Quantum Field Theory I

Lecture notes by

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ABSTRACT

The lecture notes grew out of a course given at the University of Hamburg in the summer term 2007 and the winter term 2010/11. A first version of these lecture notes were written by Andreas Bick, Lukas Buhné, Karl-Philip Gemmer, Michael Grefe, Jasper Hasenkamp, Jannes Heinze, Sebastian Jakobs, Thomas Kecker, Tim Ludwig, Ole Niekerken, Alexander Peine, Christoph Piefke, Sebastian Richter, Michael Salz, Björn Sarrazin, Hendrik Spahr, Lars von der Wense.

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1 Lecture 1: Special Relativity

1.1 Introduction

The topic of this course is an introduction to the quantum theory of relativistically invariant field theories such as scalar field theories and Quantum-Electrodynamics (QED). Applications to Particle Physics such as the computation of scattering amplitudes are also discussed. There are many excellent textbooks on the subjects, for example [2, 6, 1, 3, 4, 7]. This course closely follows [2].

Let us start by recalling various aspects of Special Relativity.

1.2 Covariance of a physical law

A physical phenomenon or process should not depend on the choice of coordinate system which is used to describe it. Therefore the physical laws should be identical in all coordinate systems. One says that a physical law has to be covariant[1] under coordinate transformations.

As a first example let us consider Newtons Mechanics which is governed by the equation of motion

$$\vec{F} = m\frac{d^2\vec{x}}{dt^2} \equiv m\ddot{\vec{x}} \ . \tag{1.1}$$

Let us choose Cartesian coordinates $\vec{F} = \sum_{i=1}^{3} F^i \vec{e}_i$, $\vec{x} = \sum_{i=1}^{3} x^i \vec{e}_i$, where $\vec{e}_i$ is an orthonormal Cartesian basis of $\mathbb{R}^3$. (1.1) is <u>invariant</u> under the Galilei-transformations

$$\begin{aligned} x^i &\to x^{i\prime} = x^i + v_0^i t + x_0^i \ , \\ t &\to t' = t + t_0 \ , \end{aligned} \tag{1.2}$$

where v_0^i, x_0^i, t_0 are constants. Indeed we can check

$$\frac{d}{dt} \to \frac{d}{dt'} = \frac{dt}{dt'}\frac{d}{dt} = \frac{d}{dt} \ , \qquad \ddot{\vec{x}} \to \ddot{\vec{x}}' = \ddot{\vec{x}} \ , \tag{1.3}$$

which shows the invariance of (1.1).

The Newton equation (1.1) is also <u>covariant</u> under rotations which can be expressed as

$$x^i \to x'^i = \sum_{k=1}^{3} D^i_k x^k \ . \tag{1.4}$$

The D^i_k are the constant matrix elements of the 3×3 rotation matrix $\mathbf{D}$ which leaves the length of a vector invariant, i.e.

$$\sum_i x^{i\prime} x^{i\prime} = \sum_{i,k,l} D^i_k x^k D^i_l x^l = \sum_j x^j x^j \ . \tag{1.5}$$

[1] In Mathematics this is often called equivariant.

This implies[2]

$$\sum_i D^i_k D^i_l = \delta_{kl} \ , \qquad \text{or in matrix notation,} \qquad D^T D = \mathbf{1} \ . \tag{1.6}$$

Under rotations not only x^i transforms as in (1.4) but all vectors (including F^i) do so that in the rotated coordinate system the Newton equation (1.1) reads

$$F^{i\prime} - m\ddot{x}^{i\prime} = \sum_k D^i_k \left(F^k - m\ddot{x}^k\right) = 0 \ . \tag{1.7}$$

We see that it is not invariant but both sides rotate identically (as vectors) so that the solutions are unaltered. One therefore calls the Newton equation "covariant under rotations".

In Newtons mechanics all coordinate systems which are related by Galilei-transformations (including rotations) are called <u>inertial reference frames</u> and in these coordinate systems the Newton equation (or rather their solutions) are identical. (This is sometimes called the Galilei principle.)

Historically the development of Special Relativity grew out of the observation that the Maxwell-equations governing Electrodynamics are not covariant under Galilei-transformations. The reason being that the speed of light c was shown to be a constant of nature in the experiments by Michelson and Moreley. The Maxwell-equations are instead covariant under Lorentz-transformations and Einstein postulated c being constant and consequently covariance with respect to Lorentz-transformations for any physical theory. (This is sometimes called the Einstein principle.) This led to the theory of Special Relativity.

1.3 Lorentz Transformations

For a 'particle' moving with the speed of light we have

$$c^2 = \sum_i \frac{dx^i}{dt}\frac{dx^i}{dt} = \sum_i \frac{dx^{i\prime}}{dt'}\frac{dx^{i\prime}}{dt'} \ , \tag{1.8}$$

where $x^{i\prime}, t'$ are the space and time coordinates of a system which moves with velocity $\vec{v}_0$ with respect to the unprimed coordinate system. Expressed infinitesimally, the line-element

$$ds^2 = c^2 dt^2 - \sum_i dx^i dx^i = c^2 dt'^2 - \sum_i dx^{i\prime} dx^{i\prime} \tag{1.9}$$

has to be constant in all coordinate systems (with $ds^2 = 0$ for light rays). It turns out that Lorentz transformations non-trivially rotate space and time coordinates into each other and are best described as rotations of a four-dimensional pseudo-Euclidean Minkowskian space-time $\mathbb{M}_4$. The coordinates of $\mathbb{M}_4$ are the 4-tupel (denoting a space-time point)

$$x^\mu = (x^0, x^i) = (x^0, x^1, x^2, x^3) \ , \quad \mu = 0, \ldots, 3, \quad i = 1, 2, 3 \ , \tag{1.10}$$

[2] Due to the six equations (1.6) the matrix D depends on 9-6=3 independent parameters which are the three rotation angles. The matrices D form the Group O(3). We will return to the group-theoretical description in QFT II.

where $x^0 = ct$. Furthermore, on $\mathbb{M}_4$ distances are measured by the line-element(1.9) which expressed in terms of x^μ reads

$$ds^2 = (dx^0)^2 - \sum_i dx^i dx^i = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu \ . \tag{1.11}$$

$\eta_{\mu\nu}$ is the (pseudo-euclidean) metric on $\mathbb{M}_4$ given by

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{1.12}$$

The inverse metric is denoted by $\eta^{\nu\rho}$ and obeys $\sum_\nu \eta_{\mu\nu}\eta^{\nu\rho} = \delta^\rho_\mu$.

Lorentz transformations can now be defined as rotation in $\mathbb{M}_4$ which leave its line element ds^2 defined in (1.11) invariant. In the linear coordinate transformations in $\mathbb{M}_4$

$$x^\mu \to x'^\mu = \sum_\nu \Lambda^\mu{}_\nu x^\nu \tag{1.13}$$

which leave ds^2 invariant, i.e. which obey

$$\sum_{\mu,\nu} \eta_{\mu\nu} dx'^\mu dx'^\nu = \sum_{\mu,\nu}\sum_{\rho,\sigma} \eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma dx^\rho dx^\sigma = \sum_{\rho,\sigma} \eta_{\rho\sigma} dx^\rho dx^\sigma \tag{1.14}$$

are Lorentz transformations. (1.14) imply the following ten conditions for the matrix elements $\Lambda^\mu{}_\rho$

$$\sum_{\mu,\nu} \eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \qquad \Leftrightarrow \qquad \Lambda^T \eta \Lambda = \eta \ , \tag{1.15}$$

where the latter is the matrix form of the former.[3]

The matrix Λ has $16 - 10 = 6$ independent matrix elements which turn out to be 3 rotation angles and 3 boost velocities. The rotation angles can be easily identified by noting that a transformation of the form (1.13) with

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \tag{1.16}$$

solve (1.15) and precisely correspond to the rotations (1.4). As we already discussed above D depends on three rotation angles.

In order to discuss the Lorentz boost let us first look at Lorentz transformations which only transform t (or rather x^0) and x^1 non-trivially. In this case Λ takes the form

$$\Lambda = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & 0 & 0 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{1.17}$$

[3]The corresponding group is the orthogonal group O(1,3), where the numbers indicates the signature of the underlying metric (1 plus, 3 minuses).

corresponding to the transformations

$$
\begin{aligned}
(ct') &= \Lambda^0{}_0 ct + \Lambda^0{}_1 x \ , \\
x' &= \Lambda^1{}_0 ct + \Lambda^1{}_1 x \ , \\
y' &= y \ , \\
z' &= z \ .
\end{aligned}
\tag{1.18}
$$

Inserted into (1.15) we obtain

$$
\begin{aligned}
(\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 &= \quad 1 \ , \\
(\Lambda^0{}_1)^2 - (\Lambda^1{}_1)^2 &= \ -1 \ , \\
\Lambda^0{}_0\Lambda^0{}_1 - \Lambda^1{}_0\Lambda^1{}_1 &= \quad 0 \ ,
\end{aligned}
\tag{1.19}
$$

which are solved by

$$
\begin{aligned}
\Lambda^0{}_0 &= \quad \cosh\theta = \Lambda^1{}_1 \ , \\
\Lambda^1{}_0 &= \ -\sinh\theta = \Lambda^0{}_1 \ ,
\end{aligned}
\tag{1.20}
$$

or in other words

$$
\Lambda = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{1.21}
$$

The parameter θ is called the rapidity. It is related to the relative velocity $v_0^1 \equiv v$ of the two coordinate systems. Let us choose the primed coordinate system to be at rest while the unprimed system moves with velocity v in the x^1-direction. Thus $x' = \Lambda^1{}_0 ct + \Lambda^1{}_1 vt = 0$ together with (1.20)implies

$$
\Lambda^1{}_0 = -\frac{v}{c}\Lambda^1{}_1 \quad \rightarrow \quad \sinh\theta = \frac{v}{c}\cosh\theta \quad \rightarrow \quad \tanh\theta = \frac{v}{c} \tag{1.22}
$$

Differently expressed we have

$$
\cosh\theta = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \equiv \gamma \ , \qquad \sinh\theta = \frac{v}{c}\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \beta\gamma \tag{1.23}
$$

or in matrix form

$$
\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ .
$$

Analogously one can determine the general solution of (1.15) for the unprimed system moving with velocity $\vec{v} = \sum_i v^i \vec{e}_i$ to be

$$
\begin{aligned}
&\Lambda^0{}_0 = \gamma \ , \qquad \Lambda^0{}_j = \gamma\,\frac{v_j}{c} = \Lambda^j{}_0 \\
&\Lambda^i{}_j = \delta^i_j + (\gamma-1)\,\frac{v_i v_j}{\vec{v}^2} \ , \qquad \gamma \equiv \frac{1}{\sqrt{1-\vec{v}^2/c^2}} \ .
\end{aligned}
\tag{1.24}
$$

On $\mathbb{M}_4$ one defines contravariant 4-vectors as

$$a^\mu = (a^0, a^i) \ , \tag{1.25}$$

with a transformation law

$$a^\mu \to a^{\mu\prime} = \sum_\rho \Lambda^\mu_\rho a^\rho \ . \tag{1.26}$$

A covariant 4-vector has a lower index and is defined as

$$a_\mu = \sum_\nu \eta_{\mu\nu} a^\nu = (a^0, -a^i) \ . \tag{1.27}$$

It transforms inversely as

$$a_\mu \to a_{\mu\prime} = \sum_\rho \Lambda^{-1T\rho}_\mu a_\rho \ , \tag{1.28}$$

where $\sum_\rho \Lambda^{-1T\rho}_\mu \Lambda^\nu_\rho = \delta^\nu_\mu$. The scalar product of two 4-vectors is defined by

$$\sum_\mu a^\mu b_\mu = \sum_{\mu,\nu} \eta_{\mu\nu} a^\mu b^\nu = a^0 b^0 - \sum_i a^i b^i \tag{1.29}$$

which is indeed a Lorentz scalar.

The partial derivatives can also be formulated as a 4-vector of $\mathbb{M}_4$. One defines

$$\begin{aligned} \partial_\mu &\equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) \ , \\ \partial^\mu &= \sum_\nu \eta^{\mu\nu}\partial_\nu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) \ . \end{aligned} \tag{1.30}$$

In this notation the wave operator $\Box$ is given by

$$\Box = \sum_\mu \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \ . \tag{1.31}$$

1.4 Relativistic wave-equations

Let us first recall the definition of the proper time τ which is the time measured in the rest-frame of a particle. Here one has

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\vec{x}^2 = (c^2 - \vec{v}^2)dt^2 \ , \tag{1.32}$$

where the second equation relates it to the time measured by a moving (with velocity $\vec{v}$) observer. Thus they are related by

$$d\tau = \sqrt{1 - \left(\frac{\vec{v}}{c}\right)^2} dt \ , \qquad \tau = \int_{t_1}^{t_2} \gamma^{-1} dt \tag{1.33}$$

The 4-momentum of a particle is defined as

$$p^\mu = m\frac{dx^\mu}{d\tau} = m\gamma\,\frac{dx^\mu}{dt} \ , \tag{1.34}$$

where m is the rest-mass, i.e. the mass measured in the rest frame. The three-momentum thus is given by

$$\vec{p} \;=\; m\gamma\,\vec{v} \;=\; m\vec{v} + \mathcal{O}(v^3/c^2) \quad \text{for} \quad |\vec{v}| \ll c\;, \tag{1.35}$$

cp^0 is identified with the energy

$$cp^0 \;\equiv\; E \;=\; m\gamma c^2 \;=\; mc^2 + \tfrac{1}{2}mv^2 \quad \text{for} \quad |\vec{v}| \ll c \tag{1.36}$$

The term mc^2 is known as the rest energy of a particle.

