The Haag-Kastler axiomatic framework

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24 juin 2012

1 Principle of locality

[1] Around the year 1830, Faraday had the idea that interactions at distance, e.g. Coulomb interaction, are not instantaneous but propagate. This suppose that each point of space participates in the physical process, that is, each point is equipped with dyanmical variables. The corresponding mathematical object are fields, functions of space-time. Locality for the law of evolution then takes the form of partial differential equations for the fields.

The next step to build our model of reality was to start from the algebra of observables instead of trying to define particles first. In the first seminar, we had seen the C^*-algebra structure of the observables but the idea of measuring a certain quantity for a given region of space-time is still missing. This is done by the following construction.

1.1 Precosheaf or net of algebras

Definition 1.1. *A category C consists of*

- *A class of objects $Ob(C)$, i.e. a collection of sets[2] which all satisfy a certain property.*
- *A class of morphisms $Hom(C)$ such that*
  - *Each morphism $f \in Hom(C)$ has a unique source object $a \in Ob(C)$ and a target object $b \in Ob(C)$. Write*

$$f : a \longrightarrow b \tag{1}$$

    The set of homomorphisms from a to b is denoted $Hom_C(a, b)$
  - *$\forall\ a, b, c \in Ob(C)$, there is an* associative *map, composition of morphisms*

$$\circ : \begin{cases} Hom_C(a,b) \times Hom_C(b,c) \longrightarrow Hom_C(a,c) \\ \quad (\quad f \quad , \quad g \quad) \qquad \longrightarrow g \circ f \end{cases} \tag{2}$$

  - *$\forall\ c \in Ob(C)$ there exist an identity $1_c \in Hom_C(c, c)$ such that $\forall f \in Hom_C(a, b)$*

$$1_b \circ f = f \circ 1_a = f \tag{3}$$

This definition just points out that the set of groups or vector spaces, have a common structure. Groups come with functions from a group to another one preserving the group operation, same for vector spaces. What is somehow disconcerting is that, abstracting from these examples, we don't require the morphisms to be applications anymore and note also that we may have $\mathrm{Hom}_C(a, b) = \emptyset$.
The category we'll deal with is the category of open sets of the Minkowski space, with inclusions as morphisms.

[1]"Nahwirkungsprinzip" in German, ([3])

[2]or another mathematical object. The idea of class contains the idea of set but overcomes Russel's paradoxe.

Definition 1.2. *A* <u>*functor*</u> *F from a category C to D is a map that associates*

- *to each $x \in Ob(C)$, an object $F(x) \in Ob(D)$*
- *to each morphism $f \in Hom_C(a,b)$, a morphism $F(f) \in Hom_D(F(a),F(b))$ such that*
  - $\forall x \in Ob(C)$
$$F(1_x) = 1_{F(x)} \tag{4}$$
  - $\forall f, g \in Hom(C)$
$$F(g \circ_C f) = F(g) \circ_D F(f) \tag{5}$$

<u>Remark :</u>

1. The very uncommon feature of this mapping is that it takes either an object and gives and object or a morphism and gives a morphism.
2. We have implicitely defined a <u>covariant</u> functor. One can define a <u>contravariant</u> functor by reversing the source and target object of $F(f) : F(b) \to F(a) \ \in \mathrm{Hom}_D(F(b), F(a))$, and
$$F(g \circ_C f) = F(f) \circ_D F(g) \tag{6}$$
3. Previously, we've pointed out the structure of a category, here we simply require mappings from a category to another to preserve the composition and identity.

