and Roberts[12, 13], who recently proposed a model of Minkowski quantum spacetime, roughly in the spirit of the »non-commutative geometry« pioneered by Alain Connes[14]. The three authors motivate their quantum spacetime by a refinement of an argument due to Pauli. It combines the Heisenberg principle with the possible generation of a microscopic black hole in the process of measurement and yields a set of two uncertainty relations between two distinct spacetime coordinates. Though of completely different origin, the work encouraged us to look for similar limitations for the measurement of spacetime lengths in linear quantum gravity.

**Storyline.** Generally, the first two chapters are to a large extent a presentation of well-known material, collected and prepared to facilitate the understanding of the original work, which is contained in chapter three. In the first chapter the classical preliminaries are covered, taking as the starting point the general theory of relativity. The Einstein equation is linearized by the assumption that the metric of the spacetime is just a small perturbation of the Minkowski metric. The harmonic gauge condition is imposed to obtain the Fierz-Pauli equation, whose solutions are interpreted as gravitational waves. To demonstrate the relevance of the linear approximation we calculate the bending of a light ray at the surface of the sun. The gauge invariant and hence physical part of a gravitational wave is shown to possess spin 2. The classical account closes with the formulation of the theory in the Lagrangian context, which is needed for its quantization. In the second chapter the transition from linear general relativity to linear quantum gravity is described. The Fierz-Pauli equation is canonically quantized to yield the Krein-Fock space of gravitons, which is an indefinite metric space, due to neglecting the gauge condition. Therefore in a second step the Gupta quantum gauge condition is imposed to yield the Hilbert-Fock space of physical gravitons. The corresponding physical operators are shown to be Riemann tensors smeared out with test functions on Minkowski space. In the third chapter we study the effect that the quantization of linear general relativity has on the measurement of length. To this end we define length operators for spacelike curves smeared out in time, keeping an eye on their gauge invariance and providing an interpretation in terms of Gauß curvature. In addition we calculate their vacuum fluctuations and obtain an uncertainty relation, restricting the proper measurement of length in space and time.

CHAPTER 1

# Gravitational Waves

## 1. General Relativity

**Relativity.** At the beginning of this century it was Albert Einstein who believed that all coordinate systems are created equal and who from this principle developed his famous general theory of relativity. Loosely speaking the principle of relativity states that physics does not care about a particular choice of coordinates. As a consequence physical objects are invariant under coordinate transformations. In this sense general coordinate transformations form the gauge group of general relativity.

**Tensors.** The coordinate invariant objects demanded by the principle of relativity are the tensors, which are multilinear maps from tensor products of tangent and cotangent spaces into $\mathbb{R}$. Consider a point $x$ in a differentiable manifold $M$ and introduce in a neighborhood $U_x$ a coordinate system $x^\mu$, thereby inducing the canonical basis

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \tag{1.1.1}$$

in the tangent space $\mathfrak{T}_x$ and the canonical cobasis $dx^\mu$ in the cotangent space $\mathfrak{T}^*_x$. A tensor $T$ is then represented by its components

$$T^{\mu\nu\cdots}{}_{\alpha\beta\cdots} = T(dx^\mu, dx^\nu, \ldots, \partial_\alpha, \partial_\beta, \ldots) \quad , \tag{1.1.2}$$

where the number $m$ of contravariant indices $\mu$, $\nu$, $\ldots$ and the number $n$ of covariant indices $\alpha$, $\beta$, $\ldots$ reflect its rank $(m, n)$. Conversely the components (1.1.2) uniquely determine the tensor

$$T = T^{\mu\nu\cdots}{}_{\alpha\beta\cdots}\partial_\mu\partial_\nu\cdots dx^\alpha dx^\beta\cdots \quad , \tag{1.1.3}$$

or equivalently

$$T = T'^{\mu'\nu'\cdots}{}_{\alpha'\beta'\cdots}\partial'_{\mu'}\partial'_{\nu'}\cdots dx'^{\alpha'}dx'^{\beta'}\cdots \tag{1.1.4}$$

if other coordinates $x'^{\mu'}$ have been chosen. From the equality of (1.1.3) and (1.1.4) it follows by use of the relation

$$dx'^{\mu'}(\partial_\mu) = \frac{\partial x'^{\mu'}}{\partial x^\mu} \tag{1.1.5}$$

that under a coordinate transformation $x'^{\mu'}(x^\mu)$ the components of the tensor $T$ change according to the law

$$T'^{\mu'\nu'\cdots}{}_{\alpha'\beta'\cdots} = T^{\mu\nu\cdots}{}_{\alpha\beta\cdots}\frac{\partial x'^{\mu'}}{\partial x^\mu}\frac{\partial x'^{\nu'}}{\partial x^\nu}\cdots\frac{\partial x^\alpha}{\partial x'^{\alpha'}}\frac{\partial x^\beta}{\partial x'^{\beta'}}\cdots \quad . \tag{1.1.6}$$

The tensor $T$ may be viewed as an equivalence class of both coordinate systems $x^\mu$ and tensor components $T^{\mu\nu\cdots}{}_{\alpha\beta\cdots}$ and is a coordinate invariant object just like the point $x$, which has a meaning irrespective of the chosen coordinates $x^\mu$.

**Spacetime.** The basic mathematical notion in general relativity is spacetime. A spacetime $(M, g)$ is a four dimensional differentiable manifold $M$, on which a Lorentz metric, i.e. a non-degenerate symmetric smooth tensor field $g$ of rank $(0, 2)$ and signature $(1, 3)$ has been defined. The manifold $M$ determines the topology and the metric $g$ the geometry of spacetime. Thus whereas the manifold $M$ is mainly the set of spacetime events, the metric $g$ contains all the relevant information about length, geodesics and curvature.

**Length.** The Lorentz metric $g$ at a point $x \in M$ defines the Lorentz product

$$g(u, v) = g(u^\mu \partial_\mu, v^\nu \partial_\nu) \tag{1.1.7}$$

$$= g_{\mu\nu} u^\mu v^\nu \tag{1.1.8}$$

of two vectors $u, v$ in the tangent space $\mathfrak{T}_x$. The Lorentz product of a vector $v$ with itself is called its Lorentz square and provides a classification into time-, light- or spacelike vectors, depending upon the Lorentz square being positive, vanishing or negative. If the tangent vectors

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda} \tag{1.1.9}$$

of a curve

$$C \colon \begin{cases} [0,1] & \to M \\ \lambda & \mapsto x \end{cases} \tag{1.1.10}$$

are all time-, light- or spacelike, then this classification applies to the curve as well. In this case it is meaningful to integrate the lengths of the tangent vectors $\dot{x}$ and define the expression

$$L(C) = \int_0^1 d\lambda \sqrt{|g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu|} \tag{1.1.11}$$

as the length of the curve $C$.

**Geodesics.** Straightest curves go by the name of geodesics. More precisely, a geodesic is a curve that parallel transports its tangent vector along itself. In a spacetime parallel transport may be introduced via the Christoffel symbol

$$\Gamma^\mu{}_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}) \quad . \tag{1.1.12}$$

Although the Christoffel symbol itself is not a tensor due to an inhomogeneous term appearing in its coordinate transformation law

$$\Gamma'^{\mu'}{}_{\alpha'\beta'} = \Gamma^\mu{}_{\alpha\beta} \frac{\partial x'^{\mu'}}{\partial x^\mu} \frac{\partial x^\alpha}{\partial x'^{\alpha'}} \frac{\partial x^\beta}{\partial x'^{\beta'}} + \frac{\partial x'^{\mu'}}{\partial x^\mu} \frac{\partial^2 x^\mu}{\partial x'^{\alpha'} \partial x'^{\beta'}} \quad , \tag{1.1.13}$$

the covariant derivative

$$\begin{aligned} \nabla_\rho T^{\mu\nu\cdots}{}_{\alpha\beta\cdots} = \partial_\rho T^{\mu\nu\cdots}{}_{\alpha\beta\cdots} &+ \Gamma^\mu{}_{\rho\sigma} T^{\sigma\nu\cdots}{}_{\alpha\beta\cdots} + \Gamma^\nu{}_{\rho\sigma} T^{\mu\sigma\cdots}{}_{\alpha\beta\cdots} + \cdots \\ &- \Gamma^\sigma{}_{\rho\alpha} T^{\mu\nu\cdots}{}_{\sigma\beta\cdots} - \Gamma^\sigma{}_{\rho\beta} T^{\mu\nu\cdots}{}_{\alpha\sigma\cdots} - \cdots \end{aligned} \tag{1.1.14}$$

of an arbitrary tensor $T$ is again a tensor. The parallel transport of a tensor $T$ along a curve $C$ as in (1.1.10) is governed by the equation

$$\dot{x}^\rho \nabla_\rho T^{\mu\nu\cdots}{}_{\alpha\beta\cdots} = 0 \quad . \tag{1.1.15}$$

For a vector $v$ this equation reads

$$\dot{x}^\alpha \nabla_\alpha v^\mu = 0 \quad , \tag{1.1.16}$$

which by inserting the covariant derivative

$$\nabla_\alpha v^\mu = \partial_\alpha v^\mu + \Gamma^\mu{}_{\alpha\beta} v^\beta \tag{1.1.17}$$

may be written as

$$\dot{v}^\mu + \Gamma^\mu{}_{\alpha\beta} \dot{x}^\alpha v^\beta = 0 \quad . \tag{1.1.18}$$

Following the definition of a geodesic given above, one simply sets $v = \dot{x}$ to obtain the geodesic equation

$$\dot{x}^\alpha \nabla_\alpha \dot{x}^\mu = 0 \quad , \tag{1.1.19}$$

or equivalently

$$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0 \quad . \tag{1.1.20}$$

A special feature of the parallel transport defined by the Christoffel symbol (1.1.12) is that it respects the Lorentz metric $g$ in the sense

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\beta{}_{\mu\alpha} g_{\nu\beta} - \Gamma^\beta{}_{\nu\alpha} g_{\mu\beta} = 0 \quad . \tag{1.1.21}$$

Hence the Lorentz square of a transported tangent vector $\dot{x}$ is preserved, which allows it to always classify a spacetime geodesic as being time-, light- or spacelike.

**Curvature.** The effect that the distance between two neighboring geodesics, though initially parallel, in general is not a constant is called geodesic deviation. The origin of this effect is the most important geometric property of spacetime, namely its curvature. Qualitatively, geodesics move closer, remain parallel or drift apart if the curvature of spacetime is positive, vanishing or negative, as illustrated in FIGURE 1. Quantitatively, the curvature of spacetime is measured by the Riemann tensor

$$R^\mu{}_{\alpha\nu\beta} = \partial_\nu \Gamma^\mu{}_{\alpha\beta} - \partial_\beta \Gamma^\mu{}_{\alpha\nu} + \Gamma^\mu{}_{\rho\nu} \Gamma^\rho{}_{\alpha\beta} - \Gamma^\mu{}_{\rho\beta} \Gamma^\rho{}_{\alpha\nu} \quad , \tag{1.1.22}$$

which is closely related to the commutator of two covariant derivatives. The Riemann tensor is antisymmetric in the first two and last two indices, satisfies the identity

$$R^\mu{}_{\alpha\nu\beta} + R^\mu{}_{\beta\alpha\nu} + R^\mu{}_{\nu\beta\alpha} = 0 \tag{1.1.23}$$

and in addition obeys the fundamental differential relation

$$\nabla_\rho R^\mu{}_{\alpha\nu\beta} + \nabla_\beta R^\mu{}_{\alpha\rho\nu} + \nabla_\nu R^\mu{}_{\alpha\beta\rho} = 0 \quad , \tag{1.1.24}$$

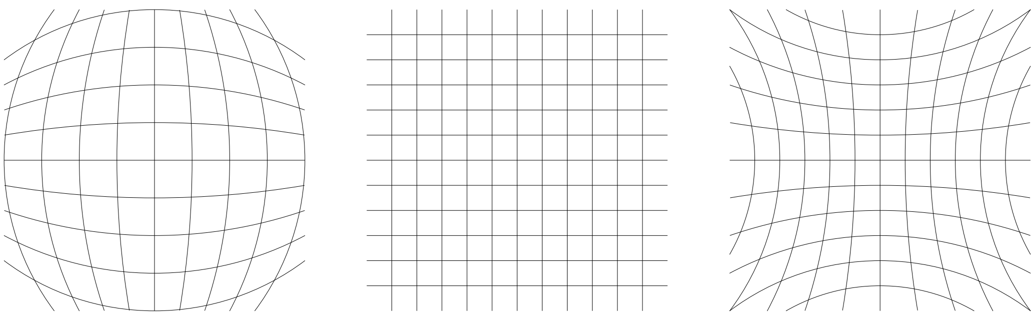


FIGURE 1. Positive, vanishing and negative curvature

known as the Bianchi identity. Given a family of geodesics

$$F : \begin{cases} [0,1] \times [0,1] & \to & M \\ (\kappa, \lambda) & \mapsto & x \end{cases} \tag{1.1.25}$$

with tangent vectors

$$\dot{x}^\mu = \frac{\partial x^\mu}{\partial \lambda} \tag{1.1.26}$$

and normal vectors

$$x'^\mu = \frac{\partial x^\mu}{\partial \kappa} \quad , \tag{1.1.27}$$

as illustrated in FIGURE 2, the equation of geodesic deviation reads

$$\dot{x}^\alpha \nabla_\alpha (\dot{x}^\beta \nabla_\beta x'^\mu) = R^\mu{}_{\alpha\nu\beta} \dot{x}^\alpha x'^\nu \dot{x}^\beta \quad . \tag{1.1.28}$$

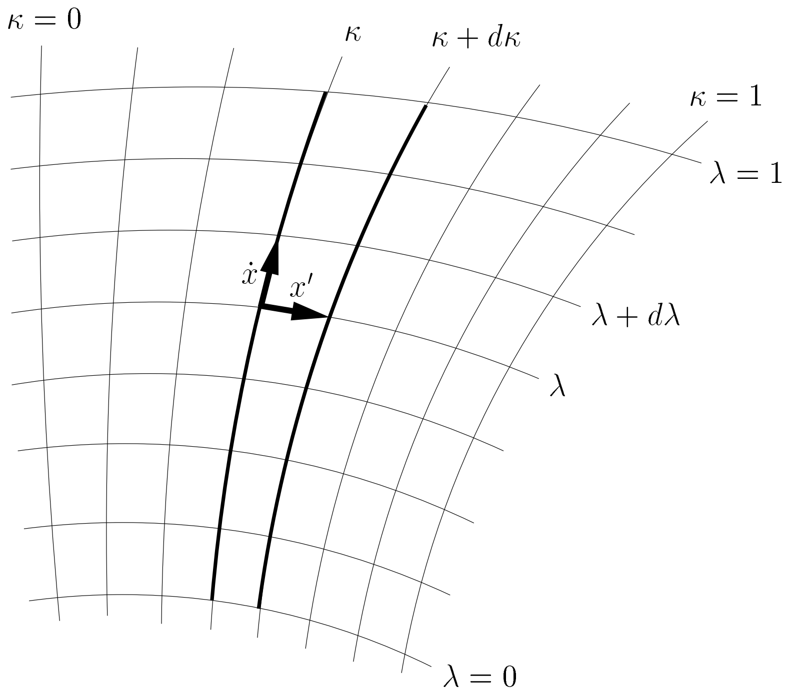



FIGURE 2. A family of geodesics

**Einstein Equation.** From the Riemann tensor (1.1.22) two other key quantities are derived by contraction, namely the Ricci tensor

$$R_{\mu\nu} = g^{\alpha\beta} R_{\mu\alpha\nu\beta} \tag{1.1.29}$$

and the scalar curvature

$$R = g^{\mu\nu} R_{\mu\nu} \quad , \tag{1.1.30}$$

which suitably combined with the Lorentz metric $g$ yield the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} \quad . \tag{1.1.31}$$

The famous Einstein equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.1.32}$$

connects the geometry of spacetime, on the left hand side, to the matter on spacetime, on the right hand side, represented by the stress-energy tensor $T$. One finds by contraction that the Einstein equation (1.1.32) is equivalent to

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \tfrac{1}{2}T g_{\mu\nu}) \tag{1.1.33}$$

with $T = g^{\mu\nu}T_{\mu\nu}$. Hence in the absence of matter the Einstein equation becomes

$$R_{\mu\nu} = 0 \quad . \tag{1.1.34}$$

**Harmonic Coordinates.** The completely antisymmetric symbol $\epsilon$ is given by

$$\epsilon_{\mu\nu\alpha\beta} = \begin{cases} \begin{matrix} +1 \\ -1 \end{matrix} & \text{if } \mu\nu\alpha\beta \text{ is an } \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \text{ permutation of } 0123 \\ 0 & \text{else} \end{cases} \tag{1.1.35}$$

and defines the Levi-Civita tensor

$$\varepsilon_{\mu\nu\alpha\beta} = \sqrt{-g}\,\epsilon_{\mu\nu\alpha\beta} \quad , \tag{1.1.36}$$

where $g$ is the determinant of the Lorentz metric matrix $g_{\mu\nu}$. Contracting both the first two and the last two indices of the Riemann tensor (1.1.22) with a Levi-Civita tensor yields the Riemann double dual[15]

$$G^{\mu\alpha\nu\beta} = \tfrac{1}{4}\varepsilon^{\mu\alpha\rho\kappa}\varepsilon^{\nu\beta\sigma\lambda}R_{\rho\kappa\sigma\lambda} \quad , \tag{1.1.37}$$

for which the Bianchi identity (1.1.24) takes on the simple form

$$\nabla_\mu G^{\mu\alpha\nu\beta} = 0 \quad . \tag{1.1.38}$$

In terms of the Riemann double dual the Einstein tensor (1.1.31) is written as the contraction

$$G^{\mu\nu} = g_{\alpha\beta}G^{\mu\alpha\nu\beta} \quad , \tag{1.1.39}$$

from which follows that the Einstein tensor satisfies the crucial relation

$$\nabla_\mu G^{\mu\nu} = 0 \quad . \tag{1.1.40}$$

The four equations, known as the contracted Bianchi identity, show that of the ten Einstein equations (1.1.32) in fact only six are independent. Given a stress-energy tensor $T$ one thus has a underdetermined system of differential equations for the ten components of the Lorentz metric $g$. The remaining four degrees of freedom however simply reflect the arbitrariness in the choice of coordinates and thus may be eliminated by the introduction of a coordinate gauge condition. A particularly useful condition is the harmonic coordinate condition

$$\partial_\mu \mathfrak{g}^{\mu\nu} = 0 \tag{1.1.41}$$

with the Lorentz metric density

$$\mathfrak{g}_{\mu\nu} = \sqrt{-g}\,g_{\mu\nu} \quad . \tag{1.1.42}$$

It can be shown that the harmonic coordinate condition (1.1.41) is equivalent to the equation

$$\nabla^\mu \nabla_\mu x^\nu = 0 \quad , \tag{1.1.43}$$

which reveals that the gauged coordinates are indeed harmonic functions on spacetime.

**Tensor Densities.** The completely antisymmetric symbol $\epsilon$ and the Lorentz metric density $\mathfrak{g}$ are not tensors but tensor densities. A tensor density $\mathfrak{T}$ of weight $W$ has components $\mathfrak{T}^{\mu\nu\cdots}{}_{\alpha\beta\cdots}$ that transform according to the law

(1.1.44) $$\mathfrak{T}'^{\mu'\nu'\cdots}{}_{\alpha'\beta'\cdots} = \mathfrak{T}^{\mu\nu\cdots}{}_{\alpha\beta\cdots} \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \frac{\partial x'^{\nu'}}{\partial x^{\nu}} \cdots \frac{\partial x^{\alpha}}{\partial x'^{\alpha'}} \frac{\partial x^{\beta}}{\partial x'^{\beta'}} \cdots \left(\frac{\partial x}{\partial x'}\right)^{W}$$

with the Jacobian $\partial x'/\partial x$ being the determinant of the Jacobi matrix $\partial x'^{\mu'}/\partial x^{\mu}$ of the coordinate transformation $x'^{\mu'}(x^{\mu})$. Tensors are thus identified as tensor densities with vanishing weight $W$. The importance of tensor densities originates in the coordinate transformation law

(1.1.45) $$dx' = \frac{\partial x'}{\partial x} dx$$

of the spacetime volume element $dx$, hence being a scalar density of weight $-1$. The determinant $g$ of the Lorentz metric matrix $g_{\mu\nu}$ is another example of a scalar density. It transforms as

(1.1.46) $$g' = \left(\frac{\partial x}{\partial x'}\right)^{2} g$$

and accordingly is of weight $2$. Combining the square root of this transformation with definition (1.1.44) of a general tensor density $\mathfrak{T}$, one obtains the transformation law

(1.1.47) $$(\sqrt{-g}')^{-W} \mathfrak{T}'^{\mu'\nu'\cdots}{}_{\alpha'\beta'\cdots} = (\sqrt{-g})^{-W} \mathfrak{T}^{\mu\nu\cdots}{}_{\alpha\beta\cdots} \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \frac{\partial x'^{\nu'}}{\partial x^{\nu}} \cdots \frac{\partial x^{\alpha}}{\partial x'^{\alpha'}} \frac{\partial x^{\beta}}{\partial x'^{\beta'}} \cdots .$$

Any tensor density $\mathfrak{T}$ of weight $W$ can thus be written in the form

(1.1.48) $$\mathfrak{T}^{\mu\nu\cdots}{}_{\alpha\beta\cdots} = (\sqrt{-g})^{W} T^{\mu\nu\cdots}{}_{\alpha\beta\cdots} \quad ,$$

where $T$ is a tensor. In general the tensor density is denoted by the letter of the corresponding tensor, yet taken from the Gothic alphabet. From equation (1.1.48) and the relation

(1.1.49) $$\Gamma^{\mu}{}_{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_{\nu} \sqrt{-g}$$

it follows that the covariant derivative of a tensor density $\mathfrak{T}$ of weight $W$ is given by

(1.1.50) $$\nabla_{\rho} \mathfrak{T}^{\mu\nu\cdots}{}_{\alpha\beta\cdots} = \nabla_{\rho} T^{\mu\nu\cdots}{}_{\alpha\beta\cdots}\Big|_{T\to\mathfrak{T}} + W\, \Gamma^{\sigma}{}_{\sigma\rho} \mathfrak{T}^{\mu\nu\cdots}{}_{\alpha\beta\cdots} \quad ,$$

i.e. as if it were a tensor $T$ barring an additional term.

**Equation of Motion.** Due to the contracted Bianchi identity (1.1.40), in combination with the Einstein equation (1.1.32), a proper stress-energy tensor $T$ obeys the equation of motion

(1.1.51) $$\nabla_{\mu} T^{\mu\nu} = 0 \quad .$$

Consider as an example the stress-energy tensor

(1.1.52) $$T^{\mu\nu} = \rho e^{\mu} e^{\nu} \quad ,$$

which is characteristic of matter in the form of dust. Here the product of mass density $\rho$ and the unit vector $e$ of the velocity field is the energy-momentum density

(1.1.53) $$p^{\mu} = \rho e^{\mu} \quad .$$

In this case the equation of motion (1.1.51) is given by

$$(\nabla_\mu p^\mu) e^\nu + p^\mu (\nabla_\mu e^\nu) = 0 \tag{1.1.54}$$

and contracted with the unit vector $e$ leads to

$$\nabla_\mu p^\mu + p^\mu (\nabla_\mu e^\nu) e_\nu = 0 \quad . \tag{1.1.55}$$

The second term vanishes because of the relation

$$(\nabla_\mu e^\nu) e_\nu = \tfrac{1}{2} \nabla_\mu (e^\nu e_\nu) = 0 \tag{1.1.56}$$

and one finds the equation of continuity

$$\nabla_\mu p^\mu = 0 \quad , \tag{1.1.57}$$

which inserted into the equation of motion (1.1.54) yields the geodesic equation

$$e^\mu \nabla_\mu e^\nu = 0 \quad . \tag{1.1.58}$$

Hence the equation of motion (1.1.51) demands that dust moves continuously along geodesics.

**DeWitt Supermetric.** One way to turn the Riemann tensor into the Einstein tensor is to take its double dual (1.1.37) and contract it as in (1.1.39). Alternatively one may first contract the Riemann tensor to the Ricci tensor (1.1.29) and then obtain the Einstein tensor via the relation (1.1.31), which by the introduction of the DeWitt supermetric[16]

$$S_{\mu\nu\alpha\beta} = \tfrac{1}{2} (g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} - g_{\mu\nu} g_{\alpha\beta}) \tag{1.1.59}$$

may be written in the form

$$G_{\mu\nu} = S_{\mu\nu}{}^{\alpha\beta} R_{\alpha\beta} \quad . \tag{1.1.60}$$

The two paths from the Riemann tensor to the Einstein tensor are illustrated in the commutative diagram in FIGURE 3. The name supermetric is derived from the fact that $S$ defines a metric

$$S(h, h') = S^{\mu\nu\alpha\beta} h_{\mu\nu} h'_{\alpha\beta} \tag{1.1.61}$$

$$\qquad = h^{\mu\nu} h'_{\mu\nu} - \tfrac{1}{2} h h' \tag{1.1.62}$$

on the ten-dimensional vector space of symmetric tensors $h$ of rank $(0, 2)$. Here the supermetric is looked upon as a non-degenerate and symmetric $10 \times 10$-matrix

$$S_{\mu\nu\alpha\beta} = S_{\alpha\beta\mu\nu} \quad , \tag{1.1.63}$$

$$\begin{tikzcd}
G^{\mu\nu} & & R_{\alpha\beta} \arrow[ll, "S^{\mu\nu\alpha\beta}"', "(1.1.60)"] \\
G^{\mu\kappa\nu\lambda} \arrow[u, "g_{\kappa\lambda}", "(1.1.39)"'] & & R_{\alpha\rho\beta\sigma} \arrow[ll, "(1.1.37)"', "\frac{1}{4}\varepsilon^{\mu\kappa\alpha\rho}\varepsilon^{\nu\lambda\beta\sigma}"] \arrow[u, "(1.1.29)", "g^{\rho\sigma}"']
\end{tikzcd}$$

FIGURE 3. From the Riemann tensor to the Einstein tensor

whose indices are in fact symmetric index pairs. In this sense the supermetric $S$ is involutive

(1.1.64) $$S^{\mu\nu}{}_{\lambda\rho}S^{\lambda\rho}{}_{\alpha\beta} = \delta^{\mu\nu}{}_{\alpha\beta} \quad ,$$

where the symmetrized Kronecker symbol

(1.1.65) $$\delta^{\mu\nu}{}_{\alpha\beta} = \tfrac{1}{2}(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} + \delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha})$$

acts as the unit matrix. In flat spacetime $(\mathbb{R}^4, \eta)$ with the Minkowski metric

(1.1.66) $$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

the supermetric is given by the expression

(1.1.67) $$H_{\mu\nu\alpha\beta} = \eta_{\mu\nu\alpha\beta} - \tfrac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta} \quad ,$$

where

(1.1.68) $$\eta_{\mu\nu\alpha\beta} = \tfrac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}) \quad .$$

With the help of the symmetric involution

(1.1.69) $$T^{\mu\nu}{}_{\alpha\beta} = \delta^{\mu\nu}{}_{\alpha\beta} - \tfrac{1}{2}\delta^{\mu\nu}\delta_{\alpha\beta}$$

we are able to diagonalize the Minkowski supermetric $H$ to the matrix

(1.1.70) $$D_{\mu\nu\alpha\beta} = T_{\mu\nu}{}^{\rho\sigma}H_{\rho\sigma\kappa\lambda}T^{\kappa\lambda}{}_{\alpha\beta}$$

(1.1.71) $$= \eta_{\mu\nu\alpha\beta} - \tfrac{1}{2}(\delta_{\mu\nu} + \eta_{\mu\nu})(\delta_{\alpha\beta} + \eta_{\alpha\beta})$$

(1.1.72) $$= \eta_{\mu\nu\alpha\beta} - 2\eta_{\mu\nu 00}\eta_{\alpha\beta 00} \quad .$$

The diagonal matrix is most conveniently written in the form

(1.1.73) $$D_{\mu\nu\alpha\beta} = \text{diag}\begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 \end{pmatrix} \quad ,$$

where each $\mu\nu$ matrix entry on the right hand side yields a diagonal component $D_{\mu\nu\mu\nu}$ on the left hand side. Taking into account the symmetry of the index pairs, we thus find that the supermetric $S$ has the signature $(6, 4)$.

