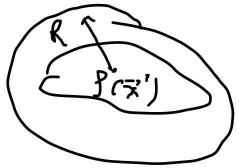
18. Multipolentwicklung

Gegeben: beliebige, lokalisierte Ladungsverteilung $\rho(\vec{x}')$

mit $\rho(\vec{x}') = \begin{cases} \text{beliebig} & \text{für } |\vec{x}'| < R \\ 0 & \text{" } |\vec{x}| \geq R \end{cases}$



und keine weitere Leiter im Raum => keine R.B.

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_{|\vec{x}'|<R} \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \quad , \quad \Delta\Phi = -\frac{\rho}{\epsilon_0}$$

Wenn Integral analytisch nicht berechenbar, kann das Fernfeld approximativ bestimmt werden.

" $\Phi(\vec{x})$ für $|\vec{x}| \gg R$

Multipolentwicklung = Entwicklung in $\frac{\vec{x}'}{|\vec{x}|} \ll 1$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{-\frac{1}{2}} = \left(x^2\left(1-\frac{x'}{x}\right)^2 + y^2\left(1-\frac{y'}{y}\right)^2 + z^2\left(1-\frac{z'}{z}\right)^2\right)^{-\frac{1}{2}}$$

$$= \left(\sum_{i=1}^{3} x_i^2 (1-a_i)^2\right)^{-1/2} \quad , \quad a_i = \frac{x_i'}{x_i} \quad , \quad x_1 = x,\ x_2 = y,\ x_3 = z$$

Taylor Entwicklung in a_i

allgemein: $$f(a_i) = f(0) + \sum_{i=1}^{3} a_i \left.\frac{\partial f}{\partial a_i}\right|_{a_i=0} + \frac{1}{2}\sum_i\sum_j a_i a_j \left.\left(\frac{\partial}{\partial a_i}\frac{\partial}{\partial a_j} f\right)\right|_{a_i=0} + O(a^3)$$

$$\left.\frac{1}{|\vec{x}-\vec{x}'|}\right|_{a_i=0} = \frac{1}{r}$$

$$\frac{\partial}{\partial a_i} \frac{1}{|\vec{X} - \vec{X}'|}\Bigg|_{a_i = 0} = \frac{X_i^2 (1 - a_i)}{\left(\sum_k X_k^2 (1 - a_k)^2\right)^{3/2}}\Bigg|_{a_i = 0} = \frac{X_i^2}{r^3}$$

$$\frac{\partial}{\partial a_j} \frac{\partial}{\partial a_i} \frac{1}{|\vec{X} - \vec{X}'|}\Bigg|_{a_i = 0} = -\frac{X_i^2 \delta_{ij}}{r^3} + \frac{3}{\cancel{2}} \frac{X_i^2 (1 - a_i)}{\left(\sum_k X_k^2 (1 - a_k)^2\right)^{5/2}} \cdot (+\cancel{2}) X_j^2 (1 - a_j)\Bigg|_{a_j = 0}$$

$$= -\frac{X_i^2 \delta_{ij}}{r^3} + 3 \frac{X_i^2 X_j^2}{r^5}$$

$$\frac{1}{|\vec{X} - \vec{X}'|} = \frac{1}{r} + \sum_i a_i \frac{X_i^2}{r^3} + \frac{1}{2} \sum_i \sum_j a_i a_j \left(3 \frac{X_i^2 X_j^2}{r^5} - \frac{X_j^2 \delta_{ij}}{r^3} \right) + O(a^3)$$

$a_i = \frac{X_i'}{X_i}$

$\downarrow$

$$= \frac{1}{r} + \sum_i \frac{X_i' X_i}{r^3} + \frac{1}{2} \sum_i \sum_j \left(3 \frac{X_i X_j X_i' X_j'}{r^5} - \frac{X_j' X_j}{r^3} \delta_{ij} \right)$$

NR: $$\sum_j \frac{x_j'^2}{r^3} = \frac{r'^2}{r^3} = \frac{r'^2 r^2}{r^5} = \frac{r'^2}{r^5}\sum_i x_i x_i = \frac{1}{r^5}\sum_i\sum_j x_i x_j r'^2 \delta_{ij}$$

$$\rightsquigarrow \quad \Phi = \frac{1}{4\pi\epsilon_0}\int_{|\vec{x}'|<R} \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3r'$$

$$\boxed{\Phi = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{1}{r^3}\sum_i x_i P_i + \frac{1}{r^5}\sum_i\sum_j x_i x_j Q_{ij} + \cdots\right)}$$

Multipolentwicklung

mit $$Q = \int \rho(\vec{x}')\, d^3x' \;\hat{=}\; \text{Gesamtladung}$$

$$P_i = \int \rho(\vec{x}')\, x_i'\, d^3x' \;\hat{=}\; \text{Dipolmoment}$$

$$Q_{ij} = \int \rho(\vec{x}')\left(3x_i'x_j' - \delta_{ij} r'^2\right) d^3x' \;\hat{=}\; \text{Quadrupolmoment}$$