Let us compute the Lorentz scalar

$$c^2 \sum_\mu p^\mu p_\mu = E^2 - c^2\vec{p}^2 = m^2c^4 \tag{1.37}$$

This is known as the relativistic energy-momentum relation

$$E \;=\; c\sqrt{m^2c^2 + \vec{p}^2} \;\approx\; \begin{cases} mc^2 + \frac{\vec{p}^2}{2m} + \ldots & \text{for } |\vec{p}| \ll mc \\ c|\vec{p}| + \ldots & \text{for } |\vec{p}| \gg mc \end{cases} \tag{1.38}$$

Inserting the quantum-theoretical correspondences

$$\vec{p} \to \frac{\hbar}{i}\nabla\;, \qquad E \to i\hbar\frac{\partial}{\partial t} \tag{1.39}$$

into (1.37) using (1.31) yields the differential operator

$$E^2 - c^2\vec{p}^2 - m^2c^4 \to -\hbar^2\frac{\partial^2}{\partial t^2} + \Delta - m^2c^4 = -c^2\hbar^2\left(\Box + \frac{mc^2}{\hbar^2}\right) = 0\;, \tag{1.40}$$

Therefore a relativistically invariant wave equation for a scalar field ϕ is given by

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0\;. \tag{1.41}$$

This is the massive Klein-Gordon-equation which we study in the next sections. In the following we also use natural units where we set $\hbar = c = 1$. In this convention the Klein-Gordon-equation reads

$$\left(\Box + m^2\right)\phi = 0\;. \tag{1.42}$$

Note that for $m = 0$ we obtain the wave equation $\Box\phi = 0$.

Henceforth we also use the summation convention which says that indices which occur twice are automatically summed over (unless stated otherwise). For example $\sum_\mu p^\mu p_\mu$ will simply be denoted by $p^\mu p_\mu$.

2 Lecture 2: Classical Scalar Field Theory

2.1 Lagrangian and Hamiltonian formalism

A scalar field $\phi(x^\mu)$ can be viewed as a map from $\mathbb{M}_4$ into $\mathbb{R}$ if ϕ is real or $\mathbb{C}$ if ϕ is complex. Thus associated with at every space-time point x^μ is number the value of ϕ at that point. Examples are the scalar potential in Electrodynamics or the Higgs-field in Particle Physics.[4]

As for mechanical systems with a finite number of degrees of freedom it is useful to develop a Lagrangian and Hamiltonian formalism for fields (systems which have an infinite number of degrees of freedom). Let us be slightly more general and discuss this formalism immediately for a collection of N scalar fields $\phi_r(x)$, $r = 1, \ldots, N$. (From now on we indicate the dependence of a field on x^μ simply by x, i.e. we use $\phi(x^\mu) \equiv \phi(x)$.)

The Euler-Lagrange equations follow from a stationary condition of an action S given by

$$S = \int dt\, L = \int_V d^4x\, \mathcal{L}(\phi_r, \partial_\mu \phi_r) \ . \tag{2.1}$$

Here $\mathcal{L}(\phi_r, \partial_\mu\phi_r)$ is a Lagrangian density which depends on the fields and their first derivatives. d^4x is the Lorentz invariant volume element and V is the volume of $\mathbb{M}_4$. Sometimes it is useful to consider the Lagrangian L which is related to its density via $L = \int d^3x\, \mathcal{L}$.

As in classical mechanics one postulate an action principle, i.e. one demands that the physical system on its trajectory extremizes S. Or in other words one demands that fluctuations of ϕ ($\phi_r \to \phi_r + \delta\phi_r$) which vary the action as $S \to S + \delta S$ are such that $\delta S = 0$ holds on the (classical) trajectory subject to the boundary condition $\delta\phi_r|_{\partial V} = 0$, where ∂V denotes the boundary of V. Applied to (2.1) one therefore obtains

$$0 = \delta S = \int_V d^4x \sum_{r=1}^{N} \left(\frac{\partial \mathcal{L}}{\partial \phi_r}\, \delta\phi_r + \sum_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\, \delta(\partial_\mu \phi_r) \right) \tag{2.2}$$

where (in complete analogy to classical mechanics) the fields ϕ_r and their derivatives $\partial_\mu\phi_r$ are treated as independent variables. Using $\delta(\partial_\mu\phi_r) = \partial_\mu(\delta\phi_r)$ the variation (2.2) can be rewritten as

$$0 = \int_V d^4x \sum_{r=1}^{N} \left[\frac{\partial \mathcal{L}}{\partial \phi_r}\, \delta\phi_r + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\, \delta\phi_r \right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \right) \delta\phi_r \right] \ , \tag{2.3}$$

where we now also used the summation convention introduced at the end of the previous section. The integral of the second term vanishes by virtue of the Gauss law and the constraint that $\delta\phi_r$ is zero at the boundary

$$\int_V d^4x\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\, \delta\phi_r \right) = \int_{\partial V} dF_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\, \delta\phi_r = 0 \ , \tag{2.4}$$

[4] From Electrodynamics you are also familiar with vector fields $\vec{E}(\vec{x}, t), \vec{B}(\vec{x}, t)$ which associate with with at every space-time point a vector (or rather two vectors). However, for pedagogical reasons we first discuss in this section a scalar field ϕ.

where dF_μ is the 3-dimensional normal surface element on ∂V. So from (2.3) we are left with

$$0 = \int_V d^4x \sum_{r=1}^{N} \left[\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \right] \delta\phi_r \tag{2.5}$$

for arbitrary and independent $\delta\phi_r$. Therefore (2.5) can be fulfilled if and only if the bracket itself vanishes. This results in the Euler-Lagrange equations for (scalar) fields

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} = 0 \qquad \text{for all } r = 1, \ldots, N \ . \tag{2.6}$$

These are the exact analogues of the Euler-Lagrange equations in classical mechanics $\frac{\partial L}{\partial q_r} - \frac{d}{dt}\frac{\partial L}{\partial(\dot{q}_r)} = 0$ for the generalized coordinates q_r.

In order to introduce the Hamiltonian formalism for fields one first defines the canonically conjugated momentum π_r of ϕ_r (strictly speaking it is a momentum *density*)

$$\pi_r \ := \ \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} \ , \qquad \text{with} \qquad \dot{\phi}_r \equiv \partial_0 \phi_r \ . \tag{2.7}$$

The Hamiltonian of a classical field theory is then given by

$$H(\phi_r, \pi_r) \equiv \int d^3x \, \mathcal{H}(\phi_r, \pi_r) := \int d^3x \sum_{r=1}^{N} \left(\pi_r \dot{\phi}_r - \mathcal{L} \right) \ . \tag{2.8}$$

$\mathcal{H}$ is called the Hamiltonian density. Note that $\mathcal{L}$ is a Lorentz scalar while $\mathcal{H}$ is not!

2.2 Real Scalar Field

We take the Lagrangian density of a real massive scalar field ϕ to be

$$\mathcal{L}(\phi, \partial_\mu \phi) = \tfrac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) \ , \tag{2.9}$$

where we have chosen 'natural' units $\hbar = c = 1$. (This convention will be used henceforth.) From the Lagrangian (2.9) we can compute

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \ , \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \, . \tag{2.10}$$

Inserted into the Euler-Lagrange equation (2.6) yields

$$\left(\Box + m^2 \right) \phi = 0 \tag{2.11}$$

which we immediately recognize as the massive Klein-Gordon equation (1.42) in natural units. Therefore, (2.9) indeed is the Lagrangian whose Euler-Lagrange equation is the Klein-Gordon equation.

In order to compute the corresponding Hamiltonian we first determine the conjugate momentum from (2.7) to be

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \ . \tag{2.12}$$

Inserted into (2.8) we arrive at

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} = \dot{\phi}^2 - \tfrac{1}{2} \left(\dot{\phi}^2 - \vec{\nabla}\phi \cdot \vec{\nabla}\phi - m^2 \phi^2 \right) = \tfrac{1}{2} \left(\pi^2 + \vec{\nabla}\phi \cdot \vec{\nabla}\phi + m^2 \phi^2 \right) \ . \tag{2.13}$$

2.3 Noether-Theorem

The Noether-Theorem states that associated with any continues symmetry there is a conserved (Noether) charge Q. This also applies to the field theories discussed here. Let us derive the theorem.

Consider an (arbitrary) infinitesimal continues symmetry which transforms the scalar fields as

$$\phi_r \to \phi_r' = \phi_r + \alpha \Delta\phi_r \ , \tag{2.14}$$

where α is the infinitesimal parameter of the transformation and $\Delta\phi$ is the deformation of the field configuration. Being a symmetry implies that the action is unchanged under the transformation (2.14), i.e.

$$S(\phi_r) = S(\phi_r') \ . \tag{2.15}$$

This implies that Lagrangian density $\mathcal{L}$ changes at most by a total divergence or in other words

$$\mathcal{L}(\phi, \partial_\mu\phi) \to \ \mathcal{L}(\phi', \partial_\mu\phi') = \mathcal{L}(\phi, \partial_\mu\phi) + \alpha\partial_\mu \mathcal{J}^\mu \ , \tag{2.16}$$

where $\mathcal{J}^\mu$ is constrained to vanish at the boundary ∂V. Inserting the transformation (2.14) into $\mathcal{L}'$ and Taylor-expanding to first order in α yields

$$\begin{aligned}
\mathcal{L}(\phi_r', \partial_\mu\phi_r') &= \mathcal{L}(\phi_r + \alpha\Delta\phi_r, \partial_\mu(\phi_r + \alpha\Delta\phi_r)) \\
&= \mathcal{L}(\phi, \partial_\mu\phi) + \alpha \sum_r \frac{\partial \mathcal{L}}{\partial \phi_r}\Delta\phi_r + \alpha \sum_r \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_r)}\partial_\mu\Delta\phi_r \\
&= \mathcal{L}(\phi, \partial_\mu\phi) + \alpha \sum_r \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_r)}\right]}_{=0,\,\text{Euler}-\text{Lagrange}} \Delta\phi_r + \alpha\partial_\mu \left(\sum_r \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_r)}\Delta\phi_r\right) \\
&\overset{!}{=} \mathcal{L}(\phi, \partial_\mu\phi) + \alpha\partial_\mu\mathcal{J}^\mu \ .
\end{aligned} \tag{2.17}$$

If one now defines the Noether current as

$$j^\mu \equiv \sum_r \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_r)}\Delta\phi_r - \mathcal{J}^\mu \ , \tag{2.18}$$

(2.17) implies the continuity equation

$$\partial_\mu j^\mu = 0 \ . \tag{2.19}$$

The associated Noether charge Q

$$Q \equiv \int d^3x\, j^0, \tag{2.20}$$

is then conserved, i.e.

$$\frac{dQ}{dt} = \int_V d^3x\, \partial_0 j^0 = \int_V d^3x\, \vec{\nabla}\cdot\vec{j} = \int_{\partial V} d\vec{F}\cdot\vec{j} = 0 \ , \tag{2.21}$$

whenever $\vec{j}|_{\partial V} = 0$ holds. Note that in the derivation of the conservation law we used the current conservation (2.19) which only holds if the Euler-Lagrange equations are satisfied.

Now we can state the Noether-Theorem for field theories which says that for every continuous symmetry there is a Noether current j^μ satisfying $\partial_\mu j^\mu = 0$ and an associated conserved Noether charge Q satisfying $\dot{Q} = 0$.

As an example consider again the real scalar field with the Lagrangian (2.9)

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2.$$

For $m = 0$ it has the shift symmetry

$$\phi \to \phi' = \phi + \alpha \ , \tag{2.22}$$

where α is a constant parameter. Comparing with (2.14) we conclude $\Delta\phi = 1$ and we cvan indeed check that (2.22) is a symmetry

$$\mathcal{L}(\phi') = \tfrac{1}{2}\partial_\mu\phi'\partial^\mu\phi' = \tfrac{1}{2}\partial_\mu(\phi+\alpha)\partial^\mu(\phi+\alpha) = \mathcal{L}(\phi) \ . \tag{2.23}$$

Comparing with (2.16) we see that $\mathcal{L}$ is invariant under the transformation (2.22) with $\mathcal{J}^\mu = 0$. Using (2.18) we compute the Noether current to be

$$j^\mu = \partial^\mu\phi \ , \tag{2.24}$$

and check its conservation by explicitly using the Euler-Lagrange equation (i.e. the massless wave equation $\Box\phi = 0$)

$$\partial_\mu j^\mu = \partial_\mu\partial^\mu\phi = \Box\phi = 0 \ . \tag{2.25}$$

As a further application let us considers constant space-time translations of the form

$$x^\mu \to x^{\mu\prime} = x^\mu + a^\mu \tag{2.26}$$

for four constant and infinitesimal parameters a^μ. The scalar field ϕ then changes according to

$$\phi(x) \to \phi(x') = \phi(x^\mu + a^\mu) = \phi(x^\mu) + a^\mu\partial_\mu\phi(x^\mu) + \mathcal{O}(a^2) \ . \tag{2.27}$$

In the previous terminology we thus have four independent transformation (three space translation and one time translation) so that $\alpha\Delta\phi = a^\mu\partial_\mu\phi$. The corresponding Noether current is the energy-momentum tensor

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\nu\phi - \delta^\mu_\nu\mathcal{L} \ . \tag{2.28}$$

It obeys the conservation law

$$\partial_\mu T^\mu_\nu = 0 \ , \tag{2.29}$$

with the four Noether charges being

$$H = \int d^3x\, T^{00} \ , \qquad p^i = \int d^3x\, T^{0i} \ , \quad i = 1,2,3 \ . \tag{2.30}$$

3 Lecture 3: Canonical quantization of a scalar field

Before we quantize the scalar field let us determine the solution of the Klein-Gordon equation (2.11)

$$(\Box + m^2)\phi(x) = 0 \ . \tag{3.1}$$

3.1 Solution of Klein-Gordon equation

This is most easily done with the help of a Fourier-Ansatz

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4}\, \tilde{\phi}(p)\, e^{-ip.x} \ , \tag{3.2}$$

where we abbreviate $p \cdot x \equiv p_\mu x^\mu$. Inserted into (3.1) we arrive at

$$(\Box + m^2)\phi(x) = \int \frac{d^4p}{(2\pi)^4}\, \tilde{\phi}(p)(m^2 - p^2)\, e^{-ip\cdot x} = 0 \ . \tag{3.3}$$

This is solved by either setting $\tilde{\phi}(p) = 0$ or $p^2 = m^2$ with $\tilde{\phi}(p)$ arbitrary. Or in other words

$$\tilde{\phi}(p) = \frac{a_{\vec{p}}}{\sqrt{2E_{\vec{p}}}}\, \delta(p^0 - E_{\vec{p}}) + \frac{a^\dagger_{\vec{p}}}{\sqrt{2E_{\vec{p}}}}\, \delta(p^0 + E_{\vec{p}}) \ , \tag{3.4}$$

where we abbreviate $E_{\vec{p}} \equiv \sqrt{|\vec{p}|^2 + m^2}$ and the coefficients in front of the δ-function are chosen for later convenience. Inserted into (3.2) we obtain the solution

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{ip\cdot x} \right)\Big|_{p^0 = E_{\vec{p}}} \ , \tag{3.5}$$

where in the second term we have replaced $\vec{p} \to -\vec{p}$ since the integration region is symmetric.

