MODULI SPACES OF CALABI-YAU COMPACTIFICATIONS

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Abstract. We review properties of Calabi-Yau compactifications of string theory, M-theory and F-theory.

1. Introduction

Calabi-Yau compactifications have played an important role in studying supersymmetric vacua of string theory. More recently they have also featured in compactifications of M-theory and F-theory. The moduli of the Calabi-Yau metric appear in the four-dimensional effective Lagrangian as scalar fields which are flat directions of the effective potential. In these lectures we focus on their moduli spaces and the corresponding couplings in in the low energy effective action and neglect other parts of the massless spectrum in our considerations.

2. A short story about string theory, F-theory and M-theory

2.1. STRING THEORY

In string theory the fundamental objects are one-dimensional strings which, as they move in time, sweep out a 2-dimensional worldsheet Σ [1]. Strings can be open or closed and their worldsheet is embedded in a D-dimensional target space of Minkowskian signature which is identified with spacetime. States in the target space appear as eigenmodes of the string and their scattering amplitudes are described by appropriate scattering amplitudes of strings. These scattering amplitudes are built from a fundamental vertex, which for closed strings is depicted in Fig. 1. It represents the splitting of a string or the joining of two strings and the strength of this interaction is governed by a dimensionless string coupling constant $g_{\rm s}$. Out of the funda-

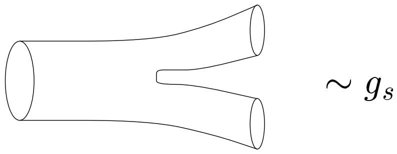

Figure 1. The fundamental closed string vertex.

mental vertex one composes all possible closed string scattering amplitudes $\mathcal{A}$, for example the four-point amplitude shown in Fig. 2. The expansion

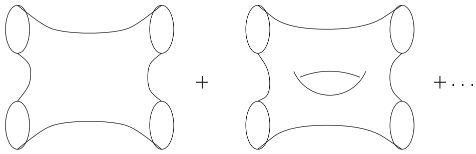

Figure 2. The perturbative expansion of string scattering amplitudes. The order of $g_{\rm s}$ is governed by the number of holes in the world sheet.

in the topology of the Riemann surface (i.e. the number of holes in the surface) coincides with a power series expansion in the string coupling constant formally written as

$$\mathcal{A} \;=\; \sum_{n=0}^{\infty} g_{\rm s}^{-\chi}\, \mathcal{A}^{(n)}\ , \tag{1}$$

where $\mathcal{A}^{(n)}$ is the scattering amplitude on a Riemann surface Σ of genus n and $\chi(\Sigma)$ is the Euler characteristic of the Riemann surface

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma R^{(2)} = 2 - 2n - b\ . \tag{2}$$

$R^{(2)}$ is the curvature on Σ and b the number of boundaries of the Riemann surface (for the four-point amplitude of Fig. 2 one has $b = 4$).[1]

In all string theories there is a massless scalar field ϕ called the dilaton which couples to $R^{(2)}$ and therefore its vacuum-expectation value determines the size of the string coupling; one finds [2, 1]

$$g_{\rm s} \;=\; e^{\langle\phi\rangle}\ . \tag{3}$$

[1]For open strings different diagrams contribute at the same order of the string loop expansion. See [1] for further details.

g_s is a free parameter since ϕ is a flat direction (a modulus) of the effective potential. String perturbation theory is defined in that region of the parameter space (which is also called the moduli space) where $g_s < 1$ and the tree-level amplitude (genus-0) is the dominant contribution with higher-loop amplitudes suppressed by higher powers of g_s. Until 1995 this was the only regime accessible in string theory.

Unitarity of the amplitudes imposes a restriction on the maximal number of spacetime dimensions and the spacetime spectrum. In these lectures we exclusively focus on string theories defined in spacetime supersymmetric backgrounds and all such string theories necessarily have $D \leq 10$. They are particularly simple in their maximal possible dimension $D = 10$ where one has only five consistent string theories: type-IIA, type-IIB, heterotic $E_8 \times E_8$ (HE8), heterotic SO(32) (HSO) and the type-I SO(32) string.[2] The first two have 32 supercharges ($q = 32$) while the other three string theories all have 16 supercharges ($q = 16$). The massless spectrum of all 5 theories is summarized as follows:

type	q	bosonic spectrum	
IIA	32	NS-NS	$G_{\mu\nu}$, $B_{\mu\nu}$, ϕ
		R-R	V_μ, $C_{\mu\nu\rho}$
IIB	32	NS-NS	$G_{\mu\nu}$, $B_{\mu\nu}$, ϕ
		R-R	$c^*_{\mu\nu\rho\sigma}$, $B'_{\mu\nu}$, ϕ'
HE8	16		$G_{\mu\nu}$, $B_{\mu\nu}$, ϕ
			A_μ in adjoint of $E_8 \times E_8$
HSO	16		$G_{\mu\nu}$, $B_{\mu\nu}$, ϕ
			A_μ in adjoint of SO(32)
I	16	NS-NS	$G_{\mu\nu}$, ϕ
		R-R	$B_{\mu\nu}$
		open string	A_μ in adjoint of SO(32)

[2] For closed strings an additional constraint arises from the requirement of modular invariance of one-loop amplitudes which results in an anomaly-free spectrum of the corresponding low-energy effective theory [3]. For open strings anomaly cancellation is a consequence of the absence of tadpole diagrams [1].

2.2. CALABI-YAU COMPACTIFICATIONS

String theories in backgrounds with $D < 10$ can be constructed either as geometrical compactifications or by specifying an appropriate conformal field theory on the string worldsheet. In these lectures we only discuss the geometrical constructions, that is we choose to compactify the 10-dimensional Minkowski space $\mathcal{M}^{(10)}$ on a compact manifold K

$$\mathcal{M}^{(10)} = \mathcal{M}^{(D)} \times K^{(10-D)} . \tag{4}$$

Consistency requires K to be Ricci-flat while preserving some supercharges implies a constraint on the holonomy group of K. One finds that part of the supersymmetry is preserved if one compactifies on Calabi-Yau manifolds Y_n . These are complex n-dimensional Ricci-flat compact Kähler manifolds with holonomy group $SU(n)$. One has

n	manifold	χ	SUSY preserved
1	Torus T^2	0	all
2	K3-surface	24	$q/2$
3	Calabi-Yau threefold Y_3	not fixed	$q/4$
4	Calabi-Yau fourfold Y_4	not fixed	$q/8$

χ is the Euler number of the Calabi-Yau manifold and we see that for $n = 1, 2$ they are topologically unique. More properties of Calabi–Yau manifolds are assembled in the Appendix.

2.3. STRING DUALITIES

The past few years have shown [4] that various string theories are interrelated by a complicated 'web' of duality relations . One distinguishes perturbative and non-perturbative dualities. Perturbative dualities already hold at weak string coupling and the map which identifies the perturbative theories does not involve the dilaton. An example is T-duality [5] which identifies different (perturbative) regions of toroidal compactifications. On the other hand non-perturbative dualities identify regions of the parameter space which are not simultaneously at weak coupling and the duality map involves the dilaton in a nontrivial way. Such non-perturbative dualities are of utmost importance since they map the strong-coupling region of a given (string) theory to the weak-coupling region of a dual theory where perturbative methods are applicable and hence the strong-coupling limit gets (at least partially) under quantitative control. The non-perturbative dualities cannot be proven at present. Rather their validity has only been checked for quantities or couplings which do not receive quantum corrections. Such couplings do exist in supersymmetric (string) theories and it is precisely for this reason that supersymmetry has played such an important (technical) role in establishing non-perturbative dualities.

