Seminar on Mathematical Aspects of Theoretical Physics by Prof. Klaus Fredenhagen

Partial Differential Equations I
Causal Structure On Lorentzian Manifolds

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Abstract

The main goal of this seminar talk is the preparation for the solution of the Cauchy problem on hyperbolic differntial manifolds. In this first part on differential equations I will give a short introduction about second order partial differential equations and motivate the importance of the hyperbolic type. I will also shortly introduce the main ideas of a solution in the distributional sense of initial and boundary value problems. In the second part, based on [Fre, §2], we will work through the standard definitions of hyperbolic manifolds to understand the foliated structure of a spacetime. Finally, based on [Tay, §5], I will demonstrate the solution of the wave equation on $(n + 1)$-dimensional Minkowski spacetime, as a preparation for its generalization arbitrary globally hyperbolic spacetimes in the next talk.

Contents

References

[Tay] M. E. Taylor "Partial Differential Equations I - Basic Theory", Applied Mathematical Sciences, Springer-Verlag, New York, 1996

[Fre] C. Bär, K. Fredenhagen "Quantum Field Theory on Curved Spacetimes: Concepts and Mathematical Foundations", Lecture Notes in Physics 786, Springer-Verlag, Berlin-Heidelberg, 2009

1 Classification of 2^{nd}-order Partial Differential Equations

Partial differential equations are known not to be unique for arbitrary boundary and initial values, therefore we will first declare what we would like our solution to be.

Definition 1.1. *An initial and boundary value problem (IBVP) is* ***well posed****, if:*

- *a solution exists*
- *the solution is unique*
- *the solution depends continuously on the initial data*

In general we have no statement about sufficient conditions for a well posed IBVP. However, one can propose statements that are well posed for a large class of initial values. For instance you know the Neumann- and Dirichlet-Problems from electrostatics. We are basically going to solve the Cauchy Problem for a generalized wave operator on globally hyperbolic spacetimes.

Definition 1.2. *For a differential operator P on a manifold M with a hypersurface $S \subset M$ and a normal vector ν on it, we call the following a* ***Cauchy Problem****:*

$$\begin{cases} Pu(x) = 0 & \text{, for all } x \in M \\ u(x) = f_0(x) & \text{, for all } x \in S \\ \nabla_\nu^k u(x) = f_k(x) & \text{, for all } x \in S, \end{cases} \tag{1.1}$$

where ∇_ν^k is the k^{th}-covariant derivative in direction of ν.

This is still not well posed, meaning we cannot take any hypersurface on an arbitrary manifold. But by demanding (1.1) to be well posed on a spacetime, we will get certain conditions on S, that need to be analysed. But let us first make some restrictions on the operator P.
A differential operator P of second order on a manifold M has the local form

$$\mathrm{P} = \sum_{i,j=0}^{n-1} a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=0}^{n-1} b^i \frac{\partial}{\partial x^i} + c \qquad , a^{ij}, b^i, c \in \mathscr{C}^0(\mathrm{M}).$$

Of course, due to Schwartz lemma $a^{ij} = a^{ji}$. To see the properties of this operator we change the coordinates $x \mapsto \xi(x)$, which yields

$$\mathrm{P}_\xi = \sum_{k,\ell=0}^{n-1} \underbrace{\left(\sum_{i,j=0}^{n-1} a^{ij} \frac{\partial \xi^k}{\partial x^i} \frac{\partial \xi^\ell}{\partial x^j} \right)}_{(1)} \frac{\partial^2}{\partial \xi^k \partial \xi^\ell} + \sum_{k=0}^{n-1} \underbrace{\left(\sum_{i,j=0}^{n-1} \frac{\partial^2 \xi^k}{\partial x^i \partial x^j} + b^i \frac{\partial \xi^k}{\partial x^i} \right)}_{(2)} \frac{\partial}{\partial \xi^k} + c$$

You can always find suitable coordinates, such that (2) vanishes (compare, for instance, the laplace operator in spherical and cartesian coordinates). So the behavior of P is governed by its highest order components. Also you see, that $a'^{k\ell} = a^{ij} \frac{\partial \xi^k}{\partial x^i} \frac{\partial \xi^\ell}{\partial x^j}$ in (1) transforms like a tensor and since a^{ij} is symmetric, you can always diagonalize. This gives rise to a classification by the (at least) locally coordinate invariant signs of the eigenvalues of a^{ij}.