The conjugate momentum is computed to be

$$\pi(x) = \frac{\partial}{\partial x^0}\, \phi(x) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_{\vec{p}}}{2}} \left(a_{\vec{p}}\, e^{-ip\cdot x} - a^\dagger_{\vec{p}}\, e^{ip\cdot x} \right) . \tag{3.6}$$

(3.5) can be inverted (see problem 2.2) to give

$$\begin{aligned} a_{\vec{p}} &= \frac{1}{\sqrt{2E_{\vec{p}}}} \int d^3x\, e^{ip\cdot x}\, (E_{\vec{p}}\, \phi(x) + i\, \pi(x)) \\ a^\dagger_{\vec{p}} &= \frac{1}{\sqrt{2E_{\vec{p}}}} \int d^3x\, e^{-ip\cdot x}\, (E_{\vec{p}}\, \phi(x) - i\, \pi(x)) \, . \end{aligned} \tag{3.7}$$

$a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ are time independent, i.e. $\partial_t\, a_{\vec{p}} = \partial_t\, a^\dagger_{\vec{p}} = 0$ (see problem 2.2).

3.2 Quantizing the scalar field

In canonical quantization $\phi(x)$ and its conjugate momentum are replaced by Hermitian operators

$$\phi(x) \to \hat{\phi}(x), \quad \hat{\phi}^\dagger(x) = \hat{\phi}(x); \qquad \pi(x) \to \hat{\pi}(x), \quad \hat{\pi}^\dagger(x) = \hat{\pi}(x) \ . \tag{3.8}$$

with canonical (equal-time) commutation relations

$$\begin{aligned} \Big[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{y},t)\Big] &= i\delta^{(3)}(\vec{x}-\vec{y}) \ , \\ \Big[\hat{\pi}(\vec{x},t), \hat{\pi}(\vec{y},t)\Big] &= 0 = \Big[\hat{\phi}(\vec{x},t), \hat{\phi}(\vec{y},t)\Big] \ . \end{aligned} \tag{3.9}$$

Inserting (3.5) and (3.6) we see that also $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ have to be promoted to operators

$$a_{\vec{p}} \to \hat{a}_{\vec{p}} \ , \qquad a^\dagger_{\vec{p}} \to \hat{a}^\dagger_{\vec{p}} \ , \tag{3.10}$$

with commutation relations which follow from (3.9). Using (3.7) and (3.9) one computes

$$\begin{aligned} \Big[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}\Big] &= \frac{i}{2\sqrt{E_{\vec{p}}E_{\vec{p}'}}} \int d^3x \int d^3y \, e^{i\vec{p}\,\vec{x}} e^{-i\vec{p}'\vec{y}} \Big(E_{\vec{p}'} \underbrace{\Big[\hat{\pi}(x), \hat{\phi}(y)\Big]}_{-i\delta^{(3)}(\vec{y}-\vec{x})} - E_{\vec{p}} \underbrace{\Big[\hat{\phi}(x), \hat{\pi}(y)\Big]}_{i\delta^{(3)}(\vec{x}-\vec{y})} \Big) \\ &= \frac{i}{2\sqrt{E_{\vec{p}}E_{\vec{p}'}}} \int d^3x \, e^{i(\vec{p}-\vec{p}')\vec{x}} \left(-iE_{\vec{p}'} - iE_{\vec{p}}\right) = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}') \ , \end{aligned} \tag{3.11}$$

where we evaluated the right hand side at $x^0 = y^0 = 0$ since we already know that $\hat{a}_{\vec{p}}$ is time-independent. Similarly one shows that the $\hat{a}_{\vec{p}}$ commute so that altogether one has

$$\begin{aligned} \big[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}\big] &= (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}') \ , \\ \big[\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}'}\big] &= 0 = \big[\hat{a}^\dagger_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}\big] . \end{aligned} \tag{3.12}$$

We thus see that $\hat{a}_{\vec{p}}$ and $\hat{a}^\dagger_{\vec{p}}$ provide an infinite set of harmonic oscillator creation and annihilation operators for each value of $\vec{p}$

We can also express the Hamiltonian $\hat{H}$ in terms of $\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}}$. Using (2.8), (3.5) and (3.7) one obtains

$$\hat{H} = \int d^3x \, \hat{\mathcal{H}} = \int \frac{d^3p}{(2\pi)^3} \, E_{\vec{p}} \left(\hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} + \tfrac{1}{2} \underbrace{\big[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}}\big]}_{(2\pi^3)\delta^{(3)}(0)} \right) . \tag{3.13}$$

(For the explicit computation see problem 2.3.) We see that it is divergent due to the summation of an infinite number of harmonic oscillator ground state energies. Since only energy differences can be measured this term is neglected and one has[5]

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} \, E_{\vec{p}} \, \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} \ . \tag{3.14}$$

[5]We will return to this issue later.

One further has

$$\left[\hat{H}, \hat{a}^\dagger_{\vec{p}\,'}\right] = \int \frac{d^3p}{(2\pi)^3}\, E_{\vec{p}}\, \hat{a}^\dagger_{\vec{p}} \underbrace{\left[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}\,'}\right]}_{(2\pi)^3\delta(\vec{p}-\vec{p}\,')} = E_{\vec{p}\,'}\, \hat{a}^\dagger_{\vec{p}\,'}\ ,$$

$$\left[\hat{H}, \hat{a}_{\vec{p}\,'}\right] = \int \frac{d^3p}{(2\pi)^3}\, E_{\vec{p}} \underbrace{\left[\hat{a}^\dagger_{\vec{p}}, \hat{a}_{\vec{p}\,'}\right]}_{-(2\pi)^3\delta(\vec{p}-\vec{p}\,')} \hat{a}_{\vec{p}} = -E_{\vec{p}\,'}\, \hat{a}_{\vec{p}\,'}\ . \tag{3.15}$$

3.3 Fock-Space

Since we have an infinite set of harmonic oscillators we can construct the space of states algebraically. One first defines a vacuum state $|0\rangle$ by demanding that it is annihilated by all $a_{\vec{p}}$[6]

$$a_{\vec{p}}\,|0\rangle = 0 \qquad \forall\ \vec{p}\ . \tag{3.16}$$

Then n-particle states are constructed by acting with the creation operators $a^\dagger_{\vec{p}}$ on $|0\rangle$

$$\text{one-particle-state:} \qquad |\vec{p}\rangle := \sqrt{2E_{\vec{p}}}\, a^\dagger_{\vec{p}}\,|0\rangle\ .$$

$$\text{N-particle-state:} \qquad |\vec{p}_1 \ldots \vec{p}_N\rangle := \sqrt{2E_{\vec{p}_1} \ldots 2E_{\vec{p}_N}}\, a^\dagger_{\vec{p}_1} \ldots a^\dagger_{\vec{p}_N}\,|0\rangle\ . \tag{3.17}$$

The corresponding energies are

$$H\,|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\,\left[H, a^\dagger_{\vec{p}}\right]|0\rangle = E_{\vec{p}}\,|\vec{p}\rangle\ ,$$

$$H\,|\vec{p}_1 \ldots \vec{p}_N\rangle = \ldots = (E_{\vec{p}_1} + \ldots + E_{\vec{p}_N})|\vec{p}_1 \ldots \vec{p}_N\rangle\ . \tag{3.18}$$

In order to compute the momentum of the states we first determine $P^{\,i}$ from (2.30) to be (for details see problem 2.3)

$$P^{\,i} = \int \frac{d^3p}{(2\pi)^3}\, p^i\, a^\dagger_{\vec{p}}\, a_{\vec{p}}\ . \tag{3.19}$$

Acting on a one-particle state yields

$$P^{\,i}\, a^\dagger_{\vec{p}}|0\rangle = \int \frac{d^3p\,'}{(2\pi)^3}\, p\,'^i\, a^\dagger_{\vec{p}\,'}\, a_{\vec{p}\,'}\, a^\dagger_{\vec{p}}|0\rangle = \int \frac{d^3p\,'}{(2\pi)^3}\, p\,'^i\, a^\dagger_{\vec{p}\,'} \underbrace{\left[a_{\vec{p}\,'}, a^\dagger_{\vec{p}}\right]}_{(2\pi)^3\delta(\vec{p}-\vec{p}\,')} |0\rangle = p^{\,i}\, a^\dagger_{\vec{p}}|0\rangle\ . \tag{3.20}$$

Thus, the state $|\vec{p}\rangle$ is eigenstate of the momentum operator $P^{\,i}$ with eigenvalue $p^{\,i}$.

Let us compute

$$\phi(x)\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{ip\cdot x}\right)|0\rangle \ = \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{\sqrt{2E_{\vec{p}}}}\, e^{ip\cdot x}\, a^\dagger_{\vec{p}}\,|0\rangle\ .$$

This corresponds to the superposition of a single particle momentum eigenstate and thus corresponds to a particle at position x^μ.

Finally we note that $\phi(x)$ and $\pi(x)$ obey the Heisenberg equations

$$i\,\partial_t\,\phi(x) = [\phi(x), H]\ , \qquad i\,\partial_t\,\pi(x) = [\pi(x), H]\ . \tag{3.21}$$

(For the computation see problem 2.4.)

[6] We drop the ^ henceforth from all operators.

4 Lecture 4: The Feynman propagator for a scalar field

Before we turn to the Dirac equation in the next few lectures let us already prepare for treating an interacting scalar field and first solve the inhomogeneous Klein-Gordon equation with the help of a Greens function.

Let first recall that an inhomogeneous partial differential equation can solved with the help of a Greens function. Let D_x be an arbitrary differential operator which differentiates with respect to x, $\Phi(x)$ the field under consideration and $f(x)$ an arbitrary source function. $\Phi(x)$ is taken to obey the differential equation

$$D_x \Phi(x) = f(x) \; . \tag{4.1}$$

The Greens function $G(x-y)$ of D_x is defined to be the solution of

$$D_x G(x-y) = -i\delta(x-y) \; . \tag{4.2}$$

In terms of $G(x-y)$ the general solution of (4.1) is given by

$$\Phi(x) = i \int dy \; G(x-y) f(y) + \Phi_{\text{hom}}(x) \; , \tag{4.3}$$

where $\Phi_{\text{hom}}(x)$ is the homogeneous solution satisfying $D_x \Phi_{\text{hom}}(x) = 0$. Indeed inserting (4.3) into (4.1) we can check

$$D_x \Phi(x) = i \int dy \; \underbrace{D_x G(x-y)}_{-i\delta(x-y)} f(y) + \underbrace{D_x \Phi_{\text{hom}}(x)}_{0} = \int dy \; \delta(x-y) f(y) = f(x) \; . \tag{4.4}$$

In the following we want to determine the Greens function $G(x-y)$ for the Klein-Gordon-operator $(\Box_x + m^2)$, i.e. solve

$$(\Box_x + m^2) G(x-y) = -i\delta^{(4)}(x-y) \tag{4.5}$$

As in the last lecture this is most easily done by a Fourier Ansatz

$$G(x-y) = \int \frac{d^4p}{(2\pi)^4} \; e^{-ip\cdot(x-y)} \; \tilde{G}(p) \; . \tag{4.6}$$

Inserted into (4.5) yields

$$\begin{aligned} (\Box_x + m^2) G(x-y) &= \int \frac{d^4p}{(2\pi)^4} \; \tilde{G}(p) \; (m^2 - p^2) e^{-ip\cdot(x-y)} \\ &\stackrel{!}{=} -i \int \frac{d^4p}{(2\pi)^4} \; e^{-ip\cdot(x-y)} = -i\delta^{(4)}(x-y). \end{aligned} \tag{4.7}$$

This implies

$$\tilde{G} = \frac{i}{p^2 - m^2} \; , \qquad \text{and} \qquad G(x-y) = \int \frac{d^4p}{(2\pi)^4} \; e^{-ip\cdot(x-y)} \; \frac{i}{p^2 - m^2}. \tag{4.8}$$

The representation (4.8) for $G(x-y)$ can be further simplified by performing the integral in p^0 with the help of the residue formula. Let us now see this in more detail by first noting that the integral in (4.8) is not well defined at the poles of the integrand $p^0 = \pm E_{\vec{p}} = \pm\sqrt{|\vec{p}|^2 + m^2}$. Thus we have to give a prescription for how to treat these poles. The introduces an ambiguity into $G(x-y)$ and, as we will see shortly, leads to four different G's.