## 2. Linear Approximation

**Linearization.** The Einstein equation is non-linear in the Lorentz metric $g$. However, there exists a linear approximation leading to reasonably accurate results if the Lorentz metric $g$ is just a small perturbation $h$ about the Minkowski metric $\eta$. More precisely, one starts with a spacetime $(\mathbb{R}^4, \tilde{g})$, where the Lorentz metric $\tilde{g}$ is of the form

(1.2.1) $$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

and satisfies the perturbative condition

(1.2.2) $$|h_{\mu\nu}| \ll 1$$

such that all non-linear terms $\mathrm{O}(h^2)$ are negligibly small. Since in the linear formalism raising and lowering of indices is understood using the Minkowski metric $\eta$, the form

$$\tilde{g}^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{1.2.3}$$

of the inverse Lorentz metric follows immediately from the relation

$$\tilde{g}^{\mu\alpha}\tilde{g}_{\alpha\nu} = \delta^{\mu}{}_{\nu} \quad . \tag{1.2.4}$$

A substitution of the Lorentz metric (1.2.1) and its inverse (1.2.3) into the Christoffel symbol (1.1.12) leads to the linearized Christoffel symbol

$$\tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \tfrac{1}{2}\eta^{\mu\nu}(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}) \quad . \tag{1.2.5}$$

Keeping only the linear terms of the Riemann tensor (1.1.22) means

$$\tilde{R}^{\mu}{}_{\alpha\nu\beta} = \partial_\nu \tilde{\Gamma}^{\mu}{}_{\alpha\beta} - \partial_\beta \tilde{\Gamma}^{\mu}{}_{\alpha\nu} \tag{1.2.6}$$

and yields the expression

$$\tilde{R}_{\mu\alpha\nu\beta} = \tfrac{1}{2}(\partial_\mu\partial_\beta h_{\alpha\nu} + \partial_\nu\partial_\alpha h_{\beta\mu} - \partial_\mu\partial_\nu h_{\alpha\beta} - \partial_\alpha\partial_\beta h_{\mu\nu}) \quad . \tag{1.2.7}$$

By contraction with the Minkowski metric $\eta$, one finds that the linearizations of the Ricci tensor (1.1.29) and the scalar curvature (1.1.30) are given by

$$\tilde{R}_{\mu\nu} = \tfrac{1}{2}(\partial_\mu\partial^\alpha h_{\alpha\nu} + \partial_\nu\partial^\alpha h_{\alpha\mu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}) \tag{1.2.8}$$

$$\tilde{R} = \partial^\mu\partial^\nu h_{\mu\nu} - \Box h \tag{1.2.9}$$

with the d'Alembertian $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and the contraction $h = \eta^{\mu\nu}h_{\mu\nu}$. The linear version of the Einstein tensor (1.1.31) thus reads

$$\tilde{G}_{\mu\nu} = \tfrac{1}{2}(\partial_\mu\partial^\alpha h_{\alpha\nu} + \partial_\nu\partial^\alpha h_{\alpha\mu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \eta_{\mu\nu}\Box h) \ . \tag{1.2.10}$$

Introducing the auxiliary variable

$$\gamma_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}h\eta_{\mu\nu} \quad , \tag{1.2.11}$$

the linearized Einstein tensor is simplified to the expression

$$\tilde{G}_{\mu\nu} = \tfrac{1}{2}(\partial_\mu\partial^\alpha\gamma_{\alpha\nu} + \partial_\nu\partial^\alpha\gamma_{\alpha\mu} - \eta_{\mu\nu}\partial^\alpha\partial^\beta\gamma_{\alpha\beta} - \Box\gamma_{\mu\nu}) \quad , \tag{1.2.12}$$

which in combination with the Einstein equation (1.1.32) yields the linearized Einstein equation

$$\Box\gamma_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta\gamma_{\alpha\beta} - \partial_\mu\partial^\alpha\gamma_{\alpha\nu} - \partial_\nu\partial^\alpha\gamma_{\alpha\mu} = -16\pi T_{\mu\nu} \quad . \tag{1.2.13}$$

In terms of the Minkowski supermetric (1.1.67), the auxiliary variable $\gamma$ is simply the dual variable

$$\gamma^{\mu\nu} = H^{\mu\nu\alpha\beta}h_{\alpha\beta} \tag{1.2.14}$$

of the metric perturbation $h$. It is also the perturbation of the inverse Lorentz metric density since the linearization

$$\sqrt{-g} \mapsto 1 + \tfrac{1}{2}h \tag{1.2.15}$$

of the density factor and the definition (1.2.3) of the inverse Lorentz metric result in the linearization

$$\tilde{\mathfrak{g}}^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu} \quad . \tag{1.2.16}$$

**Fierz-Pauli Equation.** The linearized Einstein equation (1.2.13) is simplified considerably by utilizing coordinate transformations as the gauge freedom of general relativity. In order to preserve the weakness of the gravitational field, only infinitesimal coordinate transformations are allowed. These are transformations of the form

$$x^\mu \mapsto x^\mu + v^\mu \quad , \tag{1.2.17}$$

where the vector field $v$ is of the same order as the perturbation $h$. In this case an arbitrary Lorentz metric $g$ transforms as

$$g_{\mu\nu} \mapsto g_{\mu\nu} - \mathcal{L}_v g_{\mu\nu} \quad , \tag{1.2.18}$$

where $\mathcal{L}_v$ denotes the Lie derivative with respect to the vector field $v$ given by

$$\mathcal{L}_v g_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu \quad . \tag{1.2.19}$$

For the linearized Lorentz metric $\tilde{g}$ this means a gauge transformation

$$h_{\mu\nu} \mapsto h_{\mu\nu} - \partial_\mu v_\nu - \partial_\nu v_\mu \tag{1.2.20}$$

of the perturbation $h$, or equivalently

$$\gamma_{\mu\nu} \mapsto \gamma_{\mu\nu} - \partial_\mu v_\nu - \partial_\nu v_\mu + \eta_{\mu\nu}\partial_\alpha v^\alpha \tag{1.2.21}$$

of its dual variable $\gamma$. The linearized Christoffel symbol transforms as

$$\tilde{\Gamma}^\mu{}_{\alpha\beta} \mapsto \tilde{\Gamma}^\mu{}_{\alpha\beta} + \partial_\alpha\partial_\beta v^\mu \quad , \tag{1.2.22}$$

and the linearized Riemann tensor

$$\tilde{R}_{\mu\alpha\nu\beta} \mapsto \tilde{R}_{\mu\alpha\nu\beta} \tag{1.2.23}$$

is gauge invariant under the restricted number of allowed coordinate transformations. If a solution of the equation

$$\Box v^\nu = \partial_\mu \gamma^{\mu\nu} \tag{1.2.24}$$

is taken as the vector field $v$, then the transformed perturbation $\gamma$ will satisfy the gauge condition

$$\partial_\mu \gamma^{\mu\nu} = 0 \quad , \tag{1.2.25}$$

which according to the relation (1.2.16) is simply the linearization of the harmonic gauge condition (1.1.42). The gauge condition (1.2.25) turns the linearized Einstein equation (1.2.13) into the Fierz-Pauli equation

$$\Box \gamma^{\mu\nu} = -16\pi T^{\mu\nu} \quad , \tag{1.2.26}$$

whose solutions are interpreted as gravitational radiation, where the homogeneous part represents waves coming from infinity and the inhomogeneous part represents waves generated by the matter source $T$.

**Linearized Schwarzschild Solution.** The linearized Einstein tensor (1.2.12) obviously satisfies the linearization

$$\partial_\mu \tilde{G}^{\mu\nu} = 0 \tag{1.2.27}$$

of the contracted Bianchi identity (1.1.40), from which it immediately follows that the equation of motion (1.1.51) is also linearized to

$$\partial_\mu T^{\mu\nu} = 0 \quad . \tag{1.2.28}$$

The matter sources $T$ thus move according to the Minkowski equation of motion, which is certainly satisfied for the stress-energy tensor

(1.2.29) $$T^{\mu\nu}(x) = \rho(\vec{x}) e_0^\mu e_0^\nu$$

of stationary dust. Let in addition the mass density be spherically symmetric

(1.2.30) $$\rho(\vec{x}) = \rho(|\vec{x}|)$$

and localized

(1.2.31) $$\rho(|\vec{x}|) = 0 \text{ for } |\vec{x}| > R$$

inside a region of radius $R$. As a consequence the total mass $M$ of the dust cloud is given by

(1.2.32) $$M = 4\pi \int_0^R dr\, r^2 \rho(r) \quad .$$

Since the retarded and advanced solutions

(1.2.33) $$G_\pm(x) = \frac{1}{2\pi} \delta(x^\mu x_\mu) \theta(\pm x^0)$$

(1.2.34) $$= \frac{\delta(t \mp |\vec{x}|)}{4\pi |\vec{x}|}$$

of the wave equation as Green functions satisfy the relation

(1.2.35) $$\Box G_\pm(x) = \delta(x) \quad ,$$

they can be used to obtain general solutions

(1.2.36) $$\gamma_\pm^{\mu\nu}(x) = -16\pi \int dx'\, G_\pm(x - x') T^{\mu\nu}(x')$$

of the Fierz-Pauli equation (1.2.26). Inserting the stress-energy tensor (1.2.29) and performing the time integration, we find

(1.2.37) $$\gamma^{\mu\nu}(x) = -4 e_0^\mu e_0^\nu \int d\vec{x}'\, \frac{\rho(|\vec{x}'|)}{|\vec{x} - \vec{x}'|}$$

for both the retarded and advanced solution. The angular space integration leads to

(1.2.38) $$\int d\vec{x}'\, \frac{\rho(|\vec{x}'|)}{|\vec{x} - \vec{x}'|} = 2\pi \int_0^R dr\, r^2 \rho(r) \int_{-1}^1 \frac{d(\cos\theta)}{\sqrt{|\vec{x}|^2 - 2|\vec{x}| r \cos\theta + r^2}}$$

(1.2.39) $$= \frac{2\pi}{|\vec{x}|} \int_0^R dr\, r \rho(r) \Big( \Big| |\vec{x}| + r \Big| - \Big| |\vec{x}| - r \Big| \Big) \quad ,$$

which for $|\vec{x}| \geq R$ becomes

(1.2.40) $$\frac{4\pi}{|\vec{x}|} \int_0^R dr\, r^2 \rho(r) = \frac{M}{|\vec{x}|}$$

from equation (1.2.32). In the empty space surrounding the dust cloud we thus obtain the linearized Schwarzschild solution

$$\gamma^{\mu\nu}(x) = -\frac{4M}{|\vec{x}|} e_0^\mu e_0^\nu \quad , \tag{1.2.41}$$

which using relation (1.2.11) translates into

$$h_{\mu\nu}(x) = -\frac{2M}{|\vec{x}|} \delta_{\mu\nu} \quad . \tag{1.2.42}$$

Note that the result correctly reproduces the solution within the Newtonian framework since the corresponding gravitational potential $\varphi$ is obtained as

$$\varphi(x) = \tfrac{1}{2} h_{00}(x) \tag{1.2.43}$$

from the linear approximation to general relativity.

**Light Deflection.** The mass $M_\odot$ and the radius $R_\odot$ of the sun in Planck units are given by

$$M_\odot \approx 9{,}13 \cdot 10^{37} \tag{1.2.44}$$

$$R_\odot \approx 4{,}31 \cdot 10^{43} \quad . \tag{1.2.45}$$

Inserted into the linearized Schwarzschild solution (1.2.42) these values yield the result

$$|h_{\mu\nu}| \leq \frac{2M_\odot}{R_\odot} \approx 4{,}24 \cdot 10^{-6} \quad , \tag{1.2.46}$$

indicating that the linear approximation is quite well suited to describe the gravitational physics in our solar system. Consider as an example the deflection of a light ray at the surface of the sun, as illustrated in FIGURE 4, where the linearized geodesic equation is given by

$$\dot{p}^\mu = -\tilde{\Gamma}^\mu{}_{\alpha\beta}\, p^\alpha_{\text{in}}\, p^\beta_{\text{in}} \quad . \tag{1.2.47}$$

For a light ray moving into the positive $z$-direction, i.e. $p^\mu_{\text{in}} = (k, 0, 0, k)$ with $k > 0$, and the linearized Schwarzschild solution (1.2.42) we find

$$\dot{p}^0 = -\, k\partial_3 h_{00} \tag{1.2.48}$$

$$\dot{p}^1 = -\, k\partial_1 h_{00} \tag{1.2.49}$$

$$\dot{p}^2 = -\, k\partial_2 h_{00} \tag{1.2.50}$$

$$\dot{p}^3 = -\, k\partial_0 h_{00} \quad . \tag{1.2.51}$$

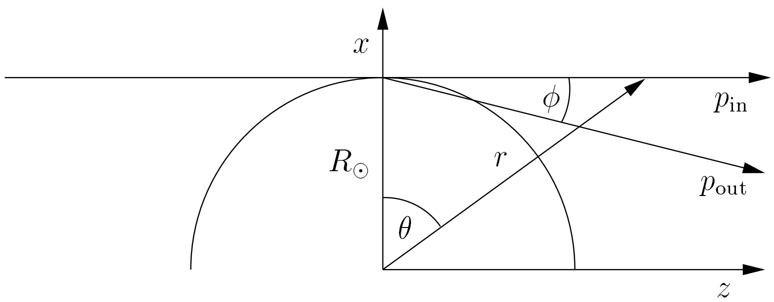



FIGURE 4. Light deflection at the surface of the sun

Therefore the change in momentum towards the sun is given by

$$\dot{p}^x = 2kM_\odot \frac{\partial}{\partial x}\frac{1}{r} \tag{1.2.52}$$

$$= -2kM_\odot \frac{\cos\theta}{r^2} \tag{1.2.53}$$

and integrated along the undeflected path of the ray yields the deflection angle

$$|\phi| \approx -\tan\phi \tag{1.2.54}$$

$$= -\frac{p^x_{\text{out}}}{p^z_{\text{in}}} \tag{1.2.55}$$

$$= -\int dz\, \frac{\dot{p}^x}{k} \tag{1.2.56}$$

$$= \frac{2M_\odot}{R_\odot}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\, \cos\theta \tag{1.2.57}$$

$$= \frac{4M_\odot}{R_\odot} \tag{1.2.58}$$

$$\approx 8{,}48\cdot 10^{-6} \quad . \tag{1.2.59}$$

This angle of approximately $1{,}75$” is about twice the value calculated from Newtonian gravity and was confirmed in 1919 by Arthur Eddington, who observed the deflection of starlight during a solar eclipse.

**Polarization.** Linearizing and gauging the vacuum Einstein equations (1.1.34) leads to the homogeneous wave equation

$$\Box h_{\mu\nu} = 0 \tag{1.2.60}$$

in combination with the supplementary condition

$$H^{\mu\nu\alpha\beta}\partial_\mu h_{\alpha\beta} = 0 \quad . \tag{1.2.61}$$

Substituting the inverse Fourier transform

$$h_{\mu\nu}(x) = \frac{1}{(2\pi)^2}\int dp\, e^{-ipx} h_{\mu\nu}(p) \tag{1.2.62}$$

into the wave equation (1.2.60) yields the corresponding equation

$$p^\alpha p_\alpha h_{\mu\nu}(p) = 0 \tag{1.2.63}$$

in energy-momentum space. Obviously a general solution is given by

$$h_{\mu\nu}(p) = 2\pi\delta(p^\alpha p_\alpha) a_{\mu\nu}(p) \quad , \tag{1.2.64}$$

where the symmetric polarization tensor $a$ is arbitrary apart from the condition

$$a_{\mu\nu}(-p) = \overline{a}_{\mu\nu}(p) \quad , \tag{1.2.65}$$

ensuring the reality of the gravitational field. The inverse Fourier transform (1.2.62) therefore becomes

$$h_{\mu\nu}(x) = \frac{1}{2\pi}\int dp\,\delta(p^\alpha p_\alpha)a_{\mu\nu}(p)e^{-ipx} \tag{1.2.66}$$

$$= \int dp\,\Big(\Gamma_-(p) + \Gamma_+(p)\Big)a_{\mu\nu}(p)e^{-ipx} \tag{1.2.67}$$

$$= \int dp\,\Gamma_+(p)\Big(\overline{a}_{\mu\nu}(p)e^{ipx} + a_{\mu\nu}(p)e^{-ipx}\Big) \quad , \tag{1.2.68}$$

where the forward and backward light cone functions

$$\Gamma_\pm(p) = \frac{1}{2\pi}\delta(p^\mu p_\mu)\theta(\pm p^0) \tag{1.2.69}$$

are formally identical to the retarded and advanced Green functions (1.2.33). Since the forward light cone function $\Gamma_+$ can be given the form

$$\Gamma_+(p) = \frac{\delta(p^0 - |\vec{p}|)}{4\pi|\vec{p}|} \quad , \tag{1.2.70}$$

the general solution of the homogeneous wave equation (1.2.60) is a linear superposition of plane waves

$$h_{\mu\nu} = a_{\mu\nu}e^{-ipx} + \overline{a}_{\mu\nu}e^{ipx} \tag{1.2.71}$$

obeying the massless dispersion relation $p^0 = |\vec{p}|$, which is to hold implicitly from now on for the energy-momentum vector $p$. The gauge condition (1.2.61) puts the constraint

$$H^{\mu\nu\alpha\beta}p_\mu a_{\alpha\beta} = 0 \tag{1.2.72}$$

on the momentum $\vec{p}$ and the polarization $a$ of a gravitational plane wave. The four relations reduce the number of independent components of the symmetric polarization tensor $a$ from ten to six. Since equation (1.2.24) determines the vector field $v$ only up to a homogeneous solution, the gauge condition (1.2.25) does not fix the gauge transformation (1.2.20) completely. Instead one may add to the vector field $v$ a plane wave solution

$$v_\mu = i\overline{b}_\mu e^{ipx} - ib_\mu e^{-ipx} \quad , \tag{1.2.73}$$

which corresponds to a gauge transformation

$$a_{\mu\nu} \mapsto a_{\mu\nu} + p_\mu b_\nu + p_\nu b_\mu \tag{1.2.74}$$

preserving the gauge condition (1.2.72). This additional four dimensional gauge freedom further reduces the number of independent components of the polarization tensor $a$ from six to two, at last truly representing its physical degrees of freedom.

**Spin.** In our discussion of the spin for a gravitational wave we closely follow Weinberg[17]. Without loss of generality consider a gravitational plane wave traveling in the positive $z$-direction, i.e. with energy-momentum $p^\mu = (k, 0, 0, k)$ for $k > 0$. The gauge condition (1.2.72) then yields the four constraints

$$a_{22} = -a_{11} \tag{1.2.75}$$
$$a_{01} = -a_{13} \tag{1.2.76}$$
$$a_{02} = -a_{23} \tag{1.2.77}$$
$$a_{03} = -\tfrac{1}{2}(a_{00} + a_{33}) \quad . \tag{1.2.78}$$

Under the transformation (1.2.74) the remaining six independent components of the polarization tensor change as

$$a_{11} \mapsto a_{11} \qquad a_{12} \mapsto a_{12} \tag{1.2.79}$$
$$a_{13} \mapsto a_{13} + ka_1 \qquad a_{23} \mapsto a_{23} + ka_2 \tag{1.2.80}$$
$$a_{00} \mapsto a_{00} + 2ka_0 \qquad a_{33} \mapsto a_{33} + 2ka_3 \quad . \tag{1.2.81}$$

Hence only the components $a_{11}$ and $a_{12}$ are gauge invariant and therefore physical. A rotation of an angle $\theta$ about the z-axis, i.e. a Lorentz transformation

$$a_{\mu\nu} \mapsto \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta a_{\mu\nu} \tag{1.2.82}$$

with the matrix

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.2.83}$$

reveals the nature of these components. Defining

$$a_\pm = a_{11} \mp ia_{12} \tag{1.2.84}$$
$$b_\pm = a_{13} \mp ia_{23} \tag{1.2.85}$$
$$c_\pm = a_{00} \mp ia_{33} \quad , \tag{1.2.86}$$

one obtains the transformations

$$a_\pm \mapsto e^{\pm i2\theta} a_\pm \tag{1.2.87}$$
$$b_\pm \mapsto e^{\pm i\theta} b_\pm \tag{1.2.88}$$
$$c_\pm \mapsto c_\pm \quad , \tag{1.2.89}$$

which identify $a_\pm$, $b_\pm$ and $c_\pm$ as polarizations pertaining to plane waves with spin 2, 1 and 0, respectively. The gauging thus provides a projection onto the physical components $a_\pm$. In this way invariance under coordinate transformations is the reason why gravitational waves possess spin 2.

## 3. Lagrangian Formulation

**Action Principle.** A standard way of presenting a physical theory is given by the Lagrangian formalism, in which the field equation is derived from a scalar

$$S = \int dx\, \mathfrak{L} \quad , \tag{1.3.1}$$

known as the action, by use of a variational principle. In general relativity the Lagrangian density $\mathfrak{L}$ is a functional of the form

$$\mathfrak{L} = \mathfrak{L}(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \partial_\alpha \partial_\beta g_{\mu\nu}, \dots) \quad . \tag{1.3.2}$$

Since the volume element $dx$ is a scalar density of weight $-1$, for the action (1.3.1) to be a scalar the Lagrangian density $\mathfrak{L}$ must be a scalar density of weight 1 and accordingly of the form

$$\mathfrak{L} = \sqrt{-g} L(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \partial_\alpha \partial_\beta g_{\mu\nu}, \dots) \quad , \tag{1.3.3}$$

where $L$ is again a scalar called the Lagrangian. The action principle demands that under a local variation of the metric

$$g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu} \tag{1.3.4}$$

the action

$$S \mapsto S + \delta S \tag{1.3.5}$$

is stationary, i.e.

$$\delta S = 0 \quad . \tag{1.3.6}$$

The relation

$$\delta S = \int dx\, \frac{\delta \mathfrak{L}}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \tag{1.3.7}$$

then yields the field equation

$$\frac{\delta \mathfrak{L}}{\delta g_{\mu\nu}} = 0 \quad , \tag{1.3.8}$$

where the functional derivative

$$\frac{\delta \mathfrak{L}}{\delta g_{\mu\nu}} = \frac{\partial \mathfrak{L}}{\partial g_{\mu\nu}} - \partial_\alpha \frac{\partial \mathfrak{L}}{\partial(\partial_\alpha g_{\mu\nu})} + \partial_\alpha \partial_\beta \frac{\partial \mathfrak{L}}{\partial(\partial_\alpha \partial_\beta g_{\mu\nu})} - \cdots \tag{1.3.9}$$

is obtained via integration by parts. Consider a Lagrangian density of the form

$$\mathfrak{L}' = \partial_\mu A^\mu(g) \quad , \tag{1.3.10}$$

where $A^\mu(g)$ is an arbitrary function of the Lorentz metric $g$. The corresponding action

$$S' = \int dx\, \partial_\mu A^\mu(g) \tag{1.3.11}$$

is converted into a surface integral at infinity using the famous theorem

$$\int_\Omega dx \partial_\mu A^\mu(g) = \int_{\partial\Omega} d\sigma_\mu A^\mu(g) \tag{1.3.12}$$

by Carl Friedrich Gauß. Since the variation of the metric $g$ supposedly vanishes at infinity, the action principle is satisfied trivially. As a result adding a divergence term such as (1.3.10) to a Lagrangian density (1.3.2) does not alter the field equation (1.3.8) and serves as a method to simplify a Lagrangian density.

**Hilbert Action.** Even before Albert Einstein in 1915 completed his general theory of relativity David Hilbert[18] found that the Lagrangian density

$$\mathfrak{L}_{\mathrm{H}} = \sqrt{-g}\, R \tag{1.3.13}$$

with the scalar curvature (1.1.30) being the Lagrangian function $L$ has the functional derivative

$$\frac{\delta \mathfrak{L}_{\mathrm{H}}}{\delta g_{\mu\nu}} = -\sqrt{-g}\, G^{\mu\nu} \tag{1.3.14}$$

reproducing the geometric side of the Einstein equation (1.1.32). In order to provide the material side, the matter Lagrangian density $\mathfrak{L}_{\mathrm{M}}$ must satisfy the relation

$$\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta g_{\mu\nu}} = 8\pi \sqrt{-g} T^{\mu\nu} \quad . \tag{1.3.15}$$

The Einstein equation (1.1.32) is thus derived from the Lagrangian density

$$\mathfrak{L}_{\mathrm{E}} = \mathfrak{L}_{\mathrm{H}} + \mathfrak{L}_{\mathrm{M}} \tag{1.3.16}$$

defined by (1.3.13) and (1.3.15), respectively. Subtracting from the Hilbert Lagrangian density

$$\mathfrak{L}_{\mathrm{H}} = \mathfrak{g}^{\mu\nu} R_{\mu\nu} \tag{1.3.17}$$

the divergence

$$\mathfrak{L}' = \partial_\mu (\mathfrak{g}^{\alpha\beta} \Gamma^{\mu}{}_{\alpha\beta} - \mathfrak{g}^{\mu\nu} \Gamma^{\alpha}{}_{\alpha\nu}) \quad , \tag{1.3.18}$$

one obtains the equivalent expression

$$\mathfrak{L}'_{\mathrm{H}} = \mathfrak{g}^{\mu\nu} (\Gamma^{\alpha}{}_{\mu\beta} \Gamma^{\beta}{}_{\nu\alpha} - \Gamma^{\alpha}{}_{\mu\nu} \Gamma^{\beta}{}_{\alpha\beta}) \quad . \tag{1.3.19}$$

Whereas the Hilbert Lagrangian density (1.3.13) is of the form

$$\mathfrak{L}_{\mathrm{H}} = \mathfrak{L}_{\mathrm{H}}(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \partial_\alpha \partial_\beta g_{\mu\nu}) \quad , \tag{1.3.20}$$

containing second derivatives of the Lorentz metric $g$, the expression (1.3.19) is of the simpler form

$$\mathfrak{L}'_{\mathrm{H}} = \mathfrak{L}'_{\mathrm{H}}(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}) \quad , \tag{1.3.21}$$

containing only first derivatives. Note that expression (1.3.19) is no longer a scalar density but has a functional derivative equal to (1.3.14) for the Hilbert Lagrangian density (1.3.13).