$\vdots$

Eigenschaften von Q_{ij}: $\quad - \quad Q_{ij} = Q_{ji}$

$$- \quad \sum_i Q_{ii} \equiv Sp(Q) = 0$$

$$= \sum_i \int \rho \left(3 x_i'^2 - \delta_{ii} r'^2\right) d^3r$$

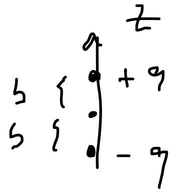
$$= \int \rho \left(3 r'^2 - 3 r'^2\right) d^3r' = 0$$

$\Rightarrow$ Q_{ij} hat 5 unabhängige Elemente $\quad (9-3-1)$

<u>Beispiele:</u> i) Punktladung am Ursprung $\quad \rho(\vec{x}') = q\,\delta(\vec{x}')$

$$\Rightarrow \quad \Phi = \frac{q}{r} = \text{Monopol}, \quad P_i = Q_{ij} = 0$$

ii)



$$\rho(\vec{x}') = q\left[\delta(\vec{x}'-\vec{b}) - \delta(\vec{x}'+\vec{b})\right], \qquad \vec{b} = b\,\vec{e}_z$$

$$Q = \int \rho(\vec{x}')\, d^3x' = q - q = 0$$

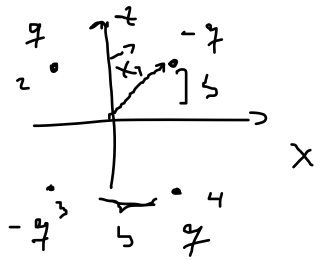
$$P_i = \int \rho(\vec{x}')\, x_i'\, d^3x' = q\left(b_i - (-b_i)\right) = 2q\, b_i$$

$$\vec{P} = 2q\vec{b}, \quad q\vec{d}, \quad \vec{d} = 2\vec{b}$$

$$Q_{ij} = \int \rho(\vec{x}') \cdot \left(3x_i' x_j' - \delta_{ij} r'^2\right) d^3x'$$

$$= q \int \left(\delta(\vec{x}'-\vec{b}) - \delta(\vec{x}'+\vec{b})\right)\left(3x_i' x_j' - \delta_{ij} r'^2\right) d^3x'$$

$$= q\left(\cancel{3b_i b_j} - \cancel{\delta_{ij}\vec{b}^2} - \cancel{3(-b_i)(-b_j)} + \cancel{\delta_{ij}(-\vec{b})^2}\right) = 0$$

iii) 

$$\rho(\vec{x}') = -q\left[\delta(\vec{x}'-\vec{x}_1) - \delta(\vec{x}'-\vec{x}_2) + \delta(\vec{x}'-\vec{x}_3) - \delta(\vec{x}'-\vec{x}_4)\right]$$

$$\vec{x}_1 = b\,(\vec{e}_x + \vec{e}_z)$$

$$\vec{x}_2 = b\,(-\vec{e}_x + \vec{e}_z)$$

$$\vec{x}_3 = b\,(-\vec{e}_x - \vec{e}_z)$$

$$\vec{x}_4 = b\,(\vec{e}_x - \vec{e}_z)$$

$$Q = 0\ , \quad P_i = -q \int \left[\delta(\vec{x}-\vec{x}_1) - \delta(\ldots\right] x_i'\, d^3x'$$

$$P_y = 0,\quad P_x = -q\,b\,(1-1-1+1) = 0\ ,\quad P_z = -q\,b\,(1+1-1-1) = 0$$

$$Q_{ij} = \int \rho(\vec{x}')\,(3x_i'x_j' - \delta_{ij}\underbrace{r'^2}_{2b^2})\,d^3x'$$

$$Q_{11} = -q\,b^2\,(1-1+1-1) = 0\ ,\quad Q_{22} = 0 = Q_{33}$$

$$Q_{12} = 0 = Q_{23}\ ,\quad Q_{13} = -3q\,b^2\,(1-(-1)1+(-1)(-1)-(1)(-1)) = \underline{\underline{-12q\,b^2}}$$

Multipolentwicklung in Kugelkoordinaten

Für $|\vec{x}| \gg R$ gilt $\Delta \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta(\vec{x}-\vec{x}') \approx 0$

$$\Rightarrow \frac{1}{|\vec{x}-\vec{x}'|} \simeq \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(b_{lm}\, r^l + \frac{c_{lm}}{r^{l+1}} \right) Y_{lm}(\theta, \varphi)$$ für $|\vec{x}| \gg R$

