Lagrange Formalismus der E-Dynamik

Mechanik: $$S = \int L(q, \dot{q}, t)\, dt$$

Wirkung ↗ ↑ Lagrangefkt.

$$\delta S = 0 \Rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$ E-L. Gl.

Verallgemeinerung auf Feldtheorie

Mechanik: $$q(t) \longrightarrow \Phi(\underbrace{\vec{x}, t}_{x^\mu}) \Rightarrow \mathcal{L}[\Phi, \partial_\mu \Phi]$$

- $t \longrightarrow \vec{x}, t \Rightarrow x^\mu$

$$S[\Phi, \partial\Phi] = \iiiint \mathcal{L}[\Phi, \partial\Phi] \underbrace{d^3x\, dt}_{\equiv d^4x}$$

$$\delta S = 0 = \int \Big\{ \underbrace{\frac{\partial \mathcal{L}}{\partial \Phi}}_{\substack{\text{Funktional}\\\text{ableiten}}} \delta \Phi + \sum_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \underbrace{\delta(\partial_\mu \phi)}_{\partial_\mu \delta\Phi} \Big\} d^4x$$

$$= \int \Big\{ \frac{\partial \mathcal{L}}{\partial \Phi} \delta\phi - \sum_\mu \Big(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\Big) \delta\phi \Big\} d^4x + \sum_\mu \underbrace{\int \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \delta\phi \Big) d^4x}_{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \underbrace{\delta\phi}_{=0} \Big|_{\text{Rand}}}$$

Annahme: $\delta\Phi\big|_{\text{Rand}} = 0$

$$0 = \int \underbrace{\left\{ \frac{\partial \mathcal{L}}{\partial \Phi} - \sum_{\mu=0}^{3} \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right\}}_{\Rightarrow\ = 0} \underline{\delta \phi} \, d^4 x$$

E-L Glchg einer Feldtheorie:

$$x^\mu = (\underset{\substack{\parallel \\ ct}}{x^0}, x, y, z)$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial \Phi} - \sum_{\mu=0}^{3} \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = 0}$$

analog zu

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

Beispiel:

$$\mathcal{L} = \frac{1}{2} \sum_{\kappa} \sum_{\nu} (\partial_\kappa \Phi)\underbrace{(\partial_\nu \Phi)\, \eta^{\kappa\nu}}_{= \partial^\kappa \Phi} = \frac{1}{2} \sum_{\kappa} (\partial_\kappa \Phi)(\partial^\kappa \Phi)$$

$$\frac{\partial \mathcal{L}}{\partial \Phi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = \frac{1}{2} \sum_{\kappa} \sum_{\nu} \left(\underbrace{\left[\frac{\partial \partial_\kappa \Phi}{\partial(\partial_\mu \Phi)}\right]}_{\delta_\kappa^\mu} \partial_\nu \Phi\, \eta^{\kappa\nu} + \eta^{\kappa\nu} \partial_\kappa \Phi \frac{\partial(\partial_\nu \Phi)}{\partial(\partial_\mu \Phi)} \right)$$

$$= \frac{1}{2} \sum_{\kappa} \sum_{\nu} \left(\underbrace{\frac{\partial(\partial_\kappa \Phi)}{\partial(\partial_\mu \Phi)}}_{\delta_\kappa^\mu} \partial_\nu \Phi\, \eta^{\kappa\nu} + (\partial_\kappa \Phi) \underbrace{\frac{\partial(\partial_\nu \Phi)}{\partial(\partial_\mu \Phi)}}_{\delta_\nu^\mu} \eta^{\kappa\nu} \right)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = \frac{1}{2}\sum_{\nu} \partial_\nu \Phi\, g^{\mu\nu} + \frac{1}{2}\sum_{\cancel{\kappa}\,\nu} (\partial_{\cancel{\kappa}\,\nu} \Phi)\, \underbrace{g^{\cancel{\kappa}\nu\mu}}_{g^{\mu\nu}}$$

