Algebraic Quantum Field Theory meets Cosmology

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SFB 676 Meeting
Bergedorf, 3rd of March 2009

Outline of the Talk

- A top-down approach: looking for a temperature!
- A bottom-up strategy: unveiling the role of the stress-energy tensor!
- A step towards a future project: finding a distinguished ground state!

Motivations - What we know

The 20^{th} century thought us a few good lessons:

1) Description of interactions leads to quantum field theory on **flat spacetime**:

- it works almost perfectly for free and electroweak forces,
- perturbative QFT, renormalization, etc...
- it involves mathematics - phenomenology and experiments.

2) Description of the gravitational interaction leads to **General Relativity**

One hand: By means of the algebraic approach, one can discuss on a rigorous basis QFT on curved backgrounds [Fredenhagen, Brunetti, Hollands, Wald, Kay, Dimock, Verch,...]

Other hand: Cosmology is

- a branch of physics which allows to unveil the structure and the dynamic of the Universe
- a natural playground to use the powerful means of QFT on curved background in the algebraic approach

Motivations - What we know - II

Modern Cosmology has some remarkable aspects:

- Classical Cosmology is modelled by rather simple solutions of Einstein's equations. Assumptions are:
  1. isotropy and homogeneity of the Universe,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \quad k = 0, \pm 1$$

  2. the behaviour of the stress-energy tensor is classical (perfect fluid)

$$T_{\mu\nu} = \rho\zeta_\mu\zeta_\nu + P\left(g_{\mu\nu} + \zeta_\mu\zeta_\nu\right), \quad \zeta^\mu\zeta_\mu = 1$$

  3. pressure P and energy density ρ are related by an equation of state

$$P = \gamma\rho$$

- it is plagued by some interpretation problems (homogeneity, flatness, the singularity problem...)

Motivations - What we would like to know

The quest to solve those problems prompted

- Modern approaches to Cosmology in which matter is often modelled by a scalar field
- Bright side
  1. it takes seriously the role of QFT as the natural playground to discuss (quantum) matter and interactions,
  2. it provides a nice exit to most of the problems of standard cosmology,
  3. models of inflation lead to testable consequences, on the temperature of CMB in particular.
- Dark side
  1. still plenty of open problems (dark matter, dark energy...),
  2. it is unclear how to derive these models from "first principles",
  3. many concepts are not so clearly defined in curved backgrounds (temperature).

On the notion of temperature - I

How to cope with temperature in curved spacetimes?

As a starting point:

- there is a good concept of thermal states ω_β in Minkowski (KMS condition)
- in this case we know how to compute expectation values, *e.g.*, for a free massless scalar field

$$\omega_\beta(:\phi^2:) = \frac{1}{12|\beta|^2} \quad |\beta| = T^{-1},$$

- we can extend it to other observables, *e.g.*,

$$\omega_\beta(\widetilde{\partial}^\mu \widetilde{\partial}^\nu :\phi^2(x):) = -\frac{(4\pi)^2}{4!} B_4 \partial^\mu \partial^\nu (\beta^2)^{-1} \doteq \alpha^{\mu\nu}(\beta),$$

$$\widetilde{\partial}^\mu :\phi^2 := \lim_{\zeta \to 0} \partial_\zeta^\mu \left(\phi(x+\zeta)\phi(x-\zeta) - \omega_{vac}(\phi(x+\zeta)\phi(x-\zeta))\mathbb{I}\right)$$

On the notion of temperature - II

On curved backgrounds M, such as FRW, what can we do?

- we have a good notion of normal ordering, *i.e.*, $:\phi^2(x):$ is meaningful,
- we seek for states ω_M whose expectation value are "coherent" with the Minkowski ones

$$\omega_M(:\phi^2(x):) = f(x) \doteq \frac{1}{12\beta^2(x)},$$

$$\omega_M(\widetilde{\partial}^\mu \widetilde{\partial}^\nu :\phi^2(x):) = \alpha^{\mu\nu}(\beta(x)),$$

and so on and so forth.