R.B.: $\lim_{|\vec{x}| \to \infty} \frac{1}{|\vec{x}-\vec{x}'|} = 0 \Rightarrow b_{lm} = 0$

$$\Rightarrow \frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m} \frac{c_{lm}}{r^{l+1}} Y_{lm}(\theta, \varphi), \qquad c_{lm} = c_{lm}(r', \theta', \varphi')$$

Da $\frac{1}{|\vec{x}-\vec{x}'|}$ sym unter $\vec{x} \leftrightarrow \vec{x}'$ muß gelten

$$c_{lm} = \underset{\overset{\shortparallel}{(-1)^m Y_{lm}}}{Y^*_{lm}}(\theta', \varphi') \left(d_{lm}\, r'^l + \frac{f_{lm}}{r'^{l+1}} \right)$$

$\lim_{r' \to 0} c_{lm}$ muß existieren, $f_{ml} = 0$

$$\leadsto \frac{1}{|\vec{x} - \vec{x}'|} = \sum_l \sum_m \underset{\substack{\uparrow \\ \text{konstant}}}{d_{lm}} \frac{r'^l}{r^{l+1}} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

<u>Satz</u>: Additionstheorem für Kugelfunktion

$$\boxed{P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi)}$$

$\gamma = \measuredangle(\vec{x}, \vec{x}')$, d.h. $\vec{x} \cdot \vec{x}' = |\vec{x}|\,|\vec{x}'| \cos\gamma(\theta, \varphi, \theta', \varphi')$

<u>Beweis</u>: [Jackson]

$\cos\gamma = 1 \quad , \quad \vec{x} \parallel \vec{x}' \quad , \quad \theta = \theta' ; \quad \varphi = \varphi'$

$$P_\ell(1) = \frac{4\pi}{2\ell+1} \sum_m Y^*_{\ell m}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi)$$

$$P_\ell(\xi) := \frac{1}{2^\ell \ell!} \frac{d}{d\xi^\ell} (\xi^2 - 1)^\ell \Big|_{\xi=1} = \frac{\ell!}{2^\ell \ell!} \Big(\underbrace{\frac{d}{d\xi}(\xi^2-1)}_{2\xi} \Big)^\ell \Big|_{\xi=1} = \frac{\ell!\, 2^\ell}{2^\ell \ell!} = 1$$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2 r r' \cos\gamma}} \underset{\cos\gamma = 1}{=} \frac{1}{\sqrt{(r - r')^2}} = \frac{1}{r - r'} = \frac{1}{r} \frac{1}{1 - \frac{r'}{r}}$$

$$= \frac{1}{r} \underbrace{\sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell}_{\frac{1}{1-\frac{r'}{r}}} = \sum_{\ell=0}^{\infty} \frac{r'^\ell}{r^{\ell+1}} \overset{!}{=} \sum_\ell \sum_m d_{\ell m} \frac{r'^\ell}{r^{\ell+1}} Y^*_{\ell m}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi)$$

geo. Reihe

$$\Rightarrow \sum_m d_{\ell m} Y^*_{\ell m}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = 1$$

$$\Rightarrow \boxed{d_{\ell m} = \frac{4\pi}{2\ell+1}}$$

$$\frac{1}{|\vec{x}-\vec{x}'|} \overset{|\vec{x}|\gg R}{=} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r'^{l}}{r^{l+1}} Y^{*}_{lm}(\theta', \varphi') \, Y_{lm}(\theta, \varphi)$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \, d^3x'$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \varphi)$$

$$q_{lm} = \sqrt{\frac{4\pi}{2l+1}} \int \rho(\vec{x}') \, r'^{l} \, Y^{*}_{lm}(\theta', \varphi') \, d^3x'$$

$$q_{00} = \sqrt{4\pi} \int \rho(\vec{x}\,') \underbrace{Y^*_{00}}_{\frac{1}{4\pi}} d^3x' = \frac{q}{\sqrt{4\pi}}$$

$$\Phi = \frac{1}{4\pi\epsilon_0} \sqrt{4\pi}\, \frac{q_{00}}{r} = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

$l=1,\ m=0, \pm 1$

$$q_{10} = \sqrt{\frac{4\pi}{3}} \int \rho(\vec{x}\,') \underbrace{r' Y^*_{10}}_{\sqrt{\frac{3}{4\pi}} \cos\Theta'} d^3x' = \int \rho(\vec{x}\,')\, z'\, d^3x = p_z$$

und

$$p_x = \frac{1}{\sqrt{2}} \left(q_{1-1} - q_{11} \right)$$

$$p_y = \frac{1}{\sqrt{2}\,i} \left(q_{11} + q_{1-1} \right)$$

$l=2,\ m=0, \pm 1, \pm 2 \Rightarrow$ 5 Quadrupolmoment q_{2m}

hängt mit Q_{ij} zusammen