Problem 2.1

Consider the generating functional $Z[J]$

$$Z[J] = \int D\phi \, e^{i\int d^4x\,(\mathcal{L}_0[\phi]+J\phi)} = Z[0]\; e^{-\frac{1}{2}\int d^4x d^4y J(x) G_F(x-y) J(y)}$$

for $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$ and $G_F(x-y) = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2-m^2}$.

a) Compute

$$\langle 0|T\{\phi(x_1)\ldots\phi(x_4)|0\rangle = \frac{1}{Z[0]}\int D\phi\,\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) e^{i\int d^4x\,\mathcal{L}_0[\phi]}$$

as a function of G_F by explicitly computing the path integral.

Hint: Follow the same steps as done in lecture 2.

b) Compute $\langle 0|T\{\phi(x_1)\ldots\phi(x_4)\}|0\rangle$ by taking appropriate derivatives of $Z[J]$ and compare the results.

Problem 2.2

Consider an interacting theory with Lagrangian $\mathcal{L} = \mathcal{L}_0 - \frac{\lambda}{4!}\phi^4$.

a) Display $Z[J]$ as a power series in λ.

b) Express $Z[J]$ with the help of appropriate derivatives of $Z[J]$ and show

$$Z[J] = e^{iS_I(\phi=-i\frac{\delta}{\delta J})}\; Z[J] \; .$$

What is S_I?

c) Show

$$\left(\frac{\delta^4 \log Z[J]}{\delta J(x_1)\ldots\delta J(x_4)}\right)_{J=0} = \langle T\{\phi(x_1)\ldots\phi(x_4)\}\rangle_{\text{conn.}} \; ,$$

for $\frac{\delta Z[J]}{\delta J}|_{J=0} = 0$.

Problem 2.3

a) Show

$$\det(e^A) = e^{\mathrm{tr}(A)}$$

for a symmetric matrix A.

b) Use a) to compute

$$\frac{\partial \det(B)}{\partial B_{ij}} \, .$$

c) Compute

$$\int dx^1 \dots dx^n e^{-f(x^1,\dots,x^n)}$$

by expanding the real function f around its minimum to second order in x^i.

Hint: Use the results of problem 1.1.