We'll need an additional property for the morphisms, which generalizes the idea of injectivity :

Definition 1.3. *A morphism $f \in Hom_C(a,b)$ is said to be a* <u>*monomorphism*</u> *if*
$$\forall \ g_1, g_2 \in Hom_C(c,a) \ , \quad f \circ g_1 = f \circ g_2 \implies g_1 = g_2 \tag{7}$$

Definition 1.4. *Let X be a topological space, denote $\mathcal{O}(X)$ the category of open sets of X with inclusions as morphisms. A C-valued* <u>*presheaf*</u>[3] *(resp.* <u>*precosheaf*</u>*) is a* contravariant *(resp. covariant) functor from the category $\mathcal{O}(X)$ to C.*

Locality Let $(\mathcal{M}, \eta_{\mu\nu})$ be the *Minkowski space* and denote $\mathcal{A}$ the category of unital C^*-algebra with C^*-algebra morphisms.

<u>Locality</u> is expressed by an $\mathcal{A}$-valued precosheaf that maps any morphisms of $\mathrm{Hom}_{\mathcal{O}(\mathcal{M})}$ to a *monomorphism* of $\mathcal{A}$. This last condition of mapping only to monomorphisms is referred to as <u>isotony</u>.The essence of it is that we associate algebras to regions, and algebras of big regions contain algebras of subregions.

It actually happens that another construction satisfies this definition, we'll check it explicitly.

[3] There is a more general definition, as always...

Definition 1.5. *A* <u>*partially ordered set*</u> *or* <u>*poset*</u> *P is a set with a binary relation $\preceq$ which is*

- reflexive, *i.e* $\forall\, p \in P\,, \quad p \preceq p$
- antisymmetric, *i.e* $\forall\, p, q \in P\,, \quad p \preceq q \;\;\&\;\; q \preceq p \;\Rightarrow\; \quad p = q$
- transitive, *i.e* $\forall\, p, q, r \in P\,, \quad p \preceq q \;\;\&\;\; q \preceq p \;\Rightarrow\; \quad p \preceq r$

The distinction with <u>totally ordered set</u> is that all elements of P are not required to be comparable.

Definition 1.6. *A* <u>*directed set*</u> *P is a poset such that*

$$\forall\, p, q \;\in P\;,\; \exists\, r \;\in P \text{ such that } p \preceq r \;\;\&\;\; q \preceq r \tag{8}$$

Definition 1.7. *Let X be a topological space and P a directed set. A* <u>*net*</u> *is a function from P to X, written $(x_p)_{p\in P}$.*

One has to interpret it as a generalization of sequences – functions from $\mathbb{N}$ to X – and notice that this structure allows us to define limits.

Net of unital C^*-algebra $(\mathcal{A}_U)_{U\in\mathcal{O}(\mathcal{M})}$ vs. $\mathcal{A}$-valued precosheaf

– The directed set $(\mathcal{O}(\mathcal{M}), \subseteq)$ is a category.

– Impose <u>isotony</u> for the net[4] : $\forall\, U_1, U_2 \;\in \mathcal{O}(\mathcal{M})$
$U_1 \subseteq U_2 \implies$ there exists an injective unital C^*-algebra morphism

$$i_{U_1,U_2} \;:\; \mathcal{A}_{U_1} \longrightarrow \mathcal{A}_{U_2} \tag{9}$$

For $U \in \mathcal{O}(\mathcal{M})$ associate $i_{U,U} = id_{\mathcal{A}_{U_1}}$, and for $U_1 \subseteq U_2 \subseteq U_3$[5]

$$i_{U_2,U_3} \circ i_{U_1,U_2} =: i_{U_1,U_3} \tag{10}$$

is an injective unital C^*-algebra morphism.
That is a covariant functor, hence a precosheaf.

– However, we haven't defined anything like a "topological space of algebras". Now we only have "local" algebras but no "global" one anymore. This is remedied by the following definition.