If we analytically continue the integral into the complex p^0 plane, we can "integrate around" the singularities. This can be done in four different ways

1. choose $\mathrm{Im}p^0 > 0$ to avoid the singularities,
2. choose $\mathrm{Im}p^0 < 0$ to avoid the singularities ,
3. avoid $p^0 = -E_{\vec{p}}$ by choosing $\mathrm{Im}p^0 < 0$ near this point and $\mathrm{Im}p^0 > 0$ near $p^0 = E_{\vec{p}}$,
4. choose the opposite path of 3.

The strategy now is to close the path in the complex p^0 plane and use the residue formula which states that the value of the integral

$$c_{-1} = \frac{1}{2\pi i}\oint f(z)dz \tag{4.9}$$

is determined by the coefficient of the single pole z_0 in the Laurent-series of $f(z)$, i.e.

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \ldots + \frac{c_{-1}(z_0)}{z-z_0} + \ldots \ . \tag{4.10}$$

Let us see how this can be done for the four different path given above.

1. For $x^0 > y^0$ the exponent $e^{-ip^0(x^0-y^0)}$ does not contribute in the integral for $p^0 \to -i\infty$. Therefore we can close the contour in the lower half-plane and both poles are enclosed. On the other hand, for $x^0 < y^0$ we have to close the contour in the upper half-plane which then includes no pole and the integral vanishes. Thus we are left to compute

$$\begin{aligned} G(x-y) &= \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \frac{i^2}{(p^0+E_{\vec{p}})(p^0-E_{\vec{p}})} e^{-ip_0(x^0-y^0)} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \\ &= -i^2 \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \left[c_{-1}(p_0 = E_{\vec{p}}) + c_{-1}(p_0 = -E_{\vec{p}})\right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \left[e^{-iE_{\vec{p}}(x_0-y_0)} - e^{iE_{\vec{p}}(x_0-y_0)}\right] , \end{aligned} \tag{4.11}$$

where the minus sign in the second line comes from integrating counter-clockwise in the p^0 plane while in (4.9) the integration is clockwise. In the third line we used

$$c_{-1}(p_0 = E_{\vec{p}}) = \frac{e^{-iE_{\vec{p}}(x_0-y_0)}}{2E_{\vec{p}}} , \qquad c_{-1}(p_0 = -E_{\vec{p}}) = -\frac{e^{iE_{\vec{p}}(x_0-y_0)}}{2E_{\vec{p}}} . \tag{4.12}$$

The final results for case 1. is called the retarded Greens function and can be written more succinctly as

$$G_{ret} = \theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[\left[e^{-ip\cdot(x-y)}\right]_{p^0=E\vec{p}} - \left[e^{-ip\cdot(x-y)}\right]_{p^0=-E\vec{p}} \right]. \quad (4.13)$$

2. In this case only $x^0 < y^0$ contributes where the path is closed in the upper half-plane. One computes analogously the advanced Greens function

$$G_{av} = -\theta(y^0 - x^0) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[\left[e^{-ip\cdot(x-y)}\right]_{p^0=E\vec{p}} - \left[e^{-ip\cdot(x-y)}\right]_{p^0=-E\vec{p}} \right]. \quad (4.14)$$

3. In this case both contours contribute. For $x^0 > y^0$ the contour is closed in the lower half-plane and only the pole at $p^0 = E_{\vec{p}}$ contributes, i.e.

$$G(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[e^{-ip\cdot(x-y)}\right]_{p^0=E_{\vec{p}}} \quad (4.15)$$

For $x^0 < y^0$ the contour is closed in the upper half-plane and only the pole at $p^0 = -E\vec{p}$ contributes, i.e.

$$G(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[e^{-ip\cdot(x-y)}\right]_{p^0=-E_{\vec{p}}} \quad (4.16)$$

Together they form the Feynman-propagator

$$\begin{aligned} G_F = {} & \theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[e^{-ip\cdot(x-y)}\right]_{p^0=E\vec{p}} \\ & + \theta(y^0 - x^0) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[e^{-ip\cdot(x-y)}\right]_{p^0=-E\vec{p}}. \end{aligned} \quad (4.17)$$

4. In this case only the sign changes compared to case 3.

The physical meaning of these Greens functions becomes more transparent if we compute the amplitude for a particle to propagate from y^μ to x^μ which is given by $D(x-y) := \langle 0|\phi(x)\phi(y)|0\rangle$. Using (3.5), (3.12) and (3.16) we compute

$$\begin{aligned} D(x-y) &= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{p}'}}} e^{-ip\cdot x} e^{ip'\cdot y} \langle 0| \underbrace{\left[a_{\vec{p}}, a^\dagger_{\vec{p}'}\right]}_{(2\pi)^3\delta(\vec{p}-\vec{p}')} |0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[e^{-ip\cdot(x-y)}\right]_{p^0=E_{\vec{p}}} . \end{aligned} \quad (4.18)$$

Analogously one finds

$$D(y-x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[e^{-ip\cdot(x-y)}\right]_{p^0=-E_{\vec{p}}}. \quad (4.19)$$

Comparing (4.13), (4.14) and (4.17) with (4.18) and (4.19) we observe

$$\begin{aligned} G_{ret} &= \theta(x^0 - y^0)\,(D(x-y) - D(y-x)) \\ G_{av} &= \theta(y^0 - x^0)\,(D(y-x) - D(x-y)) \\ G_F &= \theta(x^0 - y^0)\ D(x-y) + \theta(y^0 - x^0)\ D(y-x). \end{aligned}$$

Finally, defining the time-ordered product of two operators A, B as

$$T\{A(x)B(y)\} := \theta(x^0 - y^0)A(x)B(y) + \theta(y^0 - x^0)B(y)A(x)\ , \tag{4.20}$$

we see that the Feynman-propagator coincides with

$$G_F = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle. \tag{4.21}$$

5 Lecture 5: The Dirac equation

Historically Dirac suggested the Dirac equation as a relativistic Schrödinger equation in order to incorporate relativistic effects into quantum mechanics. This approach turned out to be erroneous and only a fully field-theoretical description as we develop it here can properly include relativistic effects. Nevertheless the Dirac equation turned out to be the appropriate field equation for fermionic fields.

5.1 γ-matrices

Since the Schrödinger equation is first order in time while the Klein-Gordon equation is second order in time, Dirac took the “square root” of the wave operator $\Box$. That is he defined an operator

$$\partial\!\!\!/ := \sum_\mu \gamma^\mu \partial_\mu \tag{5.1}$$

such that $\partial\!\!\!/^2 = \Box$. This in turn implies

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ , \tag{5.2}$$

where the curly brackets denote the anti-commutator $\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$. We immediately note that the γ's cannot be numbers but have to be matrices. Furthermore, they are not unique since any γ'^μ defined as

$$\gamma'^\mu = M\gamma^\mu M^{-1} \tag{5.3}$$

with M being a non-singular complex matrix also satisfies (5.2):

$$\{\gamma'^\mu, \gamma'^\nu\} = M\{\gamma^\mu, \gamma^\nu\} M^{-1} = 2\eta^{\mu\nu} M M^{-1} = 2\eta^{\mu\nu}. \tag{5.4}$$

With a little more work one can show that the γ^μ have to be at least 4×4 matrices. Two prominent representations are the Dirac representation

$$\gamma_D^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \ , \qquad \gamma_D^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \ , \quad i = 1, 2, 3 \ , \tag{5.5}$$

where $\mathbf{1}$ is a 2×2 identity matrix and σ^i are the standard Pauli matrices which satisfy $\sigma^i\sigma^j = \delta^{ij}\mathbf{1} + i\epsilon^{ijk}\sigma^k$. The so called chiral representation is given by

$$\gamma_c^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \ , \qquad \text{where} \qquad \sigma^\mu := (\mathbf{1}, \sigma^i), \quad \overline{\sigma}^\mu := (\mathbf{1}, -\sigma^i). \tag{5.6}$$

These two representation are related by

$$\gamma_c^\mu = M\gamma_D^\mu M^{-1} \ , \qquad \text{for} \qquad M = \begin{pmatrix} \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \ . \tag{5.7}$$

An important role is played by the γ^5 matrix defined as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{5.8}$$

which obeys

$$(\gamma^5)^2 = \mathbf{1} \ , \qquad \{\gamma^\mu, \gamma^5\} = 0 \ , \tag{5.9}$$

In the two representation just discussed they take the form

$$\gamma^5_D = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \ , \qquad \gamma^5_c = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \ . \tag{5.10}$$

Finally, one observes

$$\gamma^{0\dagger} = \gamma^0 \ , \quad \gamma^{i\dagger} = -\gamma^i \qquad \rightarrow \qquad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{5.11}$$

5.2 The Dirac equation

Now we can write down the Dirac equation

$$\sum_{b=1}^{4} \left(i\gamma^\mu_{ab}\partial_\mu - m\delta_{ab}\right)\Psi_b = 0 \ , \tag{5.12}$$

where Ψ_b is a 4-component complex spinor-field and m is a again the rest mass. The Dirac equation can be obtained as the Euler-Lagrange equation of the Lagrangian

$$\mathcal{L} \ = \sum_{a,b=1}^{4} \bar\Psi_a(i\gamma^\mu_{ab}\partial_\mu - m\delta_{ab})\Psi_b \ , \tag{5.13}$$

where $\bar\Psi = \Psi^\dagger\gamma^0$. Indeed we can compute

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\Psi_b} &= -m\bar\Psi_b \ , \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi_b)} = i\sum_a \bar\Psi_a\gamma^\mu_{ab} \\ \frac{\partial\mathcal{L}}{\partial\bar\Psi_a} &= \sum_b(i\gamma^\mu_{ab}\partial_\mu - m\delta_{ab})\Psi_b \ , \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\Psi_a)} = 0 \ . \end{aligned} \tag{5.14}$$

Inserted into (2.6) we obtain the Euler-Lagrange equations for Ψ and $\bar\Psi$

$$\sum_b(i\gamma^\mu_{ab}\partial_\mu - m\delta_{ab})\Psi_b = 0 \ , \qquad \sum_a i\partial_\mu\bar\Psi_a\gamma^\mu_{ab} + m\bar\Psi_b = 0 \ . \tag{5.15}$$

As promised the first equation is indeed the Dirac equation while the second equation can be shown to be its hermitian conjugate using (5.11).

We can also determine the Hamiltonian by first computing the conjugate momenta to be

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot\Psi} = i\bar\Psi\gamma^0 = i\Psi^\dagger \ , \qquad \bar\pi = \frac{\partial\mathcal{L}}{\partial\dot{\bar\Psi}} = 0 \ . \tag{5.16}$$

The fact that $\bar\pi$ vanishes is a problem that has to be taken care of when we come to quantizing the theory. For now we limit ourselves to classical field theory and obtain for the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \pi\dot\Psi + \bar\pi\dot{\bar\Psi} - \mathcal{L} \\ &= i\Psi^\dagger\dot\Psi - \bar\Psi(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\Psi \\ &= -i\bar\Psi\gamma^i\partial_i\Psi + m\bar\Psi\Psi \ . \end{aligned} \tag{5.17}$$

5.3 Weyl equation

In various application it will turn out to be useful to decompose the 4-component Dirac spinor Ψ into two 2-component Weyl spinors Ψ_L, Ψ_R according to

$$\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} . \tag{5.18}$$

This relation can be "inverted" with the help of two projection operators defined as

$$P_L := \tfrac{1}{2}(\mathbf{1} - \gamma_5) , \qquad P_R := \tfrac{1}{2}(\mathbf{1} + \gamma_5) , \tag{5.19}$$

which obey

$$P_L^2 = P_L , \qquad P_R^2 = P_R , \qquad P_L P_R = 0 . \tag{5.20}$$

Using γ_5 in the chiral representation (5.10) we see that

$$\Psi_L = P_L \Psi , \qquad \Psi_R = P_R \Psi \tag{5.21}$$

holds.

Continuing in the chiral representation for the γ-matrices introduced in (5.6) we obtain from (5.15)

$$i\sigma^\mu \partial_\mu \Psi_R - m\Psi_L = 0 , \qquad i\overline{\sigma}^\mu \partial_\mu \Psi_L - m\Psi_R = 0 \tag{5.22}$$

or in other words two coupled differential equations for the two Weyl spinors. For $m = 0$ they decouple and one obtains the two Weyl equations

$$i\sigma^\mu \partial_\mu \Psi_R = 0 , \qquad i\overline{\sigma}^\mu \partial_\mu \Psi_L = 0 . \tag{5.23}$$

This decoupling is the reason for using the chiral representation.