$$= \sum_{\nu} (\partial_\nu \Phi)\, g^{\mu\nu}$$

$$\sum_\mu \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = \sum_\mu \sum_\nu (\partial_\mu \partial_\nu \Phi)\, g^{\mu\nu} = \underbrace{\sum_\mu \sum_\nu g^{\mu\nu} \partial_\mu \partial_\nu}_{\Box} \Phi$$

$$= \Box \Phi$$

$\Rightarrow$ E-L Gl. $\boxed{\Box \Phi = 0}$ Wellgleichung

$$\Rightarrow \mathcal{L} = \frac{1}{2} \sum_{\mu} \sum_{\nu} (\partial_\mu \Phi)(\partial_\nu \Phi)\, \eta^{\mu\nu}$$

ist Lagrangedichte für Wellengleichung $\Box \Phi = 0$

(analog in Mechanik: $L = \frac{1}{2} m \dot{q}^2$

EL: $\ddot{q} = 0$

nächster Schritt: ED

$$\mathcal{L} = -\frac{1}{4}\sum_\mu \sum_\nu F_{\mu\nu} F^{\mu\nu} - \mu_0 \underbrace{\sum_\mu A^\mu j_\mu}_{\sum_\mu\sum_\nu A_\nu j_\mu g^{\mu\nu}}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$ Feldstärke Tensor

$$A_\mu = (c^{-1}\Phi, \vec{A})$$

E-L Gleichg:

$$\frac{\partial \mathcal{L}}{\partial A_k} - \sum_{\lambda=0}^{3} \partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_k)} = 0$$

4 Gleichg für $k = 0, 1, 2, 3$

$$\frac{\partial \mathcal{L}}{\partial A_k} = -\mu_0 \sum_\mu\sum_\nu g^{\nu\mu} j_\mu \underbrace{\frac{\partial A_\nu}{\partial A_k}}_{\delta_\nu^k} = -\mu_0 \underbrace{\sum_\mu g^{k\mu} j_\mu}_{j^k} = -\mu_0 j^k$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\kappa)} = -\frac{1}{4}\sum_\mu \sum_\nu \left\{ \frac{\partial F_{\mu\nu}}{\partial(\partial_\lambda A_\kappa)} F^{\mu\nu} + F_{\mu\nu} \frac{\partial F^{\mu\nu}}{\partial(\partial_\lambda A_\kappa)} \right\}$$

$$= -\frac{1}{2}\sum_\mu \sum_\nu \underbrace{\frac{\partial F_{\mu\nu}}{\partial(\partial_\lambda A_\kappa)}} F^{\mu\nu}$$

$$\frac{\partial F_{\mu\nu}}{\partial(\partial_\lambda A_\kappa)} = \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\lambda A_\kappa)} - \frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_\lambda A_\kappa)} = \delta_\mu^\lambda \delta_\nu^\kappa - \delta_\nu^\lambda \delta_\mu^\kappa$$

$$\curvearrowright \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\kappa)} = -\frac{1}{2}\sum_\mu \sum_\nu \left(\delta_\mu^\lambda \delta_\nu^\kappa - \delta_\nu^\lambda \delta_\mu^\kappa\right) F^{\mu\nu}$$

$$= -\frac{1}{2}\left(F^{\lambda\kappa} \underbrace{- F^{\kappa\lambda}}_{-F^{\lambda\kappa}}\right) = -F^{\lambda\kappa}$$

$$\frac{\partial \mathcal{L}}{\partial A_\kappa} - \sum_\lambda \partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\kappa)} = 0$$

$$\Rightarrow -\mu_0 j^\kappa + \sum_\lambda \partial_\lambda F^{\lambda\kappa} = 0$$

$$\Rightarrow \boxed{\sum_\lambda \partial_\lambda F^{\lambda\kappa} = \mu_0 j^\kappa}$$

inhomogene Maxwell-Gleichung (gezeigt in letzter Vorlesung)

$$\Rightarrow \boxed{\mathcal{L} = -\frac{1}{4} \sum_\mu \sum_\nu F_{\mu\nu} F^{\mu\nu} - \mu_0 \sum_\mu A^\mu j_\mu}$$

ist Lagrangedichte für E-Dynamik

homogene Maxwell-Gleichungen sind <u>keine</u> E-L Gleichungen

sondern <u>Bianchi-Identitäten</u> :