**Gauge Invariance.** The well-known theorem by Emmy Noether states that the gauge invariance of an action results in a conservation law. In general relativity the action $S$ being a scalar is invariant under coordinate transformations. Our derivation of the corresponding conservation law is held closely to d'Inverno[19]. Consider a vector field $v$ vanishing at infinity and interpret the infinitesimal coordinate transformation (1.2.17) as a local variation $\delta x = v$ of the coordinate $x$, which according to (1.2.18) results in a local variation

$$\delta g_{\mu\nu} = -\mathcal{L}_v g_{\mu\nu} \tag{1.3.22}$$

of the Lorentz metric $g$. Its substitution into relation (1.3.7) and the introduction of the abbreviation

$$\mathfrak{L}^{\mu\nu} = \frac{\delta \mathfrak{L}}{\delta g_{\mu\nu}} \tag{1.3.23}$$

leads to the variation

$$\delta S = -\int dx\, \mathfrak{L}^{\mu\nu} \mathcal{L}_v g_{\mu\nu} \quad , \tag{1.3.24}$$

which must vanish by the invariance of the action $S$ under coordinate transformations. Inserting the Lie derivative (1.2.19) one thus obtains the equation

$$\int dx\, \mathfrak{L}^{\mu\nu} \nabla_\mu v_\nu = 0 \quad , \tag{1.3.25}$$

which upon integration by parts becomes

$$\int dx\, \nabla_\mu \mathfrak{L}^{\mu\nu} v_\nu - \int dx\, \nabla_\mu (\mathfrak{L}^{\mu\nu} v_\nu) = 0 \quad , \tag{1.3.26}$$

where the second integrand is the covariant divergence $\nabla_\mu \mathfrak{u}^\mu$ of the vector density

$$\mathfrak{u}^\mu = \mathfrak{L}^{\mu\nu} v_\nu \tag{1.3.27}$$

of weight 1. As a special case of equation (1.1.50), one however finds that the covariant divergence of a vector density $\mathfrak{u}$ of weight 1 is identical to its ordinary divergence, i.e.

$$\nabla_\mu \mathfrak{u}^\mu = \partial_\mu \mathfrak{u}^\mu \quad . \tag{1.3.28}$$

Hence the second integrand in equation (1.3.26) is a divergence term, whose integral accordingly vanishes. As the vector field $v$ is arbitrary, this yields the conservation law

$$\nabla_\mu \mathfrak{L}^{\mu\nu} = 0 \quad , \tag{1.3.29}$$

which for the Hilbert Lagrangian density (1.3.13) becomes the contracted Bianchi identity (1.1.40).

**Linearization.** The linearization of the Einstein equation (1.1.32) may as well be performed within the Lagrangian framework. Restricting to lowest order terms in the simplified Hilbert Lagrangian density (1.3.19) and introducing a factor $-2$ for convenience, one obtains the quadratic Lagrangian density

$$\mathfrak{L}_{\mathrm{Q}} = \tfrac{1}{2}\partial_\mu h_{\alpha\beta} \partial^\mu h^{\alpha\beta} - \tfrac{1}{2}\partial_\mu h \partial^\mu h - \partial_\mu h_{\nu\alpha} \partial^\nu h^{\mu\alpha} + \partial_\mu h^{\mu\nu} \partial_\nu h \tag{1.3.30}$$

$$= \tfrac{1}{2}\partial_\mu \gamma_{\alpha\beta} \partial^\mu \gamma^{\alpha\beta} - \tfrac{1}{4}\partial_\mu \gamma \partial^\mu \gamma - \partial_\mu \gamma_{\nu\alpha} \partial^\nu \gamma^{\mu\alpha} \quad . \tag{1.3.31}$$

Adding the gauge fixing term

$$\mathfrak{L}_{\mathrm{G}} = \partial_\mu \gamma^{\mu\nu} \partial^\alpha \gamma_{\alpha\nu} \tag{1.3.32}$$

supplements the linearized harmonic gauge condition (1.2.25) and results in the wave Lagrangian density

$$\mathfrak{L}_{\mathrm{W}} = \mathfrak{L}_{\mathrm{Q}} + \mathfrak{L}_{\mathrm{G}} \quad , \tag{1.3.33}$$

whose functional derivative

$$\frac{\delta \mathfrak{L}_{\mathrm{W}}}{\delta h_{\mu\nu}} = -\Box \gamma^{\mu\nu} \tag{1.3.34}$$

yields the d'Alembertian term of the Fierz-Pauli equation (1.2.26). In order to provide the correct source term, the matter Lagrangian density $\mathfrak{L}_{\mathrm{M}}$ must satisfy the relation

$$\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta h_{\mu\nu}} = -16\pi T^{\mu\nu} \quad . \tag{1.3.35}$$

The Fierz-Pauli equation (1.2.26) may then be derived from the Lagrangian density

$$\mathfrak{L}_{\mathrm{FP}} = \mathfrak{L}_{\mathrm{W}} + \mathfrak{L}_{\mathrm{M}} \quad . \tag{1.3.36}$$

Subtracting the divergence term

$$\mathfrak{L}' = \partial_\mu(\gamma^{\mu\beta}\partial^\alpha\gamma_{\alpha\beta} - \gamma_{\alpha\beta}\partial^\alpha\gamma^{\mu\beta}) \tag{1.3.37}$$

simplifies the wave Lagrangian density $\mathfrak{L}_{\mathrm{W}}$ to the expression

$$\tilde{\mathfrak{L}} = \tfrac{1}{2}(\partial_\mu\gamma_{\alpha\beta}\partial^\mu\gamma^{\alpha\beta} - \tfrac{1}{2}\partial_\mu\gamma\partial^\mu\gamma) \quad , \tag{1.3.38}$$

which using relation (1.2.14) may be written in the compact form

$$\tilde{\mathfrak{L}} = \tfrac{1}{2}H^{\mu\nu\alpha\beta}\partial_\rho h_{\mu\nu}\partial^\rho h_{\alpha\beta} \quad . \tag{1.3.39}$$

**Energy-Momentum.** The canonical stress-energy tensor $\tilde{T}$ corresponding to the gravitational wave Lagrangian $\tilde{L}$ is given by

$$\tilde{T}^{\mu\nu} = \partial^\mu h_{\alpha\beta}\frac{\partial\tilde{L}}{\partial(\partial_\nu h_{\alpha\beta})} - \eta^{\mu\nu}\tilde{L} \tag{1.3.40}$$

$$= H^{\rho\sigma\kappa\lambda}\partial^\mu h_{\rho\sigma}\partial^\nu h_{\kappa\lambda} - \tfrac{1}{2}\eta^{\mu\nu}H^{\rho\sigma\kappa\lambda}\partial_\alpha h_{\rho\sigma}\partial^\alpha h_{\kappa\lambda} \tag{1.3.41}$$

$$= H^{\mu\nu\alpha\beta}H^{\rho\sigma\kappa\lambda}\partial_\alpha h_{\rho\sigma}\partial_\beta h_{\kappa\lambda} \tag{1.3.42}$$

and satisfies the conservation law

$$\partial_\mu\tilde{T}^{\mu\nu} = 0 \quad , \tag{1.3.43}$$

in concordance with the equation of motion (1.2.28). Integration over a spacelike hypersurface $S$ yields the wave energy-momentum vector

$$\tilde{P}^\nu = \int_S d\sigma_\mu\,\tilde{T}^{\mu\nu} \quad . \tag{1.3.44}$$

Under the condition that the gravitational field vanishes at spacelike infinity, this vector is independent of the particular choice of the surface since the application of the Gauß law (1.3.12) to two hypersurfaces $S$ and $S'$ with $\partial\Omega = S - S'$ leads to the difference

$$\tilde{P}^\nu - \tilde{P}'^\nu = \int_S d\sigma_\mu\,\tilde{T}^{\mu\nu} - \int_{S'} d\sigma_\mu\,\tilde{T}^{\mu\nu} \tag{1.3.45}$$

$$= \int_{S-S'} d\sigma_\mu\,\tilde{T}^{\mu\nu} \tag{1.3.46}$$

$$= \int_\Omega dx\,\partial_\mu\tilde{T}^{\mu\nu} \quad , \tag{1.3.47}$$

which vanishes according to the conservation law (1.3.43). Therefore without loss of generality consider the wave energy-momentum vector

$$\tilde{P}^\mu = \int\limits_{t=0} d\vec{x}\,\tilde{T}^{\mu 0} \quad . \tag{1.3.48}$$

In order to express the energy-momentum of gravitational waves in terms of their polarization we insert the superposition (1.2.68) of plane waves into the stress-energy tensor density (1.3.42) to obtain

$$\tilde{T}^{\mu\nu} = \int dp\,dp'\,\frac{\delta(p^0-|\vec{p}|)}{4\pi|\vec{p}|}\frac{\delta(p'^0-|\vec{p}'|)}{4\pi|\vec{p}'|}H^{\mu\nu\alpha\beta}p_\alpha p'_\beta \tag{1.3.49}$$

$$H^{\rho\sigma\kappa\lambda}\Big(e^{i(p-p')x}\overline{a}_{\rho\sigma}(p)a_{\kappa\lambda}(p') + e^{-i(p-p')x}a_{\rho\sigma}(p)\overline{a}_{\kappa\lambda}(p') \tag{1.3.50}$$

$$-\,e^{i(p+p')x}\overline{a}_{\rho\sigma}(p)\overline{a}_{\kappa\lambda}(p') - e^{-i(p+p')x}a_{\rho\sigma}(p)a_{\kappa\lambda}(p')\Big) \quad . \tag{1.3.51}$$

Because of the relation

$$\int d\vec{x}\,e^{i\vec{p}\vec{x}} = (2\pi)^3\delta(\vec{p}) \tag{1.3.52}$$

the integration (1.3.48) turns the spatial exponential functions of (1.3.50) into $\delta(\vec{p}-\vec{p}')$ and of (1.3.51) into $\delta(\vec{p}+\vec{p}')$. Since

$$H^{\mu 0\alpha\beta}p_\alpha p'_\beta = |\vec{p}|\cdot\begin{cases}(|\vec{p}|,\vec{p}) & \text{for } \vec{p}=\vec{p}' \\ (0,\vec{p}) & \text{for } \vec{p}=-\vec{p}'\end{cases} \quad \text{and } p^0=|\vec{p}|,\, p'^0=|\vec{p}'| \quad , \tag{1.3.53}$$

we find

$$\tilde{P}^\mu = \tfrac{1}{2}\int d\vec{p}\,\Big\{(|\vec{p}|,\vec{p})H^{\rho\sigma\kappa\lambda}\Big(\overline{a}_{\rho\sigma}(\vec{p})a_{\kappa\lambda}(\vec{p}) + a_{\rho\sigma}(\vec{p})\overline{a}_{\kappa\lambda}(\vec{p})\Big) \tag{1.3.54}$$

$$-\,(0,\vec{p})H^{\rho\sigma\kappa\lambda}\Big(\overline{a}_{\rho\sigma}(\vec{p})\overline{a}_{\kappa\lambda}(-\vec{p})e^{2i|\vec{p}|t} + a_{\rho\sigma}(\vec{p})a_{\kappa\lambda}(-\vec{p})e^{-2i|\vec{p}|t}\Big)\Big\} \quad , \tag{1.3.55}$$

where the polarization tensor with momentum argument is defined by

$$a_{\mu\nu}(\vec{p}) = \sqrt{\pi}\int\frac{dp^0}{\sqrt{p^0}}\,\delta(p^0-|\vec{p}|)a_{\mu\nu}(p) \quad . \tag{1.3.56}$$

The integrand (1.3.55) is an antisymmetric function, whose integral accordingly vanishes. Hence the energy-momentum vector $\tilde{P}$ of gravitational waves in terms of the polarization tensor $a$ is given by

$$\tilde{P}^\mu = \int d\vec{p}\,\vec{p}^\mu H^{\rho\sigma\kappa\lambda}\overline{a}_{\rho\sigma}(\vec{p})a_{\kappa\lambda}(\vec{p}) \quad , \tag{1.3.57}$$

where $\vec{p}^\mu = (|\vec{p}|,\vec{p})$. Since the supermetric $H$ is indefinite, the wave Hamiltonian

$$\tilde{E} = \int d\vec{p}\,|\vec{p}|H^{\rho\sigma\kappa\lambda}\overline{a}_{\rho\sigma}(\vec{p})a_{\kappa\lambda}(\vec{p}) \tag{1.3.58}$$

indicates the occurrence of gravitational waves with negative energy. This unphysical feature is removed by the introduction of the gauge condition (1.2.61). To this end consider again without loss of generality a gravitational wave traveling in the positive $z$-direction, where the gauge condition (1.2.61) imposes the four constraints (1.2.75) to (1.2.78) onto the polarization tensor $a$. As a consequence the supermetric square becomes

$$H^{\mu\nu\alpha\beta}\overline{a}_{\mu\nu}a_{\alpha\beta} = \overline{a}_+ a_+ + \overline{a}_- a_- \quad , \tag{1.3.59}$$

where the spin-2 polarizations $a_\pm$ have been defined in (1.2.87). In this way the gauge condition ensures that only the two physical polarizations contribute to the wave Hamiltonian

$$\tilde{E} = \int d\vec{p}\, |\vec{p}| \Big( \overline{a}_+(\vec{p}) a_+(\vec{p}) + \overline{a}_-(\vec{p}) a_-(\vec{p}) \Big) \tag{1.3.60}$$

and hence restores its positivity.

CHAPTER 2

# Gravitons

## 1. Canonical Quantization

**Quantization.** The other sensational discovery at the beginning of this century besides general relativity was that the laws of physics in their transition from macro- to microcosm undergo a strange morphing process known as quantization. Whereas for electrodynamics this discovery has led to the creation of a mature theory reproducing experimental results with remarkable accuracy, for gravity the situation on both the theoretical and experimental side is quite unsatisfactory. Nevertheless the quantization of the linear approximation to general relativity is straightforward and might provide a hint at more sophisticated approaches towards a unification of the two revolutionary theories.

**Commutation Relations.** The canonical quantization of gravitational waves begins by postulating the Heisenberg relation

$$[h_{\mu\nu}(t,\vec{x}), \pi^{\alpha\beta}(t,\vec{x}')] = i\delta_{\mu\nu}{}^{\alpha\beta}\delta(\vec{x}-\vec{x}') \tag{2.1.1}$$

for the perturbation $h$ and its canonical conjugate momentum

$$\pi^{\mu\nu} = \frac{\partial\tilde{\mathfrak{L}}}{\partial(\partial_0 h_{\mu\nu})} = \partial_0\gamma^{\mu\nu} \quad , \tag{2.1.2}$$

where $\tilde{\mathfrak{L}}$ is the wave Lagrangian density (1.3.39). With the help of (1.2.14) one obtains the equal-time commutation relation

$$[h_{\mu\nu}(t,\vec{x}), \partial_0 h_{\alpha\beta}(t,\vec{x}')] = iH_{\mu\nu\alpha\beta}\delta(\vec{x}-\vec{x}') \tag{2.1.3}$$

with the Minkowski supermetric $H$, as defined in (1.1.67). The transition to momentum space is performed by decomposing the plane wave solution (1.2.68) as

$$h_{\mu\nu}(x) = h^+_{\mu\nu}(x) + h^-_{\mu\nu}(x) \tag{2.1.4}$$

into the positive energy solution

$$h^+_{\mu\nu}(x) = \int dp\,\Gamma_+(p)\,e^{ipx}a^*_{\mu\nu}(p) \tag{2.1.5}$$

and the negative energy solution

$$h^-_{\mu\nu}(x) = \int dp\,\Gamma_+(p)\,e^{-ipx}a_{\mu\nu}(p) \quad . \tag{2.1.6}$$

Similar expansions in terms of the momentum argument polarization tensor (1.3.56) are given by

$$h^{+}_{\mu\nu}(x) = \int d\vec{p}\, e_{\vec{p}}(t,\vec{x}) a^{*}_{\mu\nu}(\vec{p}) \tag{2.1.7}$$

$$h^{-}_{\mu\nu}(x) = \int d\vec{p}\, \overline{e}_{\vec{p}}(t,\vec{x}) a_{\mu\nu}(\vec{p}) \quad , \tag{2.1.8}$$

with coefficient functions

$$e_{\vec{p}}(t,\vec{x}) = \frac{e^{i(|\vec{p}|t - \vec{p}\vec{x})}}{2\pi\sqrt{4\pi|\vec{p}|}} \quad . \tag{2.1.9}$$

Introducing the product

$$G(f,f') = i\int d\vec{x}\, \left( f(t,\vec{x}) \frac{\partial f'(t,\vec{x})}{\partial t} - \frac{\partial f(t,\vec{x})}{\partial t} f'(t,\vec{x}) \right) \quad , \tag{2.1.10}$$

which according to the Green theorem is independent of time if both functions $f$ and $f'$ are solutions of the wave equation, the coefficient functions are shown[20] to satisfy the orthonormality relations

$$G(e_{\vec{p}}, \overline{e}_{\vec{p}'}) = \delta(\vec{p} - \vec{p}') \tag{2.1.11}$$

$$G(e_{\vec{p}}, e_{\vec{p}'}) = 0 \quad . \tag{2.1.12}$$

The polarization tensor $a$ is expressed in terms of the field $h$ by

$$a_{\mu\nu}(\vec{p}) = G(e_{\vec{p}}, h_{\mu\nu}) \quad , \tag{2.1.13}$$

which in combination with the equal-time commutation relations (2.1.3) leads to the momentum space commutation relations

$$[a_{\mu\nu}(\vec{p}), a^{*}_{\alpha\beta}(\vec{p}')] = H_{\mu\nu\alpha\beta} G(e_{\vec{p}}, \overline{e}_{\vec{p}'}) \tag{2.1.14}$$

$$= H_{\mu\nu\alpha\beta}\delta(\vec{p} - \vec{p}') \tag{2.1.15}$$

and also

$$[a_{\mu\nu}(\vec{p}), a_{\alpha\beta}(\vec{p}')] = H_{\mu\nu\alpha\beta} G(e_{\vec{p}}, e_{\vec{p}'}) = 0 \tag{2.1.16}$$

$$[a^{*}_{\mu\nu}(\vec{p}), a^{*}_{\alpha\beta}(\vec{p}')] = H_{\mu\nu\alpha\beta} G(e_{\vec{p}'}, e_{\vec{p}}) = 0 \quad . \tag{2.1.17}$$

Substituting into expansions (2.1.7) and (2.1.8), one finds the covariant relations

$$[h^{\mp}_{\mu\nu}(x), h^{\pm}_{\alpha\beta}(x')] = iH_{\mu\nu\alpha\beta} D^{\pm}(x - x') \tag{2.1.18}$$

$$[h^{\pm}_{\mu\nu}(x), h^{\pm}_{\alpha\beta}(x')] = 0 \tag{2.1.19}$$

with the functions

$$D^{\pm}(x) = \mp\frac{i}{(2\pi)^2}\int dp\, \Gamma_{+}(p)\, e^{\mp ipx} \quad . \tag{2.1.20}$$

Adding up one finally obtains the covariant commutation relation

$$[h_{\mu\nu}(x), h_{\alpha\beta}(x')] = iH_{\mu\nu\alpha\beta} D(x - x') \quad , \tag{2.1.21}$$

where the Pauli-Jordan function

$$D(x) = D^{-}(x) + D^{+}(x) \tag{2.1.22}$$

$$= \frac{i}{(2\pi)^2}\int dp\, \Gamma_{+}(p)\, (e^{ipx} - e^{-ipx}) \tag{2.1.23}$$

is shown to be simply the difference

$$D(x) = G_-(x) - G_+(x) \tag{2.1.24}$$

$$= \frac{1}{4\pi|\vec{x}|}\Big(\delta(t+|\vec{x}|) - \delta(t-|\vec{x}|)\Big) \tag{2.1.25}$$

of the advanced and the retarded Green functions (1.2.33).

**Gravitons.** The non-commutative polarization tensor components $a_{\mu\nu}(\vec{p})$ and their complex conjugates $a^*_{\mu\nu}(\vec{p})$ are interpreted as annihilation and creation operators of gravitational field quanta, commonly known as gravitons, with momentum $\vec{p}$ and polarization $a_{\mu\nu}$ . The ground state $|0\rangle$ of the theory is defined by translation invariance and the condition

$$a_{\mu\nu}(\vec{p})|0\rangle = 0 \quad , \tag{2.1.26}$$

expressing the idea that the vacuum contains no gravitons to be annihilated. Any one-graviton state $|f\rangle$ can be written as a linear superposition

$$|f\rangle = \int d\vec{p}\, f^{\mu\nu}(\vec{p})a^*_{\mu\nu}(\vec{p})|0\rangle \tag{2.1.27}$$

with the coefficient $f$ being called a wave function. In the familiar Dirac notation the dual state given by

$$\langle f| = \int d\vec{p}\, \overline{f}^{\mu\nu}(\vec{p})\langle 0|a_{\mu\nu}(\vec{p}) \quad . \tag{2.1.28}$$

and formally combined with a second state $|f'\rangle$ using the commutation relation (2.1.14) yields the inner product

$$\langle f|f'\rangle = \int d\vec{p}\, H_{\mu\nu\alpha\beta}\overline{f}^{\mu\nu}(\vec{p})f'^{\alpha\beta}(\vec{p}) \tag{2.1.29}$$

of the one-graviton space. Due to the signature of the supermetric $H$ the graviton space is an indefinite metric space, which obviously leads to a problem with the probabilistic interpretation of the theory at this stage. However, in full analogy to the classical case, where the gravitational waves with negative energy are discarded by the introduction of a gauge condition, here simply a suitable quantum gauge condition must be imposed in order to single out the physical states of the theory.

**Krein Space.** With the indefiniteness of the metric also a problem of a more technical nature arises. The graviton space cannot be completed since its indefinite metric does not define a norm. The solution to this problem is to provide an underlying Hilbert topology. To this end consider the Hilbert space $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^4 \circledS \mathbb{C}^4)$ of square-integrable functions on $\mathbb{R}^3$ taking values in $\mathbb{C}^4 \circledS \mathbb{C}^4$, where the character $\circledS$ denotes a symmetrized tensor product. The scalar product is given by

$$(f, f') = \int d\vec{p} \sum_{\mu\nu} \overline{f}^{\mu\nu}(\vec{p})f'^{\mu\nu}(\vec{p}) \quad . \tag{2.1.30}$$

To make contact with the graviton space introduce the supermetric $H$ as an operator

$$H\colon \begin{cases} \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^4 \circledS \mathbb{C}^4) \to \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^4 \circledS \mathbb{C}^4) \\ \qquad f^{\mu\nu}(\vec{p}) \mapsto H^{\mu\nu}{}_{\alpha\beta}f^{\alpha\beta}(\vec{p}) \end{cases} \quad , \tag{2.1.31}$$

which, due to (1.1.63), is Hermitian

$$(Hf, f') = (f, Hf') \tag{2.1.32}$$

and, due to (1.1.64), also involutive

$$H^2 = 1 \quad . \tag{2.1.33}$$

Identify the Hermitian form

$$(f, Hf') = \int d\vec{p}\, H_{\mu\nu\alpha\beta}\overline{f}^{\mu\nu}(\vec{p}) f'^{\alpha\beta}(\vec{p}) \tag{2.1.34}$$

with the product (2.1.29) in one-graviton space. An indefinite metric space is called a Krein space[21] if it is complete with respect to an underlying auxiliary Hilbert topology, which is compatible in the sense of

$$\langle\,\cdot\,|\,\cdot\,\rangle = (\,\cdot\,, H\,\cdot\,) \tag{2.1.35}$$

where the operator $H$ is an Hermitian involution. Hence by the above construction one has obtained a one-graviton Krein space, which in the following will be denoted by $\mathfrak{H}_1$.