Let A and B be two perturbatively distinct string theories each with its own string coupling g_A and g_B, respectively. However, it is possible that once all quantum corrections (including the non-perturbative corrections) are taken into account A and B are equivalent as quantum theories and one has $A \equiv B$. This situation can occur in two different ways: The strong-coupling limit of A is mapped to the weak coupling limit of B or in other words $g_A \sim g_B^{-1}$. Along with this strong-weak coupling relation goes a map of the elementary excitations of theory A to the non-perturbative, solitonic excitations of theory B and vice versa. Some of these solitonic excitations have a description in string theory as open strings with Dirichlet boundary conditions ending on a fixed spatial p-dimensional hyper-plane – a Dp-brane [6]. Such Dp-branes must be regarded as dynamical objects with degrees of freedom induced by the attached open strings. A careful analysis shows that the corresponding states in spacetime are not part of the perturbative spectrum but rather correspond to non-perturbative solitonic type excitations[3]. It is precisely these states which dramatically affect the properties of string theory in its non-perturbative regime. The theories A and B are called S–dual and one also refers to this situation as a 'string–string duality'.

There is a variant of the above situation where the dilaton of theory A is not mapped to the dilaton of theory B but rather to any of the other perturbative moduli R_B of theory B. In this case one has the identifications $\phi_A \sim R_B$, $\phi_B \sim R_A$, or in other words the strong-coupling limit of A is independent of g_B. Thus the strong-coupling limit of A is again controlled by the perturbative regime of theory B and hence accessible in perturbation theory (at least in principle). The known S–dualities are summarized in the following table

D	q	duality
10	16	HSO $\sim$ I
6	16	IIA/$K3 \sim$ H/T^4
4	8	IIA/$Y_3 \sim$ H/$K3 \times T^2$
2	4	IIA/$Y_4 \sim$ H/$Y_3 \times T^2$

Another situation is encountered when the strong-coupling limit of a theory A is controlled not by a distinct theory B, but rather by a different

[3]They are non-perturbative in that their mass (or rather their tension for higher-dimensional D-branes) goes to infinity in the weak coupling limit $g_s \to 0$.

perturbative region of the same theory A. That is, the strong-coupling regime of A has an alternative weakly-coupled description within the same theory A but in terms of a different set of elementary degrees of freedom. For such a self-duality to hold the theory A has to have a nontrivial (discrete) symmetry Γ_S which maps the strong-coupling region to a region of weak coupling and simultaneously the different elementary excitations onto each other. An example of this situation is believed to be the type-IIB string in $D = 10$ which is conjectured to have $\Gamma_S = \mathrm{SL}(2, \mathbf{Z})$ [7, 8]. This exact symmetry predicts an infinite number of equivalent weakly coupled type-IIB strings which carry R-R charge; such strings have indeed been identified as appropriate D-strings [9, 10]. For later reference we need to record that the $\mathrm{SL}(2, \mathbf{Z})$ acts on the complex scalar

$$\tau \equiv \phi' + ie^{-\phi} \tag{5}$$

which is composed out of the two scalars ϕ, ϕ' of IIB. The transformation is

$$\tau \mapsto \tau' = \frac{a\tau + b}{c\tau + d} \ , \tag{6}$$

where $ad - bc = 1, \ a, b, c, d \in \mathbf{Z}$.

2.4. F-THEORY

The exact $\mathrm{SL}(2, \mathbf{Z})$ symmetry of IIB string theory inspired Vafa to construct non-perturbative string backgrounds where the dilaton is not constant [11]. More precisely he proposed to compactify IIB on the base B_n of an elliptically fibred Calabi-Yau manifold Y_{n+1}. Elliptically fibred Calabi-Yau manifolds are locally a fibre bundle with a two-torus T^2 fibred over the base B_n but on over codimension one loci the torus can degenerate. As a consequence nontrivial closed loops on B_n can induce a $\mathrm{SL}(2, \mathbf{Z})$ transformation of the complex structure of the fibre. The complex dilaton τ_{IIB} of IIB is identified with the complex structure modulus of the torus

$$\tau_{IIB} \equiv \tau_{T^2} \tag{7}$$

and thus is not constant over the compactification manifold B_n but can have $\mathrm{SL}(2, \mathbf{Z})$ monodromy [12]. It is precisely this fact which results in nontrivial (non-perturbative) string vacua inaccessible in string perturbation theory. Such vacua are termed F-theory compactifications on elliptic Calabi–Yau manifolds and each such compactification is conjectured to capture part of the non-perturbative physics of an appropriate string vacuum.[4] One finds

[4]One can alternatively define F-theory as a type IIB in a background of D7-branes or as a particular decompactification limit of type IIA [13].

[11, 14]

D	q	duality
8	16	$\mathrm{F}/K3 \sim \mathrm{H}/T^2$
6	8	$\mathrm{F}/Y_3 \sim \mathrm{H}/K3$
4	4	$\mathrm{F}/Y_4 \sim \mathrm{H}/Y_3$

2.5. M-THEORY

The various dualities discussed so far relate different perturbative string theories. In these cases the strong-coupling limit of a given string theory is controlled by another (or the same) perturbative string theory. However, not all strong-coupling limits are of this type. Instead it is possible that the strong-coupling limit of a given theory is something entirely new, not any of the other string theories [8]. This was first proposed for the strong-coupling limit of the type-IIA theory in $D = 10$. The Kaluza-Klein BPS-spectrum of this theory obeys (in the string frame)

$$M^{\mathrm{KK}} \sim \frac{|n|}{g_{\mathrm{s}}} , \tag{8}$$

where n is an arbitrary integer. These KK-states are not part of the perturbative type-IIA spectrum since they become heavy in the weak-coupling limit $g_{\mathrm{s}} \to 0$. However, in the strong-coupling limit $g_{\mathrm{s}} \to \infty$ they become light and can no longer be neglected in the effective theory. This infinite number of light states (which can be identified with D-particles of type-IIA string theory, or extremal black holes of IIA supergravity) signals that the theory effectively decompactifies with the radius R_{11} of the extra dimension being the string coupling constant

$$R_{11} \sim g_{\mathrm{s}}^{\frac{2}{3}} . \tag{9}$$

Supersymmetry is unbroken in this limit and hence the KK-states assemble in supermultiplets of the 11-dimensional supergravity. In particular the massless multiplet contains as bosonic components the 11-dimensional metric G_{MN} and a 3-form A_{MNP}. Since there is no string theory which has 11-dimensional supergravity as the low-energy limit, the strong-coupling limit of type-IIA string theory has to be a new theory, called M-theory, which cannot be a theory of (only) strings. Only limited amount of information is so far known about M-theory but it is supposed to capture all degrees of freedom of all known string theories, both at the perturbative and the non-perturbative level [15, 8, 16].[5]

[5]There exists a conjecture according to which the degrees of freedom of M-theory are captured in $\mathrm{U}(N)$ supersymmetric matrix models in the $N \to \infty$ limit [17]. These

Calabi-Yau compactifications of M-theory also correspond to particular non-perturbative limits of string theories. One has

D	q	duality
10	32	$\mathrm{M}/S^1 \sim \mathrm{IIA}$
7	16	$\mathrm{M}/K3 \sim \mathrm{H}/T^3$
5	8	$\mathrm{M}/Y_3 \sim \mathrm{H}/K3 \times S^1$
3	4	$\mathrm{M}/Y_4 \sim \mathrm{H}/Y_3 \times S^1$

2.6. THREE TRIPLETS OF DUALITIES

In the next section we will discuss some of these dualities in more detail and with particular emphasis on the map between the moduli spaces. We will organize our discussion by the number of supercharges and discuss the first two of the following three triplets of dualities:[6]