5.4 Solution of the Dirac equation

Let us first observe that any solution of the Dirac equation also solves the Klein-Gordon equation. This can be seen by acting with the operator $(i\gamma^\nu \partial_\nu + m)$ on the Dirac equation (5.15) which, using (5.2), yields

$$-(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\Psi = (\Box + m^2)\Psi = 0 . \tag{5.24}$$

This motivates the Ansatz:

$$\Psi_a^+ = u_a(p)\, e^{-ipx} , \quad \Psi_a^- = v_a(p)\, e^{ipx} , \tag{5.25}$$

where

$$u^s(p) = \frac{1}{N} \begin{pmatrix} (p\sigma + m)\, \xi^s \\ (p\overline{\sigma} + m)\, \xi^s \end{pmatrix} , \qquad v^s(p) = \frac{1}{N} \begin{pmatrix} (p\sigma + m)\, \xi^s \\ -(p\overline{\sigma} + m)\, \xi^s \end{pmatrix} , \quad s = 1, 2 , \tag{5.26}$$

with

$$p^0 = E_{\vec{p}} \geq 0 , \qquad N = \sqrt{2(E_{\vec{p}} + m)} , \qquad \xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{5.27}$$

The normalization factor N was chosen such that

$$\overline{u} = u^\dagger \gamma^0 \ , \qquad \overline{u}^\nu u^s = 2m\delta^{\nu s} \ , \qquad \overline{v}^\nu v^s = 2m\delta^{\nu s} \ , \qquad \overline{v}^\nu u^s = 0 \ . \tag{5.28}$$

This can be explicitly checked by using the chiral representation and computing

$$(i\gamma^\mu \partial_\mu - m)\Psi^\pm = (\pm\gamma^\mu p_\mu - m)\Psi^\pm = \begin{pmatrix} -m \pm \sigma^\mu p_\mu \\ \pm\overline{\sigma}^\mu p_\mu - m \end{pmatrix} \cdot \begin{Bmatrix} u(p)e^{-ipx} \\ v(p)e^{ipx} \end{Bmatrix} \ . \tag{5.29}$$

Inserting (5.26) we obtain for $u(p)$

$$\begin{pmatrix} -m & \sigma^\mu p_\mu \\ \overline{\sigma}^\mu p_\mu & -m \end{pmatrix} \begin{pmatrix} (p\sigma + m)\xi^s \\ (p\overline{\sigma} + m)\xi^s \end{pmatrix} = \begin{pmatrix} (-m(p\sigma + m) + \sigma p(\overline{\sigma} p + m))\xi^s \\ (\overline{\sigma} p(\sigma p + m) - m(p\overline{\sigma} + m))\xi^s \end{pmatrix} = 0, \tag{5.30}$$

where we used $p^2 = m^2$ and

$$\sigma^\mu p_\mu \overline{\sigma}^\nu p_\nu = \mathbf{1} p_0^2 + p_0 p_i(\sigma^i - \sigma^i) - \underbrace{\sigma^i \sigma^j}_{\delta_{ij}\mathbf{1}} p_i p_j = (p_0^2 - \vec{p}^2)\,\mathbf{1} = m^2\,\mathbf{1} \ . \tag{5.31}$$

A similar computation shows that also $v(p)$ solves the equation.

Before we proceed let us check the non-relativistic limit $p^0 \gg |\vec{p}|$. Using the Dirac representation we obtain

$$(i\gamma^\mu \partial_\mu - m) \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} i\partial_0 \phi + i\sigma^i \partial_i \chi - m\phi \\ -i\partial_0 \chi - i\sigma^i \partial_i \phi - m\chi \end{pmatrix} = 0 \tag{5.32}$$

For $p^0 \approx m$ we can split off the time dependence $\phi = e^{-imx^0}\tilde{\phi}$ with $\tilde{\phi}$ slowly varying. The second equations gives us $\chi \approx -\frac{i}{2m}\sigma^i \partial_i \phi$ which, when inserted into the first equation, yields the non-relativistic Schrödinger equation

$$i\partial_0 \tilde{\phi} \approx -\tfrac{1}{2m}\Delta\tilde{\phi} = H\tilde{\phi} \ , \tag{5.33}$$

where H is the Hamiltonian.

6 Lecture 6: Covariance of the Dirac equation

In order to make sure that the Dirac equation is a legitimate field equation in a relativistic theory we need to check its covariance under Lorentz-transformations. For the case of the Klein-Gordon equation we observed that $\Box$ is a Lorentz-invariant operator. However the Dirac operator is not Lorentz-invariant and as a consequence Ψ cannot transform as a scalar. Let us make the Ansatz for the transformation law of Ψ

$$\Psi_a \to \Psi'_a = \sum_b \left(\Lambda_{\frac{1}{2}}\right)_{ab} \Psi_b \ , \tag{6.1}$$

where $\Lambda_{\frac{1}{2}}$ also is a 4×4 matrix but does not coincide with Λ^μ_ν. Let us inspect the transformed Dirac equation

$$\begin{aligned}(i\gamma^\mu \partial'_\mu - m)\Psi' &= (i\gamma^\mu \Lambda^{-1T}{}_\mu{}^\nu \partial_\nu - m)\Lambda_{\frac{1}{2}}\Psi \\ &= \Lambda_{\frac{1}{2}}\Lambda^{-1}_{\frac{1}{2}}(i\gamma^\mu \Lambda^{-1T}{}_\mu{}^\nu \partial_\nu - m)\Lambda_{\frac{1}{2}}\Psi \\ &= \Lambda_{\frac{1}{2}}(i\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu \Lambda_{\frac{1}{2}}\Lambda^{-1T}{}_\mu{}^\nu \partial_\nu - m)\Psi \ ,\end{aligned} \tag{6.2}$$

where we used $\partial_\mu = \Lambda^{-1T}{}_\mu{}^\nu \partial_\nu$ and did not transform the γ-matrices as they have constant entries. So we see that if we can find a $\Lambda_{\frac{1}{2}}$ such that

$$\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda_\nu{}^\mu \gamma^\nu \tag{6.3}$$

we see that the Dirac equation is covariant, i.e.

$$(i\gamma^\mu \partial'_\mu - m)\Psi' \ = \ \Lambda_{\frac{1}{2}}\left(i\gamma^\mu \partial_\mu - m\right)\Psi = 0 \ . \tag{6.4}$$

$\Lambda_{\frac{1}{2}}$ can be systematically constructed using group-theoretical methods. Here we just give the result and check that is satisfies (6.3). One finds

$$\left(\Lambda_{\frac{1}{2}}\right)_{ab} = \left(e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\right)_{ab} , \qquad \text{with} \qquad S^{\mu\nu}_{ab} := \tfrac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]_{ab} \ , \tag{6.5}$$

where the six parameters of the Lorentz-transformation have been expressed in terms of an anti-symmetric 4×4 matrix $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

It is straightforward to check (6.3) infinitesimally (i.e. to first order in ω). Inserting $\Lambda_\mu{}^\nu$ and $\Lambda_{\frac{1}{2}}$ expanded to first order into (6.3) one arrives at

$$[S^{\rho\sigma}, \gamma^\mu] = i(\gamma^\rho \eta^{\sigma\mu} - \gamma^\sigma \eta^{\rho\mu}) \ , \tag{6.6}$$

which can be checked by repeated use of (5.2).

Let us also determine the transformation law of the conjugate spinor. From its definition $\overline{\Psi} := \Psi^\dagger \gamma^0$ we compute infinitesimally

$$\begin{aligned}\delta\overline{\Psi} = \delta\Psi^\dagger \gamma^0 = \Psi^\dagger \left(\tfrac{i}{2}\omega_{\mu\nu}S^{\dagger\mu\nu}\right)\gamma^0 &= \Psi^\dagger \underbrace{(\gamma^0)^2}_{=1} \left(\tfrac{i}{2}\omega_{\mu\nu}S^{\dagger\mu\nu}\right)\gamma^0 \\ = \Psi^\dagger \gamma^0 \left(\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) = \overline{\Psi}\left(\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \ ,&\end{aligned} \tag{6.7}$$

where in the second line we used $\gamma^{\dagger\mu} = \gamma^0\gamma^\mu\gamma^0$. Exponentiated this yields

$$\overline{\Psi} \to \overline{\Psi}' = \overline{\Psi}\Lambda_{\frac{1}{2}}^{-1}. \tag{6.8}$$

Thus we see that $\overline{\Psi}\Psi$ is a Lorentz-scalar as is the Lagrangian (5.13).

In the chiral representation (5.6) $S^{\mu\nu}$ is block-diagonal and given by

$$S^{0i} = \tfrac{i}{4}[\gamma^0, \gamma^i] = -\tfrac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \qquad S^{ij} = \tfrac{i}{4}[\gamma^i, \gamma^j] = \tfrac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \tag{6.9}$$

Therefore the transformation law of the two Weyl spinors defined in (5.18) reads

$$\begin{aligned} \delta\Psi_L &= (i\theta^i\sigma^i - \tfrac{1}{2}\beta^i\sigma^i)\Psi_L \\ \delta\Psi_R &= (i\theta^i\sigma^i + \frac{1}{2}\beta^i\sigma^i)\Psi_R \end{aligned} \tag{6.10}$$

where $\omega_{ij} = \epsilon_{ijk}\theta^k, \beta^i = \omega^{0i}$. We see that $\Psi_{L,R}$ both transform like two-dimensional spinors under rotations.

Finally there are 16 linearly independent spinor bilinears given by

$$\begin{aligned} 1: &\quad \overline{\Psi}\Psi \\ 4: &\quad \overline{\Psi}\gamma^\mu\Psi \\ 6: &\quad \overline{\Psi}\,[\gamma^\mu, \gamma^\nu]\,\Psi \\ 4: &\quad \overline{\Psi}\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\Psi \propto \overline{\Psi}\epsilon^{\mu\nu\rho\sigma}\gamma^\sigma\gamma^5\Psi \\ 1: &\quad \overline{\Psi}\gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}\Psi \propto \overline{\Psi}\gamma^5\Psi \end{aligned} \tag{6.11}$$

7 Lecture 7: Quantization of the Dirac field

The most general solution of the Dirac equation is a superposition of the solution given in (5.25), (5.26), i.e.

$$\begin{aligned}\psi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\sum_{s=1}^{2}\left(a^s_{\vec p}\,u^s(p)\,e^{-ipx} + b^{s\dagger}_{\vec p}\,v^s(p)\,e^{ipx}\right)_{p^0=E_{\vec p}}\\ \overline{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\sum_{s=1}^{2}\left(b^s_{\vec p}\,\overline{v}^s(p)\,e^{-ipx} + a^{s\dagger}_{\vec p}\,\overline{u}^s(p)\,e^{ipx}\right)_{p^0=E_{\vec p}}\end{aligned} \tag{7.1}$$

In order to canonically quantize the Dirac field ψ is promoted to a quantum-theoretical operator as are the Fourier-coefficients $a^s_{\vec q}, b^s_{\vec q}$. They obey the anti-commutation relation[7]

$$\begin{aligned}&\{a^r_{\vec p}\,, a^{s\,\dagger}_{\vec q}\} = (2\pi)^3\,\delta^{(3)}(\vec p-\vec q)\,\delta^{rs} = \{b^r_{\vec p}\,, b^{s\,\dagger}_{\vec q}\}\\ &\{a^r_{\vec p}\,, a^s_{\vec q}\} = \{a^{r\,\dagger}_{\vec p}\,, a^{s\,\dagger}_{\vec q}\} = \{b^r_{\vec p}\,, b^s_{\vec q}\} = \{b^{r\,\dagger}_{\vec p}\,, b^{s\,\dagger}_{\vec q}\} = 0\ ,\end{aligned} \tag{7.2}$$

This in turn implies (see problem 4.3)

$$\begin{aligned}&\{\psi_a(\vec x,t)\,,\psi^\dagger_b(\vec y,t)\} = \delta^{(3)}(\vec x-\vec y)\,\delta_{ab}\\ &\{\psi_a(\vec x,t)\,,\psi_b(\vec y,t)\} = 0 \quad = \quad \{\psi^\dagger_a(\vec x,t)\,,\psi^\dagger_b(\vec y,t)\}\end{aligned} \tag{7.3}$$

Inserting (7.2) into (5.17) we obtain

$$\begin{aligned}H &= -\int d^3x\,\left(\overline{\psi}\,\gamma^i\,\partial_i\,\psi\ -\ m\,\overline{\psi}\psi\right) = i\int d^3x\ \ \overline{\psi}\,\gamma^0\,\partial_i\,\psi\\ &= \int \frac{d^3p}{(2\pi)^3}\sum_s E_{\vec p}\left(a^{s\,\dagger}_{\vec p}\,a^s_{\vec p}\ +\ b^{s\,\dagger}_{\vec p}\,b^s_{\vec p}\right)\ ,\end{aligned} \tag{7.4}$$

where the second equation is shown to hold in problem 4.3. In the final expression again an infinite constant was dropped. We see that H is an infinite sum of fermionic harmonic oscillators.

The space of states (Fock-space) is again constructed from a vacuum state $|0\rangle$ defined as

$$a^s_{\vec p}|0\rangle = 0 = b^s_{\vec p}|0\rangle\ . \tag{7.5}$$

The one-particle states are

$$a^{s\,\dagger}_{\vec p}|0\rangle\ ,\quad b^{s\,\dagger}_{\vec p}|0\rangle\ . \tag{7.6}$$

There energy can be determined by acting with H given in (7.4)

$$\begin{aligned}&H\,a^{r\,\dagger}_{\vec q}|0\rangle = \int\frac{d^3p}{(2\pi)^3}\sum_s E_{\vec p}a^{s\,\dagger}_{\vec p}\,a^s_{\vec p}\,a^{r\,\dagger}_{\vec q}\,|0\rangle = \int\frac{d^3p}{(2\pi)^3}\sum_s E_{\vec p}a^{s\,\dagger}_{\vec p}\,\{a^s_{\vec p}\,, a^{r\,\dagger}_{\vec q}\,\}|0\rangle = E_{\vec q}a^{r\,\dagger}_{\vec q}|0\rangle\ ,\\ &H\,b^{r\,\dagger}_{\vec q}|0\rangle = \ldots = E_{\vec q}b^{r\,\dagger}_{\vec q}|0\rangle\ .\end{aligned} \tag{7.7}$$

[7] The anti-commutator of two operators A, B is defined by $\{A,B\} = AB + BA$.

Thus we see that both states have the same mass.