**Fock Space.** With the one-graviton Krein space $\mathfrak{H}_1$ at hand, the Fock construction is performed in the usual manner. Keeping in mind the underlying Hilbert topology, the $n$-graviton Krein space is defined as the symmetrized $n$-fold tensor product

$$\mathfrak{H}_n = \underbrace{\mathfrak{H}_1 \circledS \cdots \circledS \mathfrak{H}_1}_{n} \quad . \tag{2.1.36}$$

Using as the Hermitian involution the $n$-fold tensor product of the supermetric $H$ leads to the indefinite Hermitian form

$$\langle f_n | f'_n \rangle = \int d\vec{p}_1 \cdots d\vec{p}_n\, H_{\mu_1\nu_1\alpha_1\beta_1} \cdots H_{\mu_n\nu_n\alpha_n\beta_n} \tag{2.1.37}$$

$$\overline{f}_n^{\mu_1\nu_1\cdots\mu_n\nu_n}(\vec{p}_1, \ldots, \vec{p}_n) f'^{\,\alpha_1\beta_1\cdots\alpha_n\beta_n}_n(\vec{p}_1, \ldots, \vec{p}_n) \tag{2.1.38}$$

and the following definition

$$|f_n\rangle = \frac{1}{\sqrt{n!}}\int d\vec{p}_1 \cdots d\vec{p}_n\, f_n^{\mu_1\nu_1\cdots\mu_n\nu_n}(\vec{p}_1, \ldots, \vec{p}_n)\, a^*_{\mu_1\nu_1}(\vec{p}_1) \cdots a^*_{\mu_n\nu_n}(\vec{p}_n)|0\rangle \tag{2.1.39}$$

of a general $n$-graviton state in terms of its wave function, which is symmetric under an exchange of momenta and corresponding indices. With the vacuum space $\mathfrak{H}_0$ trivially defined by

$$\mathfrak{H}_0 = \mathbb{C} \quad , \tag{2.1.40}$$

the graviton Krein-Fock space $\mathfrak{H}$ is finally obtained as the direct sum

$$\mathfrak{H} = \bigoplus_{n=0}^{\infty} \mathfrak{H}_n \quad . \tag{2.1.41}$$

**Operators.** According to (2.1.27), the smeared expressions

$$a^*(f) = \int d\vec{p}\, f^{\mu\nu}(\vec{p}) a^*_{\mu\nu}(\vec{p}) \tag{2.1.42}$$

are operators mapping the vacuum space $\mathfrak{H}_0$ onto the one-graviton Krein space $\mathfrak{H}_1$. The conjugate operators

$$a(f) = \int d\vec{p}\, \overline{f}^{\mu\nu}(\vec{p}) a_{\mu\nu}(\vec{p}) \tag{2.1.43}$$

in turn map the one-graviton Krein space $\mathfrak{H}_1$ onto the vacuum space $\mathfrak{H}_0$ since

$$a(f)|f'\rangle = a(f)a^*(f')|0\rangle \tag{2.1.44}$$

$$= |0\rangle\langle 0|a(f)a^*(f')|0\rangle \tag{2.1.45}$$

$$= \langle f|f'\rangle|0\rangle \quad . \tag{2.1.46}$$

On a general $n$-graviton state (2.1.39) these fundamental operators act as

$$|f_{n+1}\rangle = a^*(f)|f_n\rangle \quad , \tag{2.1.47}$$

where the corresponding wave functions are related by

$$f_{n+1}^{\mu_1\nu_1\cdots\mu_{n+1}\nu_{n+1}}(\vec{p}_1,\ldots,\vec{p}_{n+1}) \tag{2.1.48}$$

$$= \sqrt{n+1}\Big(f_n^{\mu_1\nu_1\cdots\mu_n\nu_n}(\vec{p}_1,\ldots,\vec{p}_n) f^{\mu_{n+1}\nu_{n+1}}(\vec{p}_{n+1})\Big)_{Ⓢ} \quad , \tag{2.1.49}$$

and

$$|f_{n-1}\rangle = a(f)|f_n\rangle \quad , \tag{2.1.50}$$

where

$$f_{n-1}^{\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(\vec{p}_1,\ldots,\vec{p}_{n-1}) \tag{2.1.51}$$

$$= \sqrt{n}\int d\vec{p}_n\, f_n^{\mu_1\nu_1\cdots\mu_n\nu_n}(\vec{p}_1,\ldots,\vec{p}_n) H_{\mu_n\nu_n\mu\nu} f^{\mu\nu}(\vec{p}_n) \quad . \tag{2.1.52}$$

If in analogy to (1.3.56) one proposes the identification

$$f^{\mu\nu}(\vec{p}) = \sqrt{\pi}\int \frac{dp^0}{\sqrt{p^0}}\,\delta(p^0 - |\vec{p}|) f^{\mu\nu}(p) \quad , \tag{2.1.53}$$

then one finds

$$a^*(f) = (2\pi)^2 \int dp\, \Gamma_+(p)\, f^{\mu\nu}(p) a^*_{\mu\nu}(p) \tag{2.1.54}$$

$$= \int dp\, dx\, \Gamma_+(p) a^*_{\mu\nu}(p) e^{ipx} f^{\mu\nu}(x) \tag{2.1.55}$$

$$= \int dx\, f^{\mu\nu}(x) h^+_{\mu\nu}(x) \tag{2.1.56}$$

$$= h^+_{\mu\nu}(f) \tag{2.1.57}$$

and also

$$a(f) = h^-_{\mu\nu}(f) \quad . \tag{2.1.58}$$

Thus operators on the graviton Krein space $\mathfrak{H}$ are basically gravitational fields smeared out in spacetime. Interpreting the function (2.1.9) as an improper smearing

$$f_x^{\mu\nu}(\vec{p}) = e_{\vec{p}}(t, \vec{x}) \quad , \tag{2.1.59}$$

one obtains the gravitational field

$$a^*(f_x) = h^+_{\mu\nu}(x) \tag{2.1.60}$$

$$a(f_x) = h^-_{\mu\nu}(x) \tag{2.1.61}$$

at a single point. Yet this does not define an operator since the function $f_x$ violates the square integrability condition, which is an essential property of a wave function in the definition of the Krein space $\mathfrak{H}$. The same argument applies to the function

$$f_{\vec{p}'}^{\mu\nu}(\vec{p}) = \delta(\vec{p} - \vec{p}') \quad , \tag{2.1.62}$$

leading to the improper creation and annihilation operators

$$a^*(f_{\vec{p}}) = a^*_{\mu\nu}(\vec{p}) \tag{2.1.63}$$

$$a(f_{\vec{p}}) = a_{\mu\nu}(\vec{p}) \tag{2.1.64}$$

of a graviton with exact momentum $\vec{p}$. All these expressions are in fact operator-valued distributions, which are useful in formal manipulations and become proper operators if smeared out with test functions.

## 2. Gauge Condition

**Physical Gravitons.** The fact that the graviton Fock space $\mathfrak{H}$ is a Krein but not a Hilbert space is a result of neglecting the supplementary condition (1.2.61) in the quantization of the wave equation (1.2.60). Hence in a second step, one has to impose a quantum gauge condition that graviton states must satisfy in order to be physical. Therefore by discarding the virtual gravitons from the Krein space $\mathfrak{H}$, one obtains the Hilbert space $\mathcal{H}$ of physical gravitons.

**Gupta Condition.** As a quantum version of gauge condition (1.2.61), Suraj Gupta[22] proposed the relation

$$H^{\mu\nu\alpha\beta}\partial_\mu h^-_{\alpha\beta}|f\rangle = 0 \tag{2.2.1}$$

to hold for a physical graviton state $|f\rangle \in \mathfrak{H}$. Formally combined with the conjugate relation

$$H^{\mu\nu\alpha\beta}\partial_\mu \langle f|h^+_{\alpha\beta} = 0 \quad , \tag{2.2.2}$$

one finds the equality

$$H^{\mu\nu\alpha\beta}\partial_\mu \langle f|h_{\alpha\beta}|f\rangle = 0 \quad , \tag{2.2.3}$$

which exhibits that the classical condition (1.2.61) still holds for expectation values. In the next paragraph we will show that the Hermitian form $\langle\,\cdot\,|\,\cdot\,\rangle$ becomes positive semi-definite if restricted to the space

$$\mathfrak{H}' = \left\{\, |f\rangle \in \mathfrak{H} \,\middle|\, H^{\mu\nu\alpha\beta}\partial_\mu h^-_{\alpha\beta}|f\rangle = 0 \right\} \tag{2.2.4}$$

of gauged gravitons. Two gauged states are declared equivalent if they differ by a vector contained in the degeneracy subspace

$$\mathfrak{H}'' = \left\{ |f\rangle \in \mathfrak{H}' \,\middle|\, \langle f|f\rangle = 0 \right\} \quad . \tag{2.2.5}$$

The Hilbert space of physical gravitons is defined as the quotient space

$$\mathcal{H} = \mathfrak{H}' \Big/ \mathfrak{H}'' \quad , \tag{2.2.6}$$

on which the Hermitian form $\langle \cdot | \cdot \rangle$ becomes positive definite and therefore a scalar product. Since the Krein-Fock space $\mathfrak{H}$ is the direct sum of all $n$-graviton spaces $\mathfrak{H}_n$, a general graviton state may be written as a direct sum

$$|f\rangle = \bigoplus_{n=0}^{\infty} |f_n\rangle \tag{2.2.7}$$

of general $n$-graviton states. The Gupta condition (2.2.1) is therefore equivalent to the set of equations

$$H^{\mu\nu\alpha\beta}\partial^i_\mu h^-_{\alpha\beta}|f_n\rangle = 0 \quad \forall\, i, n \in \mathbb{N}\colon\ i \leq n \quad . \tag{2.2.8}$$

Inserting the expression (2.1.39) of a general $n$-graviton state, one obtains the equation

$$\int dp\, \overline{e}_{\vec{p}}(t, \vec{x})\vec{p}_\mu f^{\mu\nu\cdots}_n(\vec{p}, \ldots) = 0 \quad , \tag{2.2.9}$$

which is satisfied if the wave function conditions

$$\vec{p}_\mu f^{\mu\nu\cdots}_n(\vec{p}, \ldots) = 0 \quad \forall\, n \in \mathbb{N}_0 \tag{2.2.10}$$

are fulfilled. The converse is true as well since applying the product (2.1.10) to (2.2.9) yields

$$\int dp'\, G(e_{\vec{p}}, \overline{e}_{\vec{p}'})\vec{p}'_\mu f^{\mu\nu\cdots}_n(\vec{p}', \ldots) = \vec{p}_\mu f^{\mu\nu\cdots}_n(\vec{p}, \ldots) \quad , \tag{2.2.11}$$

which identically vanishes if the Gupta condition is satisfied. The Gupta condition thus holds for a state if and only if its wave functions fulfill the conditions (2.2.10).

**Positivity.** For the vacuum space $\mathfrak{H}_0 \in \mathbb{C}$ the Gupta condition is satisfied trivially. Yet its product

$$\langle f_0|f'_0\rangle = \overline{f}_0 f'_0 \tag{2.2.12}$$

is positive definite anyway and hence does not pose any problem from the beginning. The trouble starts with the one-graviton space $\mathfrak{H}_1$ and its indefinite product (2.1.29). Introducing the notation

$$f^{\mu\nu} = \begin{pmatrix} f & \vec{f}^{\,\mathrm{T}} \\ \vec{f} & \vec{\vec{f}} \end{pmatrix} \tag{2.2.13}$$

with

$$f = f^{00} \quad , \quad \vec{f} = \begin{pmatrix} f^{01} \\ f^{02} \\ f^{03} \end{pmatrix} \quad , \quad \vec{\vec{f}} = \begin{pmatrix} f^{11} & f^{12} & f^{13} \\ f^{12} & f^{22} & f^{23} \\ f^{13} & f^{23} & f^{33} \end{pmatrix} \quad , \tag{2.2.14}$$

we write the indefinite product in the form

$$D_{\mu\nu\alpha\beta} f^{\mu\nu} f'^{\alpha\beta} = \mathrm{Tr}(\vec{\vec{f}}\,\vec{\vec{f}}') - 2\vec{f}\vec{f}' - ff' \quad , \tag{2.2.15}$$

where $D$ is the, by means of (1.1.69), diagonalized supermetric

$$D(f,f') = H(Tf, Tf') \quad . \tag{2.2.16}$$

The corresponding gauge condition

$$\vec{p}_\mu T^{\mu\nu}{}_{\alpha\beta} f^{\alpha\beta} = 0 \tag{2.2.17}$$

translates into the equations

$$f - \vec{f}\vec{n} = \tfrac{1}{2}(f + \mathrm{Tr}\vec{\vec{f}}) \tag{2.2.18}$$

$$\vec{f} - \vec{\vec{f}}\vec{n} = -\tfrac{1}{2}(f + \mathrm{Tr}\vec{\vec{f}})\vec{n} \tag{2.2.19}$$

with the unit vector

$$\vec{n} = \frac{\vec{p}}{|\vec{p}|} \quad . \tag{2.2.20}$$

Introducing the projections

$$P = \vec{n} \otimes \vec{n}^{\mathrm{T}} \tag{2.2.21}$$

$$Q = \mathbb{1} - P \tag{2.2.22}$$

of rank 1 and 2, respectively, we obtain the relations

$$f^2 = \mathrm{Tr}(P\vec{\vec{f}})^2 \tag{2.2.23}$$

$$(P\vec{f})^2 = \tfrac{1}{4}\mathrm{Tr}^2(Q\vec{\vec{f}}) \tag{2.2.24}$$

$$(Q\vec{f})^2 = \mathrm{Tr}(Q\vec{\vec{f}}P\vec{\vec{f}}) \quad . \tag{2.2.25}$$

Since

$$\mathrm{Tr}(\vec{\vec{f}})^2 = \mathrm{Tr}(P\vec{\vec{f}})^2 + 2\mathrm{Tr}(P\vec{\vec{f}}Q\vec{\vec{f}}) + \mathrm{Tr}(Q\vec{\vec{f}})^2 \tag{2.2.26}$$

$$\vec{f}^2 = (P\vec{f})^2 + (Q\vec{f})^2 \quad , \tag{2.2.27}$$

we find that the one-graviton square

$$D(f,f) = \mathrm{Tr}(Q\vec{\vec{f}})^2 - \tfrac{1}{2}\mathrm{Tr}^2(Q\vec{\vec{f}}) \tag{2.2.28}$$

$$= \mathrm{Tr}^2\Big(Q\vec{\vec{f}} - \tfrac{1}{2}Q\mathrm{Tr}(Q\vec{\vec{f}})\Big) \tag{2.2.29}$$

becomes non-negative and hence the one-graviton product (2.1.29) is actually positive semi-definite if restricted to the gauged one-graviton space

$$\mathfrak{H}_1' = \Big\{ |f\rangle \in \mathfrak{H}_1 \,\Big|\, \vec{p}_\mu f^{\mu\nu}(\vec{p}) = 0 \Big\} \quad . \tag{2.2.30}$$

The degeneracy subspace is given by

$$\mathfrak{H}_1'' = \Big\{ |f\rangle \in \mathfrak{H}' \,\Big|\, f^{\mu\nu}(\vec{p}) = \vec{p}^\mu b^\nu(\vec{p}) + \vec{p}^\nu b^\mu(\vec{p}) \Big\} \tag{2.2.31}$$

and thus leads to the Hilbert space

$$\mathcal{H}_1 = \mathfrak{H}_1' \Big/ \mathfrak{H}_1'' \tag{2.2.32}$$

of a single physical graviton. The non-negativity of the gauged graviton space $\mathfrak{H}'_1$ can be extended to the $n$-graviton space $\mathfrak{H}'_n$ by the following argument. Write the one-graviton space as a kernel

$$\mathfrak{H}'_1 = \mathrm{Ker}\, p \tag{2.2.33}$$

of the map

$$p\colon f^{\mu\nu} \mapsto p_\mu f^{\mu\nu} \tag{2.2.34}$$

and the two-graviton space as

$$\mathfrak{H}'_2 = \mathrm{Ker}\,(p \circledS \mathbb{1}) \cap \mathrm{Ker}\,(\mathbb{1} \circledS p') \tag{2.2.35}$$

$$= \Big(\mathrm{Ker}\, p \circledS \mathfrak{H}_1\Big) \cap \Big(\mathfrak{H}_1 \circledS \mathrm{Ker}\, p'\Big) \quad . \tag{2.2.36}$$

However for $A \subset V$ and $B \subset V$ we have

$$(A \otimes V) \cap (V \otimes B) = A \otimes B \tag{2.2.37}$$

since any element of the left hand side can be written in the form

$$\sum_i a_i \otimes v_i = \sum_i w_i \otimes b_i \tag{2.2.38}$$

with bases $a_i \in A$, $b_i \in B$ and vectors $v_i, w_i \in V$ and a subsequent application of the cobases $\mathbb{1} \otimes (b_j, \,\cdot\,)$ and $(a_j, \,\cdot\,) \otimes \mathbb{1}$ leads to

$$v_j = \sum_i (a_j, w_i) b_i \in B \tag{2.2.39}$$

$$w_j = \sum_i (b_j, v_i) a_i \in A \quad . \tag{2.2.40}$$

Hence we obtain the relation

$$\mathfrak{H}'_2 = \mathrm{Ker}\, p \circledS \mathrm{Ker}\, p' \tag{2.2.41}$$

$$= \mathfrak{H}'_1 \circledS \mathfrak{H}'_1 \tag{2.2.42}$$

proving the positivity of the gauged two-graviton space $\mathfrak{H}'_2$ and, by iteration, of the gauged graviton Fock space $\mathfrak{H}'$. As a further consequence the Fock construction and Gupta condition commute, as illustrated in FIGURE 5.

$$\begin{tikzcd}
\mathfrak{H}_1 \arrow[r, "\text{Fock}"] \arrow[d, "\text{Gupta}"'] & \mathfrak{H} \arrow[d, "\text{Gupta}"] \\
\mathcal{H}_1 \arrow[r, "\text{Fock}"'] & \mathcal{H}
\end{tikzcd}$$

FIGURE 5. Gupta-Fock Commutative Diagram

**Physical Operators.** In our discussion of gauge invariance in this paragraph we closely follow Strocchi and Wightman[23]. In order for an operator $A$ on the graviton Krein space $\mathfrak{H}$ to yield a well-defined operator on the gauged graviton space $\mathfrak{H}'$ it has to satisfy the conditions

$$A\mathfrak{H}' \subset \mathfrak{H}' \quad \wedge \quad A^*\mathfrak{H}' \subset \mathfrak{H}' \tag{2.2.43}$$

commonly known as gauge invariance. Since one has $\mathfrak{H}'' \subset \mathfrak{H}'$ and on $\mathfrak{H}'$ the Hermitian form $\langle \cdot | \cdot \rangle$ is non-negative, the Schwarz inequality

$$|\langle f'|f''\rangle|^2 \leq \langle f'|f'\rangle\langle f''|f''\rangle \tag{2.2.44}$$

holds for two vectors $f' \in \mathfrak{H}'$ and $f'' \in \mathfrak{H}''$. Thus one finds that from $f'' \in \mathfrak{H}''$ follows its orthogonality to all vectors $f' \in \mathfrak{H}'$. The converse follows trivially from $\mathfrak{H}'' \subset \mathfrak{H}'$ and one has

$$f'' \in \mathfrak{H}'' \quad \Leftrightarrow \quad \langle f'|f''\rangle = 0 \quad \forall f' \in \mathfrak{H}'. \tag{2.2.45}$$

Hence $A^*\mathfrak{H}' \subset \mathfrak{H}'$ implies

$$\langle A^*\mathfrak{H}'|\mathfrak{H}''\rangle = 0 \quad , \tag{2.2.46}$$

which being equal to

$$\langle \mathfrak{H}'|A\mathfrak{H}''\rangle = 0 \tag{2.2.47}$$

in turn implies $A\mathfrak{H}'' \subset \mathfrak{H}''$. A gauge invariant operator $A$ thus automatically satisfies the conditions

$$A\mathfrak{H}'' \subset \mathfrak{H}'' \quad \wedge \quad A^*\mathfrak{H}'' \subset \mathfrak{H}'' \tag{2.2.48}$$

of weak gauge invariance. The conditions of gauge invariance (2.2.43) and weak gauge invariance (2.2.48) ensure that associated with a gauge invariant operator $A$ is a uniquely defined operator $\tilde{A}$ in the Hilbert space $\mathcal{H}$ of physical gravitons. As illustrated in the commutative diagram of FIGURE 6, this operator $\tilde{A}$ is given by $\tilde{A}[f'] = [Af']$ for $f' \in \mathfrak{H}'$, with $[\cdot]$ denoting an equivalence class in the Hilbert space $\mathcal{H} = [\mathfrak{H}']$.

$$\begin{tikzcd}
\mathfrak{H}' \arrow[r, "A"] \arrow[d, "{[\cdot]}"'] & \mathfrak{H}' \arrow[d, "{[\cdot]}"] \\
\mathcal{H} \arrow[r, "\tilde{A}"'] & \mathcal{H}
\end{tikzcd}$$

FIGURE 6. Physical Operators

**Riemann Operators.** If an operator $A$ and its conjugate $A^*$ both commute with the Gupta condition (2.2.1), i.e.

$$[A, H^{\mu\nu\alpha\beta}\partial_\mu h^-_{\alpha\beta}] = 0 \quad \wedge \quad [A^*, H^{\mu\nu\alpha\beta}\partial_\mu h^-_{\alpha\beta}] = 0 \quad , \tag{2.2.49}$$

then the gauge invariance (2.2.43) of the operator and hence its physicality follows immediately. A calculation of the Gupta commutators (2.2.49) for the creation and annihilation operators leads to

$$[a(f), H^{\mu\nu\alpha\beta}\partial_\mu h^-_{\alpha\beta}] = 0 \quad , \tag{2.2.50}$$

vanishing identically, and

$$[a^*(f), H^{\mu\nu\alpha\beta}\partial_\mu h^-_{\alpha\beta}] = -\, i\int d\vec{p}\, d\vec{p}'\, \overline{e}_{\vec{p}}(t,\vec{x}) H^{\mu\nu\alpha\beta}\vec{p}_\mu f^{\rho\sigma}(\vec{p}')[a^*_{\rho\sigma}(\vec{p}'), a_{\alpha\beta}(\vec{p})] \tag{2.2.51}$$

$$= i\int d\vec{p}\, \overline{e}_{\vec{p}}(t,\vec{x})\vec{p}_\mu f^{\mu\nu}(\vec{p}) \quad , \tag{2.2.52}$$

which vanishes if the condition

$$\vec{p}_\mu f^{\mu\nu}(\vec{p}) = 0 \tag{2.2.53}$$

is satisfied. The converse is true as well since

$$G(e_{\vec{p}}, [a^*(f), H^{\mu\nu\alpha\beta}\partial_\mu h^-_{\alpha\beta}]) = i\vec{p}_\mu f^{\mu\nu}(\vec{p}) \quad , \tag{2.2.54}$$

which identically vanishes if the Gupta commutator vanishes. Hence the creation and annihilation operators commuting with the Gupta condition are exactly the ones that have been smeared out with functions satisfying (2.2.53). It is reasonable to expect the quantized linearized Riemann tensor (1.2.7) to define such operators since according to (1.2.23) it is the gauge invariant quantity of the classical theory. In order to confirm this expectation we introduce the plane wave decomposition

$$\tilde{R}^+_{\mu\alpha\nu\beta}(x) = \int d\vec{p}\, e_{\vec{p}}(t,\vec{x}) A^*_{\mu\alpha\nu\beta}(\vec{p}) \tag{2.2.55}$$

$$\tilde{R}^-_{\mu\alpha\nu\beta}(x) = \int d\vec{p}\, \overline{e}_{\vec{p}}(t,\vec{x}) A_{\mu\alpha\nu\beta}(\vec{p}) \tag{2.2.56}$$

with the expression

$$A_{\mu\alpha\nu\beta}(\vec{p}) = \tfrac{1}{2}\Big(\vec{p}_\alpha\vec{p}_\beta a_{\mu\nu}(\vec{p}) + \vec{p}_\mu\vec{p}_\nu a_{\alpha\beta}(\vec{p}) - \vec{p}_\nu\vec{p}_\alpha a_{\mu\beta}(\vec{p}) - \vec{p}_\mu\vec{p}_\beta a_{\alpha\nu}(\vec{p})\Big) \quad , \tag{2.2.57}$$

which we smear out with a test function $F$ to find

$$A^*(F) = \int d\vec{p}\, F^{\mu\alpha\nu\beta}(\vec{p}) A^*_{\mu\alpha\nu\beta}(\vec{p}) \tag{2.2.58}$$

$$= \int d\vec{p}\, f^{\mu\nu}(\vec{p}) a^*_{\mu\nu}(\vec{p}) \tag{2.2.59}$$

$$= a^*(f) \tag{2.2.60}$$

with the smearing

$$f^{\mu\nu}(\vec{p}) = \tfrac{1}{2}\vec{p}_\alpha\vec{p}_\beta\Big(F^{\mu\alpha\nu\beta}(\vec{p}) + F^{\alpha\mu\beta\nu}(\vec{p}) - F^{\mu\alpha\beta\nu}(\vec{p}) - F^{\alpha\mu\nu\beta}(\vec{p})\Big) \quad . \tag{2.2.61}$$

However for this smearing the physical condition (2.2.53) is automatically satisfied. Hence even for arbitrary smearings the Riemann operators (2.2.55) and (2.2.56) define physical creation and annihilation operators.

