Problem 9.1

Show $\epsilon^{\mu\nu\rho\sigma} tr(F_{\mu\nu}F_{\rho\sigma}) = \partial_\mu \omega^\mu$, and compute ω^μ.

Problem 9.2

Show that the eight counterterms $\delta_m, \delta_{1,2,3}, \delta^c_{1,2}, \delta_1^{3g,4g}$ introduced in the renormalzed perturbation theory of a Yang-Mills theory obey at one-loop the three relations

$$\delta_1 - \delta_2 = \delta_1^{3g} - \delta_3 = \delta_1^c - \delta_2^c = \tfrac{1}{2}(\delta_1^{4g} - \delta_3) \ .$$

Problem 9.3

a) $\Lambda_{\rm QCD}$ is defined by the condition $\bar\alpha_s^{-1}(M = \Lambda_{\rm QCD}) = 0$. Show

$$\Lambda_{\rm QCD} = P\,\exp(-\frac{2\pi}{b_0\bar\alpha_S(P)}) \qquad \text{and} \qquad \frac{d\Lambda_{\rm QCD}}{dP} = 0\ .$$

b) The 2-loop correction of the β-function is proportional to b_1 and given by

$$\beta(g) = -\frac{b_0 g^3}{(4\pi)^2} - \frac{b_1 g^5}{(4\pi)^4} + \dots$$

Show that in this case the solution to the CS equation to order $(\ln(P^2/\Lambda^2))^{-2}$ results in

$$\bar\alpha(P) = \frac{4\pi}{b_0}\left[\frac{1}{\ln(P^2/\Lambda^2)} - \frac{b_1}{b_0^2}\frac{\ln\ln(P^2/\Lambda^2)}{(\ln(P^2/\Lambda^2))^2} + \dots\right]$$

Hint: In a first step do an approximate indefinite integration of the RG-flow equation and define the integration constant into the scale Λ. In a second step solve the resulting equation iteratively for $\bar\alpha$.

Problem 9.4

Consider N real scalar fields $\phi^i, i = 1, \dots, N$ with Lagrangian

$$\mathcal{L} = \sum_i \partial_\mu\phi^i\partial^\mu\phi^i - V(\phi^i)\ , \qquad V = -\tfrac{1}{2}\mu^2\sum_i \phi^i\phi^i + \tfrac{\lambda}{4}\Big(\sum_i \phi^i\phi^i\Big)^2\ , \qquad \mu^2, \lambda > 0$$

a) What is the global symmetry group of $\mathcal{L}$?

b) Compute the minimum of V and determine its symmetry group.

c) Parametrise the field space by $\phi^1 = v + h(x), \phi^2(x), \dots, \phi^N(x)$ with

$$\phi^1|_{\rm Min} = v\ , \qquad \phi^2|_{\rm Min} = \dots = \phi^N|_{\rm Min} = 0\ ,$$

and compute the masses of all N fields. Relate your result to the Goldstone-theorem.