Problem 4.1

a) For a ϕ^4-theory compute the one-loop correction to the propagator in momentum space using dimensional regularization and show that it is proportional to $m^{d-2}\Gamma(1-d/2)$.

b) Add the appropriate counterterm and determine δ_Z and δ_m.

Problem 4.2

a) By using dimensional regularization show

$$
\begin{aligned}
iV(p^2) \; &:= \tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}\frac{i}{(k+p)^2-m^2} \\
&= -\tfrac{1}{32\pi^2}\int_0^1 dx\,\Big(\frac{2}{\epsilon}-\ln\big(\frac{\Delta e^{\gamma}}{4\pi}\big)+\mathcal{O}(\epsilon)\Big)
\end{aligned}
$$

and compute Δ.

b) The four-point scattering amplitude $\mathcal{M}(p_1p_2\to p_3p_4)$ for a ϕ^4-theory is given by

$$
i\mathcal{M}(p_1p_2\to p_3p_4) = -i\lambda + (-i\lambda)^2\big(iV(s)+iV(t)+iV(u)\big) - i\delta_\lambda
$$

for $\delta_\lambda = -\lambda^2\big(V(4m^2)+2V(0)\big)$. Using the results of a) show

$$
i\mathcal{M}(p_1p_2\to p_3p_4) = -i\lambda + \tfrac{i\lambda^2}{32\pi^2}\int_0^1 dx\ln X
$$

and compute X. Check that $\mathcal{M}$ is UV-finite.

Problem 4.3

The bare Lagrangian for the Yukawa theory is given by

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m_{f0})\psi + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2 - \tfrac{1}{4!}\lambda_0\phi^4 - g_0\bar{\psi}\psi\phi \ .$$

a) Express $\mathcal{L}$ in terms of the renormalized couplings m, m_f, g, the renormalized fields ϕ_r, ψ_r and appropriate counterterms $\delta_{Z_\phi}, \delta_{Z_\psi}, \delta_{m_f}, \delta_m, \delta_\lambda, \delta_g$.

b) What are the renormalization conditions and which counterterms do they fix.

c) Compute explicitly the one-loop correction of the scalar propagator and determine

$$\delta_{Z_\phi} = \lim_{d\to 4} \frac{4g^2(d-1)}{(4\pi)^{d/2}} \int_0^1 dx x(1-x) \frac{\Gamma(2-\frac{d}{2})}{\big(\Delta(p^2=m^2)\big)^{2-\frac{d}{2}}}$$

$$\delta_m = \lim_{d\to 4} \frac{4g^2(d-1)}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(1-\frac{d}{2})}{\big(\Delta(p^2=m^2)\big)^{1-\frac{d}{2}}} + m^2\delta_{Z_\phi}$$