Problem 3.1

Define

$$\Gamma[\phi_{\rm cl}] := -E[J] - \int d^4y J(y)\phi_{\rm cl}(y)$$

where

$$E[J] = i\ln Z[J] \ , \qquad \phi_{\rm cl}(y) := -\frac{\delta E[J]}{\delta J(y)} \ .$$

a) Compute $\frac{\delta\Gamma[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x)}$ using the functional chain rule

$$\frac{\delta E[J]}{\delta\phi_{\rm cl}(x)} = \int d^4y \, \frac{\delta E[J]}{\delta J(y)} \frac{\delta J(y)}{\delta\phi_{\rm cl}(x)}$$

b) By considering $\frac{\delta}{\delta J[y]} \frac{\delta\Gamma[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x)}$ show

$$\int d^4z \, \frac{\delta^2\Gamma[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x)\delta\phi_{\rm cl}(z)} \, \frac{\delta^2 E[J]}{\delta J(z)\delta J(y)} \ = \ \delta(x-y)$$

c) How is $\frac{\delta^2\Gamma[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x)\delta\phi_{\rm cl}(z)}$ related to the quantum corrected propagator?

Problem 3.2

a) Show

$$\int (\prod_m d\theta^*_m d\theta_m) \, e^{-\sum_{ij}\theta^*_i B_{ij}\theta_j} \ = \ (\det B) \ ,$$

where $\theta_m$are Grassmann variables and B_{ij} is a constant hermitian matrix.

Hint: Diagonalize B with a unitary matrix and perform an appropriate coordinate transformation.

b) Show

$$\int (\prod_m d\theta^*_m d\theta_m) \, \theta_k\theta^*_l \, e^{-\sum_{ij}\theta^*_i B_{ij}\theta_j} \ = \ (\det B) \, B^{-1}_{kl} \ ,$$

Hint: Use the results of a) and of problem 2.3.

Problem 3.3

a) The fermion propagator is given by

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2} e^{-ip(x-y)} \ ,$$

and satisfies

$$(i\gamma^\mu \partial_\mu - m) S(x-y) = i\delta^{(4)}(x-y) \ .$$

Show

$$\begin{aligned} &i) \quad \gamma^0 S^\dagger(x-y)\gamma^0 = -S(y-x) \ , \\ &ii) \quad i\partial_x^\mu S(y-x)\,\gamma_\mu + mS(y-x) = -i\delta^{(4)}(x-y) \ . \end{aligned}$$

Hint: Use $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.

b) Show

$$Z[\bar\eta,\eta] := \int D\bar\psi D\psi \ e^{i\int d^4x\,(\mathcal{L}_0 + \bar\eta\psi + \bar\psi\eta)} \ = \ Z[0]\ e^{-\int d^4x d^4y\,\left(\bar\eta(x)\,S(x-y)\,\eta(y)\right)} \ ,$$

for $\mathcal{L}_0 = \bar\psi(i\not{\partial} - m)\psi$.

Hint: Shift $\psi(x) = \psi'(x) + a\int d^4y S(x-y)\eta(y)$ with an appropriately chosen constant a and use the results of a).

c) Compute $\langle 0|T\{\psi(x_1)\psi(x_2)\bar\psi(x_3)\bar\psi(x_4)\}|0\rangle$ by appropriately differentiating Z.