**Poincaré Lemma.** With the help of the Poincaré lemma, we will now prove that not only every Riemann operator $R(F)$ is a physical operator $h(f)$ but also the converse statement that every physical operator $h(f)$ may be written as a Riemann operator $R(F)$. To this end we rewrite the physical operator condition (2.2.53) in position space as

$$\partial_\mu f^{\mu\nu}(x) = 0 \quad . \tag{2.2.62}$$

Since the topology of the background spacetime, on which the function $f$ is defined, is trivially given by $\mathbb{R}^4$, we may apply the Poincaré lemma and conclude the existence of a solution in the form

$$f^{\mu\nu}(x) = \partial_\alpha F^{\mu\alpha\nu}(x) \quad , \tag{2.2.63}$$

with an antisymmetric function

$$F^{\mu\alpha\nu}(x) = -F^{\alpha\mu\nu}(x) \quad . \tag{2.2.64}$$

In order to obtain a Green function providing a general solution

$$F^{\mu\alpha\nu}(x) = \int dx'\, G^{\mu\alpha}{}_\rho(x-x') f^{\rho\nu}(x') \tag{2.2.65}$$

of equation (2.2.62) we first note the well-known result

$$G^{(4)}(x) = -\frac{1}{4\pi^2|x|^2} \tag{2.2.66}$$

$$\Delta^{(4)} G^{(4)}(x) = \delta(x) \tag{2.2.67}$$

for the Green function $G^{(4)}$ of the four-dimensional Laplacian $\Delta^{(4)} = \delta^{\mu\nu}\partial_\mu\partial_\nu$. As a consequence we find

$$G^\mu(x) = \frac{x^\mu}{2\pi^2|x|^4} \tag{2.2.68}$$

$$\partial_\mu G^\mu(x) = \delta(x) \quad , \tag{2.2.69}$$

for the Green function of the four-divergence $\partial$. The antisymmetric expression

$$G^{\mu\alpha}{}_\rho(x) = \delta_\rho{}^\mu G^\alpha(x) - \delta_\rho{}^\alpha G^\mu(x) \tag{2.2.70}$$

is the desired Green function for equation (2.2.62), despite a second term appearing in its divergence

$$\partial_\alpha G^{\mu\alpha}{}_\rho(x) = \delta_\rho{}^\mu \delta(x) - \partial_\rho G^\mu(x) \quad , \tag{2.2.71}$$

which however does not affect the usual reasoning since, smeared with a function $f$, it may be turned into

$$\int dx'\, \partial_\rho G^\mu(x-x') f^{\rho\nu}(x') = \int dx'\, G^\mu(x-x')\partial'_\rho f^{\rho\nu}(x') \quad , \tag{2.2.72}$$

which vanishes due to the physical condition (2.2.62). By the same argument and the symmetry of the function $f$ the divergence

$$\partial_\nu F^{\mu\alpha\nu}(x) = \int dx'\, \partial_\nu G^{\mu\alpha}{}_\rho(x-x') f^{\rho\nu}(x') \tag{2.2.73}$$

$$= \int dx'\, G^{\mu\alpha}{}_\rho(x-x')\partial'_\nu f^{\rho\nu}(x') \tag{2.2.74}$$

vanishes as well. Therefore we may apply the Poincare lemma for a second time to find the existence of a solution in the form

$$F^{\mu\alpha\nu}(x) = \partial_\beta F^{\mu\alpha\nu\beta}(x) \tag{2.2.75}$$

with an antisymmetric function

$$F^{\mu\alpha\nu\beta}(x) = -F^{\mu\alpha\beta\nu}(x) \quad . \tag{2.2.76}$$

Using the Green function (2.2.70), we also obtain the general solution

$$F^{\mu\alpha\nu\beta}(x) = -\tfrac{1}{2}\int dx'\; G^{\nu\beta}{}_{\rho}(x-x')F^{\mu\alpha\rho}(x') \quad , \tag{2.2.77}$$

which in combination with (2.2.65) becomes

$$F^{\mu\alpha\nu\beta}(x) = -\tfrac{1}{2}\int dx'\; (G*G)^{\mu\alpha\nu\beta}{}_{\rho\sigma}(x,x')f^{\rho\sigma}(x') \tag{2.2.78}$$

with the convolution square

$$(G*G)^{\mu\alpha\nu\beta}{}_{\rho\sigma}(x,x') = \int dx''\; G^{\mu\alpha}{}_{\rho}(x-x'')G^{\nu\beta}{}_{\sigma}(x''-x') \tag{2.2.79}$$

of the Green function (2.2.70). Introducing the function

$$G(x) = -\frac{1}{8\pi^2}\ln|x| \quad , \tag{2.2.80}$$

which satisfies the relation

$$\Delta^{(4)}G(x) = G^{(4)}(x) \quad , \tag{2.2.81}$$

the convolution square can explicitly be given as

$$(G*G)^{\mu\alpha\nu\beta}{}_{\rho\sigma}(x,x') \tag{2.2.82}$$

$$= (\delta_\rho{}^\mu\delta^{\alpha\kappa}\partial_\kappa - \delta_\rho{}^\alpha\delta^{\mu\kappa}\partial_\kappa)(\delta_\sigma{}^\nu\delta^{\beta\lambda}\partial_\lambda - \delta_\sigma{}^\beta\delta^{\nu\lambda}\partial_\lambda)G(x-x') \quad . \tag{2.2.83}$$

With the help of relation (2.2.81), we check the desired property

$$\partial_\alpha\partial_\beta(G*G)^{\mu\alpha\nu\beta}{}_{\rho\sigma}(x,x') = \delta_\rho{}^\mu\delta_\sigma{}^\nu\delta(x-x') + D^{\mu\nu}{}_{\rho\sigma}(x,x') \quad , \tag{2.2.84}$$

where the terms

$$D^{\mu\nu}{}_{\rho\sigma}(x,x') = (\delta^{\mu\kappa}\partial_\kappa\partial_\rho\delta_\sigma{}^\nu\Delta^{(4)} + \delta^{\nu\lambda}\partial_\lambda\partial_\sigma\delta_\rho{}^\mu\Delta^{(4)} + \delta^{\mu\kappa}\partial_\kappa\delta^{\nu\lambda}\partial_\lambda\partial_\rho\partial_\sigma)G(x) \tag{2.2.85}$$

are again of no effect since they vanish if they are smeared with a physical function $f$, as we have shown above. Equation (2.2.78) is the inverse of equation (2.2.61) in the position space form

$$f^{\mu\nu}(x) = -\tfrac{1}{2}\partial_\alpha\partial_\beta\Big(F^{\mu\alpha\nu\beta}(x) + F^{\alpha\mu\beta\nu}(x) - F^{\mu\alpha\beta\nu}(x) - F^{\alpha\mu\nu\beta}(x)\Big) \quad . \tag{2.2.86}$$

As a result every Riemann operator $R(F)$ is also a physical operator $h(f)$ and vice versa. In addition to the proof given we now present an argument in the language of differential forms. One should however note that our discussion is by no means strict since it rests on just a formal analogy. Nevertheless the »double« differential forms we introduce provide an interesting alternative view of the problem. Hence let us define the space $\mho^n_x(\mathbb{R}^4)$ of double $n$-forms over a point $x$ in Minkowski space $(\mathbb{R}^4,\eta)$ as the symmetrized tensor product

$$\mho^n_x(\mathbb{R}^4) = \Omega^n_x(\mathbb{R}^4) \circledS \Omega^n_x(\mathbb{R}^4) \tag{2.2.87}$$

of two ordinary $n$-form spaces $\Omega^n_x(\mathbb{R}^4)$ and naively observe that in case the metric perturbation

$$\boldsymbol{h} = h_{\mu\nu}\, dx^\mu \otimes dx^\nu \tag{2.2.88}$$

is looked upon as a double one-form, the linearized Riemann tensor may formally be written as a double exterior derivative

$$\tilde{\boldsymbol{R}} = \tfrac{1}{2}(d \otimes d)\boldsymbol{h} \tag{2.2.89}$$

$$= \tfrac{1}{2}\partial_\mu \partial_\nu h_{\alpha\beta}(dx^\mu \wedge dx^\alpha) \otimes (dx^\nu \wedge dx^\beta) \tag{2.2.90}$$

$$= \tilde{R}_{\mu\alpha\nu\beta}\, dx^\mu \otimes dx^\alpha \otimes dx^\nu \otimes dx^\beta \tag{2.2.91}$$

with components (1.2.7). Hence in connection with our definition of double differential forms we also introduce a set of double operations simply by performing the ordinary operations on both ordinary constituents of the double differential form. In addition to the double exterior derivative $d \otimes d$ consider the double Hodge star $* \otimes *$, which applied to the linearized Riemann tensor (2.2.90) as a double two-form yields the linearized Riemann double dual

$$\tilde{\boldsymbol{G}} = (* \otimes *)\tilde{\boldsymbol{R}} \tag{2.2.92}$$

in concordance with definition (1.1.37). Introducing the Hodge operator

$$\delta = *d* \quad , \tag{2.2.93}$$

we find as an immediate consequence of the fundamental relation

$$d^2 = 0 \tag{2.2.94}$$

that the Riemann double dual satisfies the linearized Bianchi identity

$$\delta \tilde{\boldsymbol{G}} = 0 \tag{2.2.95}$$

in agreement with equation (1.1.38). Finally defining the double wedge product $\wedge \otimes \wedge$ as the tensor product of the ordinary wedge products, we obtain an inner product

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \int \boldsymbol{A}(\wedge \otimes \wedge)(* \otimes *)\boldsymbol{B} \tag{2.2.96}$$

of two double $n$-forms $\boldsymbol{A}$ and $\boldsymbol{B}$. Introducing the smearing double one- and two-forms

$$\boldsymbol{f} = f_{\mu\nu}\, dx^\mu \otimes dx^\nu \tag{2.2.97}$$

$$\boldsymbol{F} = F_{\mu\alpha\nu\beta}\,(dx^\mu \wedge dx^\alpha) \otimes (dx^\nu \wedge dx^\beta) \quad , \tag{2.2.98}$$

we rewrite the field operators as inner products

$$\tilde{R}(F) = \langle \tilde{\boldsymbol{R}}, \boldsymbol{F} \rangle \tag{2.2.99}$$

$$h(f) = \langle \boldsymbol{h}, \boldsymbol{f} \rangle \quad , \tag{2.2.100}$$

to be identified

$$\langle \tilde{\boldsymbol{R}}, \boldsymbol{F} \rangle = \tfrac{1}{2}\langle (d \otimes d)\boldsymbol{h}, \boldsymbol{F} \rangle \tag{2.2.101}$$

$$= \tfrac{1}{2}\langle \boldsymbol{h}, (\delta \otimes \delta)\boldsymbol{F} \rangle \tag{2.2.102}$$

$$= \langle \boldsymbol{h}, \boldsymbol{f} \rangle \tag{2.2.103}$$

under the condition

$$\boldsymbol{f} = \tfrac{1}{2}(\delta \otimes \delta)\boldsymbol{F} \quad , \tag{2.2.104}$$

which in component form is simply equation (2.2.86). Since relation (2.2.94) implies

$$\delta^2 = 0 \quad , \tag{2.2.105}$$

we deduce the relation

$$(\delta \otimes \mathbb{1})\boldsymbol{f} = 0 \quad , \tag{2.2.106}$$

which is exactly the physical condition (2.2.62) and hence the statement that every Riemann operator is physical. The converse statement, i.e. the existence of a solution to equation (2.2.106) in the form of (2.2.104), is finally just the »double« Poincaré lemma.

CHAPTER 3

# Quantum Length

## 1. Length Operators

**Quantum Geometry.** From Albert Einstein we have learned that gravitation essentially has to be looked upon as geometry of space and time. Thus in a certain sense by quantizing gravitation one is led to quantum geometry. To be specific, consider a spacelike circle, whose circumference-diameter-ratio in flat space is given by the number $\pi$. In a space with positive or negative Gauß curvature this ratio is decreased or increased, respectively. If by quantizing the perturbation of the Minkowski metric the curvature of spacetime becomes an operator, then the circumference-diameter-ratio becomes an operator as well. A measurement of this operator need not necessarily yield the number $\pi$, just its vacuum expectation value is still given by the classical result.

**Length Operators.** Consider a spacelike curve

(3.1.1)
$$S\colon \begin{cases} [0,1] & \to & \mathbb{R}^4 \\ \lambda & \mapsto & s \end{cases}$$

in Minkowski spacetime $(\mathbb{R}^4, \eta)$. We may write its tangent vector

(3.1.2)
$$\dot{s}^\mu = \frac{ds^\mu}{d\lambda}$$

as a product

(3.1.3)
$$\dot{s}^\mu = \dot{s}e^\mu$$

of its length

(3.1.4)
$$\dot{s} = \sqrt{|\eta_{\mu\nu}s^\mu s^\nu|}$$

and a vector $e$ satisfying

(3.1.5)
$$\eta_{\mu\nu}e^\mu e^\nu = -1 \quad .$$

According to (1.1.11) the length $L_\eta$ of the curve $S$ is given by the expression

(3.1.6)
$$L_\eta(S) = \int_0^1 d\lambda\, \dot{s} \quad .$$

If the Minkowski metric $\eta$ is perturbed as in (1.2.1), we find that the length of the same curve $S$ becomes

$$L_{\tilde{g}}(S) = \int_0^1 d\lambda \sqrt{|\tilde{g}_{\mu\nu}(s)\dot{s}^\mu \dot{s}^\nu|} \tag{3.1.7}$$

$$= \int_0^1 d\lambda\, \dot{s} - \tfrac{1}{2}\int_0^1 d\lambda\, \frac{h_{\mu\nu}(s)\dot{s}^\mu \dot{s}^\nu}{\dot{s}} \tag{3.1.8}$$

$$= L_\eta(S) - \tfrac{1}{2}\int_0^1 d\lambda\, h_{\mu\nu}(s)\dot{s}^\mu e^\nu \quad . \tag{3.1.9}$$

Hence the metric perturbation $h$ gives rise to a length perturbation

$$L_h(S) = L_{\tilde{g}}(S) - L_\eta(S) \tag{3.1.10}$$

$$= -\tfrac{1}{2}\int_0^1 d\lambda\, h_{\mu\nu}(s)\dot{s}^\mu e^\nu \quad . \tag{3.1.11}$$

Introducing the improper smearing function

$$f_S^{\mu\nu}(q) = -\int_0^1 d\lambda \dot{s}^\mu e^\nu \delta(q - s(\lambda)) \quad , \tag{3.1.12}$$

the Minkowski length $L_\eta$ may be written as

$$L_\eta(S) = \int dq\, f_S^{\mu\nu}(q)\eta_{\mu\nu} \tag{3.1.13}$$

$$= \eta(f_S) \tag{3.1.14}$$

and the length perturbation $L_h$ as

$$L_h(S) = \tfrac{1}{2}\int dq\, f_S^{\mu\nu}(q)h_{\mu\nu}(q) \tag{3.1.15}$$

$$= \tfrac{1}{2}h(f_S) \quad . \tag{3.1.16}$$

Our classical discussion of the length perturbation applies to the quantized case as well since for spacelike separated points all tensor components of the field $h$ commute. The quantization procedure described in the last chapter simply yields an operator-valued distribution $h(f_S)$ measuring twice the quantum correction to the Minkowski length $\eta(f_S)$ of the spacelike curve $S$. We will therefore in the following call the quantity

$$L(S) = \eta(f_S) + \tfrac{1}{2}h(f_S) \tag{3.1.17}$$

loosely the length operator $L$ of the curve $S$ in linear quantum gravity.

**Physical Curves.** The question arises, whether the length operator $L(S)$ of a curve $S$ is physical or not. In analogy to condition (2.2.62) it is straightforward to call a spacelike curve $S$ physical if its smearing function $f_S$ satisfies the relation

$$\partial_\mu f_S^{\mu\nu}(q) = 0 \quad . \tag{3.1.18}$$

Using the smearing function (3.1.12) of a general spacelike curve $S$, we calculate its divergence

$$\partial_\mu f_S^{\mu\nu}(q) = -\int_0^1 d\lambda\, \dot{s}^\mu e^\nu \frac{\partial}{\partial q^\mu}\delta(q-s) \tag{3.1.19}$$

$$= \int_0^1 d\lambda\, e^\nu \dot{s}^\mu \frac{\partial}{\partial s^\mu}\delta(q-s) \tag{3.1.20}$$

$$= \int_0^1 d\lambda\, e^\nu \frac{d}{d\lambda}\delta(q-s) \tag{3.1.21}$$

$$= e^\nu \delta(q-s)\Big|_0^1 - \int_0^1 d\lambda\, \dot{e}^\nu \delta(q-s) \quad , \tag{3.1.22}$$

which we will in the following simply refer to as the divergence of the curve $S$. Consider as special cases a straight line

$$S' : \begin{cases} \dot{e}^\mu(\lambda) = 0 \\ e^\mu(\lambda) = e^\mu \end{cases} \tag{3.1.23}$$

and a smooth closed curve

$$S'' : \begin{cases} s^\mu(0) = s^\mu(1) \\ e^\mu(0) = e^\mu(1) \end{cases} \tag{3.1.24}$$

leading to the divergences

$$\partial_\mu f_{S'}^{\mu\nu}(q) = e^\nu \delta\Big(q - s(1)\Big) - e^\nu \delta\Big(q - s(0)\Big) \tag{3.1.25}$$

$$\partial_\mu f_{S''}^{\mu\nu}(q) = -\int_0^1 d\lambda\, \dot{e}^\nu \delta(q-s) \quad . \tag{3.1.26}$$

Hence in the case of a straight line only the first term of (3.1.22) comes into effect, whereas in the case of a smooth closed curve just the second term contributes. The divergences of a straight line and a circle are illustrated in FIGURE 7.

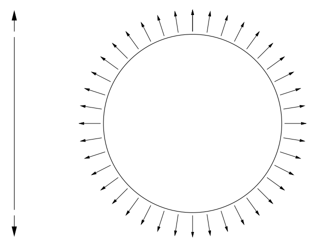


FIGURE 7. Divergence of a straight line and a circle

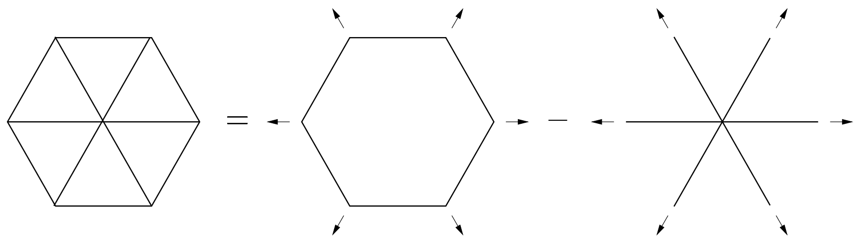


FIGURE 8. Physical Regular Hexagon $S_6$

**Regular Polygon.** Physical curves can be constructed by suitably combining ordinary curves, where a linear combination

$$S = \sum_i w_i S_i \tag{3.1.27}$$

of curves $S_i$ is defined by the linear combination

$$f_S = \sum_i w_i f_{S_i} \tag{3.1.28}$$

of its smearing functions. The simplest example is illustrated in FIGURE 8, where the divergences originating in the six corners of a regular hexagon are canceled by subtracting its three diagonals. For a regular $n$-gon with $n \neq 6$ one has to give different weight to the inner and outer lines for the divergences to cancel. This is illustrated in FIGURE 9 for the case $n = 3$, where from the $w_3''$-weighed triangle one subtracts $w_3'$-weighed lines connecting the corners of the triangle with its center. A regular polygon is shown in FIGURE 10, from which one can deduce the relation

$$\frac{l_n''}{l_n'} = 2 \sin \tfrac{\pi}{n} \tag{3.1.29}$$

for the Minkowski lengths $l_n$ of the single lines and the relation

$$\frac{w_n'}{w_n''} = 2 \cos(\tfrac{\pi}{2} - \tfrac{\pi}{n}) = 2 \sin \tfrac{\pi}{n} \tag{3.1.30}$$

for the corresponding weights $w_n$. Hence for the physical regular polygon curve

$$S_n = w_n'' S_n'' - w_n' S_n' \tag{3.1.31}$$

the Minkowski length

$$\eta(f_{S_n}) = n(w_n'' l_n'' - w_n' l_n') \tag{3.1.32}$$

identically vanishes and the length operator is given by

$$L(S_n) = \tfrac{1}{2} h(f_{S_n}) \quad . \tag{3.1.33}$$

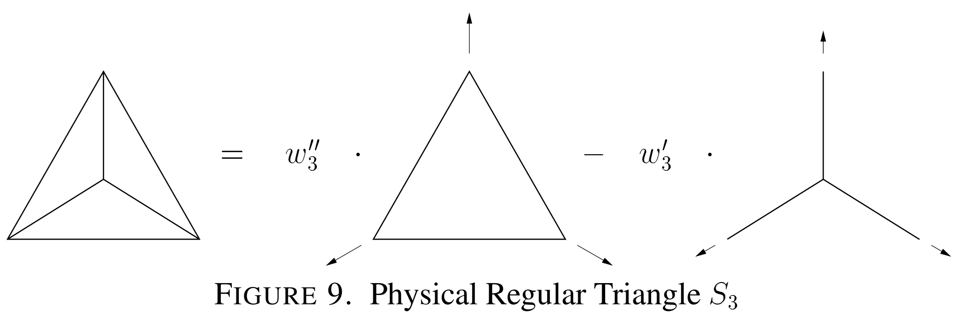



FIGURE 9. Physical Regular Triangle $S_3$

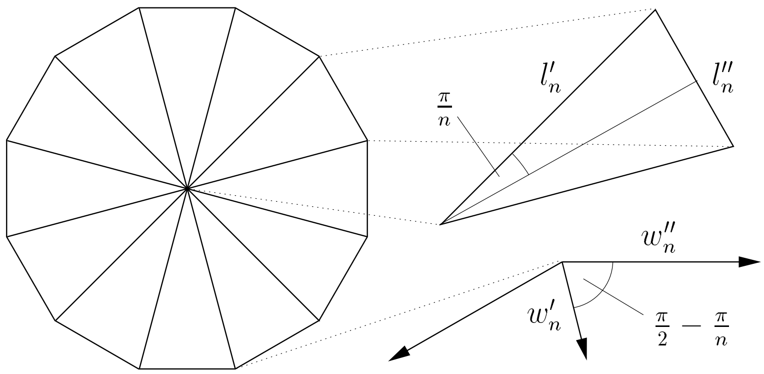



FIGURE 10. Regular Polygon

In order to obtain a rough interpretation of the physical length operators $L(S_n)$, note that the perturbation $h$ of the Minkowski metric $\eta$ gives rise to perturbations $\Delta l_i$ of the lengths $l_i$ of the straight lines. The length operator $L$ of the physical regular polygon $S_n$ may thus be written as a sum

$$L(S_n) = \sum_{i=1}^{n} L_i(S_n) \tag{3.1.34}$$

over the unphysical length perturbations

$$L_i(S_n) = w''_n \Delta l''_i - w'_n \Delta l'_i \quad . \tag{3.1.35}$$

Consider the special case of constant $L_i(S_n)$ for every side $l_i$ and define the quantity

$$K = -L_i(S_n) \quad . \tag{3.1.36}$$

In the case $K > 0$ the inner lines $l'_i$ relatively outgrow the outer lines $l'_i$ and the center of the polygon is moved to compensate for the change. In the opposite case $K < 0$ the outer lines $l'_i$ outgrow the inner lines $l'_i$ and the corners of the polygon twist in such a manner that the polygon becomes saddle shaped. Both cases $K > 0$, $K < 0$ and also the initial polygon $K = 0$ are illustrated in FIGURE 11, which suggests that the physical operators $L(S_n)$ are closely related to the Gauß curvature of the two-dimensional surface, in which the regular polygon $S_n$ is located. Notice that the $n$

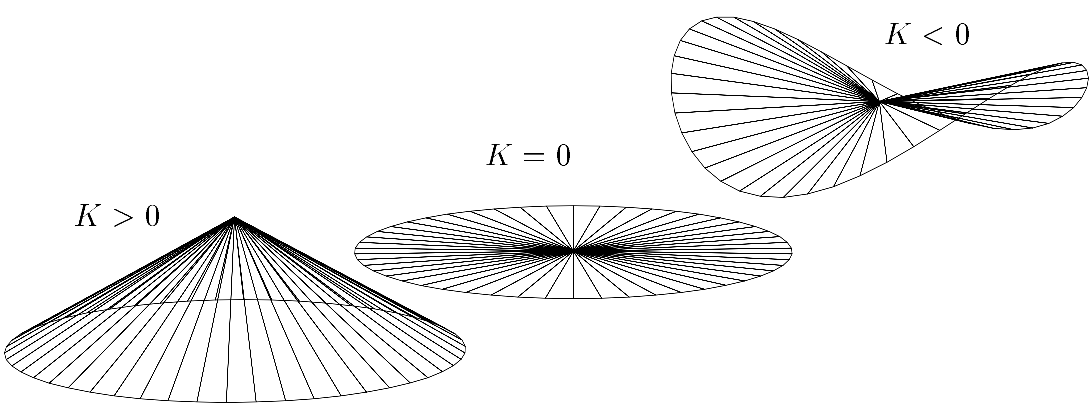



FIGURE 11. Positive, vanishing and negative Gauß curvature

angles in the center of the regular polygon for non-vanishing $K$ are not given by

$$\zeta_i = \frac{2\pi}{n} \tag{3.1.37}$$

but rather have changed by a value $\Delta\zeta_i$ with a sign opposite to $K$. This can be looked upon as a consequence of generalizing the relation (3.1.29) to

$$\frac{l''_n + \Delta l''_i}{l'_n + \Delta l'_i} = 2\sin(\tfrac{\pi}{n} + \tfrac{\Delta\zeta_i}{2}) \quad . \tag{3.1.38}$$

Since the perturbations $\Delta\zeta_i$ are small in comparison to the center angles $\zeta_i$, we have

$$2\sin(\tfrac{\pi}{n} + \tfrac{\Delta\zeta_i}{2}) - 2\sin\tfrac{\pi}{n} = \Delta\zeta_i \cos\tfrac{\pi}{n} \quad . \tag{3.1.39}$$

Using (3.1.38) and (3.1.30), we find that the left hand side is equal to

$$\frac{l''_n + \Delta l''_i}{l'_n + \Delta l'_i} - \frac{l''_n}{l'_n} = \frac{1}{l'_n w''_n}(w''_n \Delta l''_i - w'_n \Delta l'_i) \quad . \tag{3.1.40}$$

Renaming the parameters radius

$$R_n = l'_n \tag{3.1.41}$$

and weight

$$W_n = w''_n \quad , \tag{3.1.42}$$

we thus find

$$L_i(S_n) = W_n R_n \cos\tfrac{\pi}{n}\Delta\zeta_i \quad . \tag{3.1.43}$$

Introducing the total correction

$$\Delta\zeta = \sum_{i=1}^{n} \Delta\zeta_i \tag{3.1.44}$$

to the flat center angle $\zeta = 2\pi$, we finally obtain

$$L(S_n) = W_n R_n \cos\tfrac{\pi}{n}\Delta\zeta \quad . \tag{3.1.45}$$

**Circle.** By taking the number $n$ of angles to infinity, we obtain the physical circle $C$, as illustrated in FIGURE 12. In order to calculate the corresponding smearing function $f_C$, we introduce polar coordinates in the $x$-$y$-plane, i.e.

$$q^0 = t \tag{3.1.46}$$

$$q^1 = x = r\cos\varphi \tag{3.1.47}$$

$$q^2 = y = r\sin\varphi \tag{3.1.48}$$

$$q^3 = z \quad . \tag{3.1.49}$$

The straight line connecting the $i$-th corner and the center of the $n$-gon is given by

$$s^\mu(\lambda) = \lambda l'_n e^\mu_r(\alpha_i) \quad , \tag{3.1.50}$$

where $e_r(\alpha)$ is the radial unit vector

$$e^\mu_r(\alpha) = (0, \cos\alpha, \sin\alpha, 0) \tag{3.1.51}$$

and $\alpha_i$ the directional angle

$$\alpha_i = \frac{2\pi i}{n} \quad . \tag{3.1.52}$$

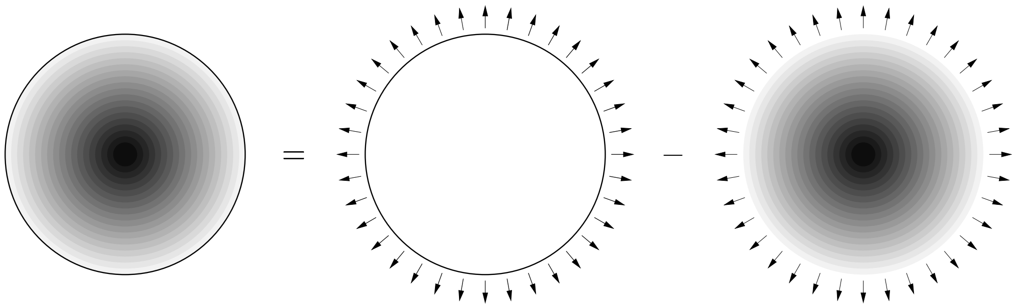



FIGURE 12. Physical Circle $C = S_\infty$

Therefore we have

$$w'_n f^{\mu\nu}_{S'_n}(q) = -\, w'_n \sum_{i=1}^{n} \int_0^1 d\lambda\, \delta\Big(q - \lambda l'_n e_r(\alpha_i)\Big) l'_n e^\mu_r(\alpha_i) e^\nu_r(\alpha_i) \tag{3.1.53}$$

$$= -\, \frac{w'_n}{r} \theta(l'_n - r)\delta(t)\delta(z) \sum_{i=1}^{n} \delta(\varphi - \alpha_i) e^\mu_r(\alpha_i) e^\nu_r(\alpha_i) \quad . \tag{3.1.54}$$

Note that for large $n$ relation (3.1.30) becomes

$$\frac{w'_n}{w''_n} = 2 \sin \tfrac{\pi}{n} \approx \frac{2\pi}{n} = \alpha_i - \alpha_{i-1} = \Delta\alpha_i \tag{3.1.55}$$

and as a consequence

$$\lim_{n\to\infty} \sum_{i=1}^{n} w'_n f(\alpha_i) = \lim_{n\to\infty} \sum_{i=1}^{n} w''_n \Delta\alpha_i f(\alpha_i) \tag{3.1.56}$$

$$= w''_\infty \int_0^{2\pi} d\alpha\, f(\alpha_i) \quad . \tag{3.1.57}$$

Hence the weighed smearing function of the inner circle $C'$ is given by the expression

$$w'_\infty f^{\mu\nu}_{C'}(q) = \lim_{n\to\infty} w'_n f^{\mu\nu}_{S'_n}(q) \tag{3.1.58}$$

$$= -\, \frac{w''_\infty}{r} \theta(l'_\infty - r)\delta(t)\delta(z) e^\mu_r e^\nu_r \quad , \tag{3.1.59}$$

whose divergence becomes

$$w''_\infty \partial_\mu f^{\mu\nu}_{C''}(q) = \frac{w''_\infty}{r} \delta(l'_\infty - r)\delta(t)\delta(z) e^\nu_r \tag{3.1.60}$$

with $e_r = e_r(\varphi)$. A similar calculation for the outer circle $C''$ leads to

$$w''_\infty f^{\mu\nu}_{C''}(q) = \lim_{n\to\infty} w''_n f^{\mu\nu}_{S''_n}(q) \tag{3.1.61}$$

$$= -\, \frac{w''_\infty l'_\infty}{r} \delta(l'_\infty - r)\delta(t)\delta(z) e^\mu_\varphi e^\nu_\varphi \quad , \tag{3.1.62}$$

where the angular unit vector $e_\varphi = e_\varphi(\varphi)$ is defined by

$$e^\mu_\varphi(\alpha) = (0, -\sin\alpha, \cos\alpha, 0) \quad . \tag{3.1.63}$$

Its divergence

$$w'_\infty \partial_\mu f^{\mu\nu}_{C'}(q) = \frac{w''_\infty}{r}\delta(l'_\infty - r)\delta(t)\delta(z)e^\nu_r \tag{3.1.64}$$

is equal to the expression (3.1.60) and thus confirms that the physical circle function

$$f^{\mu\nu}_C(q) = w''_\infty f^{\mu\nu}_{C''}(q) - w'_\infty f^{\mu\nu}_{C'}(q) \tag{3.1.65}$$

is divergence-free. Since the Minkowski length of the curve $S_n$ vanishes for any $n$, the physical circle $C$ defines a physical length operator

$$L(C) = \tfrac{1}{2}h(f_C) \tag{3.1.66}$$

with

$$f^{\mu\nu}_C(q) = \frac{W}{r}\delta(t)\delta(z)\Big(\theta(R-r)e^\mu_r e^\nu_r - R\delta(R-r)e^\mu_\varphi e^\nu_\varphi\Big) \quad , \tag{3.1.67}$$

where the parameters radius $R = l'_\infty$ and weight $W = w''_\infty$ have been renamed in analogy to (3.1.41) and (3.1.42). A rough interpretation of this operator is obtained in a straightforward manner from equation (3.1.45) in the limit

$$L(C) = \lim_{n\to\infty} L(S_n) \tag{3.1.68}$$

$$= WR\Delta\zeta \quad , \tag{3.1.69}$$

where $\Delta\zeta$ is the correction to the flat value $2\pi$ of the angle in the center of the circle, which measures the ratio between the circle's circumference and radius. Hence by setting the weight $W$ equal to unity, we obtain a physical operator $L(C)$ roughly providing a correction to the circumference $2\pi R$ of a circle with radius $R$. The exact interpretation will be given below in the paragraph about Gauß curvature.