We can also compute their charge by first observing that Dirac Lagrangian (5.13) $\mathcal{L} = \overline{\psi}\,(i\,\gamma^\mu\,\partial_\mu - m)\,\psi$ is invariant under the transformation

$$\psi \to \psi' = e^{i\alpha}\psi\ , \qquad \alpha \in \mathbb{R}\ . \tag{7.8}$$

The corresponding Noether current j^μ and Noether charge Q are determined in problem 3.2 to be

$$j^\mu = \overline{\psi}\gamma^\mu\psi\ , \qquad Q = \int d^3x\, j^0 = \int d^3x\,\psi^\dagger\psi = \int \frac{d^3p}{(2\pi)^3}\sum_s \left(a^{s\,\dagger}_{\vec{p}}a^s_{\vec{p}} - b^{s\,\dagger}_{\vec{p}}b^s_{\vec{p}}\right)\ , \tag{7.9}$$

and one can check the conservation law $\partial_\mu j^\mu = 0$.[8] Now we can compute the charge of the two one-particle states to be

$$\begin{aligned} Q\,a^{r\,\dagger}_{\vec{q}}|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\sum_s a^{s\,\dagger}_{\vec{p}}\,a^s_{\vec{p}}a^{r\,\dagger}_{\vec{q}}\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\sum_s a^{s\,\dagger}_{\vec{p}}\,\{a^s_{\vec{p}}, a^{r\,\dagger}_{\vec{q}}\}|0\rangle = a^{r\,\dagger}_{\vec{q}}|0\rangle\ , \\ Q\,b^{r\,\dagger}_{\vec{q}}|0\rangle &= \ldots = -b^{r\,\dagger}_{\vec{q}}|0\rangle\ . \end{aligned} \tag{7.10}$$

We thus see that $a^{r\,\dagger}_{\vec{q}}|0\rangle$ and $b^{r\,\dagger}_{\vec{q}}|0\rangle$ have the same mass but opposite charges. These states are therefore identified with the electron and its anti-particle the positron. In fact, the existence of anti-particle (same mass but opposite charge) is a generic prediction of the Dirac theory. Finally, using

$$\vec{p} = \int d^3x\,\psi^\dagger(-i\vec{\nabla})\psi = \int \frac{d^3p}{(2\pi)^3}\sum_s \vec{p}\left(a^{s\,\dagger}_{\vec{p}}a^s_{\vec{p}} + b^{s\,\dagger}_{\vec{p}}b^s_{\vec{p}}\right)\ , \tag{7.11}$$

we can check that $a^{r\,\dagger}_{\vec{q}}|0\rangle$ and $b^{r\,\dagger}_{\vec{q}}|0\rangle$ carry momentum $\vec{q}$.

7.1 Dirac propagator

Let us compute

$$\begin{aligned} &\langle 0|\psi_a(x)\overline{\psi_b}(y)|0\rangle = \\ &\langle 0|\int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\left(\sum_s a^s_{\vec{p}}u^s_a(p)\,e^{-ipx}\right)\left(\sum_{s'} a^{s'\,\dagger}_{\vec{p}'}\,\overline{u}^{s'}_b(p')\,e^{ip'y}\right)|0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\sum_s u^s_a(p)\overline{u}^s_b(p)\,e^{-ip(x-y)} \end{aligned} \tag{7.12}$$

In order to proceed we observe

$$\begin{aligned} \sum_s u^s_a(p)\overline{u}^s_b(p) &= \gamma^\mu_{ab}p_\mu + m\delta_{ab}\ , \\ \sum_s v^s_a(p)\overline{v}^s_b(p) &= \gamma^\mu_{ab}p_\mu - m\delta_{ab}\ , \end{aligned} \tag{7.13}$$

[8] We will see later that Q is indeed the electromagnetic charge operator.

which are proved in exercise 4.2. Inserted into (7.12) we arrive at

$$\begin{aligned}\langle 0|\psi_a(x)\overline{\psi_b}(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\gamma^\mu_{ab}p_\mu + m\delta_{ab}\right)e^{-ip(x-y)} \\ &= \left(i\gamma^\mu_{ab}\partial^x_\mu + m\delta_{ab}\right)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip(x-y)} \ .\end{aligned} \tag{7.14}$$

Analogously one obtains

$$\langle 0|\overline{\psi_b}(y)\psi_a(x)|0\rangle = -\left(i\gamma^\mu_{ab}\partial^x_\mu + m\delta_{ab}\right)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip(y-x)} \ . \tag{7.15}$$

Let us define

$$S_{R\,ab} := \Theta(x^0 - y^0)\ \langle 0|\{\psi_a(x)\,,\overline{\psi}_b(y)\}|0\rangle \tag{7.16}$$

and observe

$$S_{R\,ab} = (i\gamma^\mu\partial^x_\mu + m)\ G_{ret}(x-y)\ , \tag{7.17}$$

where $G_{ret}(x-y)$ was computed in (4.13). We can now easily check that $S_{R\,ab}$ is the Greens function of the Dirac operator in that it satisfies

$$(i\gamma^\mu\partial^x_\mu - m)S_R = \underbrace{(i\gamma^\mu\partial^x_\mu - m)(i\gamma^\mu\partial^x_\mu + m)}_{-(\Box+m^2)}\ G_{ret}(x-y) = i\,\delta^4(x-y) \tag{7.18}$$

Alternatively one can consider the Fourier-transformation of S_R

$$S_R = \int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\tilde{S_R}(p) \tag{7.19}$$

and demand (7.18) to hold. This yields

$$\tilde{S}_R = \frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2} = \frac{i}{\gamma^\mu p_\mu - m} \tag{7.20}$$

Now one has to perform the p^0 integral analogously to lecture 4 which indeed confirms (7.17).

The Feynman propagator is defined by

$$S_F(x-y) = \begin{cases} \langle 0|\psi(x)\overline{\psi}(y)|0\rangle & \text{for} \quad x^0 > y^0 \\ -\langle 0|\overline{\psi}(y)\psi(x)|0\rangle & \text{for} \quad x^0 < y^0 \end{cases} \tag{7.21}$$

One generalizes the time-ordered product to include fermionic operators by

$$T\{\hat{A}(x),\,\hat{B}(y)\} := \Theta(x^0 - y^0)\hat{A}(x)\hat{B}(y)\ \pm\ \Theta(y^0 - x^0)\hat{B}(y)\hat{A}(x)\ , \tag{7.22}$$

where the "+"-sign holds for bosonic operators while the "−"-sign is taken for fermionic operators. We see that

$$S_F(x-y) = \langle 0|T\left\{\psi(x)\overline{\psi}(y)\right\}|0\rangle \tag{7.23}$$

holds.

8 Lecture 8: Discrete Symmetries

In this section we discuss three discrete symmetries:

$$\begin{aligned} \text{parity} \quad & P: \quad (t,\vec{x}) \to (t,-\vec{x}) \ , \\ \text{time reversal} \quad & T: \quad (t,\vec{x}) \to (-t,\vec{x}) \ , \\ \text{charge conjugation} \quad & C: \quad \text{particles} \ \leftrightarrow \ \text{anti-particles} \ . \end{aligned} \tag{8.1}$$

Quantenelectrodynamics (QED) and Quantenchromodynamics (QCD) preserve individually P, C, T while the weak interaction violate them and only preserve the combination PCT. Let us start with parity.

8.1 Parity

Parity acts on the operators $a^s_{\vec{p}}$, $b^s_{\vec{p}}$ via

$$P\, a^s_{\vec{p}}\, P = \eta_a\, a^s_{-\vec{p}} \ , \qquad P\, b^s_{\vec{p}}\, P = \eta_b\, b^s_{-\vec{p}} \ , \tag{8.2}$$

i.e. it reverses $\vec{p}$ without changing the spin s. Since two parity transformations should give back the original operators up to a phase, one needs to demand $\eta_a^2 = \pm 1, \eta_b^2 = \pm 1$. For $\psi(x)$ we compute from (7.1)

$$P\psi(x)P = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1}^{2}\Big(\underbrace{Pa^s_{\vec{p}}P}_{\eta_a\, a^s_{-\vec{p}}}\ u^s(p)\, e^{-ipx} + \underbrace{Pb^{s\dagger}_{\vec{p}}P}_{\eta_b^*\, b^{s\dagger}_{-\vec{p}}}\ v^s(p)\, e^{ipx}\Big)_{p^0=E_{\vec{p}}} \tag{8.3}$$

Changing variables $p \to \tilde{p} = (p^0, -\vec{p})$ and using

$$\begin{gathered} p\cdot x = \tilde{p}\cdot(t,-\vec{x}) \ , \qquad \tilde{p}\cdot\sigma = p\cdot\bar{\sigma} \ , \qquad \tilde{p}\cdot\bar{\sigma} = p\cdot\sigma \ , \\ u(p) = \gamma^0 u(\tilde{p}) \ , \qquad v(p) = -\gamma^0 v(\tilde{p}) \ , \end{gathered} \tag{8.4}$$

we find

$$\begin{aligned} P\psi(x)P &= \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{\tilde{p}}}}}\sum_{s=1}^{2}\Big(\eta_a\, a^s_{\vec{\tilde{p}}}\gamma^0 u^s(\tilde{p})\, e^{-i\tilde{p}\cdot(t,-\vec{x})} - \eta_b^*\, b^{s\dagger}_{-\vec{\tilde{p}}}\gamma^0 v^s(\tilde{p})\, e^{i\tilde{p}\cdot(t,-\vec{x})}\Big)_{p^0=E_{\vec{p}}} \\ &= \eta_a\gamma^0\psi(t,-\vec{x}) \end{aligned} \tag{8.5}$$

for $\eta_b^* = -\eta_a$ or $\eta_a\eta_b = -1$ respectively. Note that P acts differently on particles and anti-particles.

8.2 Time-reversal

Time-reversal acts as in (8.1) and thus also inverts $\vec{p}$ and the spin (since $\vec{L} = \vec{x}\times\vec{p} \to -\vec{L}$). Thus together we have

$$t \to -t \ , \qquad \vec{p} \to -\vec{p} \ , \qquad \zeta^1 \leftrightarrow \zeta^2 \tag{8.6}$$

Let us denote the spin-flipped ζ by ζ^{-s}. Then we have

$$\zeta^{-s} = -i\sigma^2\zeta^s \ . \tag{8.7}$$

Let us also compute the spin-flipped u

$$u^{-s}(\tilde{p}) = -i\begin{pmatrix}\sigma^2 & 0\\ 0 & -\sigma^2\end{pmatrix}\big(u^s(p)\big)^* = -\gamma^1\gamma^3\big(u^s(p)\big)^* \ . \tag{8.8}$$

Analogously one finds

$$v^{-s}(\tilde{p}) = -\gamma^1\gamma^3\big(v^s(p)\big)^* \ . \tag{8.9}$$

Now we demand

$$T\, a^s_{\vec{p}}\, T = \eta_a\, a^{-s}_{-\vec{p}} \ , \qquad T\, b^s_{\vec{p}}\, T = \eta_b\, b^{-s}_{-\vec{p}} \ . \tag{8.10}$$

T acts as an anti-unitary operator, i.e.

$$T^\dagger = T^{-1} \ , \qquad Tc = c^* T \ , \quad \text{for} \quad c \in \mathbb{C} \ . \tag{8.11}$$

For the action on ψ we get

$$\begin{aligned}
T\psi(x)T &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1}^{2} T\Big(a^s_{\vec{p}}\, u^s(p)\, e^{-ipx} + b^{s\dagger}_{\vec{p}}\, v^s(p)\, e^{ipx}\Big)_{p^0=E_{\vec{p}}} T\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1}^{2} \Big(a^{-s}_{-\vec{p}}\, \big(u^s(p)\big)^*\, e^{ipx} + b^{-s\dagger}_{-\vec{p}}\, \big(v^s(p)\big)^*\, e^{-ipx}\Big)_{p^0=E_{\vec{p}}}\\
&= \gamma^1\gamma^3 \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{\vec{p}}}}}\sum_{s=1}^{2} \Big(a^{-s}_{\tilde{\vec{p}}}\, u^{-s}(\tilde{p})\, e^{i\tilde{p}\cdot(t,-\vec{x})} + b^{-s\dagger}_{\tilde{\vec{p}}}\, v^{-s}(\tilde{p})\, e^{-i\tilde{p}\cdot(t,-\vec{x})}\Big)_{p^0=E_{\vec{p}}}\\
&= \gamma^1\gamma^3\psi(-t,\vec{x}) \ ,
\end{aligned} \tag{8.12}$$

where in third equation we have again changed variables to $\tilde{p}$.

8.3 Charge conjugation

Charge conjugation C exchanges particles with anti-particles while keeping the spin unflipped. This amounts to

$$C\, a^s_{\vec{p}}\, C = b^s_{\vec{p}} \ , \qquad C\, b^s_{\vec{p}}\, C = a^s_{\vec{p}} \ . \tag{8.13}$$

Repeating analogous steps as in the previous two sections one finds

$$C\,\psi(x)\,C = -i\gamma^2\psi^* = -i\Big(\bar{\psi}\gamma^0\gamma^2\Big)^T \ . \tag{8.14}$$

8.4 Fermion bilinears

One can now compute the transformation properties of the 16 fermion bilinears introduced in (6.11). The result is summarized in table 8.4 using the convention $(-1)^\mu = 1$ for $\mu = 0$ and $(-1)^\mu = -1$ for $\mu = 1, 2, 3$.