Definition 1.3. *We call P:*
- ***elliptic*** *, if all eigenvalues of* (a^{ij}) *have the same sign*
- ***parabolic*** *, if one eigenvalue is zero and the others have the same sign*
- ***hyperbolic*** *, if one eigenvalue has the opposite sign than the rest*

You might already suspect a connection between the metric of a manifold and (a^{ij}). Thinking of the Klein-Gordon equation $(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\varphi(x) = 0$ or Dirac equation $(-\mathrm{i}\gamma^\mu\partial_\mu + m)\psi(x) = 0$ with $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ will give a more general idea of hyperbolicity, but that just as a motivational outlook.
Now a quick reminder of distributions on a manifold M. To avoid unnecessary complications we work on the Schwartz space $\mathcal{S}(\mathrm{M}, \mathbb{K}) := \{\phi \in \mathscr{C}^\infty_\mathrm{c}(\mathrm{M}, \mathbb{K})\}$ with tempered distributions $\mathcal{S}'(\mathrm{M}, \mathbb{K}) := \{\mathrm{T} : \mathcal{S}(\mathrm{M}, \mathbb{K}) \to \mathbb{K}, \text{ linear and continuous}\}$, with $\mathbb{K} = \mathbb{C}, \mathbb{R}$ as usual. Any polynomially bounded function $f \in \mathscr{C}^\infty(\mathrm{M}, \mathbb{K})$ can be seen as a tempered distribution, by identifying the map

$$\begin{aligned} f : \mathcal{S}(\mathrm{M}, \mathbb{K}) &\to \mathbb{K} \\ \varphi &\mapsto f[\varphi] := \int_\mathrm{M} f(x)\varphi(x)\, dx \end{aligned} \tag{1.2}$$

with the function itself. You also find the notation T_f as f-generated distribution.
Now let P be a differential operator on a manifold M.

Definition 1.4. *A* ***fundamental solution*** *for P at* $x \in M$ *is a distribution* $F \in \mathcal{S}'(M, \mathbb{K})$*, such that* $PF = \delta_x$*.*

For an inhomogeneous problem $\mathrm{P}u = f$, with $f \in \mathscr{C}^0(\mathrm{M}, \mathbb{K})$ and $u \in \mathscr{C}^k(\mathrm{M}, \mathbb{K})$, we have a solution $u = \mathrm{F} * f$. So for a $\varphi \in \mathcal{S}(\mathrm{M}, \mathbb{K})$ it is $u[\varphi] = \int_\mathrm{M} \mathrm{F}(x)[\varphi] f(x)\, dx$. This solves, in the distributional sense

$$\begin{aligned} \mathrm{P}u[\varphi] &= u[\mathrm{P}^*\varphi] \\ &= \int_\mathrm{M} \mathrm{F}(x)[\mathrm{P}^*\varphi] f(x)\, dx \\ &= \int_\mathrm{M} \mathrm{PF}(x)[\varphi] f(x)\, dx \\ &= \int_\mathrm{M} \delta_x[\varphi] f(x)\, dx \\ &= \int_\mathrm{M} \varphi(x) f(x)\, dx \\ &= f[\varphi]. \end{aligned}$$

Where P^* is the formal adjoint of P.

2 Causal Structure on Lorentzian Manifolds

Now we need to construct the mathematical frame for spacetimes. That is, defining the most important notions of causality and some implications in the sense of pseudo-Riemannian manifolds.

Definition 2.1. *A Lorentzian manifold* (M^m, g) *of dimension* m *is a smooth manifold with a Lorentzian metric* $g = g_{ij}dx^i \otimes dx^j$, *depending smoothly on* $x \in M$.

The metric is induced by the Lorentzian scalar product on the tangent space $\langle e_i, e_j \rangle_x =: g(\partial_i, \partial_j)$ with the identification $\mathbb{R}^m \cong \mathrm{T}_x\mathrm{M}$. Since we have only a nondegenerate inner product, which does not induce a real norm, we use a quadratic form for $X \in \mathrm{T}_x\mathrm{M}$ by $\|X\|_2 := \langle X, X \rangle$. I will just write $\|\cdot\|$ instead of $\|\cdot\|_2$, because it has a similar role.