**General Triangle.** The construction of the physical regular triangle $S_3$ can be generalized to arbitrary triangles. To this end consider a general triangle, as illustrated in FIGURE 13, with side vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ and median vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ pointing from the corners to the center of the triangle. We immediately read off the relations

$$\vec{A} + \vec{B} + \vec{C} = 0 \tag{3.1.70}$$

for the three side vectors. Since the dotted lines in FIGURE 13 leading from the triangle center to the center of the sides are known to be half as long as the corresponding median vectors, we have the equations

$$\vec{C} - \vec{B} = 3\vec{a} \qquad \vec{A} - \vec{C} = 3\vec{b} \qquad \vec{B} - \vec{A} = 3\vec{c} \quad , \tag{3.1.71}$$

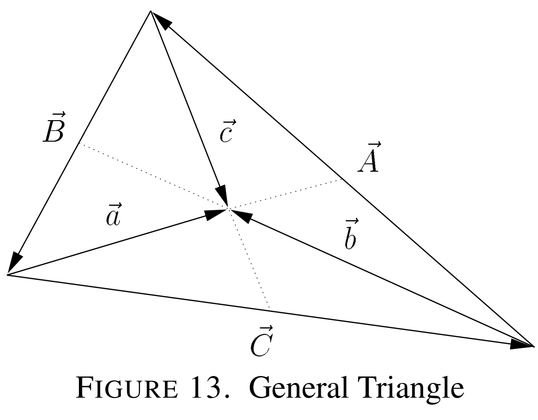



FIGURE 13. General Triangle

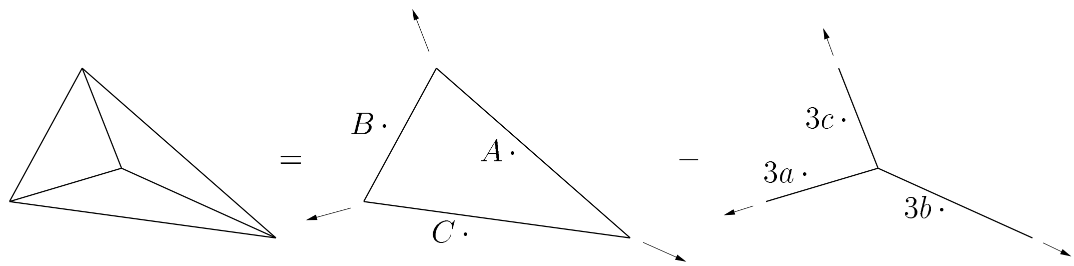



FIGURE 14. Physical General Triangle $T$

which added yield the additional relation

$$\vec{a}+\vec{b}+\vec{c}=0 \quad . \tag{3.1.72}$$

Combining relations (3.1.70) and (3.1.71), we find

$$9a^2+A^2=(\vec{B}-\vec{C})^2+(\vec{B}+\vec{C})^2=2B^2+2C^2 \tag{3.1.73}$$

and its cyclic permutations, where omitting a vector arrow denotes the Minkowski length of the corresponding vector. Adding up the cyclic permutations, we obtain the relation

$$A^2+B^2+C^2=3(a^2+b^2+c^2) \quad . \tag{3.1.74}$$

As illustrated in FIGURE 14, we construct a physical triangle $T$ by taking the three sides each weighed with its Minkowski length and subtracting the three medians each weighed with its triple Minkowski length, i.e.

$$f_T=Af_{\vec{A}}+Bf_{\vec{B}}+Cf_{\vec{C}}-3af_{\vec{a}}-3bf_{\vec{b}}-3cf_{\vec{c}} \quad . \tag{3.1.75}$$

From relation (3.1.71) follows that the curve $T$ is divergence-free in its corners and from (3.1.72) that it is in the center as well. The Minkowski length of the physical triangle $T$ is given by

$$\eta(f_T)=A\eta(f_{\vec{A}})+B\eta(f_{\vec{B}})+C\eta(f_{\vec{C}})-3a\eta(f_{\vec{a}})-3b\eta(f_{\vec{b}})-3c\eta(f_{\vec{c}}) \tag{3.1.76}$$

$$=A^2+B^2+C^2-3(a^2+b^2+c^2) \quad , \tag{3.1.77}$$

which vanishes as a consequence of equation (3.1.74). Hence with every triangle as in FIGURE 14 we associate a physical operator

$$L(T)=\tfrac{1}{2}h(f_T) \quad , \tag{3.1.78}$$

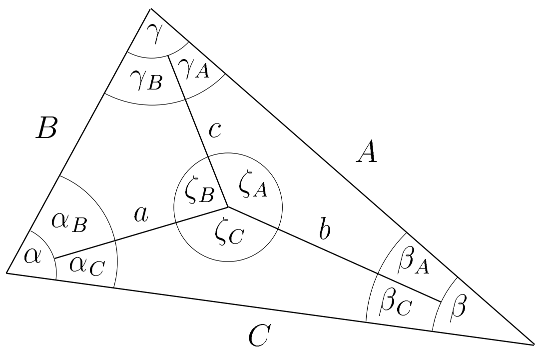



FIGURE 15. The angles of a triangle

where the function $f_T$ is given by (3.1.75). As for the regular polygon in (3.1.34) this physical length operator can be written as a sum

$$L(T) = A\Delta A + B\Delta B + C\Delta C - 3a\Delta a - 3b\Delta b - 3c\Delta c \tag{3.1.79}$$

over the weighed length perturbations of its pieces. In order to roughly interpret the operator $L(T)$ in terms of an angular perturbation, where the angles of a triangle are illustrated in FIGURE 15, we combine relations (3.1.70) and (3.1.71) to obtain

$$9a^2 - A^2 = (\vec{B} - \vec{C})^2 - (\vec{B} + \vec{C})^2 = 4BC\cos\alpha \tag{3.1.80}$$

and its cyclic permutations. Due to the length perturbation the left hand side becomes

$$9(a + \Delta a)^2 - (A + \Delta A)^2 \approx 9a^2 - A^2 + 18a\Delta a - 2A\Delta A \tag{3.1.81}$$

and the right hand side

$$4(B + \Delta B)(C + \Delta C)\cos(\alpha + \Delta\alpha) \approx \tag{3.1.82}$$

$$4BC(\cos\alpha + \Delta\alpha\sin\alpha) + 4(B\Delta C + C\Delta B)\cos\alpha \quad . \tag{3.1.83}$$

We thus obtain

$$18a\Delta a - 2A\Delta A = 4\Delta\alpha BC\sin\alpha + 4(B\Delta C + C\Delta B)\cos\alpha \tag{3.1.84}$$

and the cyclic permutations of this relation. Note that the area $D_T$ of the triangle is given by the expression

$$D_T = \tfrac{1}{2}BC\sin\alpha \quad , \tag{3.1.85}$$

and its cyclic permutations as well. Using the projective theorem

$$A\cos\beta + B\cos\alpha = C \tag{3.1.86}$$

and its cyclic permutations we find that the sum of relation (3.1.84) with its cyclic permutations can be given the form

$$L(T) = -\tfrac{2}{3}D_T(\Delta\alpha + \Delta\beta + \Delta\gamma) \quad . \tag{3.1.87}$$

Hence apart from a factor taking into account the size of the triangle, the length operator $L(T)$ roughly measures the correction to the value $\pi$ of the sum of angles of a triangle in flat spacetime. The result (3.1.87) is in concordance with the special case of the regular triangle, where (3.1.45) predicts

$$L(S_3) = \tfrac{1}{2}W_3R_3(\Delta\zeta_A + \Delta\zeta_B + \Delta\zeta_C) \quad , \tag{3.1.88}$$

since from FIGURE 15 we find

$$\Delta\zeta_A + \Delta\zeta_B + \Delta\zeta_C \approx -(\Delta\alpha + \Delta\beta + \Delta\gamma) \tag{3.1.89}$$

and from

$$D_T = \frac{3\sqrt{3}}{4}R_3^2 \qquad W_3 = \sqrt{3}R_3 \tag{3.1.90}$$

follows

$$\tfrac{1}{2}W_3R_3 = \tfrac{2}{3}D_T \quad . \tag{3.1.91}$$

Let us finally present an alternative general triangle $U$. As illustrated in FIGURE 16, this time we give unit weight to all three outer lines and subtract the three bisects with a weight of twice the cosine of the corresponding angle. In the next paragraph we will show that this »unit« triangle $U$ is physical as well.

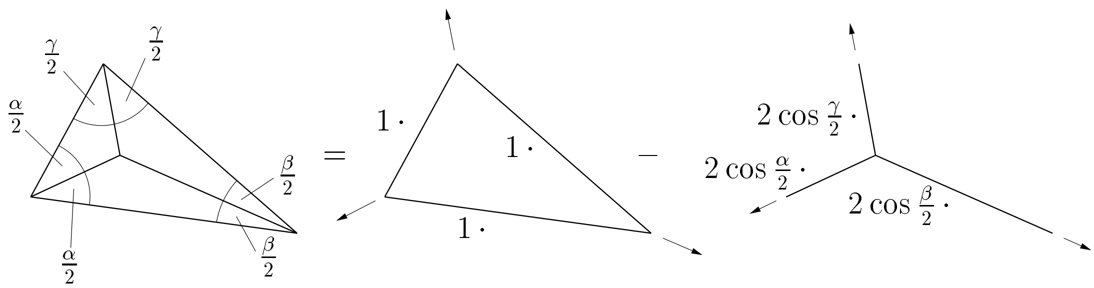



FIGURE 16. Unit Triangle $U$

**Gauß Curvature.** The Poincaré lemma states that, since the operators $L(S_n)$, $L(C)$ and $L(T)$ are physical, they can be expressed in the form of Riemann operators $\tilde{R}(F)$ with smearings $F_{S_n}$, $F_C$ and $F_T$. Since the support of the corresponding metric smearing functions $f_{S_n}$, $f_C$ and $f_T$ is localized in the $x$-$y$-plane with $z = 0$ and $t = 0$, it is reasonable to expect their Riemann smearings to be of the form

$$F^{\mu\alpha\nu\beta}(q) = \begin{cases} \delta(t)\delta(z)F(x,y) & \text{for } \mu\alpha\nu\beta = xyxy \\ 0 & \text{else.} \end{cases} \tag{3.1.92}$$

Hence effectively we are expecting the Riemann operators $\tilde{R}(F)$ to be given as an integral

$$\tilde{K}(F) = \int dxdy\, \tilde{K}(x,y)F(x,y) \tag{3.1.93}$$

over the linearized Gauß curvature

$$\tilde{K}(x,y) = \tilde{R}_{xyxy}(0,x,y,0) \quad . \tag{3.1.94}$$

In this case equations (2.2.86), defining the metric smearing

$$f^{\mu\nu}(q) = \begin{cases} \delta(t)\delta(z)f^{\mu\nu}(x,y) & \text{for } \mu\nu = xx, xy, yx, yy \\ 0 & \text{else} \end{cases} \tag{3.1.95}$$

in terms of the curvature smearing (3.1.92), are given by

$$f^{xx}(x,y) = \tfrac{1}{2}\partial_y\partial_y F(x,y) \tag{3.1.96}$$

$$f^{xy}(x,y) = f^{yx}(x,y) = -\tfrac{1}{2}\partial_y\partial_x F(x,y) \tag{3.1.97}$$

$$f^{yy}(x,y) = \tfrac{1}{2}\partial_x\partial_x F(x,y) \quad . \tag{3.1.98}$$

The functions $F$ are in fact most simply described geometrically by their graph in $x$-$y$-$F$-space. We claim that the function $F_T$ for the physical triangle $T$ is a tetrahedron,

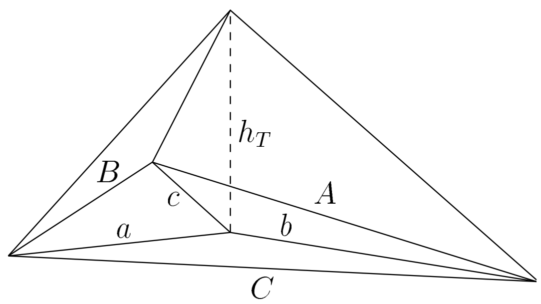



FIGURE 17. Tetrahedron $F_T$ for the physical triangle $T$

whose three bottom edges are the outer lines of the triangle and three top edges project onto the inner lines, as illustrated in FIGURE 17. In order to yield the correct weight, the volume $V_T$ of the tetrahedron has to be chosen as

$$V_T = (\tfrac{2}{3} D_T)^2 \quad , \tag{3.1.99}$$

where $D_T$ is the area of the triangle $T$ in $x$-$y$-space giving the tetrahedron a height

$$h_T = \tfrac{4}{3} D_T \quad . \tag{3.1.100}$$

In order to verify our claim we have to perform the differentiations (3.1.96) to (3.1.98), which is easiest done by choosing suitable coordinate systems in two characteristic regions of the triangle, namely the neighborhoods of an outer and an inner line of the triangle, as illustrated in FIGURE 18. For the outer line differentiation we thus have a function $F_T$, which is constant in the $y$-direction and hence leads to vanishing $f_T^{xx}$, $f_T^{xy}$ and $f_T^{yx}$. The remaining function $f_T^{yy}$ is solely determined by the behavior of the function $F$ in the $x$-direction, where it changes from vanishing to constant slope, i.e.

$$F_T(x,y) = s x \theta(x) \quad . \tag{3.1.101}$$

The slope $s$ is obtained from the condition that the plane reaches the height $h_T$ of the tetrahedron if the $x$-coordinate takes on the value $h_A$ of the distance between the center of the triangle $T$ and the side $A$, as shown in FIGURE 18. Yet this distance is given by

$$h_A = \frac{2 D_T}{3A} \quad , \tag{3.1.102}$$

as follows from the fact that the area $D_T$ of the triangle $T$ is three times the area of each triangle determined by the center point and any of the sides $A$, $B$ or $C$. Hence from the condition

$$F_T(h_A, y) = h_T \tag{3.1.103}$$

and (3.1.100) we find the value

$$s = \frac{h_T}{h_A} = 2A \quad . \tag{3.1.104}$$

Inserting the slope into (3.1.101) and calculating expression (3.1.98), we obtain the smearing function

$$f_T^{yy}(x,y) = A\delta(x) \tag{3.1.105}$$

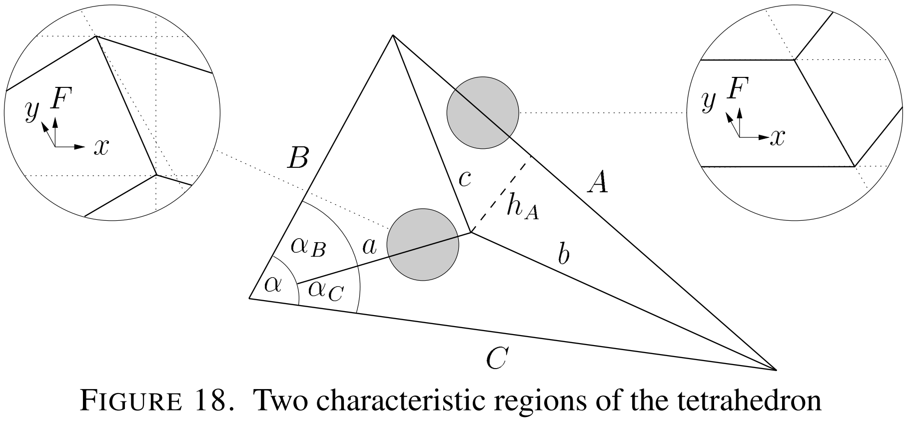



FIGURE 18. Two characteristic regions of the tetrahedron

of a straight line in the $y$-direction with weight $A$ and in analogy weights $B$ and $C$ for the remaining two outer lines, in concordance with definition (3.1.75). For the inner line differentiation things are slightly more complicated in the sense that we are dealing with two slopes in the $x$- and one in the $y$-direction. Rotating the coordinate systems from the above calculations for the outer lines $B$ and $C$ about the $z$-axis by the angles $\alpha_B$ and $-\alpha_C$, respectively, we obtain

$$F_T(x,y) = 2B(\cos\alpha_B x + \sin\alpha_B y)\theta(-x) - 2C(\cos\alpha_C x - \sin\alpha_C y)\theta(x) \tag{3.1.106}$$

for the function $F_T$ in the neighborhood of the inner line $a$. From the linearity of the function $F$ with respect to the coordinate $y$ immediately follows that the function $f_T^{xx}$ vanishes. With the help of the identity

$$B\sin\alpha_B = C\sin\alpha_C \tag{3.1.107}$$

we find that the derivative

$$\partial_x F_T(x,y) = 2B\cos\alpha_B\theta(-x) - 2C\cos\alpha_C\theta(x) \tag{3.1.108}$$

does not depend on the coordinate $y$ and hence the functions $f_T^{xy}$ and $f_T^{yx}$ vanish, too. Due to the identity

$$B\cos\alpha_B + C\cos\alpha_C = 3a \tag{3.1.109}$$

the remaining smearing function $f_T^{yy}$ yields the desired result

$$f_T^{yy}(x,y) = -3a\delta(x) \tag{3.1.110}$$

of a straight line in the $y$-direction with weight $-3a$. Hence the function $F_T$, whose graph is the given tetrahedron, in fact reproduces the physical triangle $T$. In order to find the correct interpretation for the operators $L(T)$ we make use of the well-known fact that the integral over the Gauß curvature on a geodesic triangle is equal to the difference between the sum of its angles and the flat space value $\pi$. In the linear approximation we therefore have

$$\int\limits_{\mathrm{Supp}\,F_T} dxdy\,\tilde{K}(x,y) = \Delta\alpha + \Delta\beta + \Delta\gamma \quad , \tag{3.1.111}$$

which introducing the characteristic function

$$\chi(F_T)(x,y) = \begin{cases} 1 & \text{for } (x,y) \in \mathrm{Supp}\,F_T \\ 0 & \text{else,} \end{cases} \tag{3.1.112}$$

may be rewritten as

$$\int\limits_{\mathrm{Supp}\,F_T} dxdy\,\tilde{K}(x,y) = \int dxdy\,\tilde{K}(x,y)\chi(F_T)(x,y) \quad . \tag{3.1.113}$$

As illustrated in FIGURE 19, we obtain an incomplete tetrahedron $F_T^\lambda$ from the full tetrahedron $F_T$ by chopping off its top at a height $\lambda h_T$. The derivative of the incomplete tetrahedron $F_T^\lambda$ with respect to $\lambda$ is, of course, given by the limit

$$\frac{dF_T^\lambda}{d\lambda} = \lim_{\Delta\lambda\to 0}\frac{F_T^{\lambda+\Delta\lambda} - F_T^\lambda}{\Delta\lambda} \quad , \tag{3.1.114}$$

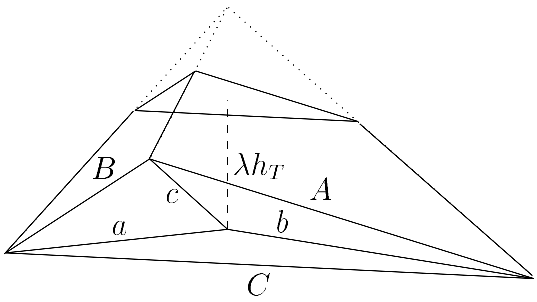



FIGURE 19. Incomplete Tetrahedron $F_T^\lambda$

which becomes the characteristic function

$$\frac{dF_T^\lambda}{d\lambda} = h_T \chi(F_T - F_T^\lambda) \quad , \tag{3.1.115}$$

making contact with our original interpretation of $L(T)$ in terms of a correction to the sum of the three angles of the triangle to the value $\pi$. From the metric point of view the differentiation (3.1.115) is seen as two affine physical triangles with opposite weights approaching each other infinitesimally. As illustrated in FIGURE 20, in the limit $\Delta\lambda \to 0$ the outer lines turn into derivatives of delta functions and the inner lines into delta function at the triangle corners. One may check this result by explicitly calculating the functions (3.1.98) to (3.1.96) for the function $F$ being the characteristic function (3.1.115). Inverting equation (3.1.115) and inserting the interpretation formula (3.1.111) we obtain

$$L(T) = -\tfrac{4}{3} D_T \int_0^1 d\lambda \, (\Delta\alpha_\lambda + \Delta\beta_\lambda + \Delta\gamma_\lambda) \quad , \tag{3.1.116}$$

where $\alpha_\lambda$, $\beta_\lambda$ and $\gamma_\lambda$ are the angles of the triangle defined by Supp$(F_T - F_T^\lambda)$. Hence the length operator $L(T)$ is in fact a measure for the correction to the sum of the three angles to the value $\pi$ but not, as we roughly interpreted before, just for the single triangle $T$ but for the sum of the affine triangles within the triangle $T$ possessing the same center. As a check consider a second special case of constant Gauß curvature $K$, where the angular corrections are of the form

$$\Delta\alpha_\lambda + \Delta\beta_\lambda + \Delta\gamma_\lambda = \lambda^2 D_T K \quad , \tag{3.1.117}$$

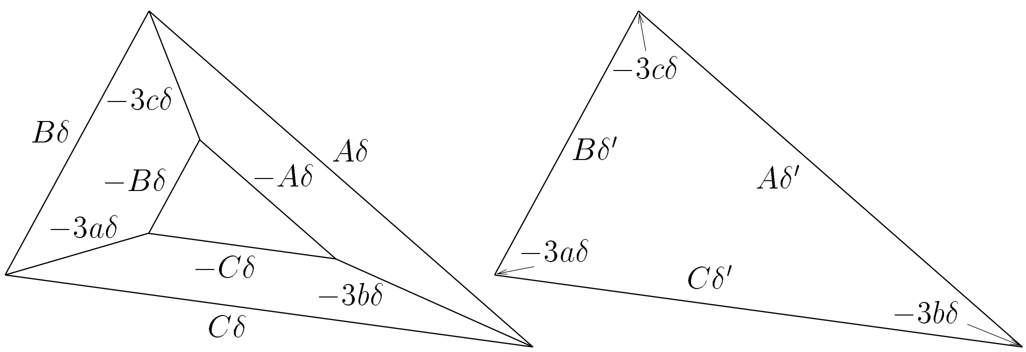



FIGURE 20. The derivative of the incomplete tetrahedron $F_T^\lambda$

leading to the expected result

$$L(T) = -V_T K \quad , \tag{3.1.118}$$

with $V_T$ being the volume of the tetrahedron $F_T$ as defined in (3.1.99). The whole analysis for the triangle $T$ can be carried through in analogy for the unit triangle $U$, yet let us confine ourselves to the remark we have to use a tetrahedron, whose function in the corresponding neighborhoods is given by

$$F_U(x, y) = 2x\theta(x) \tag{3.1.119}$$

in order to obtain a metric smearing

$$f_U^{yy}(x, y) = \delta(x) \quad , \tag{3.1.120}$$

which correctly reproduces the unit weight of the outer lines. As a consequence in the neighborhood of an inner line, e.g. the bisect of the angle $\alpha$, the curvature smearing becomes

$$F_U(x, y) = 2(\cos\tfrac{\alpha}{2}x + \sin\tfrac{\alpha}{2}y)\theta(-x) - 2(\cos\tfrac{\alpha}{2}x - \sin\tfrac{\alpha}{2}y)\theta(x) \quad , \tag{3.1.121}$$

which leads to the metric smearing

$$f_U^{yy}(x, y) = -2\cos\tfrac{\alpha}{2}\delta(x) \tag{3.1.122}$$

reproducing the expected weight and hence proving that the unit triangle $U$ is indeed physical. Similar geometrical objects as the tetrahedron are found to define functions, reproducing the physical regular $n$-gons. These objects range from the pyramid for the physical square $S_4$ to the cone for the physical circle $C$. The curvature function $F_C$ of the cone that yields the desired metric smearing functions

$$f_C^{rr}(r, \varphi) = -\frac{\theta(R - r)}{r} \tag{3.1.123}$$

$$f_C^{r\varphi}(r, \varphi) = f_C^{\varphi r}(r, \varphi) = 0 \tag{3.1.124}$$

$$f_C^{\varphi\varphi}(r, \varphi) = \delta(R - r) \tag{3.1.125}$$

is in fact simply given by

$$F_C(r, \varphi) = 2(R - r)\theta(R - r) \quad . \tag{3.1.126}$$

Note that the rough interpretation (3.1.69) for the length operator $L(C)$ can now be made precise to

$$L(C) = 2WR\int_0^R dr\, \Delta\zeta_r \quad , \tag{3.1.127}$$

where $\Delta\zeta_r$ is a correction to the center angle $\zeta$ of the circle with radius $r$ concentric to the physical circle $C$. As a consequence one might think of the operator

$$\Pi = \pi + \frac{d}{dR}\frac{L(C)}{4WR} \tag{3.1.128}$$

as the quantum-$\pi$ of a circle with radius $R$. Note that the result of the cone for physical circle suggests a method of constructing physical length operators for any spacelike closed curve $S$. As illustrated in FIGURE 21, one simply sets up a cone above the curve $S$ with an arbitrary position of the apex in $x$-$y$-$F$-space, which defines a Gauß curvature smearing $F_S$ and via equations (3.1.98) to (3.1.96) also a metric smearing $f_S$.