The "mother of all problems"

The idea is enticing but it faces a big problem

• An important observable is the stress-energy tensor $T_{\mu\nu}$, but

• for a massless scalar field in Minkowski $T \doteq Tr(T_{\mu\nu}) = 0$,

$$\omega_\beta(:T:) = 0$$

• if we look for a spacetime conformally related to Minkowski (as FRW with $k = 0$) and we take

$$\Box_g \phi - \frac{R}{6}\phi = 0, \longrightarrow T = 0$$

but, for an Hadamard (*i.e.*, ground) state ω_M

$$\omega_M(:T:) = \frac{1}{4\pi^2}\left(\frac{1}{720}(R_{ij}R^{ij} - \frac{R^2}{3} + \Box R)\right)$$

This is the trace anomaly!

The role of the trace anomaly[1]

The natural definition of temperature fails due to the trace anomaly!

Is it an accident or does it play a fundamental role, for example in cosmology?

Best arena where to investigate: semiclassical Einstein's equations!

[1] C.D., Klaus Fredenhagen, Nicola Pinamonti, Phys. Rev. D**77** (2008) 104015

A semiclassical effect - I

Let us look at our framework:

- We fix the background as an FRW spacetime with flat spatial section

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \quad M \equiv \mathbb{R} \times \mathbb{R}^3$$

- we consider for "simplicity of the talk" a scalar field on M

$$\left(\Box_g - \frac{R}{6} - m^2\right)\phi(x) = 0,$$

  which is conformally coupled to scalar curvature.

- we shall seek solutions of $G_{\mu\nu} = 8\pi\langle : T_{\mu\nu} : \rangle_\omega$, which in the FRW scenario reduces to

$$-R = 8\pi\langle : T : \rangle_\omega$$

Intermezzo: the quest for an Hadamard state

What is a good choice for ω?

A physically reasonable choice is

- an ω which is quasi-free (technical condition)
- an ω which is of Hadamard form
    - they share the same ultraviolet behaviour as the ground state in Minkowski,
    - only on these states the quantum fluctuations of $T_{\mu\nu}$ are finite,

Hence in a geodesic normal neighbourhood of any point $p \in \mathbb{R} \times \mathbb{R}^3$, the integral kernel of the two-point function is

$$\omega(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y) \ln \frac{\sigma_\epsilon(x,y)}{\lambda} + W(x,y)$$

Intermezzo - II

Let start again from

$$\omega(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y)\ln\frac{\sigma_\epsilon(x,y)}{\lambda} + W(x,y)$$

One can prove that

- U, V, W are all smooth scalar functions,
- in Minkowski $U = 1$ and $V = \frac{m^2}{2\sqrt{m^2(x-y)^2}} J_1(\sqrt{m^2(x-y)^2})$, whereas in curved backgrounds they are a series

  $$U(x,y) = \sum_{n=0}^{\infty} u_n(x,y)\sigma^n, \quad V(x,y) = \sum_{n=0}^{\infty} v_n(x,y)\sigma^n,$$

  determined out of recursion relations,
- the singular part, namely U and V, depends only on geometric quantities such as $R, R^2, R_{\mu\nu}R^{\mu\nu}\ldots$
- the choice of a quantum state of Hadamard form lies only in W

A semiclassical effect - II

Let us *assume* to take an Hadamard state! Then

$$\langle :T: \rangle_\omega = -m^2 \frac{W(x,x)}{8\pi^2} + \frac{v_1(x,x)}{4\pi^2},$$
$$v_1(x,x) = \frac{1}{720}\left(R_{ij}R^{ij} - \frac{R^2}{3} + \Box R\right) + \frac{m^4}{8}.$$

Plugging it in the semiclassical Einstein's equations, shaking them a little bit, we end up with

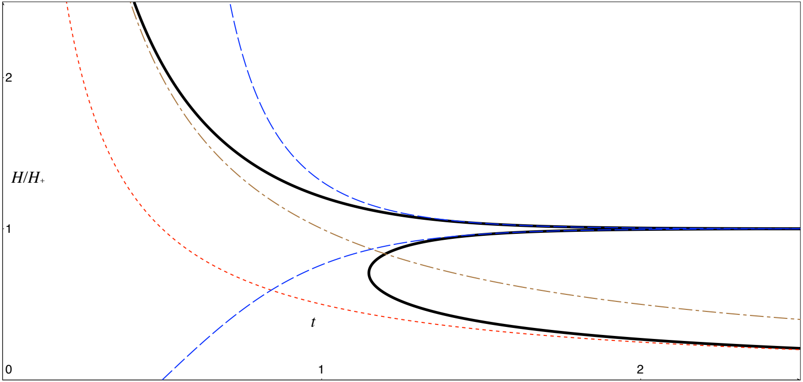
$$-6\left(\dot{H} + 2H^2\right) = -8\pi m^2 \langle :\phi^2: \rangle_\omega + \frac{1}{\pi}\left(-\frac{1}{30}(\dot{H}H^2 + H^4 + \frac{m^4}{4}\right),$$

where $H = \frac{\dot{a}(t)}{a(t)}$.