D	q	duality
8	16	$\mathrm{F}/K3 \sim \mathrm{H}/T^2$
7	16	$\mathrm{M}/K3 \sim \mathrm{H}/T^3$
6	16	$\mathrm{IIA}/K3 \sim \mathrm{H}/T^4$
6	8	$\mathrm{F}/Y_3 \sim \mathrm{H}/K3$
5	8	$\mathrm{M}/Y_3 \sim \mathrm{H}/K3 \times S^1$
4	8	$\mathrm{IIA}/Y_3 \sim \mathrm{H}/K3 \times T^2$
4	4	$\mathrm{F}/Y_4 \sim \mathrm{H}/Y_3$
3	4	$\mathrm{M}/Y_4 \sim \mathrm{H}/Y_3 \times S^1$
2	4	$\mathrm{IIA}/Y_4 \sim \mathrm{H}/Y_3 \times T^2$

3. The $q = 16$ triplet

Let us first discuss toroidal compactifications of the heterotic string H/T^n where $n = 10 - D$. The massless multiplets are the gravitational multiplet GR containing the spacetime metric $G_{\mu\nu}$ an antisymmetric tensor $B_{\mu\nu}$, n Abelian graviphotons γ_μ and a real scalar ϕ. The second massless multiplet is the vector multiplet V which contains a (non-Abelian) vector A_μ and n

matrix models have been known for some time [18] and were also known to describe supermembranes [19] in the light-cone gauge [20]. The same quantum-mechanical models describe the short-distance dynamics of N D-particles, caused by the exchange of open strings [10]. For a review see, for example, [21].

[6] The case $q = 4$ is still under active investigation and therefore its discussion is postponed to some later occasion.

real scalars Z in the adjoint representation of the gauge group G

$$\begin{aligned} GR: &\quad (G_{\mu\nu}, B_{\mu\nu}, n \times \gamma_\mu, \phi) \ , \\ V: &\quad (A_\mu, n \times Z) \ , \qquad\qquad \mu = 0, \ldots, D-1 \ . \end{aligned}$$

The scalars in the Cartan-subalgebra of G are flat directions of the effective potential and parameterize the Coulomb-branch of the theory where G is broken to its maximal Abelian subgroup $G \to U(1)^r$ ($r = \mathrm{rank}(G)$). In the heterotic string one has $r = 16$ and therefore a massless spectrum of

$$1\, GR \ + \ (16+n)\, V \ . \tag{10}$$

The moduli space spanned by the scalars in the Cartan-subalgebra is given by [22]

$$\mathcal{M}_{H/T^n} = R^+ \times \left. \frac{SO(16+n,n)}{SO(16+n) \times SO(n)} \right/ \Gamma_T \ , \tag{11}$$

where Γ_T is the T-duality group

$$\Gamma_T = SO(16+n, n, \mathbf{Z}) \ . \tag{12}$$

The factor R^+ is spanned by the dilaton ϕ (the scalar in the gravitational multiplet) and supersymmetry does not allow any mixing with the other moduli. At special points in this moduli space the gauge group is enhanced to non-Abelian subgroups of $E_8 \times E_8$ or $SO(32)$.

Let us now turn to type IIA compactified on K3 or IIA/K3 for short. These theories live in $D = 6$ and the massless spectrum is determined by the zero modes of the Laplacian on K3. Some details are collected in the Appendix or better in ref. [23]. One finds in the NS-NS sector the graviton $G_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$, 20 scalars from the $(1,1)$-deformations of the Calabi-Yau metric $\delta G_{i\bar{j}}$, 38 scalars from the deformations of the complex structure δG_{ij}, 20 scalars from the $(1,1)$-forms $B_{i\bar{j}}$, 2 scalars from the $(2,0)$ and $(0,2)$-forms B_{ij} and the dilaton ϕ. These are altogether 81 scalars in the NS-NS sector. In the R-R sector one has a vector A_μ, a three-form $A_{\mu\nu\rho}$, 20 vectors $A_{\mu i\bar{j}}$, and 2 vectors $A_{\mu ij}$. In $D = 6$ a three-form is Poincare dual to a vector

$$\epsilon^{\mu_1 \cdots \mu_6} \partial_{\mu_1} A_{\mu_2\mu_3\mu_4} \sim \partial^{\mu_5} A^{\mu_6} \ , \qquad \text{or} \qquad dA_3 \sim *dA_1 \ , \tag{13}$$

so that there are altogether 24 gauge fields in the R-R sector. These fields nicely assemble into $1\, GR + 20\, V$ with a moduli space [24, 23]

$$\mathcal{M}_{IIA} = R^+ \times \frac{SO(20,4)}{SO(20) \times SO(4)} / \Gamma_T \ , \tag{14}$$

where

$$\Gamma_T = SO\,(20, 4, \mathbf{Z}) \ , \tag{15}$$

and the R^+-factor is again spanned by the field in the GR multiplet – the type IIA dilaton ϕ_{IIA}.

The string theories IIA/K3 and H/T^4 are conjectured to be S–dual [25, 7, 8, 26, 27]. Both theories have the same representation of supersymmetry with exactly the same massless spectrum. Furthermore, from (11), (14) one learns that also the moduli spaces (including the discrete identifications Γ_T) of the two string compactifications coincide

$$\mathcal{M}_{H/T^4} = \mathcal{M}_{IIA} \ . \tag{16}$$

The effective actions of the two perturbative theories agree if one identifies [8]

$$\begin{aligned} \phi_{\rm H} &= -\phi_{\rm IIA} \ , \\ H_{\rm H} &= e^{-2\phi_{\rm IIA}} * H_{\rm IIA} \ , \\ (g_{\mu\nu})_{\rm H} &= e^{-2\phi_{\rm IIA}} (g_{\mu\nu})_{\rm IIA} \ , \end{aligned} \tag{17}$$

where H is the field strength of the antisymmetric tensor. The first equation in (17) implies a strong-weak coupling relation while the second is the equivalent of an electric-magnetic duality. Further evidence for this S-duality arises from the observation that the zero modes in a solitonic string background of the type-IIA theory compactified on $K3$ have the same structure as the Kaluza–Klein modes of the heterotic string compactified on T^4 [26, 27].

The non-Abelian gauge symmetry enhancement is a simple Higgs mechanism in the heterotic vacuum. In the type IIA vacuum it is more intriguing and related to the singularities of the K3. Whenever an effective theory becomes singular at special points (or submanifolds) of the moduli space it signals the breakdown of the effective description. Heavy modes can become light and should no longer be excluded from the low energy effective theory. This is precisely what happens at the orbifold singularities of K3 where 2-cycles collapse. A D2-brane can wrap around such a 2-cycle generating a non-Abelian gauge boson. These singularities follow an A-D-E classification and thus the corresponding gauge bosons can be mapped to the gauge bosons of the heterotic string.

Let us now turn to the next duality in one dimension higher $D = 7$. On the heterotic side of the previous duality it is simple to decompactify one dimension. On the type IIA side this is impossible for K3 but recall that the strong coupling limit of the ten-dimensional type IIA string is governed by a theory in one dimension higher, M-theory. Thus one is led to consider $M/K3$ as the possible dual of H/T^3.

