This problem set counts towards the bonus if you hand it in by 8.30 on 6.7.11. (Note that handing it in jointly does not count.)

Problem 11.1

The group $O(N)$ is the group of orthogonal matrices O satisfying $OO^T = 1$.

a) Which property do the generators t^a satisfy?

Hint: Die t^a are defined by $O = \mathbf{1} + i\sum_a \alpha^a t^a + \mathcal{O}(\alpha^2)$.

b) How many generators t^a do exist?

c) Give a basis for the t^a in the fundamental 3-dimensional representation (denoted by $\mathbf{3}$) of $O(3)$.

d) Compute in this basis $c(\mathbf{3})$ and $c_2(\mathbf{3})$ and verify $d(\mathbf{3})c_2(\mathbf{3}) = d(\mathbf{G})c(\mathbf{3})$.

Problem 11.2

Consider a $N \times N$ matrix of Dirac fermions $\psi_{ij}, i, j = 1, \ldots, N$ with a transformation law

$$\psi_{ij} \to \psi'_{ij} = \sum_{kl} U_{ik} U_{jl} \psi_{kl} \ ,$$

with $U(x)$ being an x-dependent unitary matrix.

a) Give the covariant derivative of ψ_{ij} and compute its transformation law.

Hint: The gauge bosons transform as

$$A_\mu \to A'_\mu = U A_\mu U^\dagger - \tfrac{i}{g}(\partial_\mu U)U^\dagger \ .$$

b) Give the renormalizable action for this theory and show explicitly the invariance of the fermion mass term.

Problem 11.3

Consider an $SU(N)$ gauge theory with n_f massless Weyl fermions in representation $\mathbf{r_f}$ and n_s massless complex scalars in representation $\mathbf{r_s}$ (without Yukawa interactions).

a) Which divergent 1 PI one-loop diagrams contribute to the correction of the propagators of the gauge boson, the Weyl fermions and the complex scalars respectively.

   Hint: In the first case you should draw 5 diagrams and 1 diagram each in the second and third case.

b) Which divergent 1 PI one-loop diagrams contribute to the correction of the fermion-fermion-gauge boson vertex?

   Hint: You should draw 2 diagrams.

c) For this theory the one-loop coefficient of the β-function is given by

$$b_0 = -\tfrac{11}{3}c_2(G) + \tfrac{2}{3}n_f c(\mathbf{r_f}) + \tfrac{1}{3}n_s c(\mathbf{r_s}) \ .$$

   Compute b_0 for the following cases:

   (i) $SU(N)$ gauge theory with 1 family of Weyl-fermions in the adjoint representation.

   (ii) $SU(N)$ gauge theory with 4 families of Weyl-fermions and 3 complex scalars all in the adjoint representation.

   (iii) $SU(N)$ gauge theory with n_f families of Weyl-fermions and n_f complex scalars all in the fundamental N-dimensional representation.

   Hint: $c(\mathbf{N}) = 1/2,\ c_2(G) = N,\ d(\mathbf{r})c_2(\mathbf{r}) = d(G)c(\mathbf{r})$.

Problem 11.4

Consider a $U(2)$ gauge theory with a Higgs doublet $\phi^i, i = 1, 2$, i.e. in the fundamental representation. The generators are $t^a = \frac{1}{2}(\mathbf{1}, \vec{\sigma})$.

a) Give the gauge invariant and renormalizable Lagrangian of this theory including the Higgs potential and the covariant derivatives.

b) What is the condition for spontaneous symmetry breaking? Parameterize the field space by $\phi^1, \phi^2 = \frac{1}{\sqrt{2}}(v + h + i\sigma)$ and determine v.

c) Identify the Goldstone-bosons by computing the scalar masses. Compute additionally the mass matrix of the gauge bosons.