Problem 10.1

Consider an $SU(2)$ gauge theory with real scalar fields $\phi^a, a = 1, 2, 3$ which transform in the adjoint representation of $SU(2)$. The Lagrangian is given by

$$\mathcal{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \tfrac{1}{2} D_\mu \phi^a D^\mu \phi^a - V(\phi^a) \ ,$$

where

$$V = -\tfrac{1}{2}\mu^2 \phi^a \phi^a + \tfrac{1}{4}\lambda\big(\phi^a \phi^a\big)^2 \ , \qquad \mu^2, \lambda > 0 \ , \qquad D_\mu \phi^a = \partial_\mu \phi^a + g \epsilon_{abc} A^b_\mu \phi^c$$

a) Minimize the potential and parameterize the field space by $\phi^1, \phi^2, \phi^3 = v + h(x)$.

b) Identify the Goldstone bosons and compute the mass of the Higgs boson.

c) Identify the broken and unbroken generators and the unbroken gauge group.

d) Compute the mass matrix of the gauge bosons.

Problem 10.2

Consider the GSW-model with gauge group $SU(2) \times U(1)_Y$. A fermion ψ transforms in the **2** representation of $SU(2)$ and carries hypercharge y.

a) Give the covariant derivative of ψ in terms of the original gauge bosons A^a_μ and B_μ.

b) Rewrite the covariant derivative in terms of $W^\pm_\mu, Z^0_\mu$ and γ_μ and show

$$D_\mu \psi = \Big(\partial_\mu - \frac{ig}{\sqrt{2}}(W^+_\mu \sigma^+ + W^-_\mu \sigma^-) - \frac{ig}{\cos\theta_w} Z^0_\mu (\tfrac{1}{2}\sigma^3 - \sin^2\theta_w Q) - ieQ\gamma_\mu\Big)\psi \ .$$

c) Give the covariant derivative for an $SU(2)$ singlet with hypercharge y in both cases.

Problem 10.3

The Yukawa interactions in the GSW model are given by

$$\mathcal{L}_{\text{Yuk}} = \lambda\big(\phi^i \bar{E}^i_L e_R + \phi^{*i} \bar{e}_R E^i_L\big) \ .$$

a) Show that $\phi^i E^j_L \sigma^2_{ij}$ is $SU(2)$ invariant.

b) Introduce a new field ν_R which is neutral under $SU(2) \times U(1)_Y$ and use a) to give an aditional $SU(2) \times U(1)_Y$ invariant Yukawa interaction.

c) Compute the resulting neutrino mass m_ν.

d) Add a further term $\Delta\mathcal{L}_M = M\nu_R\nu_R + h.c.$ to $\mathcal{L}_{\text{Yuk}}$ and determine the eigenvalues of the resulting 2×2 neutrino mass matrix in the limit $M \gg m_\nu$.

Problem 10.4

Consider an $SU(2)\times U(1)_Y$ gauge theory with two Higgs doublets ϕ_1, ϕ_2 (both with hypercharge $1/2$).

a) Give the kinetic terms of this theory including explicitly the covariant derivatives of ϕ_1, ϕ_2.

b) Assume that both Higgs doublets have parallel vacuum expectation values, i.e. obey $\phi^i_{1,2}|_{min} = \frac{1}{\sqrt{2}}\binom{0}{v_{1,2}} + \ldots$, and compute the masses of the gauge bosons.

c) How many physical Higgs bosons, how many Goldstone bosons are in the theory and what is the unbroken gauge group?