	$\overline{\Psi}\Psi$	$\overline{\Psi}\gamma^\mu\Psi$	$\overline{\Psi}\,[\gamma^\mu,\gamma^\nu]\,\Psi$	$\overline{\Psi}\gamma^\sigma\gamma^5\Psi$	$\overline{\Psi}\gamma^5\Psi$
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$
C	$+1$	$+1$	-1	$+1$	-1
CPT	$+1$	$+1$	-1	-1	$+1$

Table 8.0: Transformation properties of fermion bilinears

9 Lecture 9: Perturbation theory

Let us start with a simple example of an interacting field theory

$$\mathcal{L}(\phi,\partial_\mu\phi) = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\,\phi^4\ , \qquad \lambda\in\mathbb{R}\ , \tag{9.1}$$

where ϕ is again a real scalar field. Compared to the Lagrangian (2.9) a self-interaction term ϕ^4 has been added and the strength of the interaction is governed by a real dimensionless coupling constant λ. We can repeat the steps of section 2 and compute

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi - \tfrac{1}{3!}\lambda\phi^3\ , \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi\ , \qquad \pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi\ . \tag{9.2}$$

Inserted into the Euler-Lagrange equation (2.6) yields

$$(\Box + m^2)\,\phi = -\tfrac{1}{3!}\lambda\phi^3 \tag{9.3}$$

while from (2.8) we obtain

$$\mathcal{H} = \pi\dot\phi - \mathcal{L} = \tfrac{1}{2}\,(\pi^2 + \vec\nabla\phi\cdot\vec\nabla\phi + m^2\phi^2) + \tfrac{1}{4!}\lambda\,\phi^4\ . \tag{9.4}$$

Our final goal is to compute scattering cross sections and decay rates in an interacting quantum field theory. Let us start by computing the Greens function $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$ where $|\Omega\rangle$ is the ground state auf the interacting theory which does not coincide with the ground state $|0\rangle$ of the free theory, i.e. $|\Omega\rangle\neq|0\rangle$. For arbitrary λ this is a difficult computation. What we can do is evaluate the Greens function perturbatively in λ which gives good results for $\lambda\ll 1$. We split the Hamiltonian

$$\begin{aligned} H &= H_0 + H_{\rm int} \\ H_0 &= \tfrac{1}{2}\int d^3x\big(\pi^2 + \vec\nabla\phi\cdot\vec\nabla\phi + m^2\phi^2\big) \\ H_{\rm int} &= \tfrac{1}{4!}\lambda\int d^3x\phi^4\ . \end{aligned} \tag{9.5}$$

Before we proceed let us recall the relation of operators in the Heisenberg versus Schrödinger picture. They are related by[9]

$$\phi_H(t,\vec x) = e^{iH(t-t_0)}\phi_S(t_0,\vec x)e^{-iH(t-t_0)}\ , \tag{9.6}$$

[9]In the previous sections we always used a time-dependent $\phi(t,\vec x)$, i.e. a Heisenberg operator.

where ϕ_H is the time-dependent Heisenberg operator while ϕ_S is the time-independent Schrödinger operator. ϕ_H solves the Heisenberg equation

$$i\partial_t \phi_H = -H\phi_H + \phi_H H = [\phi_H, H] \ . \tag{9.7}$$

An operator in the interaction picture is defined by

$$\phi_I(t,\vec{x}) = e^{iH_0(t-t_0)}\phi_S(t_0,\vec{x})e^{-iH_0(t-t_0)} \quad = \int \frac{d^3p}{(2\pi)^3}\, \tfrac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{ip\cdot x}\right)\Big|_{p^0=E_{\vec{p}}} \ , \tag{9.8}$$

where the second equation holds since we have already solved the free theory. Inverting (9.8) and inserting into (9.6) we arrive at

$$\phi_H(t,\vec{x}) = U^\dagger(t,t_0)\phi_I U(t,t_0) \ , \qquad \text{with} \qquad U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \ , \tag{9.9}$$

where one can easily check $U^\dagger U = 1$ and $U(t_0,t_0) = 1$. In order to determine U we compute

$$\begin{aligned} i\partial_t U &= -e^{iH_0(t-t_0)}H_0 e^{-iH(t-t_0)} + e^{iH_0(t-t_0)}He^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\ &= H_I U \ , \end{aligned} \tag{9.10}$$

where

$$H_I = e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)} \ . \tag{9.11}$$

This can be turned into the integral equation

$$\int_{t_0}^t dt'\partial_{t'}U(t',t_0) = U(t,t_0) - U(t_0,t_0) = -i\int_{t_0}^t dt' H_I(t')U(t',t_0) \ , \tag{9.12}$$

and thus

$$U(t,t_0) = 1 - i\int_{t_0}^t dt' H_I(t')U(t',t_0) \ . \tag{9.13}$$

(9.13) can now be "solved" iteratively by inserting (9.13) repeatedly into itself

$$\begin{aligned} U(t,t_0) &= 1 - i\int_{t_0}^t dt' H_I(t')\Big(1 - i\int_{t_0}^{t'} dt'' H_I(t'')U(tt'',t_0)\Big) \\ &= 1 + (-i)\int_{t_0}^t dt' H_I(t') + (-i)^2\int_{t_0}^t dt'\int_{t_0}^{t'} dt'' H_I(t')H_I(t'') \\ &\quad + (-i)^3\int_{t_0}^t dt'\int_{t_0}^{t'} dt''\int_{t_0}^{t''} dt''' H_I(t')H_I(t'')H_I(t''') + \dots \ . \end{aligned} \tag{9.14}$$

(9.14) can be written as a time-ordered product which for an arbitrary number of operators is defined as

$$\begin{aligned} T\{H(t_1)\dots H(t_n)\} :=&\theta(t_1-t_2)\theta(t_2-t_3)\dots\theta(t_{n-1}-t_n)\, H(t_1)\dots H(t_n) \\ &+ n!\ \text{permutations} \end{aligned} \tag{9.15}$$

Let us first consider

$$\begin{aligned}
&\tfrac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\{H(t_1)H(t_2)\} \\
&\quad = \tfrac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, \theta(t_1-t_2)H(t_1)H(t_2) + \tfrac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, \theta(t_2-t_1)H(t_2)H(t_1) \\
&\quad = \tfrac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H(t_1)H(t_2) + \tfrac{1}{2}\int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 H(t_2)H(t_1) \\
&\quad = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H(t_1)H(t_2)\ ,
\end{aligned} \tag{9.16}$$

where in the last step we changed the integration variables $t_1 \leftrightarrow t_2$. An analogous computation shows

$$\tfrac{1}{n!}\int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t} dt_n\, T\{H(t_1)\ldots H(t_n)\} = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} H(t_1)\ldots H(t_n)\ . \tag{9.17}$$

Thus we can write (9.14) in terms of time-ordered products as

$$U(t,t_0) = T\{e^{-i\int_{t_0}^{t} dt' H_I(t')}\}\ . \tag{9.18}$$

In order to proceed we need the generalization

$$U(t,t') = T\{e^{-i\int_{t'}^{t} dt'' H_I(t'')}\}\ . \tag{9.19}$$

which satisfies the properties of a time-evolution operator

$$U(t,t') = U(t,t_0)U(t_0,t')\ . \tag{9.20}$$

However, this requires a modification of (9.9)

$$U(t,t') = e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)}\ , \tag{9.21}$$

which agrees with (9.9) for $t' = t_0$ and obeys

$$U(t,t')U(t',t'') = U(t,t'')\ . \tag{9.22}$$

So far we rewrote ϕ_H in terms of the known ϕ_I and H_I. We are left with expressing the ground state $|\Omega\rangle$ of the interacting theory in terms of the ground state $|0\rangle$ of the free theory. The two ground states are defined by

$$H_0|0\rangle = E_0^0|0\rangle = 0\ , \qquad H|\Omega\rangle = E_0|\Omega\rangle\ , \tag{9.23}$$

where we inserted our choice $E_0^0 = 0$. Let us compute

$$e^{-iHT}|0\rangle = \sum_n e^{-iHT}|n\rangle\langle n|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle\ , \tag{9.24}$$

where we inserted a complete set of states. Now we take the somewhat unconventional limit $T \to \infty(1-i\epsilon)$ which allows us to single out the first term in the sum

$$\lim_{T\to\infty(1-i\epsilon)} e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \text{subleading} \ , \tag{9.25}$$

where we used $E_0 < E_{n\neq 0}$. (9.25) implies

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{-iHT}|0\rangle}{e^{-iE_0T}\langle\Omega|0\rangle} \tag{9.26}$$

In order to express $|\Omega\rangle$ in terms of U we shift $T \to T + t_0$ and insert $e^{iH_0(T+t_0)}$ to obtain

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{-iH(T+t_0)}e^{iH_0(T+t_0)}|0\rangle}{e^{-iE_0(T+t_0)}\langle\Omega|0\rangle} = \lim_{T\to\infty(1-i\epsilon)} \frac{U(t_0,-T)|0\rangle}{e^{-iE_0(T+t_0)}\langle\Omega|0\rangle} \ . \tag{9.27}$$

Analogously one derives

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,t_0)}{e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle} \ . \tag{9.28}$$

The normalization $\langle\Omega|\Omega\rangle = 1$ implies

$$\lim_{T\to\infty(1-i\epsilon)} e^{-2iE_0T}|\langle 0|\Omega\rangle|^2 = \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,-T)|0\rangle \ . \tag{9.29}$$

Now we are prepared to compute the correlator $\langle\Omega|\phi(x)\phi(y)|\Omega\rangle$. Let us first choose $x^0 > y^0 > t_0$ and use

$$\begin{aligned}\phi(x)\phi(y) &= U^\dagger(x^0,t_0)\phi_I(x)U(x^0,t_0)U^\dagger(y^0,t_0)\phi_I(y)U(y^0,t_0)\\ &= U^\dagger(x^0,t_0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,t_0) \ ,\end{aligned} \tag{9.30}$$

where in the last step we used (9.9). Inserting (9.27), (9.28), and (9.30) we arrive at

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,t_0)U^\dagger(x^0,t_0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,t_0)U(t_0,T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}\\ &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle} \ .\end{aligned} \tag{9.31}$$

Note that the numerator is time-ordered so that together with (9.18) we arrive at

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_I(x)\phi_I(y)U(T,x^0)U(x^0,y^0)U(y^0,-T)\}|0\rangle}{\langle 0|U(T,-T)|0\rangle}\\ &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_I(x)\phi_I(y)U(T,-T)\}|0\rangle}{\langle 0|U(T,-T)|0\rangle}\\ &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_I(x)\phi_I(y)e^{-i\int_T^T dtH_I(t)}\}|0\rangle}{\langle 0|T\{e^{-i\int_T^T dtH_I(t)}\}|0\rangle} \ .\end{aligned} \tag{9.32}$$

For $y^0 > x^0$ one derives a similar fomula so that altogether one has

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_I(x)\phi_I(y)e^{-i\int_T^T dtH_I(t)}\}|0\rangle}{\langle 0|T\{e^{-i\int_T^T dtH_I(t)}\}|0\rangle} \ . \tag{9.33}$$

Thus we see that the correlator can indeed be compactly expressed in terms of ϕ_I, H_I and $|0\rangle$. For an arbitrary product of operators one derives analogously

$$\langle\Omega|T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_I(x_1)\dots\phi_I(x_n)e^{-i\int_T^T dtH_I(t)}\}|0\rangle}{\langle 0|T\{e^{-i\int_T^T dtH_I(t)}\}|0\rangle} \ . \tag{9.34}$$

This formula will now be used as the starting point of perturbation theory.

10 Lecture 10: Wick Theorem

As stated at the end of the last section, the goal is to compute amplitudes of the form $\langle 0|T\{\phi_I(x_1)\dots\phi_I(x_n)e^{-i\int dtH_I}\}|0\rangle$. We already did compute $\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\}|0\rangle = G_F(x_1-x_2)$ in section 4 which is the Feynman propagator(cf. (4.21)). The strategy will be to express ϕ in terms of the operators $a, a^\dagger$ and commute them appropriately. Wicks Theorem provides a systematic way to do this.

It is useful to introduce the notation $\Phi_I = \Phi_I^+ + \Phi_I^-$ with

$$\Phi_I^+ = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\,e^{-ipx}a_{\vec p}\ , \qquad \Phi_I^- = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\,e^{ipx}a^\dagger_{\vec p}\ , \tag{10.1}$$

which obey

$$\Phi_I^+|0\rangle = 0\ , \qquad \text{and} \qquad \langle 0|\Phi_I^- = 0\ . \tag{10.2}$$

Now we define the <u>normal order $N\{a\}$</u> by arranging all $a^\dagger$-operators on the left and all a-operators on the right, i.e.

$$N\{\underbrace{a_{\vec p_1}a^\dagger_{\vec k_1}\cdots a^\dagger_{\vec k_n}a_{\vec p_r}}_{\text{arbitrary order}}\} := \overbrace{a^\dagger_{\vec k_1}\cdots a^\dagger_{\vec k_n}}^{\text{arbitrary order}}\cdot\underbrace{a_{\vec p_1}\cdots a_{\vec p_r}}_{\text{arbitrary order}} \tag{10.3}$$

This implies immediately

$$\langle 0|N\{\phi_I(x_1)\dots\phi_I(x_n)\}|0\rangle = 0\ . \tag{10.4}$$

Next we define the <u>Wick contraction</u> by

$$\overline{\Phi(x)\Phi}(y) := \Theta(x^0-y^0)\left[\Phi^+(x),\Phi^-(y)\right] + \Theta(y^0-x^0)\left[\Phi^+(y),\Phi^-(x)\right] \tag{10.5}$$

By inserting (10.1) and using (4.17) one shows

$$\overline{\Phi(x)\Phi}(x) = G_F(x-y)\ . \tag{10.6}$$

The Wick contraction is precisely the difference between time ordering and normal ordering and one has

$$T\{\Phi(x)\Phi(y)\} = N\{\Phi(x)\Phi(y) + \overline{\Phi(x)\Phi}(y)\} \tag{10.7}$$

In order to prove (10.7) we insert (10.1) into the definition (10.3) and obtain

$$\begin{aligned}
T\{\Phi(x)\Phi(y)\} &= \Theta(x^0-y^0)\Big(\Phi^+(x)\Phi^+(y) + \Phi^-(x)\Phi^-(y) + \left[\Phi^+(x),\Phi^-(y)\right] \\
&\qquad + \Phi^-(x)\Phi^+(y) + \Phi^-(y)\Phi^+(x)\Big) + \Theta(y^0-x^0)\Big(x\leftrightarrow y\Big) \\
&= \Theta(x^0-y^0)\left(N\{\Phi(x)\Phi(y)\} + \left[\Phi^+(x)\Phi^-(y)\right]\right) \\
&\quad + \Theta(y^0-x^0)\left(N\{\Phi(x)\Phi(y)\} + \left[\Phi^+(y)\Phi^-(x)\right]\right) \\
&= N\{\Phi(x)\Phi(y) + \overline{\Phi(x)\Phi}(y)\}\ ,
\end{aligned} \tag{10.8}$$

where in the last step we used (10.5) and the fact that G_F is a c-number and thus can be moved inside the normal order.

