Spezielle Relativitätstheorie (Einstein 1905)

Frage: Was sind die IS der Elektrodynamik?

Postulat: c = Naturkonstante

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu}\, dx^\mu dx^\nu = ds'^2 = \sum_\mu \sum_\nu \eta_{\mu\nu}\, dx'^\mu dx'^\nu$$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \qquad x^\mu = (ct, x^i)$$

$i = 1, 2, 3$

Koordinate einer 4-dim Raum-Zeit

Minkowski-Raum M_4

pseudo-euklidische Metrik η

Lorentztransformation

$$x^\mu \to x'^\mu = \sum_\nu \Lambda^\mu{}_\nu \, x^\nu$$

↖ 4x4 Matrix

$$(*) \qquad \Lambda^T \eta \Lambda = \eta$$

10 Bedingungen an Λ

$\Rightarrow$ Λ hat 6 Parameter

3 Drehwinkel:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}$$

↖ Drehmatrix

3 Geschwindigkeiten (der beiden KSysteme gegeneinander)

Spezialfall: $\quad \Lambda = \begin{pmatrix} a & b & & 0 \\ c & d & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

$\curvearrowright$ (*): $\quad a^2 - c^2 = 1 = d^2 - b^2 \quad , \quad ab - cd = 0$

$\Rightarrow \quad a^2 = d^2 = \cosh^2 \varphi \quad , \quad b^2 = c^2 = \sinh^2 \varphi$

$\nwarrow$ Rapidität

<u>Zusätzliche Forderung:</u>

i) $\Lambda^0{}_0 > 0$ (kein Zeitumkehr)

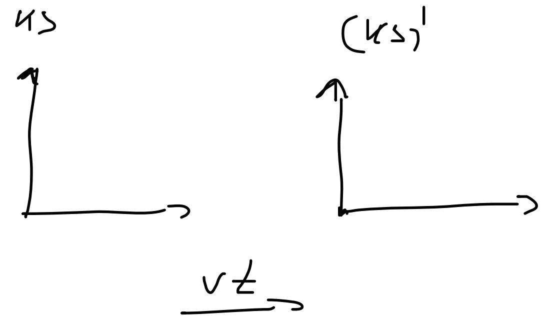
$\Rightarrow \quad a = \cosh \varphi$

ii) $\det(\Lambda^T \eta \Lambda) = \underbrace{\det \Lambda^T}_{\det \Lambda} \underbrace{\det \eta}_{-1} \det \Lambda \overset{!}{=} \underbrace{\det \eta}_{-1}$

(*) $\Rightarrow \det \Lambda = \pm 1 \qquad \det \Lambda = 1$ keine Raumspiegelung

$\Rightarrow d = a$

Konvention $b = c = -\sinh\varphi$



Ursprung von KS'

$$x' = 0 \,, \quad x = vt$$

L.T.: $$x' = \Lambda^1{}_0 \underbrace{ct}_{x^0} + \Lambda^1{}_1 \underbrace{x}_{x^1}$$

$$0 = \Lambda^1{}_0\, ct + \Lambda^1{}_1\, vt$$

$$\Rightarrow \frac{v}{c} = -\frac{\Lambda^1{}_0}{\Lambda^1{}_1} = -\frac{b}{d} = \frac{\sinh\varphi}{\cosh\varphi} = \tanh\varphi$$

$$\cosh\varphi = \frac{1}{\sqrt{1-\tanh^2\varphi}} = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \equiv \gamma > 1 \quad \text{für } v < c$$

$$\sinh\varphi = \frac{\tanh\varphi}{\sqrt{1-\tanh^2\varphi}} = \frac{v}{c}\cdot\gamma \,, \quad \beta \equiv \frac{v}{c}$$

$$ct' = \Lambda^0{}_0\, ct + \Lambda^0{}_1\, x = \cosh\psi\, ct - \sinh\psi\, x$$

$$= \gamma ct - \frac{v}{c}\gamma x = \boxed{\gamma\left(ct - \frac{v}{c}x\right) = ct'}$$

$$x' = \Lambda^1{}_1\, x + \Lambda^1{}_0\, ct = \gamma x - \frac{v}{c}\gamma tc = \boxed{\gamma(x - vt) = x'}$$