Definition 1.8. *The* <u>*inductive limit*</u> *or* <u>*direct limit*</u> *of $\big((\mathcal{A}_U)_{U\in\mathcal{O}(\mathcal{M})}, (i_{U,V})_{U,V\in\mathcal{O}(\mathcal{M})}\big)$ is the set*

$$\mathcal{A}(\mathcal{M}) := \lim_{\rightarrow} \mathcal{A}_U := \frac{\bigsqcup_{U\in\mathcal{O}(\mathcal{M})} \mathcal{A}_U}{\sim} \tag{11}$$

where $\forall\, a_U \in \mathcal{A}_U,\; a_V \in \mathcal{A}_V$

$$a_U \sim a_V \iff \exists\, W \;\in \mathcal{O}(\mathcal{M}) \text{ s.t. } i_{U,W}(a_U) = i_{V,W}(a_V) \tag{12}$$

There is a canonical unital C^*-algebra morphism for all $U \in \mathcal{O}(\mathcal{M})$

$$i_U : \mathcal{A}_U \longrightarrow \mathcal{A}(\mathcal{M}) \tag{13}$$

that sends any element $a \in \mathcal{A}_U$ to its equivalence class in $\mathcal{A}(\mathcal{M})$.

[4] cf. direct system

[5] $(U_2 \subseteq U_3) \circ (U_1 \subseteq U_2)$

However one has to complete $\mathcal{A}(\mathcal{M})$ to make it into a C^*-algebra[6], the so-called quasilocal algebra.

2 Haag-Kastler axioms

The goal of an axiomatic framework is to build a theory from a minimal number of assumptions. There had been several attemps to define axioms for quantum field theory, but none of them is completely satisfying. Rudolf Haag and Daniel Kastler introduced in 1964 a set of axioms for the net of algebras. Recall that the framework for quantum field theory is special relativity (cf. Appendix), the Minkowski space-time $\mathcal{M}$ with Poincaré group $\mathfrak{P}$, though generalization to curved space-time exist. The axioms are the following :

Locality [7] It expresses the independance of algebras associated to space-like separated regions :

$$\begin{aligned} &\forall\, U, V \in \mathcal{O}(\mathcal{M}),\ \text{space-like separated} \\ &[a, b] = 0\ ,\quad \forall a \in \mathcal{A}_U,\ b \in \mathcal{A}_V \end{aligned} \tag{14}$$

Recall that two systems $\mathcal{A}_1$ and $\mathcal{A}_2$ are called independant if the whole system is isomorphic to $\mathcal{A}_1 \otimes \mathcal{A}_2$. Requiring commutativity is a priori weaker.

Covariance Poincaré group acts on the net : $\forall\ \mathbf{g} \in \mathfrak{P}$, $U \in \mathcal{O}(\mathcal{M})$, there is an isomorphism

$$\alpha^U_{\mathbf{g}} : \mathcal{A}_U \longrightarrow \mathcal{A}_{\mathbf{g}\cdot U} \tag{15}$$

such that $\forall\ V \subseteq U$

$$\alpha^U_{\mathbf{g}}\ |_V = \alpha^V_{\mathbf{g}} \tag{16}$$

and $\forall\ \mathbf{g}_1, \mathbf{g}_2 \in \mathfrak{P}$

$$\alpha^{\mathbf{g}\cdot U}_{\mathbf{g}_1} \circ \alpha^U_{\mathbf{g}_2} = \alpha^U_{\mathbf{g}_1\cdot\mathbf{g}_2} \tag{17}$$

(15) says that algebras of regions related by a Poincaré transformation are related. This is actually very strong since all pairs of points are related by translations. (16) allows us to define the action of the Poincaré group on the inductive limit.

More generally, the above construction is to be compared with the action of a group G on functions. Let (X, α), (Y, β) be two actions of G, then $\forall\ f \in \mathcal{F}(X, Y)$ and $\mathbf{g} \in G$

$$\mathbf{g} \cdot f := \ \beta(\mathbf{g}) \circ f \circ \alpha^{-1}(\mathbf{g}) \tag{18}$$

defines an action[8] of G on $\mathcal{F}(X, Y)$. In particular, it sends functions with compact support in $U \subset X$, to functions with compact support in $\alpha(\mathbf{g}) \cdot U$.