However for the corresponding operator to give uniform weight to the lengths of the different parts of the curve the slope at the basis of the cone must have a constant value. For the cone this is certainly true only in the case of a circular base curve leading to the length operator $L(C)$ of the physical circle $C$. However equations (3.1.119) and (3.1.120) show that in fact any geometrical object may be set up above the curve $S$ as long as the slope at its base has the value two. As a result it is definitely possible to construct physical length operators containing the correction to the classical length of any closed spacelike curve $S$ but the part one obtains in addition to the unphysical smearing function $f_S$ is, of course, highly non-unique.

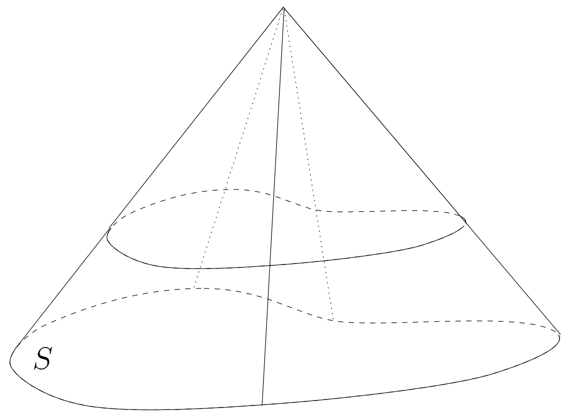



FIGURE 21. The cone over a closed spacelike curve

## 2. Vacuum Fluctuations

**Variance.** The fundamental unpredictability inherent to quantum theory has the amazing consequence that although a pure state represents the optimum of attainable knowledge about a physical system, the repeated measurement of an observable in this state in general does not render a single value but rather a range of values. In order to get a first impression of the spectrum of possible outcomes for the measurement of an operator $A$ in a state $|\varphi\rangle$, one usually calculates the expectation value

$$\langle A\rangle_\varphi = \langle\varphi|A|\varphi\rangle \tag{3.2.1}$$

and the corresponding variance

$$(\Delta A)^2_\varphi = \langle A^2\rangle_\varphi - \langle A\rangle^2_\varphi \quad . \tag{3.2.2}$$

Since we are solely interested in the expectation value and variance of a single state, namely the vacuum state $|0\rangle$, we will in the following simply omit the state index $\varphi$. Since the vacuum expectation value of the of the gravitational perturbation $h$ vanishes, the vacuum expectation value of a length operator (3.1.17) is trivially given by its Minkowski length

$$\langle L(S)\rangle = \eta(f_S) \quad . \tag{3.2.3}$$

In order to obtain a formula for the corresponding variance

$$\Delta_S^2 = \Big(\Delta L(S)\Big)^2 \tag{3.2.4}$$

we take the vacuum expectation value of equation (2.1.18), leading to the two-point function

(3.2.5) $$\langle 0|h_{\mu\nu}(x)h_{\alpha\beta}(x')|0\rangle = iH_{\mu\nu\alpha\beta}D^{+}(x-x') \quad ,$$

which is of course rather of distributional type and hence meant in the sense of

(3.2.6) $$\langle 0|h(f)h(f')|0\rangle = iD^{+}(f,f')$$

with

(3.2.7) $$D^{+}(f,f') = \int dx dx'\, H_{\mu\nu\alpha\beta}f^{\mu\nu}(x)f^{\alpha\beta}(x')D^{+}(x-x') \quad .$$

Therefore the variance of a length operator $L(S)$ with respect to the vacuum state is given by

(3.2.8) $$\Delta_S^2 = \tfrac{i}{4}D^{+}(f_S,f_S) \quad .$$

**Straight Line.** Consider a spacelike straight line $L$ from the origin of Minkowski spacetime to the point

(3.2.9) $$l^{\mu} = le^{\mu} = (0,l\vec{e}) = (0,\vec{l}) \quad .$$

The corresponding smearing function is given by

(3.2.10) $$f_L^{\mu\nu}(q) = -\int_0^1 d\lambda\, le^{\mu}e^{\nu}\delta(q-\lambda l)$$

(3.2.11) $$= -\int_0^l d\lambda\, e^{\mu}e^{\nu}\delta(q-\lambda e)$$

and yields a Minkowski length

(3.2.12) $$\eta(f_L) = l \quad ,$$

a quantum correction

(3.2.13) $$h(f_L) = -\int_0^l d\lambda\, h_{\mu\nu}(\lambda e)e^{\mu}e^{\nu}$$

and hence a length operator

(3.2.14) $$L(L) = l - \tfrac{1}{2}\int_0^l d\lambda\, h_{\mu\nu}(\lambda e)e^{\mu}e^{\nu} \quad .$$

A conceptual problem associated with the length operator $L(L)$ of the straight line $L$ is of course that it is not physical. However, the technical problems in the following calculations can relatively easy be treated and the characteristic features of the variance are already contained in this simple example. Using the supermetric contraction

(3.2.15) $$H_{\mu\nu\alpha\beta}e^{\mu}e^{\nu}e^{\alpha}e^{\beta} = \tfrac{1}{2}$$

the variance of the straight line length operator $L(L)$ becomes

$$\Delta_L^2(l) = \tfrac{i}{8}\int_0^l d\lambda \int_0^l d\lambda' \int dq dq'\, \delta(q-\lambda e)\delta(q'-\lambda' e) D^+(q-q') \tag{3.2.16}$$

with

$$D^+(q-q') = -\frac{i}{(2\pi)^3}\int \frac{d\vec{p}}{2|\vec{p}|}\, e^{-i|\vec{p}|(t-t')+i\vec{p}(\vec{q}-\vec{q}')} \quad . \tag{3.2.17}$$

However, due to the singular character of the line, this variance diverges. For this reason we will smear out the spacelike straight line $L$ in time by substituting

$$\delta(q-\lambda e) = \delta(t)\delta(\vec{q}-\lambda\vec{e}) \quad \longrightarrow \quad \delta_a(t)\delta(\vec{q}-\lambda\vec{e}) \quad , \tag{3.2.18}$$

where for simplicity we have chosen a normalized Gauß smearing

$$\delta_a(t) = \frac{e^{-\frac{t^2}{2a^2}}}{a\sqrt{2\pi}} \quad , \tag{3.2.19}$$

which in the limit

$$\lim_{a\to 0} \delta_a(t) = \delta(t) \tag{3.2.20}$$

restores the singular line. The integrations over time may then be performed as

$$\int dt dt'\, \delta_a(t)\delta_a(t') e^{-i|\vec{p}|(t-t')} = e^{-\vec{p}^2 a^2} \quad . \tag{3.2.21}$$

The momentum integration is handled by introducing spherical polar coordinates. Carrying out the angular integrations first, we obtain

$$\int \frac{d\vec{p}}{2|\vec{p}|}\, e^{i\vec{p}(\vec{q}-\vec{q}')} e^{-\vec{p}^2 a^2} = 2\pi \int_0^\infty dk\, \frac{\sin(k\rho)}{\rho} e^{-k^2 a^2} \tag{3.2.22}$$

with the variable $\rho = |\vec{q}-\vec{q}'|$, which by the space integrations over the delta functions is turned into $\rho = |\lambda - \lambda'|$. This leaves the variance

$$\Delta_L^2(b) = \frac{1}{32\pi^2}\int_0^b d\lambda \int_0^b d\lambda'\, F(|\lambda-\lambda'|) \tag{3.2.23}$$

$$= \frac{1}{16\pi^2}\int_0^b d\rho\, (b-\rho) F(\rho) \tag{3.2.24}$$

with the new variable

$$b = \frac{l}{a} \tag{3.2.25}$$

and the function

$$F(\rho) = \int_0^\infty dk\, \frac{\sin(k\rho)}{\rho} e^{-k^2} \quad . \tag{3.2.26}$$

Inserting a Taylor expansion for the sine function and performing the Gauß integrations yields

$$F(\rho) = \tfrac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n\, n!}{(2n+1)!}\, \rho^{2n} \quad , \tag{3.2.27}$$

leading to

$$\Delta_L^2(b) = \frac{1}{32\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,(n-1)!}{(2n-1)(2n)!}\, b^{2n} \quad . \tag{3.2.28}$$

Hence for small values of $b$ the variance $\Delta_L^2$ is approximated by the quadratic function

$$\Delta_L^2(b) \approx \frac{b^2}{64\pi^2} \quad , \tag{3.2.29}$$

which is shown in FIGURE 22 as a dashed line together with a numerical calculation of the variance drawn as a solid line. The derivative of the variance is given by

$$\frac{d\Delta_L^2}{db} = \frac{1}{16\pi^2}\int_0^b d\rho\, F(\rho) \quad , \tag{3.2.30}$$

where the integral in the limit of $b$ going to infinity becomes

$$\int_0^\infty dk \int_0^\infty d\rho\, \frac{\sin(k\rho)}{\rho} e^{-k^2} = \frac{\pi}{2}\int_0^\infty dk\, e^{-k^2} \tag{3.2.31}$$

$$= \frac{\pi\sqrt{\pi}}{4} \quad . \tag{3.2.32}$$

Hence for large values of $b$ the variance $\Delta_L^2$ is approximated by the linear function

$$\Delta_L^2(b) \approx \frac{b}{64\sqrt{\pi}} \quad , \tag{3.2.33}$$

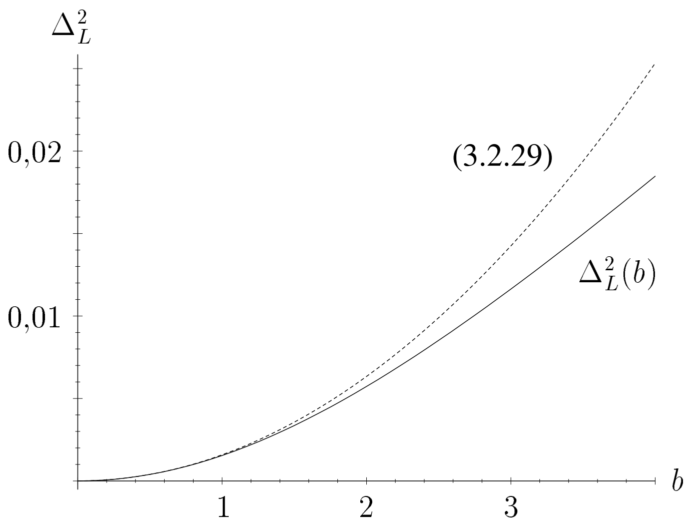



FIGURE 22. Quadratic approximation

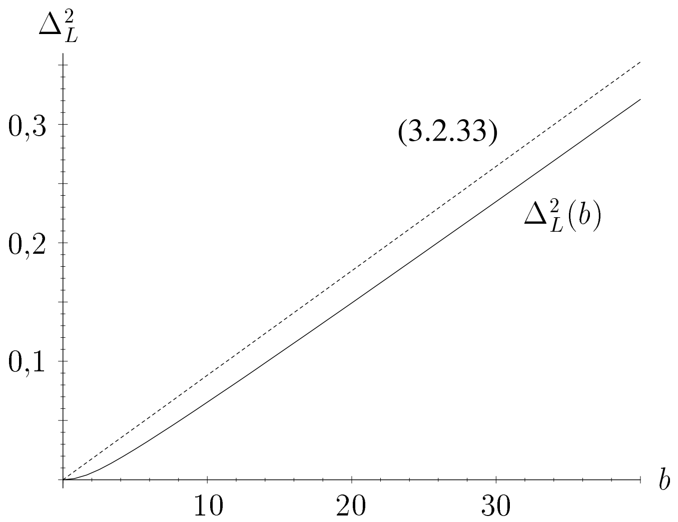



FIGURE 23. Asymptotic linear approximation

which is illustrated in FIGURE 23. In fact a more thorough analysis produces the asymptotic function

$$\Delta_L^2(b) = \frac{b}{64\sqrt{\pi}} - \frac{1}{16\pi^2}\Big(\ln b + \frac{\gamma}{2} + 1\Big) + \mathrm{O}(b^{-2}) \tag{3.2.34}$$

with $\gamma$ being Euler's constant. As illustrated in FIGURE 24, this function yields a much better asymptotic approximation to the variance.

**Uncertainty Relations.** The function $\Delta_L^2$ calculated above describes the strength of the vacuum fluctuations occurring in the measurement of the length of a straight line $L$. For these fluctuations about the classical length $l$ to become finite, we had to introduce a time interval $a$ reflecting the duration of the measurement process. This time smearing can be replaced by a space smearing without changing the results. To this end note that the substitution

$$\delta_a(t)\delta(\vec{q} - \lambda\vec{e}) \quad \longrightarrow \quad \delta(t)\delta_a(\vec{q} - \lambda\vec{e}) \tag{3.2.35}$$

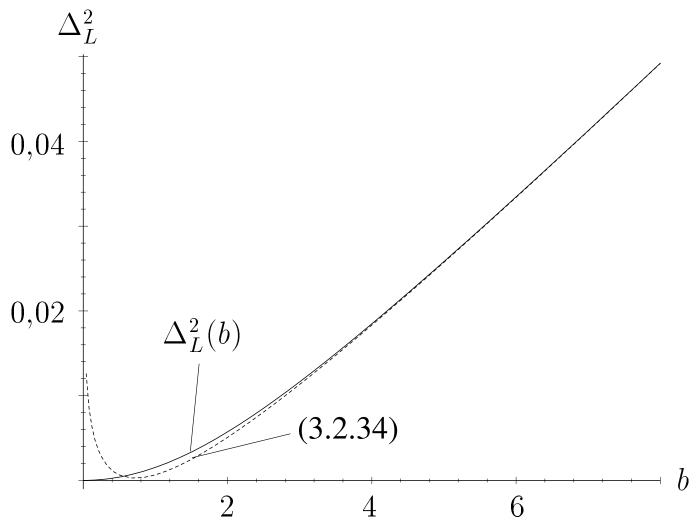



FIGURE 24. Asymptotic logarithmic approximation

due to the relation

$$\int d\vec{q}d\vec{q}'\, \delta_a(\vec{q} - \lambda\vec{e})\delta_a(\vec{q}' - \lambda'\vec{e})e^{i\vec{p}(\vec{q}-\vec{q}')} = e^{-\vec{p}^2a^2}e^{i\vec{p}\vec{e}(\lambda-\lambda')} \tag{3.2.36}$$

also leads to the integral (3.2.22). Therefore the calculated variance function $\Delta_L^2(b)$ applies as well to the case of an instantaneous length measurement of a fuzzy straight line. It should however be clear from FIGURE 25 that in order to interpret the object $L$ as a line and the corresponding operator $L(L)$ as a length the condition $b \gg 0$ must be satisfied. As a consequence the linear approximation (3.2.33) applies and we have $\Delta_L^2$ in the order of

$$\Delta_L^2 \approx \frac{l}{a} \tag{3.2.37}$$

as the variance of a fuzzy straight line $L$ with spacelike length $l$ and width $a$. Since it makes no sense to assign to a line $L$ a length $l$, whose fluctuations $\Delta_L$ are of the same order as the designated value, we demand

$$\Delta_L^2 \ll l^2 \tag{3.2.38}$$

from a proper length measurement. This condition is also the vacuum expectation value of the relation

$$h(f_L)^2 \ll \eta(f_L)^2 \quad , \tag{3.2.39}$$

which is just the perturbative condition (1.2.2) of linearized general relativity applied to the case of the straight line $L$ and may serve as a justification for using the linear approximation. Combining relations (3.2.37) and (3.2.38) we find the condition

$$la \gg 1 \tag{3.2.40}$$

as a limit to the applicability of length measurements. However, this relation (3.2.40) is remarkably similar to the space-space uncertainty relation

$$\Delta x\Delta y + \Delta x\Delta z + \Delta y\Delta z \gtrsim 1 \tag{3.2.41}$$

proposed by Doplicher, Fredenhagen and Roberts[12, 13] for their Minkowski quantum spacetime motivated by the Heisenberg principle and general relativity. In fact, the choice $\Delta x = l$, $\Delta y = \Delta z = a$ leads to

$$la \gtrsim 1 \quad . \tag{3.2.42}$$

In addition the three authors propose a time-space uncertainty relation

$$\Delta t(\Delta x + \Delta y + \Delta z) \gtrsim 1 \quad , \tag{3.2.43}$$

which for the choice $\Delta t = a$, $\Delta x = l$, $\Delta y = \Delta z = 0$ also leads to inequality (3.2.42), this time matching our original interpretation of the parameter $a$ as the duration of

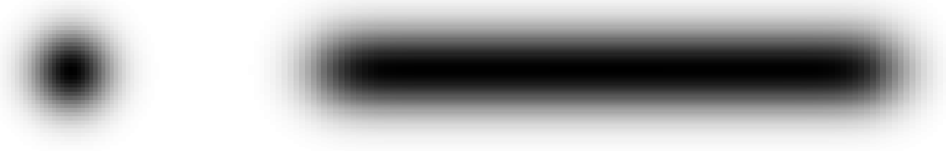


FIGURE 25. Fuzzy lines for $b = 0{,}2$ and $b = 20$

the measurement process. Hence as far as Gauß smearings are concerned, we find that the domain of applicability for measurements of space and time intervals lies well within the limits set forth by Doplicher, Fredenhagen and Roberts. Although we have used the variance $\Delta_L^2$ for the unphysical straight line $L$ to obtain the result (3.2.40), the uncertainty relation is nevertheless gauge invariant since it has been derived in the asymptotic linear limit of large $l$, where the unphysical ends of the straight line $L$ are moved to spacelike infinity.

**Circle.** As for the variance $\Delta_L^2$ of the unphysical straight line $L$, we will now perform the corresponding analysis for the variance $\Delta_C^2$ of the physical circle $C$, which is described by the function (3.1.67). In order to obtain a finite variance $\Delta_C^2$, we will again introduce a Gauß smearing in time and use the function

$$f_C^{\mu\nu}(q) = \frac{1}{r}\delta_a(t)\delta(z)\Big(\theta(R-r)e_r^\mu e_r^\nu - R\delta(R-r)e_\varphi^\mu e_\varphi^\nu\Big) \quad , \tag{3.2.44}$$

where weight $W$ is set equal to unity. For the supermetric contractions we find

$$H_{\mu\nu\alpha\beta}e_r^\mu e_r^\nu e_{r'}^\alpha e_{r'}^\beta = (\vec{e}_r\vec{e}_{r'})^2 - \tfrac{1}{2}\vec{e}_r^{\,2}\vec{e}_{r'}^{\,2} \tag{3.2.45}$$

$$= \cos^2(\varphi-\varphi') - \tfrac{1}{2} \tag{3.2.46}$$

$$= \tfrac{1}{2}\cos(2\varphi-2\varphi') \tag{3.2.47}$$

and similarly

$$H_{\mu\nu\alpha\beta}e_r^\mu e_r^\nu e_{\varphi'}^\alpha e_{\varphi'}^\beta = -\tfrac{1}{2}\cos(2\varphi-2\varphi') \tag{3.2.48}$$

$$H_{\mu\nu\alpha\beta}e_\varphi^\mu e_\varphi^\nu e_{r'}^\alpha e_{r'}^\beta = -\tfrac{1}{2}\cos(2\varphi-2\varphi') \tag{3.2.49}$$

$$H_{\mu\nu\alpha\beta}e_\varphi^\mu e_\varphi^\nu e_{\varphi'}^\alpha e_{\varphi'}^\beta = \tfrac{1}{2}\cos(2\varphi-2\varphi') \quad . \tag{3.2.50}$$

Hence the variance becomes

$$\Delta_C^2(R,a) = \tfrac{i}{8}\int dt dt' dz dz'\; \delta_a(t)\delta_a(t')\delta(z)\delta(z') \tag{3.2.51}$$

$$\cdot\int_0^\infty dr dr'\,\Big(\theta(R-r)+R\delta(R-r)\Big)\Big(\theta(R-r')+R\delta(R-r')\Big) \tag{3.2.52}$$

$$\cdot\int_0^{2\pi} d\varphi d\varphi'\,\cos(2\varphi-2\varphi')D^+(q-q') \quad . \tag{3.2.53}$$

As for the straight line, the integrations over time are again given by (3.2.21) and the angular integrations over the momentum variable by (3.2.22). Keeping in mind the $z$-$z'$-integrations, we find that the variable $\rho = |q-q'|$ in terms of the polar coordinates is determined by the relation

$$\rho^2 = r^2 + r'^2 - 2rr'\cos\vartheta \tag{3.2.54}$$

with $\vartheta = \varphi - \varphi'$. Introducing the variable

$$b = \frac{R}{a} \quad , \tag{3.2.55}$$

we may therefore write the circle circumference variance in the form

$$\Delta_C^2(b) = \Big(1 + b\frac{\partial}{\partial b}\Big)\Big(1 + b'\frac{\partial}{\partial b'}\Big)T(b,b')\Big|_{b=b'} \tag{3.2.56}$$

with the function

$$T(b,b') = \frac{1}{16\pi}\int_0^b dr \int_0^{b'} dr' \int_0^{2\pi} d\vartheta\, \cos(2\vartheta) F(\rho) \quad , \tag{3.2.57}$$

where $F(\rho)$ is defined by (3.2.26). By a functional theoretic argument or a table of integrals[24], we find that this function is also given by

$$F(\rho) = \tfrac{1}{2}\int_0^1 dk\, e^{-\frac{\rho^2}{4}(1-k^2)} \quad . \tag{3.2.58}$$

We insert relation (3.2.54) and perform the angular integration[25]

$$\int_0^{2\pi} d\vartheta\, \cos(2\vartheta) e^{\frac{1}{2}rr'(1-k^2)\cos\vartheta} = 2\pi I_2\Big(\tfrac{1}{2}rr'(1-k^2)\Big) \tag{3.2.59}$$

to obtain

$$T(b,b') = \frac{1}{16}\int_0^b dr \int_0^{b'} dr' \int_0^1 dk\, I_2\Big(\tfrac{1}{2}rr'(1-k^2)\Big) e^{-\frac{1}{4}(r^2+r'^2)(1-k^2)} \quad , \tag{3.2.60}$$

with the Bessel function

$$I_2(z) = -J_2(iz) \tag{3.2.61}$$

of an imaginary argument. Introducing Taylor expansions for the Bessel function[26]

$$I_2\Big(\tfrac{1}{2}rr'(1-k^2)\Big) = \sum_{n=1}^{\infty} \frac{r^{2n} r'^{2n} (1-k^2)^{2n}}{16^n (n-1)!(n+1)!} \tag{3.2.62}$$

and the exponential function

$$e^{-\frac{1}{4}(r^2+r'^2)(1-k^2)} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (-\tfrac{1}{4})^{i+j} \frac{r^{2i}}{i!}\frac{r'^{2j}}{j!}(1-k^2)^{i+j} \quad , \tag{3.2.63}$$

we are able to perform the integration

$$\int_0^1 dk\, (1-k^2)^{2n+i+j} = \frac{4^{2n+i+j}(2n+i+j)!^2}{(4n+2(i+j)+1)!} \quad . \tag{3.2.64}$$

Since

$$\Big(1 + b\frac{\partial}{\partial b}\Big)\int_0^b dr\, r^{2n+2i} = \frac{2(n+i+1)}{2(n+i)+1} b^{2n+2i+1} \quad , \tag{3.2.65}$$

we finally obtain

$$\Delta_C^2(b) = \tfrac{1}{4}\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{n+i+1}{2(n+i)+1}\,\frac{n+j+1}{2(n+j)+1} \tag{3.2.66}$$

$$\cdot\,\frac{(2n+i+j)!^2}{(4n+2(i+j)+1)!}\,\frac{(-1)^{i+j}(b^2)^{2n+i+j+1}}{i!j!(n-1)!(n+1)!} \quad , \tag{3.2.67}$$

which by the introduction of the summation index $m = 2n+i+j$ can be rewritten in the form

$$\Delta_C^2(b) = \tfrac{1}{4}\sum_{m=2}^{\infty}\frac{(-1)^m m!^2}{(2m+1)!}b^{2(m+1)}\sum_{n=1}^{[\frac{m}{2}]}\frac{1}{(n-1)!(n+1)!} \tag{3.2.68}$$

$$\cdot\sum_{i=0}^{m-2n}\frac{1}{(m-2n-i)!}\,\frac{m-n-i+1}{2(m-n-i)+1}\,\frac{n+i+1}{2(n+i)+1}\,\frac{1}{i!} \quad , \tag{3.2.69}$$

where the Gauss bracket $[\,\cdot\,]$ of a number indicates rounding off to the next integer. Hence for small values of $b$ the variance $\Delta_C^2$ is approximated by the function

$$\Delta_C^2(b) \approx \frac{b^6}{540} \quad , \tag{3.2.70}$$

which is shown in FIGURE 26 as a dashed line together with the numerical calculation of the variance drawn as a solid line. In order to determine the behavior for large values of $b$, we have to examine expression (3.2.56) in more detail. The double derivative term can be solved exactly to[27]

$$T''(b) = bb'\frac{\partial}{\partial b}\frac{\partial}{\partial b'}T(b,b')\bigg|_{b=b'} \tag{3.2.71}$$

$$= \frac{b^2}{16}\int_0^1 dk\,\frac{e^{-\frac{1}{2}b^2k}}{\sqrt{1-k}}I_2\left(\tfrac{1}{2}b^2k\right) \tag{3.2.72}$$

$$= \frac{b^6}{960}\,{}_2F_2\left(\tfrac{5}{2},3;5,\tfrac{7}{2};-b^2\right) \quad , \tag{3.2.73}$$

where ${}_2F_2$ is a generalized hypergeometric function, which is related to a confluent hypergeometric function ${}_1F_1$ via the differential identities

$$\left(3+z\frac{d}{dz}\right)\left(4+z\frac{d}{dz}\right){}_2F_2\left(\tfrac{5}{2},3;5,\tfrac{7}{2};z\right) = 12\,{}_1F_1\left(\tfrac{5}{2};\tfrac{7}{2};z\right) \tag{3.2.74}$$

$$\left(\tfrac{5}{2}+z\frac{d}{dz}\right){}_2F_2\left(\tfrac{5}{2},3;5,\tfrac{7}{2};z\right) = \tfrac{5}{2}\,{}_1F_1(3;5;z) \quad . \tag{3.2.75}$$