Für $v \ll c$: $\gamma = 1 + O\left(\frac{v^2}{c^2}\right)$

$$\Rightarrow \underbrace{t' = t \;, \quad x' = x - vt}_{\text{Galilei Transformation}}$$

Verallgemeinerung $\quad \vec{v} = \sum_i v_i \vec{e}_i$ (ohne Herleitung)

$$\Lambda^0{}_0 = \gamma = \frac{1}{\sqrt{1 - \frac{|\vec{v}|^2}{c^2}}}$$

$$\Lambda^i{}_0 = -\gamma \frac{v_i}{c} = \Lambda^0{}_i$$

$$\Lambda^i{}_j = \delta^i{}_j + \frac{v_i v_j}{|\vec{v}|^2}(\gamma - 1)$$

$$\Lambda = \left(\begin{array}{c|c} \gamma & -\gamma \frac{\vec{v}}{c} \\ \hline -\gamma \frac{\vec{v}}{c} & \underbrace{\Lambda^i{}_j}_{3 \times 3} \end{array} \right)$$

3 Geschwindigkeit v_i

$$ct' = \Lambda^0{}_0 ct + \sum_i \Lambda^0{}_i \cancel{x}^i = \gamma \left(ct - \frac{\vec{x} \cdot \vec{v}}{c} \right)$$

$$x'^i = \Lambda^i{}_0 ct + \sum_j \Lambda^i{}_j x^j = x^i + \frac{(\vec{v} \cdot \vec{x})\, v^i (\gamma - 1)}{|\vec{v}|^2} - v^i \gamma t$$

Transformation der Geschwindigkeit

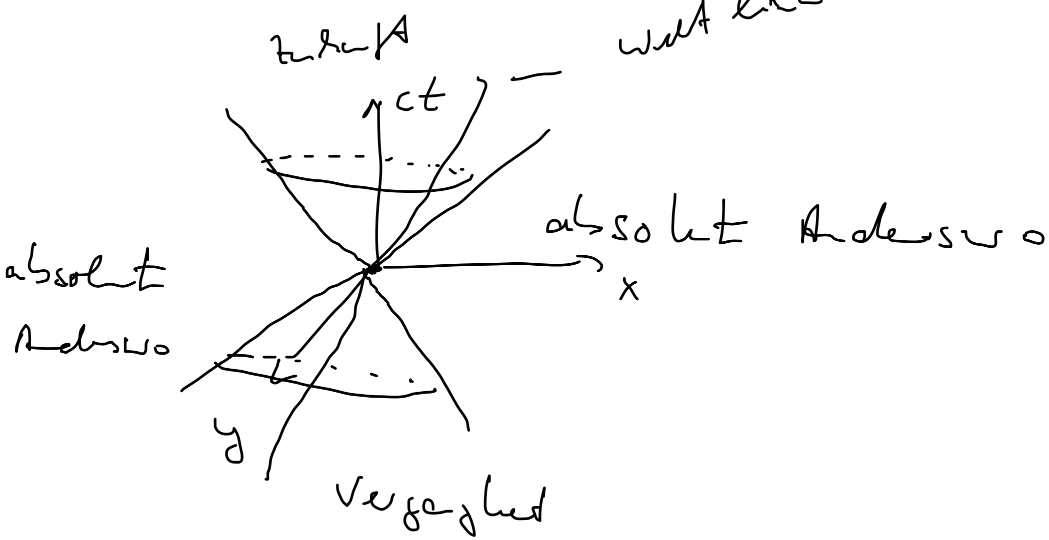
$$dx' = \gamma\, dx - \gamma v\, dt \qquad dt' = \gamma\, dt - \gamma \frac{v}{c^2} dx$$

$$\frac{dx'}{dt'} = \frac{\cancel{\gamma} dx - \cancel{\gamma} v\, dt}{\cancel{\gamma} dt - \cancel{\gamma} \frac{v}{c^2} dx} = \frac{dx - v\, dt}{dt\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{1}{\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)}\left(\frac{dx}{dt} - v\right)$$