[6] cf. [1, p. 14]

[7] or "Micro-causality" in some QFT lessons

[8] α^{-1} such that we have a left action.

Time-slice axiom Let $U \in \mathcal{O}(\mathcal{M})$ be an open set which has a Cauchy surface Γ_U then

$$\mathcal{A}_{U''} \cong \mathcal{A}_U \tag{19}$$

Notice that algebras are associated to open sets, what Cauchy surfaces are not[9], but this axiom does indeed express the idea that the algebra "associated to a Cauchy surface" contains as much information as the algebra of its causal completion. That is precisely the property of hyperbolic equations whose solutions are uniquely defined by initial conditions on a Cauchy surface, however the axiom states it prior to any dynamic, any law for the evolution of the system.

Stability condition or Spectrum condition It requires a more specific framework, namely :

- the existence of a representation $(\mathcal{H}, \pi)$ of $\mathcal{A}(\mathcal{M})$, i.e. to each element $a \in \mathcal{A}(\mathcal{M})$ is associated a bounded operator $\pi(a)$ on the Hilbert space $\mathcal{H}$ such that $\pi(\lambda\, a^* + b) = \lambda\, \pi(a)^\dagger + \pi(b)$ and $\big\|\pi(a)^\dagger \circ \pi(a)\big\| = \|\pi(a)\|^2$.
- previous representation be such that the action of the translation group[10] on the algebra can be written as follows : $\forall\ \mathbf{g} \in \mathbb{R}^4$, and $a \in \mathcal{A}(\mathcal{M})$

$$\pi(\alpha_{\mathbf{g}}(a)) = U(\mathbf{g}) \circ \pi(a) \circ U^{-1}(\mathbf{g}) \tag{20}$$

  where U is a unitary operator on $\mathcal{H}$ and $\mathbf{g} \to U(\mathbf{g})$ is continuous under the strong topology for the operator space.

The stability condition now states that the joint spectrum of the generators of translations[11] lies in the forward causal cone. Those generators being the 4-impulsion observables, this condition expresses in a covariant way that particles have positive energy and travel slower than the speed of light.

3 Example

3.1 A toy example

I'll present a somehow artificial example starting with just an algebra and adding conditions successively.

Canonical Anticommutation Relation (CAR) algebra Let $\mathcal{H}$ be a Hilbert space with basis $\{f_i\}_{i\in I}$ and an anti-unitary involution Γ, the pro-

[9] cf. submanifold of codim ≥ 1.

[10] cf. Poincaré group.

[11] in the representation $(\mathcal{H}, U)$ of the translation group on the Hilbert space. We implicitely use the fact that a representation of a Lie group also gives a representation of its Lie algebra and vice versa.

totype being complex conjugation : $\forall g, h \in \mathcal{H},\ \lambda, \mu \in \mathbb{C}$

$$\Gamma(\lambda g + \mu h) = \bar{\lambda}\ \Gamma(g) + \bar{\mu}\ \Gamma(h) \tag{21}$$
$$\langle \Gamma(g), \Gamma(h) \rangle = \langle g, h \rangle \tag{22}$$
$$\Gamma^2 = \mathbf{1} \tag{23}$$

Now for all $i \in I$ associate an abstract element $a(f_i)$ and define an antilinear[12] map :

$$a : \begin{cases} \mathcal{H} \longrightarrow \ \mathrm{Span}\,(a(f_i),\ i \in I) \\ \ f \longrightarrow \ a(f) \end{cases} \tag{24}$$

Then extend $\mathrm{Span}\,(a(f_i),\ i \in I)$ into a unital C^*-algebra $\mathcal{A}(\mathcal{H})$ such that $\forall f, g \in \mathcal{H}$

$$\begin{cases} a(f)^* = a(\Gamma(f)) \\ \{a(f), a(g)\} = 0 \\ \{a(f), a^*(g)\} = (f, g)\mathbf{1} \end{cases} \tag{25}$$

Since in an complex algebra one can add, multiply two elements, and also multiply by a complex number, the CAR algebra $\mathcal{A}(\mathcal{H})$ is generated by polynomials in the annihilation and creation operators $a(f)$ and $a^*(f)$. A theorem [4, **Th. 5.2.5**] states that this freely generated algebra with anticommutation relations (25) is uniquely defined up to isomorphisms.