Definition 2.2. *Tangent vectors* $X \in T_xM$ *are called:*
- ***timelike*** *,if* $\|X\| > 0$
- ***lightlike*** *, if* $\|X\| = 0$ *and* $X \neq 0$
- ***spacelike*** *, if* $\|X\| < 0$ *or* $X = 0$
- ***causal*** *, if* X *is timelike or lightlike*

The set of all timelike vectors $\mathcal{I}(x) := \{X \in \mathrm{T}_x\mathrm{M}, \textit{timelike}\} \subset \mathrm{T}_x\mathrm{M}$ at a point x has two connected components, namely the interior of the future or past directed light cone, as known from special relativity. On a manifold the concept is quite similar, by choosing one component to be **future directed**, e.g. saying all X, with $X^0 > 0$. As we will mention later according to theorem 2.1, this is alwas possible! The **past directed** component is defined as the non-future directed component.

Definition 2.3. *We define the following sets:*
- $\mathcal{I}_+(x) := \{X \in T_xM$ *, timelike, future directed*$\}$
- $\mathcal{J}_+(x) := \overline{\mathcal{I}_+(x)}$
- $\mathcal{C}_+(x) := \partial\mathcal{I}_+(x)$

The past directed analoguous sets are defined as $\mathcal{I}_-(x)$, $\mathcal{J}_-(x)$ *and* $\mathcal{C}_-(x)$

The concept of timelike curves on flat Minkowski space is very intuitive. It translates the same way on curved spacetimes.

Definition 2.4. *A* $\mathscr{C}^1$*-curve* $\tau : \mathbb{R} \to M$*, is called timelike, lightlike, causal, spacelike, future or past directed, if* <u>*all*</u> *its tangent vectors are timelike, lightlike, causal, spacelike, futur or past directed, respectively. I.e.* $\dot{\tau}(0) \in \mathcal{I}_{+/-}(x)$*, et cetera.*

Remember $\tau(x) \in \mathrm{M}$ can be parametrized, such that $\dot{\tau}(0) \in \mathrm{T}_x\mathrm{M}$. With the concepts in the tangent space, which locally looks like Minkowski space, we define relations on the manifold itself.

Definition 2.5. *For two points* $x, y \in M$ *we define the relations:*
- $x \ll y$ *, there is a future-directed timelike curve in* M *from* x *to* y
- $x < y$ *, there is a future-directed causal curve in* M *from* x *to* y
- $x \leqslant y$ *,* $x < y$ *or* $x = y$

The counterpart of the components of the light cone in tangent space is then given by the following definitions:

Definition 2.6. *For $x, y \in M$ and $A \subset M$ define:*

- *the **chronological future** of x* $\quad \mathcal{I}_+^M(x) := \{y \in M | x < y\}$
- *the chronological future of A* $\quad \mathcal{I}_+^M(A) := \bigcup_{x \in A} \mathcal{I}_+^M(x)$
- *the **causal future** of x* $\quad \mathcal{J}_+^M(x) := \{y \in M | x \leqslant y\}$
- *the causal future of A* $\quad \mathcal{J}_+^M(A) := \bigcup_{x \in A} \mathcal{I}_+^M(x)$

The chronological and causal past are defined analogously.

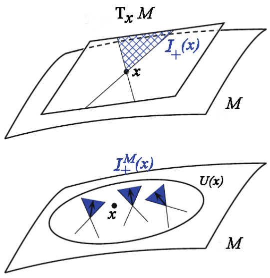

Figure 1: Definitions 2.3 and 2.6 and their difference in tangent space and on the manifold. The lower picture shows the sets of different points in U(x). Source: [Fre, §2]

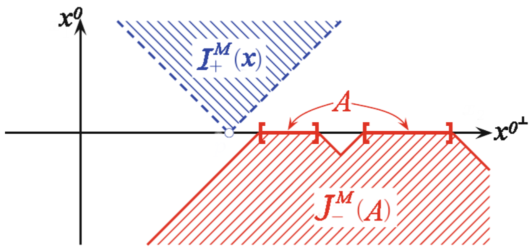

Figure 2: Some sets of Defintion 2.6 illustrated for e.g. 2-dimensional Minkowski space. Source: [Fre, §2]

With all this fundamental notions we may now examine the causal structure of Lorentzian manifolds. We shall shortly recall the notion of an orientation on M, which is the choice of a sign.