The massless spectrum of $M/K3$ contains the 7-dimensional spacetime metric $G_{\mu\nu}$, 58 deformations of the Calabi-Yau metric $\delta G_{i\bar{j}}, \delta G_{ij}$, a 3-form $A_{\mu\nu\rho}$ and 22 vectors $A_{\mu i\bar{j}}, A_{\mu ij}$. Note that there is no dilaton and no antisymmetric tensor $B_{\mu\nu}$ in this compactification. However, in $D = 7$ a 3-form is dual to an antisymmetric tensor

$$dA_3 = *dB_2 \ . \tag{18}$$

The massless fields of M/K3 assemble into $1\,GR$ including $(G_{\mu\nu}, B_{\mu\nu}, 3A_\mu, \phi)$ and $19\,V$ including $(A_\mu, 3Z)$. The moduli space is determined by the moduli space of K3-surfaces [23]

$$\mathcal{M}_{M/K3} = R^+ \times \frac{SO\,(19,3)}{SO\,(19) \times SO\,(3)} \Big/ SO\,(19,3,\mathbf{Z}) \ , \tag{19}$$

where the R^+-factor is spanned by the ϕ in the gravitational multiplet which is related to the volume of K3. From eqs. (11), (19) we see

$$\mathcal{M}_{M/K3} = \mathcal{M}_{H/T^3} \ , \tag{20}$$

including the discrete identifications. A more detailed comparison of the respective effective actions [8] reveals that the 7-dimensional heterotic string coupling g_H^7 is related to the volume of K3 measured in the 11-dimensional M-theory metric by

$$\left(g_H^7\right)^{4/3} = Vol_M\,(K3) \ . \tag{21}$$

g_H^7 in turn can be related to the heterotic string couplings in $D = 6, 10$ via

$$\frac{1}{\left(g_H^6\right)^2} = \frac{R}{\left(g_H^7\right)^2} = \frac{Vol\,(T^4)}{\left(g_H^{10}\right)^2} \ , \tag{22}$$

where R is the radius of 7th dimension measured in the 7-dimensional string metric. The low energy description of M-theory in terms of 11-dimensional supergravity is valid for large $Vol_M\,(K3)$. From eq. (21) one infers that in this limit the heterotic string becomes strongly coupled.

The non-Abelian gauge symmetry enhancement in M/K3 has the same explanation as before: a shrinking 2-cycle of K3 corresponds to a massless gauge boson on the heterotic side.

We already discussed F-theory compactifications in section 2.4 where we defined them IIB compactifications on the base of an elliptic Calabi-Yau manifold. Let us also relate them to M-theory compactifications. Consider M/T^2 which is the strong coupling limit of IIA/S^1. The latter theory is

T-dual to IIB/S^1 with the following relation of parameters (measured in the M-theory metric)

$$g_{IIA} = R_{11}^{3/2}, \qquad g_{IIB} = \frac{R_{11}}{R_{10}}, \qquad R_{IIB} = \frac{1}{\sqrt{R_{11}}R_{10}} \ . \tag{23}$$

Thus we can view 10-dimensional IIB theory as the following limit

$$IIB \ \sim \lim_{R_{10},R_{11}\to 0} M/T^2 \qquad \text{with } \ g_{IIB} \ \text{ fixed} \ . \tag{24}$$

Thus the size of T^2 shrinks but the complex structure $\tau_{T^2} = \tau_{IIB}$ is kept finite (c.f. (7)).

With this relation in mind one can employ what is called the adiabatic argument [28]. Consider the compactification $IIB/B_n \times S^1$. By the previous argument this theory is related to $M/B_n \times T^2$. For large B_n the manifold $B_n \times T^2$ is locally the same as an elliptic Calabi-Yau Y_{n+1} and thus adiabatically one has

$$F/Y_{n+1} := IIB/B_n = \lim_{T^2\to 0} M/Y_{n+1} \ . \tag{25}$$

This can be immediately related to T^2-compactifications of the heterotic string. We already established $M/K3 \sim H/T^3$ with $Vol_M\,(K3) = (g_H^7)^{4/3}$. For an elliptic K3 and using the adiabatic argument this implies

$$\left(g_H^7\right)^{4/3} = \left(\frac{g_H^8}{\sqrt{R_8}}\right)^{4/3} = Vol_M(B) \cdot Vol(T^2) \ . \tag{26}$$

Thus a shrinking T^2 corresponds to the decompactification $R_8 \to \infty$ and therefore

$$F/K3 \sim \lim_{T^2\to 0} M/K3 \sim H/T^2 \ . \tag{27}$$

Or in words: F-theory compactified on an elliptic $K3$ yields an 8-dimensional vacuum with 16 supercharges which is dual to the heterotic string compactified on T^2 [11, 29].

The previous 'back-of-an-envelope' argument can be made more precise [11, 30]. The moduli space of elliptic K3's with a zero size fibre is [30]

$$\mathcal{M}_{F/K3} \ = \ R^+ \times \frac{SO\,(18,2)}{SO\,(18) \times SO\,(2)} \Big/ SO\,(18,2,\mathbf{Z}) \ , \tag{28}$$

which coincides with the moduli space of H/T^2 (c.f. (11)). The R^+-factor is spanned by the volume of the base B which for elliptic K3 necessarily is a $\mathbf{P}^1$.

4. The $q = 8$ triplets

The important new feature for string vacua with $q = 8$ is the fact that the massless spectrum and the gauge group is no longer uniquely fixed. Let us again first discuss the three heterotic theories $H/K3 \times T^{0,1,2}$ and then the corresponding dualities.

The massless multiplets in $D = 6, q = 8$ are the gravitational multiplet which contains the metric and a selfdual antisymmetric tensor $B^+_{\mu\nu}$, the vector multiplet V which only contains a vector and no scalars,[7] the tensor multiplet T containing an anti-selfdual antisymmetric tensor $B^-_{\mu\nu}$ and a real scalar ϕ and the hypermultiplets H featuring 4 real scalars q

$$\begin{aligned} GR: &\quad (G_{\mu\nu}, B^+_{\mu\nu}) \\ V: &\quad (A_\mu) \\ T: &\quad \left(B^-_{\mu\nu}, \phi\right) \\ H: &\quad (4q) \ . \end{aligned}$$

In order to preserve 8 supercharges the compactification manifold must be a K3 but in addition the vector bundle on K3 has to be holomorphic and stable, i.e. [1]

$$F_{ij} = F_{\bar{i}\bar{j}} = 0 = g_{i\bar{j}} F^{i\bar{j}} \ . \tag{29}$$

On K3 these conditions coincide with the instanton condition $F = \tilde{F}$.

In addition the chiral fermions in $D = 6$ lead to gauge and gravitational anomalies. The anomalies cancel if the following conditions are satisfied [31, 32]

- the number of hypermultiplets n_H, vector multiplets n_V and tensor multiplets n_T satisfy

$$n_H - n_V + 29 n_T - 273 = 0 \ . \tag{30}$$

- A Green-Schwarz anomaly cancellation mechanism can be employed with a modified 2-form field strength

$$H = dB + \omega_L - \sum_a v_a \, \omega^a_{YM} \ , \tag{31}$$

  where $\omega_L (\omega^a_{YM})$ are gravitational (Yang-Mills) Chern-Simons terms and v_a some numerical coefficients. The modified definition of H implies the consistency condition

$$0 = \int_{K3} dH = \int_{K3} trR \wedge R - \sum_a \int_{K3} tr\,(F \wedge F)_a = 24 - \sum_a n_a \ , \tag{32}$$

[7]Thus there is no Coulomb branch in $D = 6$.

where n_a is the instanton number. Thus there necessarily has to be a a non-trivial instanton background on K3.