Fock representation One can always build an algebra starting from a vector space, namely the tensor algebra. The Fock space of a Hilbert space $\mathcal{H}$ is defined in the same way

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n \geq 0} \left(\bigotimes_{i=1}^{n} \mathcal{H} \right) = \bigoplus_{n \geq 0} \mathcal{H}^n \tag{26}$$

but has an additional hermitian product on each subspace $\mathcal{H}^n$. $\forall\ \psi \in \mathcal{F}(\mathcal{H})$, write $\psi^{(n)}$ its orthogonal projection on $\mathcal{H}^n$. However, we are just interested in the vector space structure of $\mathcal{F}(\mathcal{H})$ as a representation space of $\mathcal{A}(\mathcal{H})$.

$\forall\ f \in \mathcal{H}$ define the action of the creation and annihilation operators

on $\mathcal{H}^0$ by

$$\mathrm{a}(f)\ \psi^0 = \ 0 \tag{27}$$
$$\mathrm{a}^*(f)\ \psi^0 = \ \psi^0 f \quad ^{13} \tag{28}$$

[12] It is a convention from [4], where the hermitian product is antilinear in the first argument.

and on $\mathcal{H}^n,\ n \geq 1$

$$\begin{aligned} \mathrm{a}(f)\ (g_1 \otimes \cdots \otimes g_n) = &\ \sqrt{n}\ (f, g_1)\ g_2 \otimes \cdots \otimes g_n && (29)\\ \mathrm{a}^*(f)\ (g_1 \otimes \cdots \otimes g_n) = &\ \sqrt{n+1}\ f \otimes (g_1 \otimes \cdots \otimes g_n) && (30) \end{aligned}$$

Define the antisymmetric projector on each subspace $\mathcal{H}^n$
$\forall\ f_1 \otimes \cdots \otimes f_k \in \mathcal{H}^k$

$$P_-(f_1 \otimes \cdots \otimes f_k) := \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} \epsilon(\sigma) f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)} \tag{31}$$

The Fermi Fock space is

$$\mathcal{F}_-(\mathcal{H}) := P_-\left(\mathcal{F}(\mathcal{H})\right) \tag{32}$$

When (and only when!) restricted and projected onto the Fermi Fock space

$$\tilde{a}(f) :=\ P_- \circ\ \mathrm{a}(f) \circ P_-\ ,\quad \tilde{a}^*(f) :=\ P_- \circ\ \mathrm{a}^*(f) \circ P_- \tag{33}$$

the operators satisfy (25).

Indeed

$$\tilde{a}^*(g)(h_1 \otimes \cdots \otimes h_n) = \sqrt{n+1}\ \frac{1}{(n+1)!} \sum_{\sigma \in \mathcal{S}_{n+1}} \epsilon(\sigma)\ l_{\sigma(1)} \otimes \cdots \otimes l_{\sigma(n+1)} \tag{34}$$

where $l_1 := g$, and $l_i := h_{i-1}$ for $i \neq 1$. Then

$$\begin{aligned} \mathrm{a}(f)\ \tilde{a}^*(g)\ (h_1 \otimes \cdots \otimes h_n) &= \frac{n+1}{(n+1)!} \sum_{\sigma \in \mathcal{S}_{n+1}} \epsilon(\sigma)\ \Big(f, l_{\sigma(1)}\Big)\ l_{\sigma(2)} \otimes \cdots \otimes l_{\sigma(n+1)} \\ &= \sum_{i=1}^{n+1} \Big(f, l_i\Big) \left(\frac{1}{n!} \sum_{\sigma \in \mathcal{S}^i_{n+1}} \epsilon(\sigma)\ l_{\sigma(2)} \otimes \cdots \otimes l_{\sigma(n+1)} \right) \qquad (35) \\ &= \tilde{a}(f)\ \tilde{a}^*(g)(h_1 \otimes \cdots \otimes h_n) \end{aligned}$$