Definition 2.7. *A manifold N^n is **orientable** provided there exists a colletion $\mathscr{O}$ of coordinate systems in N which domains cover N and such that for each $\xi, \zeta \in \mathscr{O}$ the Jacobian determinant function $J(\xi, \zeta) = \det(\partial \xi^i / \partial \zeta^j)$ is positive. $\mathscr{O}$ is called an **orientation Atlas** for N.*

So we are choosing a sign + or − (right or left handed) of $\mathrm{T}_x\mathrm{N} \cong \mathbb{R}^n$ and the change of coordinates must preserve that sign.
Time orientation follows a similar idea. Except the time orientation depends on the Lorentzian metric, unlike the orientability of a manifold, which only depends on the topology. However, we basically time-orient the tangent space by choosing one connected component of the light cone + or − (future or past directed) and we want to travel further into the future, once we headed in that direction, and not suddenly go backwards in time.

Definition 2.8. *A Lorentzian manifold (M, g) is **time-orientable**, if there exists a continuous timelike vektorfield $X \in \mathfrak{X}(M)$, id est $X(x) \in \mathcal{I}_+(x) \subset T_xM$ for all x. We call a connected, time-oriented Lorentzian manifold **spacetime**.*

We need some more conditions to prevent us from unphysical events. Imagine an periodic perturbation on a small support in the spacetime. Since the solution of the wave equation propagates in time, it will spread over the spacetime. If we had closed causal curves (meaning time travel), the solution could propagate to the initial spacetime point and disturb its own solution. Then the new and any following solutions would be equivalent and we loose uniqueness (well posedness).

Definition 2.9. *A spacetime satisfies the* ***weak causality condition****, if it does not contain any closed causal curves.*
A spacetime satisfies the ***strong causality condition****, if it does not contain any almost closed causal curves.*
Or, for each point $x \in M$ and each open neighborhood $U(x) \subset M$ there exists an open neighborhood $V(x) \subset U(x)$, such that each causal curve in M starting and ending in V is entirely contained in U.

Consequently the strong causality condition forbids closed causal curves, that have only one point missing. Figure 3 shows a cylinder of 2-dimensional Minkowski space. The sets $\mathcal{I}_{\pm}(x)$ cover all of the cylinder for all x, so we can easily draw closed causal curves. If we cut out two stripes G_1 and G_2, such that their endpoints can just be connected by a lightlike curve, it is not possible to have closed causal curves anymore. But we can get infinitely close to closing the curve, depending on how narrow we take the turn around the endpoints of G_1 and G_2.

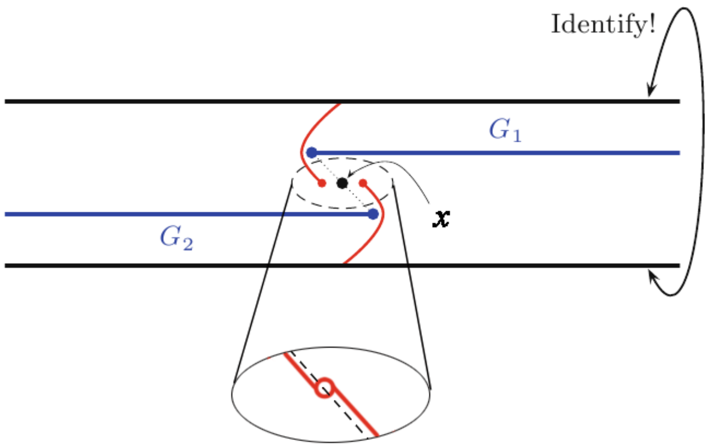

Figure 3: Sliced Minkowski cylinder with an almost closed causal curve. The lower part is a zoom of the dotted circle. Source: [Fre, §2]

Definition 2.10. *A spacetime (M, g) is called* ***globally hyperbolic****, if it satisfies the strong causality condition and for all $x, y \in M$ the set $\mathcal{J}_+^M(x) \cap \mathcal{J}_-^M(y)$ is compact.*

Note that the second condition implies, that no gravitational singularities without an event horizon exist (also called naked singularities), which would allow matter to collapse into a single point.
With this kind of manifold, uniqueness for hyperbolic differential operators will be provided. Next I will give the notion of a Cauchy hypersurface on globally hyperbolic manifolds, which happens to be exactly the kind of Cauchy surface we need for well posedness of the Cauchy problem. There is no easy way to proove this, so you will have to take the statements for granted.

