Unruh Effect

by

Thomas Reichenbächer

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Contents

The Unruh effect [5] is a name for the fact that uniformly accelerated observers in Minkowski spacetime associate a KMS state (i.e. a thermal state) to the vacuum state seen by initial Minkowski observers. In other words, the particle content of a field theory is observer dependent [3].

But we do not want to stress here the possibly philosophical implications of the Unruh effect, but rather want to derive and explain it. To this end we first have to get familiar with field theory in curved spacetimes and its ramifications in section 1. We proceed introducing appropriate coordinates in Minkowski spacetime in section 2 until we can finally derive the Unruh effect in section 3. Our notation will be as follows: as we are working in four dimensional spacetime we usually use four vectors which are denoted by $x, x', \ldots$ If we consider only their spatial part (i.e. on the Cauchy hypersurface) we write $\vec{x}, \vec{x}', \ldots$ Furthermore we make use of a metric with signature $(+,-,-,-)$. Finally, we suppress any factors of 2π in the integral measures as they are only introduced for normalization reasons.

1 Static Spacetimes

1.1 Our Framework

We consider an *ultrastatic spacetime* which is a four-dimensional globally hyperbolic manifold $\mathbb{R} \times \Sigma$ with causal structure and a metric of the form

$$g = dt^2 - h \quad , \tag{1}$$

where h is a time-independent metric on the remaining three-dimensional space Σ. Moreover we want Σ to be a *Cauchy hypersurface*, i.e. a closed hypersurface which is intersected by each inextendible timelike curve only once [3].

To build a field theory in this spacetime—in our case the simplest one, containing only scalar fields—we need to define a differential operator in the (probably curved) spacetime. The d'Alembertian is in this case given by

$$\Box_g = \partial_t^2 - \Delta_h \tag{2}$$

and the Laplacian on the Cauchy hypersurface Σ is given by $\Delta_h = \frac{1}{\sqrt{|h|}}\partial_i h^{ij}\sqrt{|h|}\partial_j$, with h the determinant of the metric.

Now one can simply add the square of the mass and arrives at the famous Klein-Gordon equation. But in our case it is more convenient to define the operator $A = -\Delta_h + m^2$ so that the Klein-Gordon equation reads

$$(\Box_g + m^2)\varphi(x) = (\partial_t^2 + A)\varphi(x) = 0 \quad , \tag{3}$$

where $\varphi(x)$ is assumed to be a solution to this equation.

1.2 Two-Point Function

If one studies the solutions of the Klein-Gordon equation in Minkowski spacetime one usually builds them out of eigenfunctions of the energy-momentum operator k, i.e. $e^{\pm ikx}$. After that one can define the vacuum state as the one being annihilated by all annihilation operators which are in turn those eigenfunctions with positive frequency ($e^{-ik_0x_0}$, $k_0 > 0$). Applying the eigenfunctions with negative frequency to the vacuum state one defines states with a well-defined number of particles. But in this construction one makes essential use of the time translation symmetry of Minkowski spacetime in order to define positive and negative frequency functions.

In the general case of a curved spacetime one has no obvious time translation symmetry to do that. But one can define an inner product of solutions to the Klein-Gordon equation, quantize them and search for a complete set of solutions (complete w.r.t. the inner product), such that one can define one half of them to be of positive frequency. Expanding a general solution in these functions one finds creation and annihilation operators as coefficients with which one can then define a vacuum state (which depends heavily on the choice of positive frequency functions).[1]

We want to put the cart before the horse and state the result for a general state in the sense of the algebraic approach to quantum field theory. Then we prove its main properties to motivate to call it a state (cf. the talk on Algebraic Quantum Mechanics for the definition and properties of a state).