In perturbative heterotic vacua there is only one dilaton and one antisymmetric tensor and thus one always has $n_T = 1$. The moduli space for this class of vacua reads

$$\mathcal{M} = R^+ \times \mathcal{M}_H \ , \tag{33}$$

where R^+ is the factor spanned by the scalar in the tensor multiplet (the heterotic dilaton) while $\mathcal{M}_H$ is the moduli space spanned by the scalars in the hypermultiplets. It includes the 80 moduli of K3 and the moduli of Yang-Mills instantons on K3 (which are parameterized by their size, position on K3 and orientation in the gauge group G). Supersymmetry requires $\mathcal{M}_H$ to be a quaternionic manifold. For $n_T \neq 1$ (which can occur in non-perturbative vacua of the heterotic string) one finds [33]

$$\mathcal{M} = \frac{O\,(1, n_T)}{O\,(n_T)} \times \mathcal{M}_H \ . \tag{34}$$

$\mathcal{M}_H$ has singularities when the size ρ of an instanton shrinks $\rho \to 0$ [34]. For the $SO(32)$ heterotic string these singularities are caused by non-perturbative gauge bosons becoming massless [35]. For k instantons shrinking at the same point on K3 the perturbative gauge group $SO(32)$ is enhanced beyond the perturbatively allowed rank to

$$G_{NP} = SO\,(32) \times Sp\,(k) \tag{35}$$

with an additional hypermultiplet in the $(\mathbf{32}, \mathbf{2k})$ representation of G_{NP}.

For the $E_8 \times E_8$ heterotic string a different explanation is employed. A small instanton is associated with a five-brane of M-theory (more precisely of $M/K3 \times S^1/\mathbf{Z}_2$) with additional tensor multiplets living on the worldvolume of the five-brane. In this case one has $n_T \geq 1$ and a non-perturbatively different situation compared to the heterotic $SO(32)$ case [36, 14, 32, 30].

Next we turn to $H/K3 \times S^1$ in $D = 5$. The massless multiplets in this case are:

$$\begin{aligned} GR: &\quad (G_{\mu\nu}, A_\mu) \\ V: &\quad (A_\mu, Z) \\ T: &\quad (B_{\mu\nu}, \phi) \\ H: &\quad (4q)\ . \end{aligned}$$

In $D = 5$ an antisymmetric tensor is dual to a vector $dB_2 \sim *dA_1$ and thus the tensor multiplet is dual to a vector multiplet $T \sim V$. Furthermore, due to the presence of the scalar in the vector multiplet a Coulomb branch exists and the moduli space has the form

$$\mathcal{M} = \mathcal{M}_H \times \mathcal{M}_V \ . \tag{36}$$

Supersymmetry dictates that locally the moduli space is a direct product. Since the hypermultiplets are the same as in $D = 6$ also $\mathcal{M}_H$ is unchanged. $\mathcal{M}_V$ is known at the tree level and one has [37]

$$\mathcal{M}_V^{(0)} = R^+ \times \frac{SO\,(1, r+1)}{SO\,(r+1)} \ , \tag{37}$$

where $r = \text{rank}(G)$ and the extra vector multiplet corresponds to the radius of S^1. At the quantum level only $\mathcal{M}_V$ is corrected in string theory since the dilaton is part of a tensor multiplet (or the dual vector multiplet). The corrections are such that there is only a perturbative correction at 1-loop and non-perturbative corrections [38, 37]. This non-renormalization theorem is dictated by supersymmetry.

Finally, we consider $H/K3 \times T^2$ which has $D = 4, q = 8\,(N = 2)$. The massless multiplets in this case are

$$\begin{aligned} GR: &\quad (G_{\mu\nu}, A_\mu) \\ V: &\quad (A_\mu, Z) \\ H: &\quad (4q) \\ VT: &\quad (B_{\mu\nu}, A_\mu, \phi) \ . \end{aligned}$$

where Z now is a complex scalar in the adjoint representation of G. VT is the vector-tensor multiplet which contains the heterotic dilaton ϕ [39]. In $D = 4$ the duality relates $dB_2 = *da_0$ and thus the vector-tensor multiplet is dual to an (Abelian) vector multiplet $VT \sim V$. The moduli space reads

$$\mathcal{M} = \mathcal{M}_H \times \mathcal{M}_V \ , \tag{38}$$

where $\mathcal{M}_H$ is the same (quaternionic) space as in $D = 6, 5$ while $\mathcal{M}_V$ is a special Kähler manifold [40]. Special Kähler manifolds have a Kähler metric

$$G_{i\bar{j}} = \frac{\partial}{\partial Z^i}\frac{\partial}{\partial \bar{Z}^{\bar{j}}}\, K(Z, \bar{Z}) \ , \qquad i, j = 1, \ldots, n_V \ , \tag{39}$$

with a Kähler potential K determined by a holomorphic prepotential F

$$K = -\ln\left[X^I \bar{F}_I(\bar{X}) + \bar{X}^I F_I(X)\right] , \qquad I = 0, 1, \ldots, n_V \ , \tag{40}$$

where

$$F_I \equiv \frac{\partial F}{\partial X^I} \ , \qquad Z^i = \frac{X^i}{X^0} \ , \qquad F(\lambda X) = \lambda^2 F(X) \ . \tag{41}$$

The gauge kinetic terms have the following structure

$$\mathcal{L} = -\frac{1}{4} g_{IJ}^{-2} F_{\mu\nu}^{I} F_{\mu\nu}^{J} + \frac{\theta_{IJ}}{2\pi} F_{\mu\nu}^{I} \tilde{F}_{\mu\nu}^{J} + \ldots \tag{42}$$

where

$$\begin{aligned} g_{IJ}^{-2} &\sim \mathcal{N}_{IJ} - \bar{\mathcal{N}}_{IJ} \,, \qquad \theta_{IJ} \sim \mathcal{N}_{IJ} + \bar{\mathcal{N}}_{IJ} \,, \\ \mathcal{N}_{IJ} &= \frac{1}{4}\bar{F}_{IJ} - \frac{N_{IK} Z^K N_{JL} Z^L}{Z^K N_{KZ} Z^L} \,, \qquad N_{IJ} = \frac{1}{4}(F_{IJ} + \bar{F}_{IJ}) \,. \end{aligned} \tag{43}$$

The gauge group in $H/K3 \times T^2$ string vacua contains three $U(1)$ vector multiplets, two from T^2 (denoted by T, U) and the dual of the vector-tensor multiplet denoted by S. At the tree level the holomorphic prepotential is uniquely determined by [41, 39]

$$F^{(0)} = (X^0)^2 \cdot S\,(TU - \Phi^a \Phi^a) \ , \tag{44}$$

where Φ^a are the $U(1)$ multiplets which span the Cartan subalgebra of the heterotic gauge group. This prepotential corresponds to the moduli space

$$\mathcal{M}_V^{(0)} = \frac{SU\,(1,1)}{U\,(1)} \times \frac{SO\,(2, r+2)}{SO\,(r+2) \times SO\,(2)} \ , \tag{45}$$

where the first factor is spanned by the dilaton multiplet. Quantum corrections induce only one-loop and non-perturbative corrections to F [39]

$$F = F^{(0)} + F^{(1)} + F^{(NP)} \ . \tag{46}$$

The quaternionic moduli space of the hypermultiplets is the same as in D=6,5 and receives no quantum corrections since the dilaton cannot couple to $\mathcal{M}_H$.