Definition 2.11. *A subset S of a spacetime is called* ***achronal*** *(or* ***acausal****), if and only if each timelike (resp. causal) curve meets S at most once. We call S a* ***Cauchy hypersurface****, if each inextendible timelike curve in M meets S at exactly one point.*

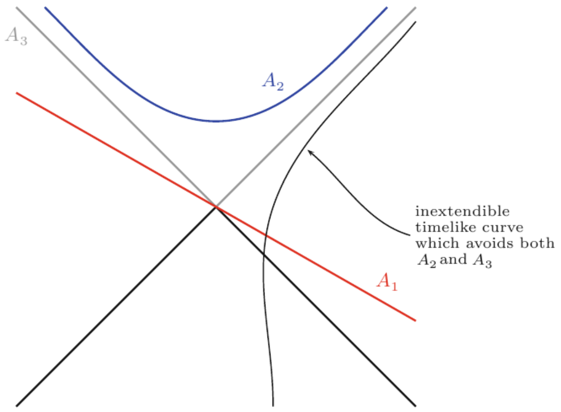

Figure 4: Achronal hypersurfaces in Minkowski space. A_1 is the only Cauchy hypersurface. Source: [Fre, §2]

To picture this, see Figure 4. The existence of such a surface is strongly connected with globally hyperbolic spacetimes and even provides a nice picture of them.

Theorem 2.1. *For a spacetime (M, g), the following are equivalent:*
(1) M is globally hyperbolic
(2) there exists a Cauchy hypersurface S in M
(3) M is isometric to $\mathbb{R} \times S$ with metric $g = -\beta dt \otimes dt + g_R$, where $\beta \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^+)$ and g_R is a Riemannian metric on S depending smoothly on $t \in \mathbb{R}$ and each $\{t\} \times S$ is a smooth spacelike Cauchy hypersurface.

Point (3) of this theorem was just proven in 2003 by Bernal and Sánchez[1]. It provides us with a metric, so we can choose a pure time direction and hence a foliation of cauchy hypersurfaces (see Figure 5). We summarize the results by

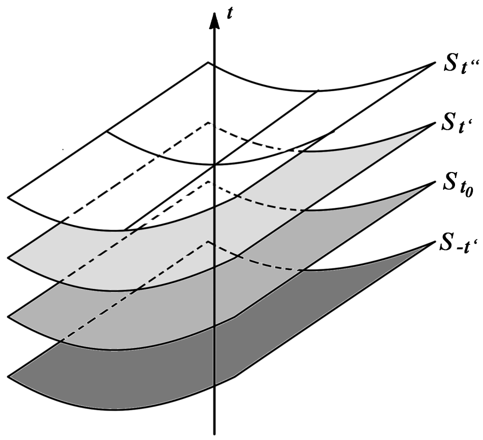

Figure 5: Schematic foliation of a globally hyperbolic spacetime.

[1]A. N. Bernal, M. Sánchez, ”On Smooth Cauchy Hypersufraces and Geroch’s Splitting Theorem”, Commun. Math. Phys., 243 (2003) 461-470

Proposition 2.1. *Let P be hyperbolic, (M, g) a globally hyperbolic spacetime with Cauchy surface S, as defined before. Then the Cauchy problem is well defined:*

$$\begin{cases} Pu(x) = 0 & \text{, for all } x \in M \\ u(x) = f_0(x) & \text{, for all } x \in S \\ \nabla_{x^0} u(x) = f_k(x) & \text{, for all } x \in S. \end{cases} \tag{2.1}$$

Where ∇_{x^0} is in time-direction, due to (3) of theorem 2.1.

3 Wave Equation on Minkowski Space

Now we solve the Cauchy problem for the most simple case of Minkowski spacetime, which is of course globally hyperbolic with the spacial dimensions as the Cauchy hypersurfaces and hence $\mathbb{M}^{n+1} \cong \mathbb{R} \times \mathbb{R}^n$. The problem then states

$$\begin{cases} (\partial_t^2 - \Delta_x)\, u(x) & = 0 \\ u(0,x) & = f(x) \\ \partial_t u(0,x) & = g(x) \end{cases}$$

We solve it in the distributional sense, so let $f, g \in \mathcal{S}'(\mathbb{R}^n)$ and we want a solution $u \in \mathscr{C}^\infty(\mathbb{R}, \mathcal{S}'(\mathbb{R}^n))$.
Performing a Fourier transformation over the spacial part yields an ordinary initial value problem. Remark, that Fourier transformation is a homeomorphism from $\mathcal{S}(\mathbb{R}^n)$ to itself and gives an equivalent problem.

$$\begin{cases} (\partial_t^2 - |\xi|^2)\, \hat{u}(\xi) & = 0 \\ \hat{u}(0,\xi) & = \hat{f}(\xi) \\ \partial_t \hat{u}(0,\xi) & = \hat{g}(\xi) \end{cases}$$