Für e-m Welle gilt $\frac{dx}{dt} = c$

$$\rightarrow \frac{dx'}{dt'} = \frac{c - v}{1 - \frac{v}{c^2} c} = c$$

Minkowski-Raum



Lichtkegel

Zeit Dilatation (bewegte Uhren gehen langsamer)

Uhr ruht KS' $dx^{i\,\prime} = 0$, Eigenzeit der Uhr $d\tau^2 = (dt')^2 = \frac{ds^2}{c^2}$

Beobachtet in KS $\frac{ds^2}{c^2} = d\tau^2 = dt^2 - \frac{1}{c^2}dx^2 = dt^2 \underbrace{\left(1 - \frac{v^2}{c^2}\right)}_{1/\gamma^2} = \frac{1}{\gamma^2}dt^2$

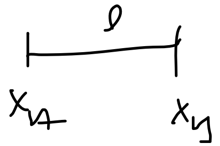
Laborzeit $\uparrow$ (dt) $\frac{dx}{dt} = v$

$$d\tau = d\tau' \iff dt' = \frac{dt}{\gamma} \implies \frac{dt}{dt'} = \gamma > 1$$

$\implies$ In KS vergeht die Zeit zwischen 2 Schlägen langsamer!

Längenkontraktion

Stab der Länge $\ell = x_A - x_B$



$$\ell' = x'_A - x'_B = \gamma(x_A - vt) - \gamma(x_B - vt) = \gamma(x_A - x_B) = \gamma\ell$$

$$\ell' = \gamma\ell \Rightarrow \ell' > \ell$$

Im ruhenden IS wird ein kürzere Länge gemessen!

Minkowski-Raum ist Vektorraum

Elemente sind 4er Vektoren $V^\mu = (V^0, V^1, V^2, V^3)$

$V \in M_4$

transformiert unter L.T. $V^\mu \to V^{\mu'} = \sum_\nu \Lambda^\mu{}_\nu V^\nu$

Skalarprodukt in M_4: $\sum_\mu \sum_\nu \eta_{\mu\nu} V^\mu V^\nu$ ist invariant unter L.T.

(Verallgemeinerung von $\vec{V} \cdot \vec{V} = \sum_i \sum_j \delta_{ij} V^i V^j$ in $\mathbb{R}^3$

$= \sum_i V^i V^i$

$$\sum_\mu \sum_\nu \eta_{\mu\nu} V'^\mu V'^\nu = \underbrace{\sum_\mu \sum_\nu}_{} \sum_\rho \sum_\sigma \underbrace{\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma}_{\eta_{\rho\sigma}} V^\rho V^\sigma = \sum_\rho \sum_\sigma \eta_{\rho\sigma} V^\rho V^\sigma$$

Schreibweise: kontravariante Vektor $V^\mu = (V^0, V^i)$

kovariante Vektor $V_\nu := \sum_\mu \eta_{\nu\mu} V^\mu$

$= (V^0, -V^i)$

Innenprodukt: $\boxed{\sum_\mu V_\mu V^\mu = \sum_\mu \sum_\nu \eta_{\mu\nu} V^\nu V^\mu}$

Tensor 2. Stufe: $T^{\mu\nu}$ (kontravariant)

transformiert $T^{\mu\nu} \to T'^{\mu\nu} = \sum_\rho \sum_\sigma \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma}$

$T_{\mu\nu} = \sum_\rho \sum_\sigma \eta_{\mu\rho} \eta_{\nu\sigma} T^{\rho\sigma}$ kovariante Tensor 2. Stufe)

$$T^{\mu}{}_{\nu} = \sum_{\rho} \eta_{\rho\nu}\, T^{\rho\mu}$$ gemischter Tensor 2. Stufe