Define a general state ω via the so-called n-point function of the ground state $\omega_n(f_1, \ldots, f_n) = \omega(\varphi(f_1) \cdots \varphi(f_n))$, which has to be viewed in a distributional sense, i.e. evaluated with testfunctions f_i. In our case it is sufficient to restrict this general treatment to two-point functions, defined as $\omega(\varphi(f)\varphi(h)) = \int d^4x\, d^4x'\, \omega_2(x, x')f(x)h(x')$. And for φ being a general solution to Eq. (3) we present the resulting state

$$\omega_2(t, \vec{x}, t', \vec{x}') = \frac{1}{2\sqrt{A}} e^{-i\sqrt{A}(t-t')}(x, x') \tag{4}$$

which is taken from [1]. The squareroot of A in this expression is well-defined, because A is positive as it can be identified with ω^2 (via the spectral theorem, see below). Again, this has to be thought of in a distributional sense. In order to motivate this solution we have to check that this defines really a state. Thus, we have to check the positivity of ω_2 and that ω_2 really solves Eq. (3).

1.3 Check 1: Solution of the Klein-Gordon-Equation

To start with the latter, we get

$$\begin{aligned} & \int d^4x\sqrt{|g|}\, d^4x'\sqrt{|g|}\, (\partial_t + A)\frac{1}{2\sqrt{A}} e^{-i\sqrt{A}(t-t')} f(x)h(x') \\ = & \int d^4x\sqrt{|g|}\, d^4x'\sqrt{|g|} \left(-\frac{A}{2\sqrt{A}} e^{-i\sqrt{A}(t-t')} + \frac{A}{\sqrt{A}} e^{-i\sqrt{A}(t-t')} \right) f(x)h(x') \\ = & \; 0 \quad \forall f, h \end{aligned}$$

Obviously, this also holds for taking the derivative with respect to t'.

1.4 Check 2: Positivity

Proving the positivity of ω_2 takes more effort and needs a bit preparation. Namely, the *spectral theorem* states, that for a self-adjoint operator A on a

[1]This procedure is described and executed in more detail in [3].

Hilbert space $\mathcal{H}$ with eigenvectors ϕ_k of A (i.e. $A\phi_k = \omega_k^2 \phi_k$) one can define an arbitrary function f of A acting on $h \in \mathcal{H}$ by

$$f(A)h = \int f(\omega_k^2) dP_k(h) = \int f(\omega_k^2)\phi_k(\phi_k, h) d\mu_k \quad . \tag{5}$$

Here, P_k is an orthogonal projector, $(\cdot, \cdot)$ denotes the scalar product on $\mathcal{H}$ and $d\mu_k$ is an integration measure.

As we are working in a square integrable (L^2) Hilbert space with basis $\{\phi_{\vec{k}}\}$ we can write $\omega_2(x, x') = \int d\vec{k} \, \frac{1}{2\sqrt{A}} e^{-i\sqrt{A}(t-t')} \phi_{\vec{k}}(\vec{x})\phi^*_{\vec{k}}(\vec{x'})$. The spectral theorem then allows us to identify A with ω^2 (if acting on $\phi_{\vec{k}}$) and hence

$$\begin{aligned} \omega_2(f, f^*) &= \int d^4x \sqrt{|g|}\, d^4x' \sqrt{|g|}\, \frac{d\vec{k}}{2\omega}\, e^{-i\omega t} \phi_{\vec{k}}(\vec{x}) f(x) e^{i\omega t'} \phi^*_{\vec{k}}(\vec{x}') f^*(x') \\ &= \int d^4x \sqrt{|g|}\, d^4x' \sqrt{|g|} \frac{d\vec{k}}{2\omega}\, \psi_{\vec{k}}(x) f(x) \psi^*_{\vec{k}}(x') f^*(x') \\ &= \int \frac{d\vec{k}}{2\omega}\, |c(\vec{k})|^2 \geq 0 \quad , \end{aligned}$$

where we introduced functions $\psi_{\vec{k}}(x) = e^{-i\omega t}\phi_{\vec{k}}(\vec{x})$ in the second line and numbers $c(\vec{k}) = \int d^4x \sqrt{|g|}\, \psi_{\vec{k}}(x) f(x)$ in the last line.

All in all our two-point function is a solution to the Klein-Gordon equation and is positive, hence defines a state.

2 Rindler Spacetime

Our aim in this section is to introduce and understand the Rindler spacetime.