The solution is straight foreward and analogous to the non-distributional case:

$$\hat{u}(t,\xi) = \hat{g}(\xi)\frac{\sin(t|\xi|)}{|\xi|} + \hat{f}(\xi)\cos(t|\xi|).$$

The $1/|\xi|$ is artificial, but leaves the solution still well defined and unique (in non-distributional sense). It is convention and will be handy for finding a closed form for the solution. We already see that this solution can be turned into a distribution, by construction given in chapter 1.2, but it is not sure to be unique. We now get the fundamental solution by setting $f = 0$ and $g = \delta_x$, giving $\hat{g} = (2\pi)^{-n/2}$. We call it Riemann distribution and write it a little differently:

$$\begin{aligned} \hat{R}(t,\xi) &= (2\pi)^{-n/2}|\xi|^{-1}\sin(t|\xi|) \\ &= \mathfrak{Im}\left[(2\pi)^{-n/2}|\xi|^{-1}\mathrm{e}^{\mathrm{i}t|\xi|}\right] \end{aligned}$$

For the back transformation we use the following statement:

Proposition 3.1. *For $T \in \mathbb{R}$, $T > 0$, $\xi \in \mathbb{R}^n$, $n \geqslant 2$, it holds:*

$$\begin{aligned} & \hat{F}(T,\xi) = (2\pi)^{-n/2}|\xi|^{-1}e^{-T|\xi|} \\ \Leftrightarrow \qquad & F(T,x) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{n-1}{2}}}\left(T^2 + |x|^2\right)^{-\frac{n-1}{2}} \end{aligned}$$

You find the proof in [Tay, §5]. Remark that $\hat{F}$ is holomorphic in T, so it is legitimate to choose $T \in \{z \in \mathbb{C} | \mathfrak{Re}\, z > 0\}$, we take $-T = \mathrm{i}(t + \mathrm{i}\varepsilon)$. Conclude, that

$$R(t,x) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{n+1}{2}}} \lim_{\varepsilon \searrow 0} \mathfrak{Im}\left[\left(|x|^2 - (t + \mathrm{i}\varepsilon)^2\right)^{-\frac{n-1}{2}}\right] \tag{3.1}$$

With the artificial $|\xi|^{-1}$ from above, the solution can now, provided that the limit exists, be stated compactly as

$$u = R * g + \frac{\partial}{\partial t} R * f$$

Now let us discuss the qualitative behavior. Only if $|x| = |t|$, we have trouble, since the limit only exists in distributional sense. Otherwise the limit for functions exists, which then generates the distribution. Let us start with the easy parts:
$|x| > |t|$:
We take the limit and obtain $\left(\sqrt{|x|^2 - t^2}\right)^{-(n-1)}$, which is real for any integer, so $R(t,x)$ vanishes. This represents the finite propagation speed.
$|x| < |t|$:
Take the limit in polar representation

$$\left(|x|^2 - (t + \mathrm{i}\varepsilon)^2\right)^{-\frac{n-1}{2}} = \left(\sqrt{(|x|^2 - t^2 + \varepsilon^2)^2 + 4\varepsilon^2 t^2} \cdot \mathrm{e}^{\mathrm{i}\varphi}\right)^{-\frac{n-1}{2}},$$

where the phase φ is:

$$\varphi = \begin{cases} \arctan\left(\frac{2\varepsilon t}{t^2 - |x|^2 - \varepsilon^2}\right) + \pi & \text{, for } t < 0 \\ \arctan\left(\frac{2\varepsilon t}{t^2 - |x|^2 - \varepsilon^2}\right) - \pi & \text{, for } t > 0. \end{cases}$$

Hence in the Limit we have

$$R(t,x) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{n+1}{2}}} \left(t^2 - |x|^2\right)^{-\frac{n-1}{2}} \mathfrak{Im}\,(\pm \mathrm{i})^{n-1},$$

where $+$ is for $t > 0$ and $-$ for $t < 0$ now.
For odd n the right part $(\pm\mathrm{i})^{n-1}$ becomes real and $R(t,x)$ vanishes. For even n we have

$$R(t,x) = (-1)^{\frac{n-2}{2}} \operatorname{sign}(t) \frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{n+1}{2}}} \left(t^2 - |x|^2\right)^{-\frac{n-1}{2}} \tag{3.2}$$