Tensor (n,m)-ter Stufe $\quad T^{\overbrace{\mu_1 \dots \mu_n}^{n}}{}_{\underbrace{\nu_1 \dots \nu_m}_{m}}$

Was folgt

$$T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m} = \sum_{\rho_1 \dots \rho_m} \eta_{\nu_1 \rho_1} \cdots \eta_{\nu_m \rho_m}\, T^{\mu_1 \dots \mu_n \nu_1 \dots \nu_m}$$

$$T^{\mu_1 \dots \nu_m} \longrightarrow T'^{\mu_1 \dots \nu_m} = \sum_{\rho_1 \dots \sigma_m} \Lambda^{\mu_1}{}_{\rho_1} \cdots \Lambda^{\nu_m}{}_{\sigma_m}\, T^{\rho_1 \dots \nu_m}$$

Ableitung

$$\frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right) \equiv \partial_\mu$$

$$\frac{\partial}{\partial x_\mu} = \sum_\nu \eta^{\mu\nu} \frac{\partial}{\partial x^\nu} \equiv \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

inverse Metrik, d.h. es gilt

$$\sum_\rho \eta^{\mu\rho} \eta_{\rho\nu} = \delta^\mu_{\ \nu}$$

$$\Rightarrow \quad \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \eta_{\mu\nu}$$

$$\frac{\partial}{\partial x^\mu} x^\nu = \begin{pmatrix} \frac{\partial}{\partial ct} ct & \frac{\partial}{\partial ct} x^1 & \cdots \\ \vdots & \frac{\partial}{\partial x^1} x^1 & \cdots \\ \vdots & \vdots & \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = \delta^\nu_\mu$$

$$\sum_\mu \partial_\mu \partial^\mu = \sum_\mu \sum_\nu g^{\mu\nu} \partial_\mu \partial_\nu = g^{00} \partial_0 \partial_0 + \sum_{ij} \underbrace{g^{ij}}_{-\delta^{ij}} \partial_i \partial_j$$

$$= \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \underbrace{\sum_i \partial^i \partial_i}_{\Delta} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta = \Box$$

$$\boxed{\Box = \sum_\mu \partial_\mu \partial^\mu}$$

Lorentzinvariant

<u>Def</u>. 4er Geschwindigkeit

$$u^\mu := \frac{dx^\mu}{d\tau} = (u^0, u^i)$$

$$u^0 = \frac{dct}{d\tau} = c\gamma$$

$$u^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt} \frac{dt}{d\tau} = v^i \gamma$$

Def 4er Impuls $p^\mu := m\, u^\mu = (p^0, p^i)$

$\uparrow$ konstant

$$p^0 \equiv \frac{E}{c} = m\, u^0 = m c \gamma$$

$$\sum_\mu p_\mu p^\mu = \sum_{\mu\nu} p^\mu p^\nu \eta_{\mu\nu} = m^2 c^2 \gamma^2 - \underbrace{\sum_i p^i p^i}_{\vec{p}^{\,2}} = \frac{E^2}{c^2} - \vec{p}^{\,2}$$

$$= m^2 c^2 \gamma^2 - m^2 \gamma^2 \sum_i v^i v^i = m^2 c^2 \underbrace{\gamma^2 (1 - \vec{v}^{\,2})}_{1} = m^2 c^2$$

rel. Energie-Impuls Beziehung: $\boxed{E^2 = c^2 \vec{p}^{\,2} + m^2 c^4}$

Ruheenergie

kin. Energie $T = E - m c^2 = \sqrt{c^2 p^2 + m^2 c^4} - m c^2 \underset{v \ll c}{\simeq} \frac{1}{2} m v^2$

$$E = mc^2 + \frac{1}{2} m v^2 + \cdots \mathcal{O}\left(\frac{v^4}{c^2}\right)$$

$$\vec{p} = m\vec{v} + \mathcal{O}\left(\frac{v^3}{c^2}\right)$$

masselose Teilchen $\quad E = c\,|\vec{p}|$