Let is summarize the moduli spaces which appear in K3 compactifications of the heterotic string

$$\begin{aligned} D = 6: &\quad \mathcal{M} = \mathcal{M}_H \times R^+ \\ D = 5: &\quad \mathcal{M} = \mathcal{M}_H \times \left(R^+ \times \tfrac{SO(1,r+1)}{SO(r+1)} + q.c.\right) \\ D = 4: &\quad \mathcal{M} = \mathcal{M}_H \times \left(\tfrac{SU(1,1)}{U(1)} \times \tfrac{SO(2,r+2)}{SO(r+2)\times SO(2)} + q.c.\right) \ , \end{aligned}$$

where $q.c$ indicates that there are quantum corrections.[8]

Let us now turn to the discussion of the dual vacua. In $D = 4$ the dual of the heterotic vacua are vacua constructed as IIA string theory compactified

[8] These quantum corrections are not additive; rather they generically destroy the tree level factorization.

on Calabi-Yau threefolds Y_3, IIA/Y_3 for short [42]. In the NS-NS sector the massless spectrum contains $G_{\mu\nu}, B_{\mu\nu}, \phi$, the deformations of the Calabi-Yau metric and the antisymmetric tensor on the Calabi-Yau manifold. The deformations of the metric are given by the deformations of the Kähler form and the deformations of the complex structure. The former can be expanded in terms of harmonic $(1,1)$-forms $e^A_{i\bar{j}}$ on Y_3 [43]

$$\delta G_{i\bar{j}} = \sum_{A=1}^{h_{1,1}} M^A(x)\, e^A_{i\bar{j}} \ . \tag{47}$$

The deformations of complex structure are expanded as

$$\delta G_{ij} = \sum_{\alpha=1}^{h_{1,2}} q^\alpha(x) b^\alpha_{ij} \ , \tag{48}$$

where $b^\alpha_{ij} = \Omega_j^{\bar{j}\bar{k}} \chi_{i\bar{j}\bar{k}}$ and $\chi_{i\bar{j}\bar{k}}$ are the $(1,2)$-forms, Ω_{ijk} is the $(3,0)$-form. For the antisymmetric tensor one has

$$B_{i\bar{j}} = \sum_{A=1}^{h_{1,1}} B^A(x)\, e^A_{i\bar{j}} \ . \tag{49}$$

In the R-R sector one finds

$$A_{ijk} = C^0(x)\Omega_{ijk} \ , \quad A_{i\bar{j}\bar{k}} = \sum_{\alpha=1}^{h_{1,2}} C^\alpha(x)\chi^\alpha_{i\bar{j}\bar{k}} \ , \quad A_{\mu i\bar{j}} = \sum_{A=1}^{h_{1,1}} A^A_\mu(x) e^A_{i\bar{j}} \ , \tag{50}$$

where C^0, C^α are complex.

The massless fields assemble in the following $N=2$ supermultiplets

$$\begin{array}{ll} GR: & (G_{\mu\nu}, A_\mu) \\ T: & (B_{\mu\nu}, \phi, C^0) \\ V: & (A^A_\mu, Z^A) \\ H: & (q^\alpha, C^\alpha) \end{array}$$

where $Z^A = M^A + iB^A$. Thus altogether one has a spectrum of

$$1GR + 1T + h_{1,1}V + h_{1,2}H \ . \tag{51}$$

Using the (Poincare) duality $T \sim H$ one can express the moduli space as

$$\mathcal{M} = \mathcal{M}_H \times \mathcal{M}_V \ . \tag{52}$$

$\mathcal{M}_V$ is not quantum corrected since the dilaton ϕ sits in T or H, respectively. This component of the moduli space is "known" in the sense that the prepotential obeys the following general structure [44, 45]

$$F = (X^0)^2 \Big[d_{ABC} Z^A Z^B Z^C + \chi\zeta(3) + \sum_{d_A} n_{d_A} Li_3(e^{-2\pi d_A Z^A}) \Big] \ , \qquad (53)$$

where

$$d_{ABC} = \int_{Y_3} e^A \wedge e^B \wedge e^C \ , \qquad Li_3\,(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^3} \quad . \qquad (54)$$

The d_{ABC} are the Calabi-Yau intersection numbers and the n_{d_A} are integers which count the number of rational curves on Y_3.

On the hypermultiplet side only $\mathcal{M}_H^{(0)}$ is known. One has [46]

$$\begin{aligned} \mathcal{L} \;&=\; e^{-2\phi} \left(-\frac{1}{2} R + 2\,(\partial\phi)^2 - G_{\alpha\beta} \partial q^\alpha \partial \bar{q}^\beta - \frac{1}{6} H^2_{\mu\nu\rho} \right) \qquad (55) \\ &+\; \epsilon^{\mu\tau\rho\sigma} H_{\mu\tau\rho} \left(C R^{-1} \partial_\sigma \bar{C} - \frac{1}{2} C R^{-1} \partial_\sigma N R^{-1}\,(C+\bar{C}) - C \leftrightarrow \bar{C} \right) \\ &-\; \frac{1}{2} \left(\partial_\mu C - \frac{1}{2}\,(C+\bar{C})\, R^{-1} \partial_\mu \mathcal{N} \right) R^{-1} \left(\partial_\mu \bar{C} - \frac{1}{2}\,(C+\bar{C})\, R^{-1} \partial_\mu \bar{\mathcal{N}} \right) . \end{aligned}$$

$G_{\alpha\beta}$ is a special Kähler metric and $R \equiv \mathrm{Re}\mathcal{N}$ so that both are determined by a holomorphic prepotential. The reason for this special feature is that in IIB compactifications one finds a massless spectrum

$$G + T + h_{1,2}\ V + h_{1,1}\ H \ . \qquad (56)$$

The role of $h_{1,1}$ and $h_{1,2}$ is exactly reversed compared to the IIA case. It is believed that for any given Y_3 there exists a mirror partner $\widetilde{Y}_3$ with the property [47, 48]

$$h_{1,1}(\widetilde{Y}_3) = h_{1,2}(Y_3) \ , \qquad h_{1,2}(\widetilde{Y}_3) = h_{1,1}(Y_3) \qquad (57)$$

such that the Euler number is reversed $\chi(Y_3) = -\chi(\widetilde{Y}_3)$. Thus, in string theory this mirror symmetry leads to a perturbative duality

$$IIB/\widetilde{Y}_3 \equiv IIA/Y_3 \ . \qquad (58)$$

In IIB compactifications the q^α reside in vector multiplets and thus one infers that they have to be coordinates on a special Kähler manifold. More generally, the duality implies a map (the c-map) between the moduli spaces which acts as [49]

$$c : \mathcal{M}_V \to \mathcal{M}_H^{(0)} \ . \qquad (59)$$

The quantum corrections to the hypermultiplet geometry are not fully known yet. Let us start our discussion with the case $h_{1,2} = 0$ so that only the (universal) tensor multiplet is present. A string 1-loop computation has been performed which determined the correction to the Einstein term [50]

$$\mathcal{L} = -\frac{1}{2}(e^{-2\phi} + \chi)R + \dots \ . \tag{60}$$

Then supersymmetry uniquely determines the corrections [51]

$$\begin{aligned}\mathcal{L} \;\; = \;\; & -\frac{1}{2}(e^{-2\phi} + \chi)(R - \frac{1}{6}H^2_{\mu\nu\rho}) + 2e^{-4\phi}(\chi + e^{-2\phi})^{-1}(\partial\phi)^2 \\ & -\partial_\mu C \partial_\mu \bar{C} + \epsilon^{\mu\tau\rho\sigma} H_{\mu\tau\rho}\left(C\partial_\sigma \bar{C} - \bar{C}\partial_\sigma C\right) \ . \end{aligned} \tag{61}$$

This Lagrangian has a perturbative Peccei-Quinn symmetry which originates from the fact that C appears in the R-R sector

$$C \to C + \text{const.} \ . \tag{62}$$

This symmetry is exact in string perturbation theory and forbids any higher loop correction. Thus the 1-loop Lagrangian is perturbatively exact and we have a “new” non-renormalization theorem. (Non-perturbative corrections do exist [52].)

The antisymmetric tensor in the Lagrangian (61) can be dualized to a scalar $\tilde{\phi}$ and in this dual basis the 1-loop corrected metric is found to be Kähler with a Kähler potential

$$K = -\ln\left(S + \bar{S} + 2\chi - C\bar{C}\right), \qquad S \equiv e^{-2\phi} + i\tilde{\phi} + C\bar{C} \ . \tag{63}$$

In this dual basis the metric appears to be corrected at all loops and does agree with the metric conjectured by Strominger [50].