$|x| = |t|$:
This case is quite subtle. The limit does not exist for functions, however it does exist in distributional sense, i.e. $R[\phi]$ exists. For even n, the solution as a function covers this case. Making R a distribution and integrating over $\xi := (t,x) \in \mathbb{M}^{n+1}$,

$$\begin{aligned} R[\phi] &= \int\limits_{\operatorname{supp} R \subset \mathbb{M}^{n+1}} \operatorname{sign}(t) \frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{n+1}{2}}} \|\xi\|^{-\frac{n-1}{2}} \phi(\xi)\, d\xi \\ &= \int\limits_{|x|<|t|} \ldots \mathrm{d}\xi \\ &= \int\limits_{|x|\leqslant|t|} \ldots \mathrm{d}\xi \end{aligned}$$

yields no difference.
For odd n, use $\mathfrak{Im}(z) = \frac{1}{2\mathrm{i}}(z - z^*)$ and the Plemelj jump relation[2], which states

[2] see also [Tay] or some books on distributions

$\lim\limits_{\varepsilon\searrow 0}\left((x+i\varepsilon)^{-k}-(x-i\varepsilon)^{-k}\right)=2\pi i\frac{\delta^{(k-1)}}{(k-1)!}$, where $\delta^{(k-1)}$ is the $(k-1)^{\text{th}}$-derivative in distributional sense, to solve (3.1). This needs to be worked out case by case, but you will see it once below in the example part.
Put together, we have that

$$\operatorname{supp}R(t,x)\subset\begin{cases}\{x\in\mathbb{R}^n \mid |x|=|t|\} & \text{, for odd } n\\ \{x\in\mathbb{R}^n \mid |x|\leqslant|t|\} & \text{, for even } n\end{cases}\tag{3.3}$$

We also get a decomposition $R(t,x)=R_+(t,x)+R_-(t,x)$, where $R_\pm$ is defined on $\mathcal{J}_\pm^{\mathbb{M}^{n+1}}(0)$, respectively. These are the retarded R_+ and the advanced R_+ Riemann distribution. In figure 6 the qualitative results are shown.

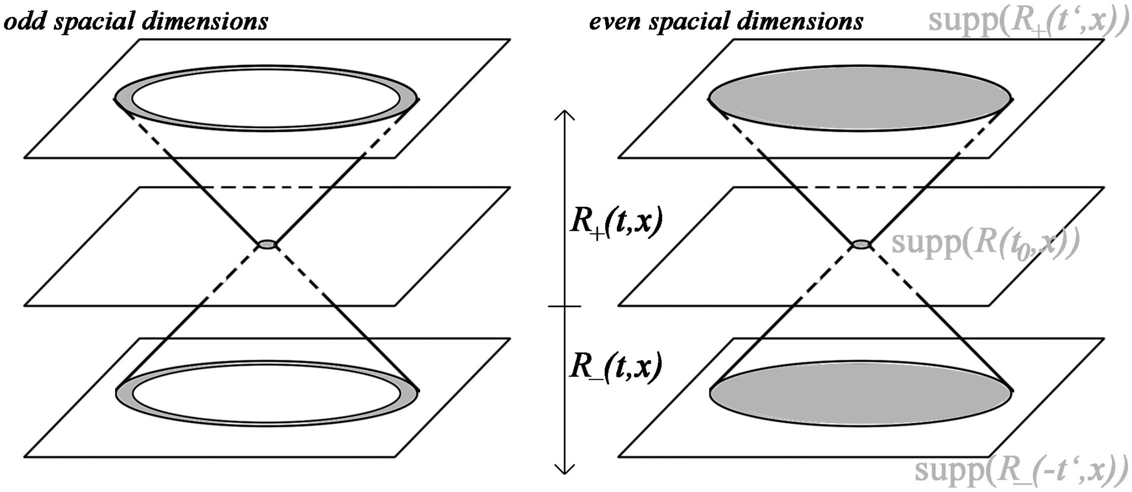

Figure 6: Difference in propagation of the support in odd and even dimensions.