One can compute that for $m^2 \gg R$ and $m^2 \gg H$, $\langle :\phi^2: \rangle_\omega = \frac{1}{32\pi^2} m^2 + \beta R$.

A semiclassical effect - III

$$\dot{H} = \frac{-H^4 + H_+^2 H^2}{H^2 - \frac{H_+^2}{4}}, \quad H_+^2 = 360\pi - 2880\pi^2 m^2 \beta.$$



Has someone ever met an Hadamard state?

The bottom-up strategy seems to bear fruit but

- is the result stable if we consider another kind of matter field[2]?
- Are all our assumptions robust enough?

Particularly does an Hadamard state exist an FRW spacetime?

- Hadamard states are the building block for perturbation theory,
- we ultimately need to tackle interacting models,
- many cosmological predictions of models such as inflation come from quantum effect and from perturbation theory.

[2] C.D., Thomas Hack, Nicola Pinamonti, work in progress, almost on-line

Yes, they exist - I

We need a strategy to identify an Hadamard state on a FRW spacetime

1. construct states of low energy and prove they are Hadamard,[3]
2. direct construction of this state: possible, but tricky and time consuming,
3. circumvent the obstacle seeking an alternative approach.

A large class of FRW possesses a distinguished (cosmological) horizon.

Can we use it to implement a bulk-to-boundary correspondence? We know

- it works perfectly in AdS spacetimes via AdS/CFT,
- it fits in the picture of algebraic quantum field theory[4],
- it can be implemented in asymptotically flat spacetimes.

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[3] H. Olbermann, Class. Quant. Grav. **24** (2007) 5011

[4] see also P. L. Ribeiro, arXiv:0712.0401 [math-ph].

Yes, they exist - II

Let us consider an FRW spacetime with[5]

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] = a^2(\tau)[-d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)],$$

let us restrict the class of scale factors as:

$$a(\tau) = -\frac{1}{H\,\tau} + O\left(\tau^{-2}\right)\ ,$$

$$\frac{da(\tau)}{d\tau} = \frac{1}{H\,\tau^2} + O\left(\tau^{-3}\right)\ , \frac{d^2a(\tau)}{d\tau^2} = -\frac{2}{H\,\tau^3} + O\left(\tau^{-4}\right).$$

- they all posses a cosmological horizon $\Im^- \sim \mathbb{R} \times \mathbb{S}^2$ in the past,
- If $a(\tau) = -\frac{1}{H\tau}$ then $\tau = -e^{-Ht}$, hence cosmological de-Sitter spacetime.
- as $\tau \to -\infty$, the background **"tends to"** de Sitter, Hence we are dealing with an exponential acceleration in the proper time t. This is the the prerequisite of all inflationary models.

[5] C.D., Nicola Pinamonti, V. Moretti Comm. Math. Phys **285** (2009), 1129

Yes, they exist - III

Let us consider the usual real scalar field

$$P\Phi = 0, \qquad P = -\Box + \xi R + m^2 \text{ and } \xi R + m^2 > 0$$

with compactly supported initial data on a Cauchy surface,

- Each solution Φ is a smooth function on M, *i.e.*, $\Phi \in C^\infty(M)$,
- The set of solutions $S(M)$ of our equation is a symplectic space if endowed with the Cauchy-independent nondegenerate symplectic form:

$$\sigma(\Phi_1, \Phi_2) \doteq \int_\Sigma (\Phi_1 \nabla_N \Phi_2 - \Phi_2 \nabla_N \Phi_1)\, d\mu_g^{(\Sigma)},$$

- each $\phi \in S(M)$ can be extended to a unique smooth solution of the same equation on $M \cup \Im^- \longrightarrow \Gamma\phi \doteq \phi|_{\Im^-} \in C^\infty(\Im^-)$.