The confluent hypergeometric function ${}_1F_1$ is known[28] to asymptotically behave as

$${}_1F_1(a;c;-z) = \frac{\Gamma(c)\,z^{-a}}{\Gamma(c-a)}\,{}_2F_0(a,a-c+1;z^{-1}) + \mathrm{O}(e^{-z}) \tag{3.2.76}$$

leading to

$$ {}_2F_2\left(\tfrac{5}{2}, 3; 5, \tfrac{7}{2}; -z\right) = \frac{30\sqrt{\pi}}{z^{\frac{5}{2}}} - \frac{120}{z^3} + \frac{120}{z^4} + \mathrm{O}(e^{-z}) \tag{3.2.77} $$

and hence

$$ T''(b) = \frac{\sqrt{\pi}}{32}b - \frac{1}{8} + \frac{1}{8b^2} + \mathrm{O}(e^{-z}) \quad . \tag{3.2.78} $$

Due to the symmetry

$$ T(b, b') = T(b', b) \tag{3.2.79} $$

the sum of the two single derivative terms of expression (3.2.56) is equal to

$$ T'(b) = 2b\frac{\partial}{\partial b}T(b, b')\bigg|_{b=b'} \quad . \tag{3.2.80} $$

From the general rule

$$ \frac{\partial}{\partial b}T(b, b) = \left(\frac{\partial}{\partial b} + \frac{\partial}{\partial b'}\right)T(b, b')\bigg|_{b=b'} \tag{3.2.81} $$

it follows that the remaining term

$$ T(b) = T(b, b) \tag{3.2.82} $$

is determined by the relation

$$ b\frac{\partial}{\partial b}T(b) = T'(b) \quad . \tag{3.2.83} $$

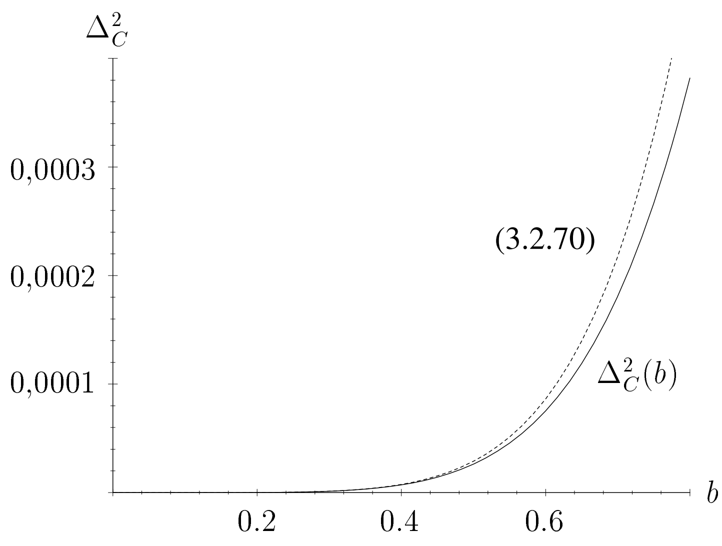



FIGURE 26. Sixth order approximation

As a crude upper bound for the two single derivative terms serves the limit[29, 30, 31]

$$T'_\infty(b) = \lim_{b'\to\infty} 2b\frac{\partial}{\partial b}T(b,b') \tag{3.2.84}$$

$$= \frac{b}{8}\int_0^1 dk \int_0^\infty dr\, I_2\Big(\tfrac{1}{2}rb(1-k^2)\Big)e^{-\frac{1}{4}(r^2+b^2)(1-k^2)} \tag{3.2.85}$$

$$= \frac{b\sqrt{\pi}}{8}\int_0^1 \frac{dk}{\sqrt{1-k^2}}\, I_1\Big(\tfrac{1}{8}rb(1-k^2)\Big)e^{-\frac{1}{8}(r^2+b^2)(1-k^2)} \tag{3.2.86}$$

$$= \frac{b^3\pi^{\frac{3}{2}}}{512}\, {}_2F_2\Big(\tfrac{3}{2},\tfrac{3}{2};3,2;-\tfrac{1}{4}b^2\Big) \quad . \tag{3.2.87}$$

From the differential equation[32]

$$\left\{\frac{d}{dz}\Big(2+z\frac{d}{dz}\Big)\Big(1+z\frac{d}{dz}\Big) - \Big(\tfrac{3}{2}+z\frac{d}{dz}\Big)\Big(\tfrac{3}{2}+z\frac{d}{dz}\Big)\right\}{}_2F_2\Big(\tfrac{3}{2},\tfrac{3}{2};3,2;z\Big) = 0 \tag{3.2.88}$$

we deduce the generalized hypergeometric function's asymptotic behavior

$$(\pi z)^{\frac{3}{2}}{}_2F_2\Big(\tfrac{3}{2},\tfrac{3}{2};3,2;-z\Big) = c_1 \ln z + c_2 + \mathrm{O}\Big(\frac{\ln z}{z}\Big) \quad , \tag{3.2.89}$$

where an additional numerical analysis suggests the values

$$c_1 = 8 \tag{3.2.90}$$

$$c_2 = 16e^{-1} \tag{3.2.91}$$

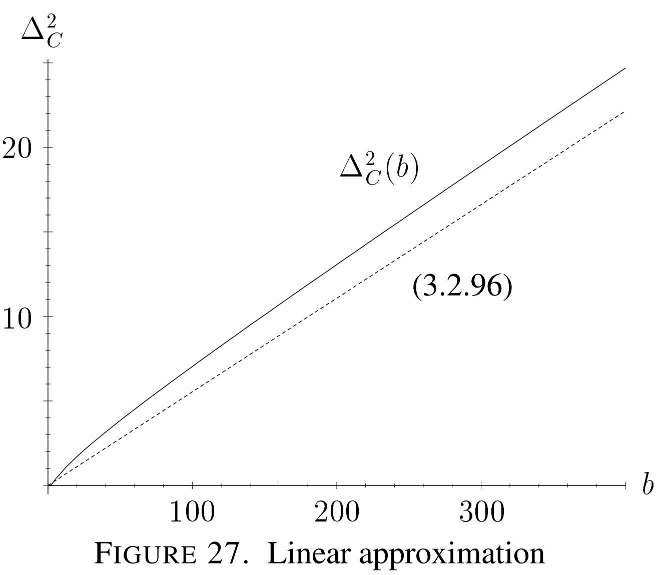



FIGURE 27. Linear approximation

leading to

$$T'_\infty(b) = \tfrac{1}{4}\ln b + \tfrac{1}{4}(e^{-1} - \ln 2) + \mathrm{O}\Big(\frac{\ln b}{b^2}\Big) \quad . \tag{3.2.92}$$

It follows from relation (3.2.83) that the non-derivative term $T(b)$ is bounded from above by

$$T_\infty(b) = \int \frac{db}{b}\, T'_\infty(b) \tag{3.2.93}$$
$$= \tfrac{1}{8}(\ln b)^2 + \tfrac{1}{4}(e^{-1} - \ln 2)\ln b + \mathrm{O}(\text{const.}) \quad . \tag{3.2.94}$$

As a consequence the asymptotic behavior of the variance

$$\Delta_C^2(b) = T(b) + T'(b) + T''(b) \tag{3.2.95}$$

is dominated by the linear function of the double derivative term (3.2.78). Hence for large values of $b$, we have found the approximation

$$\Delta_C^2(b) \approx \frac{\sqrt{\pi}}{32} b \quad , \tag{3.2.96}$$

which is illustrated in FIGURE 27. Further analysis reveals the surprising fact that our rough estimate for the non-linear correction doubles the exact result to a high degree of accuracy such that the asymptotic variance becomes

$$\Delta_C^2(b) = \frac{\sqrt{\pi}}{32} b + \frac{(\ln b)^2}{16} + \tfrac{1}{8}(e^{-1} - \ln 2 + 1)\ln b + \mathrm{O}(\text{const.}) \quad . \tag{3.2.97}$$

FIGURE 28 shows the derivative of the linear and logarithmic approximations together with the numerical solution.

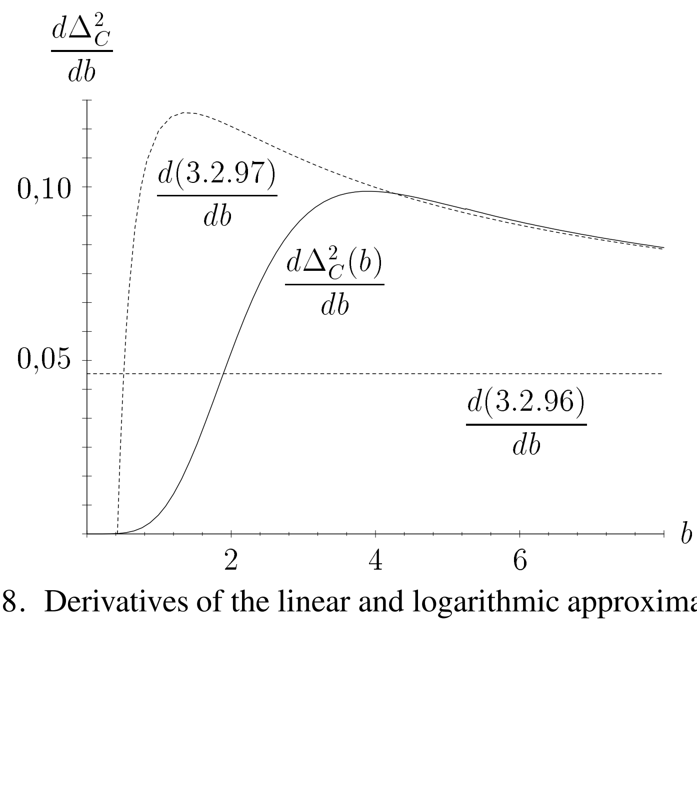



FIGURE 28. Derivatives of the linear and logarithmic approximations

**Asymptotic Linearity.** The linear behavior in the asymptotic regime $b \gg 0$ seems to be a generic feature of the variances $\Delta_L^2$ and $\Delta_C^2$. In order to prove this asymptotic linearity without reference to a particular form of time smearing, consider a function $\phi$, which is normalized

$$\int dt\, \phi(t) = 1 \quad , \tag{3.2.98}$$

for simplicity centered around the origin

$$\int dt\, t\phi(t) = 0 \tag{3.2.99}$$

and scaled with a parameter $a$ defines a smearing function

$$\phi_a(t) = \frac{\phi(\frac{t}{a})}{a} \tag{3.2.100}$$

of scale $a$. Up to this point we used as $\phi$ the Gauß function

$$\delta(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \quad , \tag{3.2.101}$$

not to be confused with the delta function, which in this notation is written as $\delta_0$ and in fact is any limit $\phi_0$. Entering the line of argument for the straight line in equation (3.2.21), the integration over time becomes

$$\int dt dt'\, \phi_a(t)\phi_a(t') e^{-i|\vec{p}|(t-t')} = 2\pi |\tilde{\phi}_a(|\vec{p}|)|^2 \quad . \tag{3.2.102}$$

Since for the Fourier transform $\tilde{\phi}_a$ the relation

$$\tilde{\phi}_a(|\vec{p}|) = \tilde{\phi}(a|\vec{p}|) \tag{3.2.103}$$

holds, we can move on and obtain

$$F(\rho) = 2\pi \int_0^\infty dk\, \frac{\sin(k\rho)}{\rho} |\tilde{\phi}(k)|^2 \tag{3.2.104}$$

for the function (3.2.26), which following (3.2.30) leads to the limit

$$\lim_{b\to\infty} \frac{d\Delta_L^2}{db} = \frac{1}{8\pi} \int_0^\infty dk \int_0^\infty d\rho\, \frac{\sin(k\rho)}{\rho}\, |\tilde{\phi}(k)|^2 \tag{3.2.105}$$

$$= \frac{1}{16} \int_0^\infty dk\, |\tilde{\phi}(k)|^2 \tag{3.2.106}$$

$$= \frac{1}{32} \int dt\, \phi(t)^2 \quad . \tag{3.2.107}$$

One may check this equation for the Gauß function (3.2.101) and compare it with the former result (3.2.33). Since the integral (3.2.107) is non-zero and finite, we have shown that irrespective of the chosen smearing function the asymptotic behavior of the straight line variance $\Delta_L^2$ is linear. In the case of the physical circle $C$ the analogous calculations are a little more complex. Inserting (3.2.104) into (3.2.57), we find that the double derivative term (3.2.71), which contained the asymptotic linear term in the calculation for the Gauß function, is given by

$$T''(b) = \frac{b}{16}\int_0^\infty dk \int_0^{2\pi} d\vartheta\, \cos 2\vartheta\, \frac{\sin\left(2bk \sin\frac{1}{2}\vartheta\right)}{\sin\frac{1}{2}\vartheta}\, |\tilde{\phi}(k)|^2 \tag{3.2.108}$$

$$= \frac{b}{16}\int_0^\infty dk\, |\tilde{\phi}(k)|^2\, C(bk) \quad . \tag{3.2.109}$$

with the function[33]

$$C(z) = 4\int_0^1 du\, \sin(2zu)\left(\frac{1}{u\sqrt{1-u^2}} - 8u\sqrt{1-u^2}\right) \tag{3.2.110}$$

$$= 4\pi z\left({}_1F_2\left(\tfrac{1}{2}; \tfrac{3}{2}, 1; -z^2\right) - {}_1F_2\left(\tfrac{3}{2}; \tfrac{3}{2}, 3; -z^2\right)\right) \quad . \tag{3.2.111}$$

The second hypergeometric function can be rewritten

$${}_1F_2\left(\tfrac{3}{2}; \tfrac{3}{2}, 3; -z^2\right) = {}_0F_1(3; -z^2) \quad , \tag{3.2.112}$$

where the hypergeometric function ${}_0F_1$ satisfies the relation[34]

$$z^{c-1}{}_0F_1(c; -z^2) = \Gamma(c) J_{c-1}(2z) \quad . \tag{3.2.113}$$

Since the Bessel function $J_2$ is an oscillating function, whose amplitude in the limit[35] of large $z$ approaches

$$|J_2(z)| \lessapprox \sqrt{\frac{2}{\pi z}} \quad , \tag{3.2.114}$$

we find

$$|{}_0F_1(3; -z^2)| \lessapprox \frac{2}{\sqrt{\pi}} z^{-\frac{5}{2}} \tag{3.2.115}$$

such that in the limit of large $z$ the second hypergeometric function of (3.2.111) does not significantly contribute to the function $C(z)$. However, for the first hypergeometric function the asymptotic approximation

$${}_1F_2\left(\tfrac{1}{2}; \tfrac{3}{2}, 1; -z^2\right) \approx \frac{1}{2z} \tag{3.2.116}$$

holds, which leads to

$$\lim_{z\to\infty} C(z) = 2\pi \tag{3.2.117}$$

and hence to the desired result

$$\lim_{b\to\infty} \frac{T''(b)}{b} = \frac{\pi}{8}\int_0^\infty dk\, |\tilde{\phi}(k)|^2 \tag{3.2.118}$$

$$= \frac{\pi}{16}\int dt\, \phi(t)^2 \quad , \tag{3.2.119}$$

proving the asymptotic linearity for the term $T''(b)$. Unfortunately we have not yet been able to show that the remaining terms $T'(b)$ and $T(b)$ are of non-leading order. However, the fact that in the case of the Gauß function (3.2.101) the linear coefficient (3.2.119) originating from the term $T''(b)$ already produces the correct result (3.2.96) strongly supports the idea that the remaining terms do not contribute to the asymptotic linearity.

**Observability.** From the asymptotic linearity of the variance $\Delta_L^2$ follows that the uncertainty relations (3.2.40) can be generalized to

$$la \gg \frac{c(\phi)}{32} \quad , \tag{3.2.120}$$

or equivalently

$$la \gtrsim c(\phi) \quad , \tag{3.2.121}$$

where the integral

$$c(\phi) = \int dt\, \phi(t)^2 \tag{3.2.122}$$

obviously depends on the specific form $\phi$, which has been chosen for the smearing function $\phi_a$. However, in order to interpret the scaling parameter $a$ as a maximal extension we may assume

$$\text{Supp}\phi_a = \Big[ -\tfrac{a}{2}, \tfrac{a}{2} \Big] \quad , \tag{3.2.123}$$

or equivalently

$$\text{Supp}\phi \ = \Big[ -\tfrac{1}{2}, \tfrac{1}{2} \Big] \quad . \tag{3.2.124}$$

From the Schwarz inequality we find

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} dt dt'\, \phi(t)^2 \geq \Big( \int_{-\frac{1}{2}}^{\frac{1}{2}} dt\, \phi(t) \Big)^2 \tag{3.2.125}$$

which because of the support condition (3.2.124) is equal to

$$\int dt\, \phi(t)^2 \geq \Big( \int dt\, \phi(t) \Big)^2 \tag{3.2.126}$$

and hence

$$c(\phi) \geq 1 \quad . \tag{3.2.127}$$

This result applies as well to the Gauß function (3.2.101), which is in fact excluded by the assumption (3.2.123) but sufficiently quickly decreases outside of the interval (3.2.124). Due to equation (3.2.127) we can now replace inequality (3.2.121) by the universal uncertainty relation

$$la \gtrsim 1 \quad , \tag{3.2.128}$$

which introducing mks-units becomes

$$l[\text{m}] \cdot a[\text{s}] \gtrsim \lambda_{\text{P}} \tau_{\text{P}} \tag{3.2.129}$$

with the Planck length $\lambda_{\mathrm{P}} = 1{,}616 \cdot 10^{-35}$m and time $\tau_{\mathrm{P}} = 5{,}391 \cdot 10^{-44}$s. As with most other results from quantum gravity, due to the smallness of

$$\lambda_{\mathrm{P}}\tau_{\mathrm{P}} \approx 10^{-80} \quad , \tag{3.2.130}$$

the uncertainty relation is not expected to be directly observable in the near future, though one might hope to discover a consequence, which is open to measurement. As a measure for the observability of the straight line length fluctuations $\Delta_L$ consider the dimensionless relative magnitude

$$M = \frac{\Delta_L}{l} \quad , \tag{3.2.131}$$

whose square we have found to be given by

$$M^2 = \frac{c(\phi)}{la} \quad . \tag{3.2.132}$$

The lower bound (3.2.127) for the function $c(\phi)$, which has been a consequence of the support condition (3.2.124), is actually attained for a uniform distribution

$$\phi(t) = \theta\Big(t + \tfrac{1}{2}\Big) - \theta\Big(t - \tfrac{1}{2}\Big) \quad . \tag{3.2.133}$$

In order for $c(\phi)$ to become significantly greater than unity, the function $\phi(t)$ itself must become large at some point and as a consequence of (3.2.98) possess one or several narrow peaks. However, in this case the support of the function $\phi$ is not the most useful indicator of its localization. Therefore in order to correctly interpret the scaling parameter $a$ as the extension of the smearing $\phi_a$, we have to choose a smearing form $\phi$ such that

$$c(\phi) \approx 1 \quad . \tag{3.2.134}$$

Since by a standard argument the energy $E$ involved in the resolution of a space or time extension $d$ is approximately given by

$$E_d \approx \frac{1}{d} \quad , \tag{3.2.135}$$

we find that the relative magnitude $M$ of the length fluctuation effect is related to the energies involved as

$$M^2 \approx E_l E_a \quad . \tag{3.2.136}$$

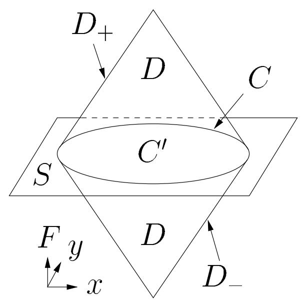
Inserting the Planck energy $\varepsilon_{\mathrm{P}} = 1{,}222 \cdot 10^{19}$GeV leads to the result

$$M \approx \sqrt{E_l[\mathrm{GeV}] \cdot E_a[\mathrm{GeV}]} \cdot 10^{-19} \quad . \tag{3.2.137}$$

Taking into account the upper limit $E \approx 1000$GeV for today's accelerators, we finally obtain a value of $M \approx 10^{-16}$ reflecting the direct observability of the length fluctuation effect.

# Outlook

**Physical Length Operators.** Within the framework of linear quantum gravity we have defined length operators of spacelike curves, which in general have turned out to be gauge variant and hence unphysical. Yet we were able to construct certain physical length operators as suitable combinations of unphysical ones. We further showed that every length operator of a closed spacelike curve may be turned into a physical operator by adding an operator located in the interior of the closed curve. This correction is, of course, not unique and has to be specified according to the desired interpretation. Let us just mention one interesting prescription for the construction of such a physical operator. To this end consider a closed spacelike curve $C$ and choose any two-dimensional spacelike surface $S$, in which the curve is contained. Interprete a cross section in the trivial real line bundle $B = S \times \mathbb{R}$ as a smearing function $F$ on the surface $S$, which may be used to define a Gauß curvature operator $K(F)$ as in equation (3.1.93). Introduce on the bundle space $B$ the flat Lorentzian metric with diagonal $(-1, -1, 1)$, taking as the timelike direction the fibres. Consider the three-dimensional region $D \subset B$ that is causally dependent on the curve $C$ and its interior $C'$. Define two surfaces $D_+$ and $D_-$ as sets of points, through which timelike curves respectively leave and enter the region $D$. The two surfaces $E_\pm = D_\pm \cup S \setminus C'$ may each be looked upon as a cross section in the bundle $B$ and accordingly define two smearing functions $F_\pm$. From equations (3.1.119) and (3.1.120) of the analysis given for unit triangle $U$ it should be clear that the edge of the two surfaces $E_\pm$ above the curve $C$ possesses just the right slope for the Gauß operator $K(F_+) - K(F_-)$ to contain the length operator $L(C)$ of the closed curve $C$.




**Asymptotic Linearity.** Since the length operators we have defined are in fact operator-valued distributions, we had to introduce a smearing in time in order to obtain finite results for the calculation of their vacuum fluctuations. In the special case of a Gauß smearing we extensively studied the fluctuations for the unphysical straight line $L$ and the physical circle $C$. We noticed in our calculations that in the limit of a large length $l$ or radius $R$ in comparison to the extension $a$ of the timelike smearing the variances $\Delta_L^2$ and $\Delta_C^2$ both become linear. We were finally able to generalize this result to an arbitrary normalized time smearing $\phi_a$ with extension $a$ and obtained

$$\Delta_L^2 = \frac{l}{32} \int dt\, \phi_a^2(t) \tag{3.2.138}$$

$$\Delta_C^2 = \frac{2\pi R}{32} \int dt\, \phi_a^2(t) \tag{3.2.139}$$

for the spacelike straight line $L$ of length $l \gg a$ and the spacelike physical circle $C$ with radius $R \gg a$. Let us at this point formulate the conjecture

$$\Delta_S^2 = \frac{s}{32}\int dt\,\phi_a^2(t) \tag{3.2.140}$$

for the variance $\Delta_S^2$ of the length operator $L(S)$ of a spacelike curve $S$ with an expected length $s$ much greater than the duration $a$ of the measurement.

**Uncertainty Relation.** The asymptotic linear formula of the variance $\Delta_L^2$ for the straight line $L$, though being unphysical, is nevertheless a gauge invariant result since it has been derived in the limit of large $l$, where the unphysical ends of the straight line $L$ are moved to spacelike infinity. Splitting the smearing function $\phi_a$ into its form $\phi$ and scale $a$ as in equation (3.2.100), we arrive at the original formula

$$\Delta_L^2 = \frac{1}{32}\frac{l}{a}\int dt\,\phi^2(t) \quad , \tag{3.2.141}$$

of asymptotic linearity for the variable $\frac{l}{a}$. We have argued that the scale $a$ has absolute meaning only in the case of a quite uniformly distributed smearing function, i.e.

$$\int dt\,\phi^2(t) \approx 1 \quad . \tag{3.2.142}$$

In combination with the reasonable assumption that for a proper length measurement the fluctuations $\Delta_L$ must be smaller than the designated value $l$, we have derived the relation

$$la \gtrsim 1 \quad . \tag{3.2.143}$$

The observation that this relation coincides with the limit for the validity of the linear approximation to general relativity has in retrospect served as a justification for the application of linear quantum gravity to the problem of calculating vacuum fluctuations of lengths. As an astounding feature of the relation we have mentioned its similarity to the uncertainty relations derived by Doplicher, Fredenhagen and Roberts in an entirely different approach to quantizing Minkowski space. It should hence be a matter of great interest to find out more about the existence of such relations in linear quantum gravity.

**Generalizations.** Finally we would like to mention three obvious generalizations in connection to our studies. The first generalization is the dimension of the geometric objects and quantities under study. Besides length operators for curves, one should in a straightforward manner be able to define area operators for surfaces and volume operators for hypersurfaces in terms of the metric perturbation $h$. Accordingly the gauge condition should be geometrically implemented and the vacuum fluctuations of physical quantities be examined. The second generalization to be mentioned is that of the state, in which the fluctuations are calculated. Other Fock states or even a state of thermal equilibrium should be considered. The third and last generalization concerns the background spacetime, which in fact should not be confined to Minkowski space. A spacetime of particular interest is that of a black hole, where the object to be quantized is a small perturbation of the Schwarzschild metric. One might for example want to calculate the fluctuations of the area operator for the horizon. Since the radius of the black hole is defined via this quantity, one is effectively studying fluctuations of the size of the black hole. It is safe to conclude that linear quantum gravity is a vast area of study with lots of interesting research to be done.

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