2.1 Rindler Coordinates

We begin with Minkowski spacetime where one considers the (flat) manifold $M = \mathbb{R} \times \mathbb{R}^3$ with metric

$$g = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad . \tag{6}$$

This is an ultrastatic and globally hyperbolic spacetime according to the above definitions. As probably known this spacetime is also invariant under Lorentz boosts of the form

$$\begin{aligned} x^0 \quad &\longrightarrow \quad x^{0\prime} = x^0 \cosh z + x^1 \sinh z \\ x^1 \quad &\longrightarrow \quad x^{1\prime} = x^0 \sinh z + x^1 \cosh z \end{aligned}$$

with rapidity z. This already suggests the coordinate transformation

$$\begin{aligned} x^0 &= e^\lambda \sinh \theta \\ x^1 &= e^\lambda \cosh \theta \\ x^2 &= y \\ x^3 &= z \end{aligned} \tag{7}$$

to *Rindler coordinates* θ, λ, y, z where worldlines represent constantly accelerated observers. To calculate the metric in these coordinates we need

$$\begin{aligned} dx^0 &= e^\lambda (\sinh \theta d\lambda + \cosh \theta d\theta) \\ dx^1 &= e^\lambda (\cosh \theta d\lambda + \sinh \theta d\theta) \end{aligned}$$

$$\begin{aligned} \Rightarrow (dx^0)^2 - (dx^1)^2 \quad &= \quad e^{2\lambda} \left[\sinh^2 \theta (d\lambda)^2 + \cosh^2 \theta (d\theta)^2 + 2 \sinh \theta \cosh \theta \, d\lambda \, d\theta \right. \\ &\qquad \left. - \cosh^2 \theta (d\lambda)^2 - \sinh^2 \theta (d\theta)^2 - 2 \sinh \theta \cosh \theta \, d\lambda \, d\theta \right] \\ &= \quad e^{2\lambda} \left[(\sinh^2 \theta - \cosh^2 \theta)(d\lambda)^2 + (\cosh^2 \theta - \sinh^2 \theta)(d\theta)^2 \right] \quad . \end{aligned}$$

So that using the relation $\cosh^2 \theta - \sinh^2 \theta = 1$ the metric reads

$$g = e^{2\lambda} (d\theta)^2 - e^{2\lambda} (d\lambda)^2 - (dy)^2 - (dz)^2 \quad \text{(Rindler metric)} \quad . \tag{8}$$

This kind of metric is now called a *static spacetime* as the time coordinate appears with a constant prefactor. Additionally, the Rindler spacetime is globally hyperbolic with Cauchy hypersurfaces $\theta = const$. Note that, doing this coordinate transform we restricted ourselves into the region $\{x \in M | \, |x^0| < x^1\}$ of Minkowski spacetime, also called right *Rindler wedge*.

2.2 Acceleration in General Relativity

As a translation in the Rindler coordinate θ can be interpreted as a constant acceleration a in x^1 direction, we want to compute this acceleration: In general the acceleration is the time derivative of the (four) velocity v. In the case of relativity this becomes

$$\begin{aligned} & a = \frac{d}{dt}(\gamma v) = const. \\ \Rightarrow \quad & at + b = \gamma v \quad . \end{aligned} \tag{9}$$

With the initial condition $v(0) = 0$ we can set $b = 0$. Hence,

$$\begin{aligned} & at = \gamma v = \frac{v}{\sqrt{1 - v^2}} \\ \Rightarrow \quad & v = \frac{at}{\sqrt{1 + a^2 t^2}} = \frac{dx}{dt} \\ \Rightarrow \quad & x = \int dt \, \frac{at}{\sqrt{1 + a^2 t^2}} = \frac{1}{a} \sqrt{1 + a^2 t^2} + c \quad , \end{aligned}$$

where the integration constant c can be set to zero by an appropriate shift/redefinition of x. To this end we are left with

$$x^2 - t^2 = \frac{1}{a^2} \tag{10}$$

$$\Rightarrow \quad e^{2\lambda}(\cosh^2 \theta - \sinh^2 \theta) = e^{2\lambda} = \frac{1}{a^2} \quad . \tag{11}$$

The acceleration is thus given by $a = e^{-\lambda}$ in Rindler coordinates.

3 Unruh Effect

We will next calculate an expression for the state defined by ω_2 on Minkowski space, then transform it to Rindler coordinates and observe very interesting consequences.