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \qquad \text{(letzte Vorlesung)}$$

$$= \partial_\lambda \left(\cancel{\partial_\mu A_\nu} - \bcancel{\partial_\nu A_\mu} \right) + \partial_\mu \left(\partial_\nu A_\lambda - \cancel{\partial_\lambda A_\nu} \right) + \partial_\nu \left(\bcancel{\partial_\lambda A_\mu} - \partial_\mu A_\lambda \right) = 0$$

ist identisch erfüllt, also eine Identität

Bemerkungen

(i) $\sum_\kappa \partial_\kappa \left(\sum_\lambda \partial_\lambda F^{\lambda\kappa} \right) = \sum_\kappa \sum_\lambda \partial_\kappa \partial_\lambda F^{\lambda\kappa} = 0$

$= \mu_0 \sum_\kappa \partial_\kappa j^\kappa$

$\Rightarrow \boxed{\sum_\kappa \partial_\kappa j^\kappa = 0}$ Kontinuitätsgleichung

$\vec{\nabla} \vec{j} + \frac{\partial \rho}{\partial t} = 0$

ii) $\mathcal{L}$ ist Lorentz invariant

iii) S ist ein invariant

Eichtrafo. $A_\mu \to A'_\mu = A_\mu - \partial_\mu \Lambda$

leicht verifiz.: $F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu}$

$$\mathcal{L} \to \mathcal{L}' = -\frac{1}{4} \sum_\mu \sum_\nu F'_{\mu\nu} F'^{\mu\nu} - \mu_0 \sum_\mu A'^\mu j_\mu$$

$$= \underbrace{-\frac{1}{4} \sum_\mu \sum_\nu F_{\mu\nu} F^{\mu\nu} - \mu_0 \sum_\mu A^\mu j_\mu}_{\mathcal{L}} + \mu_0 \sum_\mu (\partial^\mu \Lambda) j_\mu$$

$$= \mathcal{L} + \mu_0 \sum_\mu (\partial^\mu \Lambda) j_\mu$$

d.h. $\mathcal{L}$ ist nicht eich invariant aber $\quad S \to S' = \int \mathcal{L}' \, d^4x$

$$= \underbrace{\int \mathcal{L} \, d^4x}_{S} + \mu_0 \sum_\mu \int (\partial^\mu \Lambda) \, \hat{j}_\mu \, d^4x$$

$$= S - \mu_0 \sum_\mu \int \Lambda \, \underbrace{(\partial^\mu \hat{j}_\mu)}_{\Rightarrow \text{ Kontinuitätsgl.}} \, d^4x$$

$$+ \mu_0 \underbrace{\sum_\mu \Lambda \, j_\mu \Big|_{\text{Rand}}}_{= 0}$$

$$= S$$

$\Rightarrow$ S ist eich invariant $\Rightarrow$ E-L. Gl. sind eichinvariant !

Klausur

Termin: 19.2., 10–12 Uhr, HS II

Hilfsmittel: – 2 DIN A4 Blätter, hand beschrieben
– kein Handy, kein Taschenrechner, kein Computer

Papier mitbringen

Vorklausur:
- auf homepage Freitag / Montag
- Besprechung 17.2. 10^{30} HS III

Vorbereitung:
(ohne Gewähr)

Blatt 1–3 komplett, (4 / 1,2), (5 / 1,2,4)
(6 / 1,3), (7 / 1), (8 / 1,3), (9 / 1,2), (10 / 1,3,4)
(11 / komplett), (12 / 1,2,4), (13 / 2,3)