where $\mathcal{S}^i_{n+1} \cong \mathcal{S}_n$ stands for permutations of $n+1$ elements such that $\sigma(1) = i$. Similarly

$$\begin{aligned} \tilde{a}^*(g)\ \tilde{a}(f)\ (h_1 \otimes \cdots \otimes h_n)\ &=\ \tilde{a}^*(g) \left(\sqrt{n}\ \sum_{i=1}^{n} \Big(f, h_i\Big)\ \frac{1}{n!} \sum_{\pi \in \mathcal{S}^i_n} \epsilon(\pi)\ h_{\pi(2)} \otimes \cdots \otimes h_{\pi(n)} \right) \\ =\ & \sum_{i=1}^{n} \Big(f, l_{i+1}\Big)\ \frac{1}{n!} \sum_{\sigma \in \tilde{\mathcal{S}}^{i+1}_{n+1}} \epsilon(\sigma) \left(\frac{1}{(n-1)!} \sum_{\pi \in \mathcal{S}^i_n} \epsilon(\pi)\ l_{\sigma(1)} \otimes l_{\sigma(\pi(2)+1)} \otimes \cdots \otimes l_{\sigma(\pi(n)+1)} \right) \\ =\ & -\sum_{i=2}^{n+1} \Big(f, l_i\Big) \left(\frac{1}{n!} \sum_{\sigma \in \mathcal{S}^i_{n+1}} \epsilon(\sigma)\ l_{\sigma(2)} \otimes \cdots \otimes l_{\sigma(n+1)} \right) \qquad (36) \end{aligned}$$

[13] $\mathcal{H}^0 = \mathbb{C}$, i.e. ψ^0 is a complex number.

where $\tilde{\mathcal{S}}^{i+1}_{n+1}$ are permutations such that $\sigma(i+1) = i+1$. Finally

$$\Big(\tilde{a}(f)\ \tilde{a}^*(g) + \tilde{a}^*(g)\ \tilde{a}(f)\Big)\,(h_1 \otimes \cdots \otimes h_n) = \Big(f, g\Big)\ \ P_-\ \Big(h_1 \otimes \cdots \otimes h_n\Big) \tag{37}$$

The other anticommutation calculus is straightforward.

The following calculous $\forall\, g_1 \otimes \cdots \otimes g_{n-1} \in \mathcal{H}^{n-1},\ h_1 \otimes \cdots \otimes h_n \in \mathcal{H}^n$

$$\begin{aligned}
\Big(g_1 \otimes \cdots \otimes g_{n-1}, a(f)(h_1 \otimes \cdots \otimes h_n)\Big)_{\mathcal{F}(\mathcal{H})} &= \sqrt{n}\ (f, h_1)\ \prod_{i=1}^{n-1} (g_i, h_{i+1}) \\
\Big(a^*(f) g_1 \otimes \cdots \otimes g_{n-1}, (h_1 \otimes \cdots \otimes h_n)\Big)_{\mathcal{F}(\mathcal{H})} &= \sqrt{n}\ (f, h_1)\ \prod_{i=1}^{n-1} (g_i, h_{i+1})
\end{aligned} \tag{38}$$

shows that $\tilde{a}^*(f)$ is adjoint to $\tilde{a}(f)$ for the inner product on $\mathcal{F}(\mathcal{H})$. Therefore a map sending $a(f)$ and $a^*(f)$ to $\tilde{a}(f)$ and $\tilde{a}^*(f)$ respectively preserves the CAR and is a C^*-algebra morphism, i.e. the Fermi Fock space is a representation of $\mathcal{A}(\mathcal{H})$[14].