3.1 The Minkowski Two-Point Function ...

In section 1 we defined the differential operator $A = -\Delta + m^2$ now acting on the Hilbert space $L^2(\mathbb{R}^3)$. A is only essential self-adjoint, but can be extended to an self-adjoint operator which will be denoted by A, too. As aforementioned the solutions of $A\phi_{\vec{k}} = \omega_{\vec{k}}^2 \phi_{\vec{k}}$ are $\phi_{\vec{k}} = e^{i\vec{k}\cdot\vec{x}}$ with $\omega_{\vec{k}}^2 = \vec{k}^2 + m^2$. Additionally, we define the scalar product to be $(f, h) = \int d\vec{x}\, f^*(\vec{x}) h(\vec{x})$ and choose the integral measure $d\vec{k}$. With the help of the spectral theorem we can then express arbitrary functions $f(A)$ of the operator A by

$$f(A)h(\vec{x}) = \int d\vec{k}\, f(\omega_{\vec{k}}^2) e^{i\vec{k}\cdot\vec{x}} \int d\vec{x}'\, e^{-i\vec{k}\cdot\vec{x}'} h(\vec{x}') = \int d\vec{k}\, d\vec{x}'\, f(\omega_{\vec{k}}^2) e^{-i\vec{k}\cdot(\vec{x}'-\vec{x})} h(\vec{x}') \tag{12}$$

and can thus identify

$$f(A)(\vec{x}, \vec{x}') = \int d\vec{k}\, e^{-i\vec{k}\cdot(\vec{x}'-\vec{x})} \tag{13}$$

as the kernel of a distribution. This procedure applied to the state ω_2 of Eq. (4) and making use of the Minkowski metric $\eta = \text{diag}(1, -1, -1, -1)$ we get

$$\omega_2(t, \vec{x}, t', \vec{x}') = \int d\vec{k}\, \frac{e^{-i\omega(t-t')}}{2\omega} e^{-i\vec{k}\cdot(\vec{x}'-\vec{x})} = \int d\vec{k}\, \frac{1}{2\omega} e^{-ik\cdot(x-x')} \quad , \tag{14}$$

defining $k = (\omega, \vec{k})$ and $x = (t, \vec{x})$.

3.2 ... becomes a Rindler Two-Point Function

With this result we perform a coordinate change to Rindler coordinates according to Eqs. (7):

$$\begin{aligned} \omega_2(\theta, \lambda, y, z, \theta', \lambda', y', z') \quad = \quad & \int \frac{d\vec{k}}{2\omega} \exp[-i\omega(e^{\lambda} \sinh\theta - e^{\lambda'} \sinh\theta') \\ & + ik_1(e^{\lambda} \cosh\theta - e^{\lambda'} \cosh\theta') + ik_2(y - y') + ik_3(z - z')] \end{aligned}$$

Now one can check—which we will actually do in a moment—that this defines a KMS state. Again this needs a short preparation.

3.3 KMS States

Recall from the second talk on the KMS Condition that the KMS condition for a free field, i.e. our two-point function, takes the form

$$\omega_2(\theta, \lambda, y, z, \theta', \lambda', y', z') = \omega_2(\theta', \lambda', y', z', \theta + i\beta, \lambda, y, z) \quad . \tag{15}$$

If this holds, ω_2 is invariant under the one parameter automorphism group of time translations and thus defines a thermal state with inverse temperature β.

To prove that Eq (15) really holds we begin with a translation

$$\theta \longrightarrow \theta - \frac{\theta+\theta'}{2} = \frac{\theta-\theta'}{2}, \theta' \longrightarrow \theta' - \frac{\theta+\theta'}{2} = -\frac{\theta-\theta'}{2}$$

corresponding to a Lorentz transformation in x_1-direction as seen before. After that, ω_2 becomes

$$\begin{aligned}\omega_2 &= \int \frac{d\vec{k}}{2\omega} \exp[-i\omega(e^{\lambda}\sinh\frac{\theta-\theta'}{2} + e^{\lambda'}\sinh\frac{\theta-\theta'}{2}) \\ &\quad +ik_1(e^{\lambda}\cosh\frac{\theta-\theta'}{2} - e^{\lambda'}\cosh\frac{\theta-\theta'}{2}) + ik_2(y-y') + ik_3(z-z')] \quad .\end{aligned}$$