Now let us solve the problem explicitly up to $n=3$.
<u>$n=1$:</u>
This case in not covered, but we can solve it classically. The problem states:

$$\begin{cases}\left(\partial_t^2-\partial_x^2\right)u(t,x) & =0\\ u(0,x) & =f(x)\\ \partial_t u(0,x) & =g(x)\end{cases}$$

By coordinate change $(t,x)\mapsto(x+t,x-t)$, the problem turns into $\partial_{y_1}\partial_{y_2}u(t,x)=0$. With $u(t,x)=w_1(x+t)+w_2(x-t)$ and initial conditions

$$\begin{cases}u(0,x) & =w_1(x)+w_2(x)=f(x)\\ \partial_t u(0,x) & =w_1'(x)-w_2'(x)=g(x)\end{cases}$$

$$\Rightarrow\begin{cases}w_1(x)+w_2(x)=f(x)\\ w_1(x)-w_2(x)=\int\limits_0^x g(\xi)\,d\xi+C\end{cases}$$

Hence we obtain

$$u(t,x) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)\,d\xi$$

$$\Leftrightarrow \qquad u(t,x) = \frac{1}{2}\int_{x-t}^{x+t} g(\xi)\,d\xi + \frac{1}{2}\frac{\partial}{\partial t}\int_{x-t}^{x+t} f(\xi)\,d\xi.$$

In terms of distributions, we conclude, that

$$R(t,x) = \operatorname{sign}(t)\,\frac{1}{2}\left(\theta(|x|-t) + \theta(|x|+t)\right).$$

Where θ is the Heaviside distribution. So supp$R(t,x) \subset \{x \in \mathbb{R} \,|\, |x| = |t|\}$ and

$$\begin{aligned} R_+(t,x) &= \frac{1}{2}\theta(|x|-t) &&, \text{ for } t \geqslant 0 \\ R_-(t,x) &= -\frac{1}{2}\theta(|x|+t) &&, \text{ for } t \leqslant 0. \end{aligned}$$

<u>$n = 2$:</u>
We already worked out the closed form for even spacial dimensions, so with $n = 2$:

$$R(t,x) = \operatorname{sign}(t)\,\frac{1}{2\pi}\left(t^2 - |x|^2\right)^{-\frac{1}{2}}$$

Here supp$R(t,x) \subset \{x \in \mathbb{R}^2 \,|\, |x| \leqslant |t|\}$ with

$$R_+(t,x) = \begin{cases} \frac{1}{2\pi}\left(t^2 - |x|^2\right)^{-\frac{1}{2}} & , \text{ for } t \geqslant 0 \\ 0 & , \text{ for } t < 0 \end{cases}$$

$$R_-(t,x) = \begin{cases} 0 & , \text{ for } t > 0 \\ -\frac{1}{2\pi}\left(t^2 - |x|^2\right)^{-\frac{1}{2}} & , \text{ for } t \leqslant 0 \end{cases}$$

<u>$n = 3$:</u>
We start again from (3.1) and follow the hints.

$$\begin{aligned} R(t,x) &= \frac{1}{2\pi^2}\lim_{\varepsilon\searrow 0}\frac{1}{2\mathrm{i}}\left[\left(|x|^2 - (t+\mathrm{i}\varepsilon)^2\right)^{-1} - \left(|x|^2 - (t+\mathrm{i}\varepsilon)^2\right)^{-1}\right] \\ &= \frac{1}{2\pi^2}\lim_{\varepsilon\searrow 0}\frac{1}{2\mathrm{i}}\left[\left(|x|^2 - t^2 + \varepsilon^2 - 2\mathrm{i}t\varepsilon\right)^{-1} - \left(|x|^2 - t^2 + \varepsilon^2 + 2\mathrm{i}t\varepsilon\right)^{-1}\right] \\ &= \frac{1}{2\pi^2}\frac{1}{2\mathrm{i}}(\operatorname{sign}(t)\,2\pi\mathrm{i})\delta(|x|^2 - t^2) \\ &= \operatorname{sign}(t)\,\frac{1}{2\pi}\frac{1}{2|t|}\left(\delta(|x|-t) + \delta(|x|+t)\right) \\ &= \frac{1}{4\pi t}\left(\delta(|x|-t) + \delta(|x|+t)\right) \end{aligned}$$

The sign comes in, because $-2\mathrm{i}t\varepsilon$ in the second line changes its sign and hence the Plemelj jump changes its sign, too. We have supp$R(t,x) \subset \{x \in \mathbb{R}^3 \,|\, |x| = |t|\}$ with

$$\begin{aligned} R_+(t,x) &= \frac{1}{4\pi t}\delta(|x|-t), \text{ for } t \geqslant 0 \\ R_-(t,x) &= \frac{1}{4\pi t}\delta(|x|+t), \text{ for } t \leqslant 0 \end{aligned}$$