For $h_{1,2} \neq 0$ the quantum corrections are not fully known. One does know [50, 51]

$$\begin{aligned}\mathcal{L} \;\; = \;\; & -\frac{1}{2}(e^{-2\phi} + \chi)R - \frac{1}{2}\left(e^{-2\phi} - \chi\right) G_{\alpha\beta}\partial q^\alpha \partial \bar{q}^\beta \\ & -2\chi\epsilon^{\mu\nu\rho\sigma} H_{\mu\nu\rho} V_\sigma + \dots \end{aligned} \tag{64}$$

where

$$V_\sigma \equiv \frac{\partial K}{\partial q^\alpha}\partial_\sigma q^\alpha - \frac{\partial K}{\partial \bar{q}^\alpha}\partial_\sigma \bar{q}^\alpha \ . \tag{65}$$

However, the $N = 2$ supersymmetric (i.e. quaternionic) completion of $\mathcal{L}$ is not completely known yet [53]. Nevertheless, the presence of $2(h_{1,2} + 1)$ continuous PQ-symmetries suggests that also for the case $h_{1,2} \neq 0$ the 1-loop correction is exact in perturbation theory and there is a perturbative non-renormalization theorem.

It is conjectured that IIA/Y_3 is dual to $H/K3 \times T^2$ [42]. In particular this implies that spectrum and the respective moduli spaces have to agree

$$\mathcal{M}_H^{IIA} = \mathcal{M}_H^{\text{Het}} \ , \qquad \mathcal{M}_V^{IIA} \equiv \mathcal{M}_V^{\text{Het}} \ . \tag{66}$$

There have been very little checks on $\mathcal{M}_H$ so far. The validity of this duality has been only been checked for $\mathcal{M}_V$ but for quite a number of dual string vacua [42, 54, 55]. One has to find

$$F^{IIA} = F^{\text{het}} \ , \tag{67}$$

which implies

$$d_{ABC} Z^A Z^B Z^C + \chi\zeta\,(3) + \sum_{d_A} n_{d_A} Li_3 = S\,(TU - \Phi^a\Phi^a) + F^{(1)} + F^{(NP)} \ . \tag{68}$$

This is a condition on the intersection numbers d_{ABC}. They have to obey

$$d_{SSS} = d_{SS\hat{A}} = 0 \ , \qquad \text{sign}(d_{S\hat{A}\hat{B}}) = (+,-,...,-) \ , \tag{69}$$

where $\hat{A}$ denotes all moduli except the dual of the heterotic dilaton. These conditions are the statement that the Calabi–Yau manifold has to be a $K3$-fibration [54, 28, 56]. That is, the Y_3 manifold is fibred over a $\mathbf{P^1}$ base with fibres that are $K3$ manifolds. The size of the $\mathbf{P^1}$ is parameterized by the modulus which is the type II dual of the heterotic dilaton. Over a finite number of points on the base, the fibre can degenerate to something other than $K3$ and such fibres are called singular. The other Kähler moduli are either moduli of the $K3$ fibre or of the singular fibres. In general one finds

$$\text{sign}(d_{S\hat{A}\hat{B}}) = (+,-,\ldots,-,0,\ldots,0) \ , \tag{70}$$

where the non-vanishing entries correspond to moduli from generic $K3$ fibres while the zeros arise from singular fibres. Since a $K3$ has at most 20 Kähler moduli the non-vanishing entries have to be less than 20. From eq. (68) one concludes that type II Calabi-Yau compactifications in the large radius limit can be the dual of perturbative heterotic vacua if they are $K3$-fibrations with all moduli corresponding to generic fibres. This class of type II vacua is automatically consistent with the heterotic bound on the rank of the gauge group. The $(1,1)$ moduli of singular fibres have no counterpart in perturbative heterotic vacua. If there were heterotic moduli with such couplings they would not couple properly to the (heterotic) dilaton and furthermore violate the bound on the rank of the gauge group. However, we already discussed the possibility that in $D = 6$ the gauge group can be non-perturbatively enhanced at singular points in the moduli space [35]. It was further shown that these non-perturbative gauge fields do not share the canonical coupling to the dilaton. Upon compactification to $D = 4$ the scalars of these non-perturbative vector multiplets couple precisely like type II moduli corresponding to singular fibres [57, 58].

Let us now turn to vacua of M/Y_3 (which have $D = 5, q = 8$). We expect this theory to be dual to $H/K3 \times S^1$ by the following chain of arguments. M/S^1 is the dual of IIA in $D = 10$. Thus compactifying both theories on Y_3 one expects the dual pair $M/Y_3 \times S^1 \sim IIA/Y_3$. However, the previous discussion also suggests $IIA/Y_3 \sim H/K3 \times T^2$ and thus one is lead to conjecture $M/Y_3 \sim H/K3 \times S^1$ [38, 37].[9]

The massless spectrum of M/Y_3 contains the metric $G_{\mu\nu}$, $h_{1,1}$ deformations of the Kähler form $\delta G_{i\bar{j}}$, $h_{1,2}$ (complex) deformations of the complex structure δG_{ij}, a three-form $A_{\mu\nu\rho}$, $h_{1,1}$ vectors $A_{\mu i\bar{j}}$, one complex scalar A_{ijk} and $h_{1,2}$ complex scalars $A_{ij\bar{k}}$. The duality in $D = 5$ relates the 3-form to a scalar

$$dA_3 = * d\tilde{\phi} \ , \tag{71}$$

and so altogether one has the spectrum

$$1GR + (h_{1,1} - 1)\, V + (h_{1,2} + 1)\, H \ , \tag{72}$$

where

$$\begin{array}{ll} GR: & (G_{\mu\nu}, A_\mu) \\ V: & (A_\mu, Z) \\ H: & (4q) \quad . \end{array}$$

Z is real and the volume of Y_3 resides not in a vector multiplet but rather in a hypermultiplet. (This accounts for the ± 1 in the counting.) The moduli space is again

$$\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H \ . \tag{73}$$

A more detailed comparison of the effective actions of M/Y_3 and $H/K3 \times S^1$ reveals

$$\frac{1}{(g_h^5)^2} = \frac{[Vol_M(\mathbf{P}_1)]^{\frac{3}{2}}}{[Vol_M(Y_3)]^{\frac{1}{2}}} = \frac{Vol_M(\mathbf{P}_1)}{[Vol_M(K3)]^{\frac{1}{2}}} \ , \tag{74}$$

where $\mathbf{P}_1$ is again the base of the K3-fibrations and the second equation used the adiabatic argument. From (74) we immediately infer that large $\mathbf{P}_1$ corresponds to weak heterotic coupling while a large K3 corresponds to strong heterotic coupling. Eq. (74) can also be 'derived' by 'fibering' the duality $M/K3 \sim H/T^3$ over $\mathbf{P}_1$. This implies $M/K3 \times \mathbf{P_1} \sim H/T^3 \times \mathbf{P}_1$

[9]For perturbative heterotic vacua one expects again that Y_3 is K3-fibred.

and using the adiabatic argument also $M/Y_3 = H/K3 \times S^1$. In terms of the couplings one has

$$\frac{1}{\left(g_h^5\right)^2} = \frac{Vol_H\left(\mathbf{P}_1\right)}{\left(g_h^7\right)^2} = \frac{Vol_M\left(\mathbf{P_1}\right)}{\left[Vol_M\left(K3\right)\right]^{\frac{1}{2}}}\ , \tag{75}$$

where we used $\left(g_h^7\right)^2 = \left[Vol_M\left(K3\right)\right]^{\frac{3}{2}}$.