Implementation of a net of algebras Choose $\mathcal{H} := L^2(\mathcal{M})$ and $\forall\ \psi \in \mathcal{F}(\mathcal{H})$ [15] define

$$(a(f)\ \psi)^{(n)}(x_1, \ldots, x_n)) \ := \ \sqrt{(n+1)} \int \bar{f}(y) \psi^{(n+1)}(y, x_1, \ldots, x_n) \tag{39}$$

$$(a^*(f)\ \psi)^{(n)}(x_1, \ldots, x_n)) \ := \ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(x_1) \psi^{(n-1)}(x_1, \ldots, \hat{x}^i, \ldots, x_n) \tag{40}$$

This is an explicit realization of the annihilation and creation operators. Restricting to the Fermi Fock space then yields the CAR algebra.

For each open set $U \subseteq \mathcal{M}$ associate the vector space of L^2 functions with compact support in U, $L^2(U) \subset L^2(\mathcal{M})$. Then a CAR algebra can be built over that Hilbert space which finally leads to a net of algebras. The isotony property comes from the fact that functions with support in $U \subseteq V$ are functions with support in V, the identity is injective.

Haag Kastler axioms The question is now whether some axioms are automatically satisfied, do impose extra conditions in previous construction, or are even compatible at all.

[14]In the case of CCR there is the Stone-von Neumann uniqueness theorem, it doesn't seem to be the case for CAR algebra.

[15]It actually means $\psi^n(x_1, \ldots, x_n)$ is of the form $h_1(x_1) \cdots h_n(x_n)$, however (anti)commutation will still be satisfied on the bigger space of multivariable function space.

- **Microcausality :** For $U, V \in \mathcal{O}(\mathcal{M})$ space-like, there exist $W \in \mathcal{O}(\mathcal{M})$, $U, V \subseteq W$. From previous calculous (35) and (36) we see that $\forall\ f \in L^2(U) \subseteq L^2(W)$, $\forall\ g \in L^2(V) \subseteq L^2(W)$ and $(h_1 \otimes \cdots \otimes h_n) \in (L^2(W))^n$

$$\begin{aligned}
&[\tilde{a}(f), \tilde{a}^*(g)](h_1 \otimes \cdots \otimes h_n) = \\
&\qquad \underbrace{(f,g)}_{=0} \left(\frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \epsilon(\pi)\ h_{\sigma(1)} \otimes \cdots \otimes h_{\pi(n)} \right) \\
&\quad +\ \ 2 \sum_{i=2}^{n+1} \underbrace{(f, l_i)}_{\neq 0} \left(\frac{1}{n!} \sum_{\sigma \in \mathcal{S}_{n+1}^i} \epsilon(\sigma)\ l_{\sigma(2)} \otimes \cdots \otimes l_{\sigma(n+1)} \right)
\end{aligned} \tag{41}$$

  This quantity is not vanishing in general, neither are $[\tilde{a}(f), \tilde{a}(g)]$ and $[\tilde{a}^*(f), \tilde{a}^*(g)]$ which is in sharp contrast with the CCR algebra case!! However it doesn't mean that CAR algebras are incompatible with microcausality, the failure comes from the implementation and more precisely from the injection $i_{U,W} : \mathcal{A}_U \to \mathcal{A}_W$. The notion of precosheaf requires the existence of such injection for $U \subseteq W$ but not unicity of it. The microcausality axiom constrains the choice of those injective morphisms but in our implementation, those injection were implicitely given by $L^2(U) \subseteq L^2(W)$.
- **Covariance :** The action of the Poincaré group is yet to be defined but with (15) and (18) in mind, we see that the axiom imposes a restriction on the (X, α) of (18). Recall the "scalar field" (physics terminology) representation of the Poincaré group on the Hilbert space of functions $\forall\ \mathbf{g} \in \mathfrak{P}$, $\forall\ \varphi \in L^2(U)$

$$\rho^U_{\mathbf{g}} \cdot \varphi(x) := \varphi(\mathbf{g}^{-1} \cdot x) \tag{42}$$