Then

$$\begin{aligned}\omega_2(\theta',\lambda',y',z',\theta+i\beta,\lambda,y,z) &= \int \frac{d\vec{k}}{2\omega}\exp[-i\omega(e^{\lambda'}+e^{\lambda})\sinh\frac{\theta'-\theta-i2\pi}{2} \\ &\quad +ik_1(e^{\lambda'}-e^{\lambda})\cosh\frac{\theta'-\theta-i2\pi}{2} \\ &\quad +ik_2(y'-y)+ik_3(z'-z)] \\ &= \int \frac{d\vec{k}}{2\omega}\exp[i\omega(e^{\lambda'}+e^{\lambda})\sinh\frac{\theta'-\theta}{2} \\ &\quad -ik_1(e^{\lambda'}-e^{\lambda})\cosh\frac{\theta'-\theta}{2} \\ &\quad -ik_2(y-y')-ik_3(z-z')] \\ &= \int \frac{d\vec{k}}{2\omega}\exp[-i\omega(e^{\lambda}+e^{\lambda'})\sinh\frac{\theta-\theta'}{2} \\ &\quad +ik_1(e^{\lambda}-e^{\lambda'})\cosh\frac{\theta-\theta'}{2} \\ &\quad +ik_2(y-y')+ik_3(z-z')] \\ &= \omega_2(\theta,\lambda,y,z,\theta',\lambda',y',z')\end{aligned}$$

setting $\beta = 2\pi$ in the first, using trigonometric relations[2] in the second and changing variables[3] $k_{2,3} \longrightarrow -k_{2,3}$ in the third line. Thus, ω_2 defines indeed a KMS state with temperature $T = \frac{1}{\beta} = \frac{1}{2\pi}$.

Observe that we considered a in the calculation above (see Eq. (9)) to be constant, hence also λ is constant (and of course y and z are). Thus the eigentime for an accelerated observer in Rindler coordinates is

$$\begin{aligned} &(d\tau)^2 = e^{2\lambda}(d\theta)^2 \\ \Rightarrow\quad &d\tau = e^{\lambda}d\theta \quad . \end{aligned} \tag{16}$$

[2]Those are in particular

$$\begin{aligned}&\sinh(x-y) = \sinh x\cosh y - \cosh x\sinh y \quad , \\ &\cosh(x-y) = \cosh x\cosh y - \sinh x\sinh y \quad , \\ &\cosh(ix) = \cos x \quad , \\ &\sinh(ix) = i\sin x \quad .\end{aligned}$$

[3]This leaves the integral invariant as the minus sign from $dk_{2,3} \longrightarrow -dk_{2,3}$ will be "eaten" by again changing the integration boundaries to get their right order back.

Hence, in eigentime τ of the observer, one has to substitute $\tau \longrightarrow \tau + i\frac{\beta}{a}$ instead of $\theta \longrightarrow \theta + i\beta$. That is the reason why we have to substitute 2π by $\frac{2\pi}{a}$ for β.

To summarize, one can say that an accelerated observer sees the vacuum as a thermal bath with temperature $T = \frac{a}{2\pi}$ which is known as the Unruh effect.

3.4 Remarks

We finally want to comment on two observations: Firstly, ω_2 in Minkowski spacetime defines a pure state whereas the restriction to the Rindler wedge makes it a mixed state. Secondly, as the Unruh effect is a consequence of mathematical definitions it does not need an experimental verification by its own. Although it can be helpful to interpret experimental results as seen by a Rindler observer and not an inertial Minkowski observer. Moreover the temperature seen by such an Rindler observer is—at least for macroscopic objects and usual accelerations—far from being detectable: e.g. a moving object with acceleration $a = 5\,ms^{-2}$ would be seen to be in a thermal bath with temperature $T \sim 10^{-19}K$. Thus, this effect could become relevant for linear particle accelerators, because one has to keep in mind that the above computation is valid only for linear accelerations. Ultimately, harder computations show that the Unruh effect is even valid in interacting theories and not only for scalar particles [3].

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