Problem 7.1

Consider a gauge theory with Lagrangian

$$\mathcal{L} = \sum_j \left(\bar{\psi}_j i\gamma^\mu (D_\mu \psi)_j - m\bar{\psi}_j \psi_j\right) - \tfrac{1}{4c}\,\mathrm{tr} F_{\mu\nu}F^{\mu\nu} \ ,$$

with

$$(D_\mu\psi)_i = \partial_\mu \psi_i - ig\sum_j (A_\mu)_{ij}\psi_j \ , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \ .$$

a) Show

$$\sum_j [D_\mu, D_\nu]_{ij}\,\psi_j \ = \ -ig\sum_{j=1} (F_{\mu\nu})_{ij}\,\psi_j \ .$$

b) By using $(A_\mu)_{ij} = \sum_a A^a_\mu t^a_{ij}$ show

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{bca} A^b_\mu A^c_\nu \ .$$

c) Show that

$$D_\rho F_{\mu\nu} := \partial_\rho F_{\mu\nu} - ig[A_\rho, F_{\mu\nu}]$$

is a covariant derivative. Determine $D_\rho F^a_{\mu\nu}$.

d) Show that Euler-Lagrange equations are given by

$$i\gamma^\mu (D_\mu\psi)_i - m\psi_i = 0 \ , \qquad D^\mu F^a_{\mu\nu} = g j^a_\nu \ ,$$

and compute j^a_ν.

e) Compute the Noether current J^a_ν including both ψ_i and A^a_μ by using the global limit of the gauge transformation and show its conservation by using d).

f) Show

$$D_\rho F_{\mu\nu} + D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} = 0 \ .$$

Problem 7.2

a) Show that the matrices $(t^a)_{bc} := if^{bac}$ are a representation of the Lie algebra in that they satisfy

$$[t^a, t^b] = i \sum_c f^{abc} t^c \ , \qquad a, b, c = 1, \ldots, d(G)$$

Hint: Use the Jacobi identity.

b) Define

$$T^2_{ij} := \sum_k \sum_a t^a_{ik} t^a_{kj} \ , \qquad i, j, k = 1, \ldots, d(r)$$

and show

$$[T^2, t^b] = 0 \qquad \forall b \ .$$

c) Use $T^2_{ij} = c_2(r)\delta_{ij}$ and $\mathrm{tr}(t^a t^b) = c(r)\delta^{ab}$ to show

$$c(r)\, d(G) = c_2(r)\, d(r) \ . \qquad (*)$$

d) Check $(*)$ in the fundamental representation of $SU(2)$ (denoted by $\mathbf{2}$) where

$$t^a := \tfrac{1}{2}\sigma^a \ , \qquad [t^a, t^b] = i \sum_c \epsilon^{abc} t^c \ ,$$

and σ^a are the Pauli-matrices.

Problem 7.3

a) The complex conjugate representation $\bar{r}$ is generated by $t^a_{\bar{r}} = -(t^a_r)^T$. Derive this relation and show that $t^a_{\bar{r}}$ satisfies the Lie algebra.

b) A representation is called real if

$$t^a_{\bar{r}} = S t^a_r S^{-1} \quad \forall a.$$

Show that the $\mathbf{2}$ of $SU(2)$ is real and find S.