Finally we turn to F-theory compactified on an elliptic Calabi–Yau threefold F/Y_3. Such vacua have $D = 6, q = 8$ and are conjectured to be dual to the heterotic string compactified on $K3$ [14]. Recall that via the adiabatic argument one has for any elliptic Calabi-Yau manifold

$$F/Y_n = \lim_{T^2 \to 0} M/Y_n\ . \tag{76}$$

For threefolds Y_3 one also has the duality $M/Y_3 = H/K3 \times S^1$ and thus from (74)

$$\frac{1}{\left(g_h^5\right)^2} = \frac{R_6}{\left(g_h^6\right)^2} = \frac{\left[Vol_M\left(\mathbf{P}_1\right)\right]^{\frac{3}{2}}}{\left[Vol_M\left(Y_3\right)\right]^{\frac{1}{2}}} = \frac{\left[Vol_M\left(\mathbf{P}_1\right)\right]^{\frac{3}{2}}}{\left[Vol_M\left(B_2\right)\right]^{\frac{1}{2}}\left[Vol_M\left(T^2\right)\right]^{\frac{1}{2}}}\ . \tag{77}$$

Thus the limit $T^2 \to 0$ sends $R_6 \to \infty$ with

$$\frac{1}{\left(g_h^6\right)^2} = \frac{\left[Vol_M\left(P_1\right)\right]^{3/2}}{\left[Vol_M\left(B_2\right)\right]^{\frac{1}{2}}}\ . \tag{78}$$

As before there is an alternative 'derivation' by fibering the $D = 8$ duality $F/K3 \sim H/T^2$ over $\mathbf{P}_1$. This gives $F/K3 \times \mathbf{P}_1 \sim H/T^2 \times \mathbf{P}_1$ and via the adiabatic argument $F/Y_3 \sim H/K3$.

The spectrum of F/Y_3 features

$$1GR + \left(h_{1,2}(Y_3) + 1\right) H + \left(h_{1,1}(B_2) - 1\right) T + n_V V \tag{79}$$

with B_2 being the base of the elliptic fibration. Since the gauge group G can be non-Abelian n_V is generically not determined. However, for the rank of G one has [14][10]

$$\mathrm{r}(G) = h_{1,1}\left(Y_3\right) - h_{1,1}\left(B_2\right) - 1. \tag{80}$$

The F-theory duals of the perturbative heterotic string are constructed from threefolds Y_3 which are elliptically and K3-fibred at the same time

[10] This can be derived from compactification to $D = 5$ where one has a Coulomb branch and $n_V^5 = \mathrm{r}(G) + n_T + 1 = h_{1,1}\left(Y_3\right) - 1$ vector multiplets.

[14]. This determines the base B_2 to be the Hirzebruch surface $\mathbb{F}_k$. Such manifolds have $h_{1,1}(\mathbb{F}_k) = 2$ and thus $n_T = 1$ as required for the perturbative heterotic string. In fact there is a beautiful correspondence between the heterotic vacua labelled by the instanton numbers (n_1, n_2) and elliptically fibred Calabi-Yau manifolds with the base being the Hirzebruch surfaces $\mathbb{F}_{n_2-12}$ [14].

Blown up $\mathbb{F}_k$ have $h_{1,1}(\mathbb{F}_k) > 2$ and thus $n_T > 1$. These F-theory vacua thus capture non-perturbative physics of the heterotic vacua including the possibility of additional tensor multiplets, the transitions between the various branches of moduli space and subspaces of symmetry enhancement [14, 57, 59].

A. Calabi-Yau manifolds

In this appendix we briefly recall a few facts about Calabi–Yau manifolds which we frequently use in the main text. (For a more extensive review see for example [1, 23, 48].)

A Calabi–Yau manifold Y_n is a Ricci-flat Kähler manifold of vanishing first Chern-class. Its holonomy group is $SU(n)$ where n is the complex dimension of Y_n. The simplest Calabi–Yau manifolds are tori of complex dimension 1. For $n = 2$ all Calabi–Yau manifolds are topologically equivalent to the $K3$ surface, while for $n = 3, 4$ one finds many topologically distinct Calabi–Yau manifolds. Such manifolds are of interest in string theory since they break some of the supersymmetries when a ten-dimensional string theory is compactified on Y_n.

The massless modes of a string vacuum are directly related to the zero modes of the Laplace operator on Y_n. These zero modes are the non-trivial differential k-forms on Y_n and they are elements of the cohomology groups $H^k(Y)$. On a compact Kähler manifold one can decompose any k-form into (p,q)-forms with p holomorphic and q antiholomorphic differentials $(p+q=k)$. Analogously, the associated cohomology groups decompose according to

$$H^k(Y) = \oplus_{p+q=k} H^{p,q}(Y). \tag{81}$$

The dimension of $H^{p,q}(Y)$ is called the Hodge number $h_{p,q}$ $(h_{p,q} = \dim H^{p,q})$; it is symmetric under the exchange of p and q, i.e. $h_{p,q} = h_{q,p}$, and Poincaré duality identifies $h_{p,q} = h_{n-p,n-q}$. Finally, the Euler number is given by

$$\chi = \sum_{p,q} (-1)^{p+q} h_{p,q} \,. \tag{82}$$

For K3 the Hodge numbers are

$$\begin{array}{ccccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} .$$

The Euler number is $\chi = 24$.

The moduli space of non-trivial metric deformations which preserve the Calabi-Yau condition is parameterized by 20 deformations of Kähler form $\delta G_{i\bar{j}}$ and 19 + 19 deformations of complex structure δG_{ij}. They are the coordinates of the homogeneous space

$$\mathcal{M} = R^+ \times \frac{SO\,(3,19)}{SO\,(3) \times SO\,(19)} / SO\,(3,19,\mathbf{Z}) \ , \tag{83}$$

where R^+ is spanned by the volume of K3.

For a Calabi-Yau threefold Y_3 one has the Hodge diamond

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

where $h_{1,1}$ and $h_{1,2}$ are arbitrary and the Euler number is $\chi(Y_3) = 2(h_{1,1} - h_{1,2})$. $h_{1,1}$ counts the number of Kähler deformations of the metric while $h_{1,2}$ counts the number of deformations of the complex structure. The moduli space is locally a direct product of the Kähler moduli space and the complex structure moduli space

$$\mathcal{M} = \mathcal{M}_{h_{1,1}} \times \mathcal{M}_{h_{1,2}} \ . \tag{84}$$

Each factor is a special Kähler manifold and the corresponding K obeys eq. (40). For $\mathcal{M}_{h_{1,2}}$ one finds

$$K_{1,2} = -\ln \int_{Y_3} \Omega \wedge \bar{\Omega} \tag{85}$$

where $\Omega(\bar{\Omega})$ is the unique $(3,0)$-form ($(0,3)$-form) on Y_3. In the large volume limit one has for $\mathcal{M}_{h_{1,1}}$

$$K_{1,1} = -\ln Vol\,(Y_3) = -\ln d_{ABC} M^A M^B M^C \ . \tag{86}$$

It is believed that most Calabi–Yau threefolds (if not all) have a mirror partner [47, 48]. That is, for a given Calabi–Yau threefold Y_3 with given $h_{1,1}(Y_3)$ and $h_{1,2}(Y_3)$ there exists a mirror manifold $\tilde{Y}_3$ with $h_{1,1}(\tilde{Y}_3) = h_{1,2}(Y_3)$ and $h_{1,2}(\tilde{Y}_3) = h_{1,1}(Y_3)$ which implies $\chi(Y_3) = -\chi(\tilde{Y}_3)$.

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