  Since the association of an algebra to an open set was achieved through functions, an extension of this action to the algebra fulfilling the axiom can be written

$$\alpha^U_{\mathbf{g}} \cdot a(f) := \gamma(\mathbf{g}) \cdot a(\rho(\mathbf{g}) \cdot f) \tag{43}$$

  where γ is an action on operators, i.e. similar to (18).
- **Time-slice** In our implementation, an algebra of a big region is always strictly bigger than the algebra of a subregion, infringing the axiom. The latter can be seen as the counterpart of the isotony, algebra of a subregion is contained in the algebra of a bigger region but for some specific bigger regions, the algebra does contain anything more then the one of the smaller region. One can certainly built a net of algebra which intrinsically incorporates the time-slice axiom but the usual procedure is to start with a quasilocal algebra that has too many elements and then restrict it which is the rôle of "dynamic".

- **Stability condition** (43) corresponds to (20) with $U = \gamma$ (once a representation of the quasilocal algebra is given). This axiom contrains the choice for γ

Examples from QFT are given in [2, Chapter II, section 3.]

3.2 Some fundamental notions

This framework allows us to give a very precise definition of some fundamental notions in physics. I'll just give a list without developping.

Definition 3.1. *Observables are self-adjoint elements of the C^*-algebra.*

C^*-algebras can always be represented by bounded operators on some Hilbert space but there can be a continuum of inequivalent representations in the case of simple C^*-algebras for example. The notion of particle is a choice of particular representations :

Definition 3.2 (Wigner)**.** *A particle is an irreducible, strongly continuous positive energy representation of the Poincaré group.*

The link with the more naive idea of particle is that irreducible representations of the Poincaré group can be labelled by the eigenvalue of the Casimir of the Poincaré algebra, and the latter turns out to be $P^2 = m^2$, the mass squared. Another elementary notion is that of state.

Definition 3.3. *A state is a positive normalized linear functional on the C^*-algebra.*

Given a state, one can actually build a probability distribution on the spectrum of a self-adjoint operator, cf. [1, p.8]. Every norm 1 vector of the representation space is a state, but the definition allows for more sophisticated ones.

Definition 3.4. *The vacuum sector is a representation which has a vector, the vacuum, which is invariant under the Poincaré group (or the relevant symmetry group of the system).*

There is in general not unicity of such representation and not even existence in the case of symmetry breaking.

Definition 3.5. *For a system with n undistinguishable particles, bosons and fermions are respectively the trivial and signature representation of the permutation group.*

The meaning of "indistinguishable" is that permutation is a symmetry of the system.

The representation of observables can be further restricted : assume the representation of the C^*-algebra is decomposed into irreducible components, the so called called superselection sectors, then the observables are required to be bloc diagonal with respect to the decomposition. As a consequence, the "relative phase" θ in a superposition $e^{i\theta}\psi_1 + \psi_2$ of two states ψ_1 and ψ_2 belonging to two distinct superselection sectors is not observable.

4 Appendice : the special relativity framework

Definition 4.1. *A piecewise smooth curve $c : [a, b] \to \mathcal{M}$ is called causal if all its tangent vector are time-like or lightlike.*

Definition 4.2. *A subset $\mathcal{S}$ of the Minkowski space is a Cauchy hypersurface if every inextendible causal curve intersets $\mathcal{S}$ exactly once.*

Definition 4.3. $\forall\ U \in \mathcal{M}$ *its causal complement is*

$$U^{'} := \{y \in \mathcal{M} \ \textit{s.t.}\ y - x \ \textit{is space-like},\ \forall x \in U\} \tag{44}$$

Its causal completion

$$U^{''} := \left\{z \in \mathcal{M} \ \textit{s.t.}\ z - y \ \textit{is space-like},\ \forall y \in U^{'}\right\} \tag{45}$$

Références

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