Yes, they exist - IV

Bulk) A Weyl C^*-algebra $\mathcal{W}(M)$ can be associated to $(\mathcal{S}(M), \sigma)$. This is, up to $*$-isomorphisms, unique and its non vanishing generators $W_M(\phi)$ satisfy:

$$W_M(-\phi) = W_M(\phi)^*, \quad W_M(\phi) W_M(\phi') = e^{\frac{i}{2}\sigma(\phi,\phi')} W_M(\phi + \phi'),$$

Horizon) The symplectic space of real wavefunctions is:

$$\mathcal{S}(\Im^-) = \left\{ \psi \in C^\infty(\mathbb{R} \times \mathbb{S}^2) \mid \psi \in L^\infty, \partial_\ell \psi \in L^1, \widehat{\psi} \in L^1, k\widehat{\psi} \in L^\infty \right\},$$

$$\sigma_{\Im^-}(\psi, \psi') = \int_{\mathbb{R}\times\mathbb{S}^2} \left(\psi \frac{\partial \psi'}{\partial \ell} - \psi' \frac{\partial \psi}{\partial \ell} \right). \quad \forall \psi, \psi' \in \mathcal{S}(\Im^-),$$

Algebra) Since $\sigma_{\Im^-}$ is nondegenerate, we can construct a Weyl C^*-algebra $\mathcal{W}(\Im^-)$ as

$$W_{\Im^-}(\psi) = W^*_{\Im^-}(-\psi), \qquad W_{\Im^-}(\psi) W_{\Im^-}(\psi') = e^{\frac{i}{2}\sigma_{\Im^-}(\psi,\psi')} W_{\Im^-}(\psi + \psi').$$

Yes, they exist - V

We can introduce a *distinguished* state $\omega : \mathcal{W}(\Im^-) \to \mathbb{C}$ as

$$\omega\left(W(\psi)\right) = e^{-\frac{\mu(\psi,\psi)}{2}}, \quad \forall W(\psi) \in \mathcal{W}(\Im^-)$$

where $\forall \psi, \psi' \in \mathcal{S}(\Im^-)$

$$\mu(\psi, \psi') = \int\limits_{\mathbb{R}\times S^2} 2k\Theta(k)\overline{\widehat{\psi}(k,\theta,\varphi)}\widehat{\psi}'(k,\theta,\varphi)dkdS^2(\theta,\varphi),$$

being $\psi(k), \psi'(k)$ the Fourier-Plancherel transform

$$\psi(k) = \int\limits_{\mathbb{R}} dl\ \frac{e^{ikl}}{\sqrt{2\pi}}\psi(l,\theta,\varphi).$$

- ω is quasifree and pure,

Yes, they exist - VI

Proposition

For all $\Phi \in S(M)$ and $m^2 + \xi R > 0$, then

- $\Gamma\Phi \in \mathcal{S}(\Im^-)$,
- $\sigma_{\Im^-}(\Gamma\phi, \Gamma\phi') = H^2\sigma(\phi, \phi')$,
- $\exists\imath : \mathcal{W}(M) \to \mathcal{W}(\Im^-)$ as an isometric $*$-homomorphism.

Consequence:

- Any state $\widetilde{\omega} : \mathcal{W}(\Im^-) \to \mathbb{C}$ can be pulled back to

$$\imath^*(\widetilde{\omega}) : \mathcal{W}(M) \to \mathbb{C}.$$

Endgame[6]

Particularly the preferred state

$$\omega_M(a) := \omega(\imath(a)). \quad \forall a \in \mathcal{W}(M)$$

Main Result

The state ω_M

- is always of Hadamard form,
- is the Bunch-Davies state in de Sitter spacetime,
- ω_M represents a natural distinguished cosmological "ground (vacuum) state" to tackle the study of linear perturbations,
- it is invariant under the natural action of any bulk isometry (acting on the algebra).

[6] C.D., Nicola Pinamonti, V. Moretti 0812.4033 [gr-qc]

Conclusions

We have realised

- the original top-down idea $\rightarrow$ the conformal anomaly as key ingredient,
- the bottom-up strategy brings,
  1. the existence of late time stable solutions for the semiclassical Einstein's equations,
  2. the identification of a distinguished ground state for free scalar field theories on certain FRW spacetimes.

And now,

- we can look for thermal states of minimum energy,[7]
- we can look at inflation from a mathematical-physics point of view as a tool to rule out a few dozen models,
- we can try to understand the role quantum effects of all kind of fields (spinors, bosons,...) when used to explain dark matter, dark energy,...

[7] M. Kusku, arXiv:0901.1440 [hep-th] (Ph.D. Thesis - Hamburg).