The Wick Theorem generalizes (10.7) for an arbitrary product of field operators and states:

$$T\{\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\} = N\{\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n) + \text{all Wick contractions}\}\ . \tag{10.9}$$

As example consider

$$\begin{aligned} T\{\Phi_1\Phi_2\Phi_3\Phi_4\} = N\{&\Phi_1\Phi_2\Phi_3\Phi_4 + \wick{\c{\Phi}_1\c{\Phi}_2}\Phi_3\Phi_4 + \wick{\c{\Phi}_1\Phi_2\c{\Phi}_3}\Phi_4 + \wick{\c{\Phi}_1\Phi_2\Phi_3\c{\Phi}_4} \\ &+ \Phi_1\wick{\c{\Phi}_2\c{\Phi}_3}\Phi_4 + \wick{\c1{\Phi}_1\c1{\Phi}_2\c2{\Phi}_3\c2{\Phi}_4} + \wick{\c1{\Phi}_1\c2{\Phi}_2\c1{\Phi}_3\c2{\Phi}_4} + \wick{\c1{\Phi}_1\c2{\Phi}_2\c2{\Phi}_3\c1{\Phi}_4}\}\ , \end{aligned} \tag{10.10}$$

where we abbreviated $\Phi(x_i) = \Phi_i$.

Wicks Theorem is particularly useful for computing vacuum expectation values (VEVs) of products of operators since all normal ordered terms drop out due to the property (10.4). For example, taking the VEV of (10.10) and using (10.6) one obtains

$$\begin{aligned} \langle 0|T\{\Phi_1\Phi_2\Phi_3\Phi_4\}|0\rangle &= \langle 0|\ \wick{\c1{\Phi}_1\c1{\Phi}_2\c2{\Phi}_3\c2{\Phi}_4} + \wick{\c1{\Phi}_1\c2{\Phi}_2\c1{\Phi}_3\c2{\Phi}_4} + \wick{\c1{\Phi}_1\c2{\Phi}_2\c2{\Phi}_3\c1{\Phi}_4}\ |0\rangle \\ &= G_F(x_1-x_2)G_F(x_3-x_4)\ + G_F(x_1-x_3)G_F(x_2-x_4)\ + G_F(x_1-x_4)G_F(x_2-x_3)\ , \end{aligned} \tag{10.11}$$

and everything is expressed in terms of G_F only.

One can prove Wicks theorem by induction. Assume that it holds for $n-1$ operators and then prove that it also holds for n operators

$$\begin{aligned} T\{\Phi(x_1)\ldots\Phi(x_n)\} &= \Theta(x_1^0-x_2^0)\,\Theta(x_2^0-x_3^0)\ \ldots\ \Theta(x_{n-1}^0-x_n^0)\,\Phi_1\ldots\Phi_n \\ &\qquad + \text{all permutations} \\ &= \begin{cases} \Phi_1 T\{\Phi_2\ldots\Phi_n\} & \text{for } x_1^0 \text{ largest} \\ \Phi_2 T\{\Phi_1\Phi_3\ldots\Phi_n\} & \text{for } x_2^0 \text{ largest} \\ \ldots & \end{cases} \end{aligned}$$

For each possibility we can now use the assumption that the theorem holds for $n-1$ operators, i.e.

$$\begin{aligned} \Phi_1 T\{\Phi_2\ldots\Phi_n\} &= \Phi_1 N\{\Phi_2\ldots\Phi_n + \text{all contractionen without}\Phi_1\} \\ &= N\{\Phi_1^-\ ((\Phi_2\ldots\Phi_n) + \text{all contractionen without}\Phi_1) + \\ & N\{\underbrace{[\Phi_1^+,\Phi_2]}_{\wick{\c{\Phi}_1\c{\Phi}_2}}\Phi_3\ldots\Phi_n + \Phi_2\underbrace{[\Phi_1^+,\Phi_3]}_{\wick{\c{\Phi}_1\c{\Phi}_3}}\ldots\Phi_n + \Phi_2\Phi_3[\Phi_1^+,\Phi_4]\ldots\Phi_n + \ldots\} \\ &= N\{\Phi_1\Phi_2\ldots\Phi_n + \text{ all contractionen}\} \end{aligned} \tag{10.12}$$

Before we continue let us also state the Wick-theorem for fermions. We already defined the time-ordered product for fermions in (7.22) and expressed the Feynman propagator in terms of a time-ordered product in (7.21) and (7.23). The crucial difference compared to bosonic operators is an extra minus sign steming from the anti-commutation property of fermions. Analogously one defines for n fermionic operators an extra minus sign for each permutation

$$\begin{aligned} T\{\Psi(x_1)\ldots\Psi(x_n)\} = & \ \Theta(x_1^0 - x_2^0)\Theta(x_2^0 - x_3^0)\ldots\Theta(x_{n-1}^0 - x_n^0)\,\Psi(x_1)\ldots\Psi(x_n) \\ & - \ \Theta(x_2^0 - x_1^0)\Theta(x_1^0 - x_3^0)\ldots\Theta(x_{n-1}^0 - x_n^0)\,\Psi(x_2)\Psi(x_1)\ldots\Psi(x_n) \\ & + \ldots \\ & + (-1)^{n_i} \text{ for } n_i \text{ permutations} + \ldots \end{aligned} \tag{10.13}$$

Similarly, the normal ordering is defined with a analogous minus sign, i.e.

$$N(a_{\vec{p}_1} a_{\vec{p}_2} a^\dagger_{\vec{q}_1}\ldots) = (-1)^k a^\dagger_{\vec{q}_1}\ldots a^\dagger_{\vec{q}_r} a_{\vec{p}_1}\ldots a_{\vec{p}_n} = (-1)^{k+1} a^\dagger_{\vec{q}_1}\ldots a^\dagger_{\vec{q}_r} a_{\vec{p}_2} a_{\vec{p}_1}\ldots a_{\vec{p}_n}\ , \tag{10.14}$$

for r raiusing and n fermionic lowering operators.

The Wick-theorem for fermions again states

$$T\{\Psi_1\bar{\Psi}_2\Psi_3\ldots\} = N(\Psi_1\bar{\Psi}_2\Psi_3\ldots + \text{ all Wick contractions})\ , \tag{10.15}$$

where a Wick contraction now is defind by

$$\overbrace{\Psi_1\bar{\Psi}_2} := \begin{cases} \{\Psi^+(x), \bar{\Psi}^-(y)\} & \text{for} \quad x^0 > y^0 \\ -\{\bar{\Psi}^+(y), \Psi^-(x)\} & \text{for} \quad y^0 > x^0 \end{cases} \tag{10.16}$$

where $\Psi = \Psi^+ + \Psi^-$ analogously to $\Phi = \Phi^+ + \Phi^-$. Comparing to (7.21) we see $\langle 0|\overbrace{\Psi_1\bar{\Psi}_2}|0\rangle = S_F(x-y)$ holds.

11 Lecture 11: Feynman diagrams

Let us compute

$$\mathcal{A} = \langle\Omega|T\{\Phi(x)\Phi(y)\}|\Omega\rangle = \frac{1}{N}\langle 0|T\{\Phi(x)\Phi(y)\cdot\exp\left(-i\int_{-\infty}^{+\infty}H_I(t)\mathrm{d}t\right)\}|0\rangle \tag{11.1}$$

where

$$N = \langle 0|T\{\exp\left(-i\int_{-\infty}^{+\infty}H_I(t)\mathrm{d}t\right)\}|0\rangle\ . \tag{11.2}$$

Since H is time independent it is a Schrödinger operator and thus for a Φ^4-theory we have

$$\int \mathrm{d}t\int \mathrm{d}^3 z H_I(t) = \tfrac{\lambda}{4!}\int d^4 z\Phi_I^4(z)\ . \tag{11.3}$$

Expanding in λ yields

$$\langle\Omega|T\{\Phi(x)\Phi(y)\}|\Omega\rangle = \tfrac{1}{N}\langle 0|T\{\Phi(x)\Phi(y)\}|0\rangle - \tfrac{i\lambda}{4!N}\int \mathrm{d}^4 z\langle 0|T\{\Phi_x\Phi_y\Phi_I^4\}|0\rangle + O(\lambda^2) \tag{11.4}$$

Using Wicks theorem one obtains three distinct terms plus permutations which merely result in multiplication factor

$$\begin{aligned}\langle\Omega|T\{\Phi(x)\Phi(y)\}|\Omega\rangle &= \tfrac{1}{N}\langle 0|\Phi_I(x)\Phi_I(y)|0\rangle \\ &+ 3\left(\tfrac{-i\lambda}{4!N}\right)\int \mathrm{d}^4 z\langle 0|T\{\Phi_I(x)\Phi_I(y)\Phi_I(z)\Phi_I(z)\Phi_I(z)\Phi_I(z)\}|0\rangle \\ &+ 4\cdot 3\left(\tfrac{-i\lambda}{4!N}\right)\int \mathrm{d}^4 z\langle 0|T\{\Phi_I(x)\Phi_I(z)\Phi_I(z)\Phi_I(z)\Phi_I(z)\Phi_I(y)\}|0\rangle \\ &+ \mathcal{O}(\lambda^2)\end{aligned} \tag{11.5}$$

In terms of Feynman propagators this corresponds to

$$\begin{aligned}\mathcal{A} = \langle\Omega|T\{\Phi(x)\Phi(y)\}|\Omega\rangle = \tfrac{1}{N}\Big[G_F(x-y) &- \tfrac{i\lambda}{8}\int \mathrm{d}^4 z(G_F(x-y)G_F(z-z)G_F(z-z) \\ &- \tfrac{i\lambda}{2}\int \mathrm{d}^4 z G_F(x-z)G_F(y-z)G_F(z-z)) + O(\lambda^2)\Big]\end{aligned} \tag{11.6}$$

Now one associates a set diagrammatic rules (the Feynman rules) with such expressions:

1. For each propagator $G_F(x-y) =$
2. For each vertex $-i\lambda\int \mathrm{d}z =$ 
3. For each external point $1 =$ 
4. Finally one needs to compute the symmetry factor (which is not encoded in the diagram)

In terms of such diagrams the expression (11.6) corresponds to

$\mathcal{A} = \frac{1}{N}\Big($  $\Big)$ (11.7)

Before we continue recall that in expressions like (9.34) we encounter the limit

$$\lim_{T\to\infty(1-i\epsilon)} \int_{-T}^{T} \mathrm{d}z^0 \int \mathrm{d}^3 z \, e^{-i(p_1^0+p_2^0+p_3^0-p_4^0)z^0} \ . \tag{11.8}$$

The integrand diverges at either $\pm\infty$ unless p^0 is also taken to be slightly imaginary $p^0 \to p^0(1+i\epsilon)$. This precisely corresponds to taking Feynman propagator.

The second diagram in (11.7) is called a disconnected diagram. Such diagrams arise when $\Phi(x), \Phi(y)$ and $e^{\int H_I}$ are contracted seperately among themselves. Thus one has schematically

$$\mathcal{A} = \frac{1}{N}\big(\text{all connected diagrams}\big)\big(e^{(\text{disconnected pieces})}\big) \tag{11.9}$$

Since N is given by (11.2) we see that is also is proportional to $(e^{(\text{disconnteced pieces})}$ and thus cancel the second factor in (11.9). We are left with

$$\mathcal{A} = \text{sum of all connected diagrams} \tag{11.10}$$

This observation is readily generalized to an arbitrary n-point amplitude

$$\begin{aligned} \mathcal{A}_n &= \langle\Omega|T\{\Phi(x_1)\dots\Phi(x_n)\}|\Omega\rangle \\ &= \text{sum of all connected diagrams with n external points} \end{aligned} \tag{11.11}$$

12 Lecture 12: S-Matrix

A typical experiment in particle physics consists of a collision of two particle beams and the subsequent "detection" of the decay products. The measured and computed quantity is the cross section σ defined as

$$\sigma = \frac{\text{number of scattering events}}{\varrho_{\mathcal{A}} \cdot l_{\mathcal{A}} \cdot \varrho_{\mathcal{B}} \cdot l_{\mathcal{B}} \cdot A} \ , \tag{12.1}$$

where the two colliding particle bunches $\mathcal{A}$ and $\mathcal{B}$ have densities $\varrho_{\mathcal{A}}$ and $\varrho_{\mathcal{B}}$ and length $l_{\mathcal{A}}$ and $l_{\mathcal{B}}$. A is the common area.

In order to compute σ one assumes that at $t = \pm\infty$ the particles are non-interacting and can be described by the the state

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \phi(\vec{k}) |\vec{k}\rangle \ . \tag{12.2}$$

Thus the intial state is given by

$$|\phi_{\mathcal{A}}\phi_{\mathcal{B}}\rangle_{\text{in}} = \int \frac{d^3k_{\mathcal{A}}}{(2\pi)^3} \int \frac{d^3k_{\mathcal{B}}}{(2\pi)^3} \frac{\phi_{\mathcal{A}}(\vec{k}_{\mathcal{A}})\phi_{\mathcal{B}}(\vec{k}_{\mathcal{B}})}{\sqrt{(2E_{\mathcal{A}})(2E_{\mathcal{B}})}} e^{-i\vec{b}\vec{k}_{\mathcal{B}}} |\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle_{\text{in}} \tag{12.3}$$

where $\vec{b}$ is the impact parameter. The final state reads

$$_{\text{out}}\langle\phi_1\phi_2\cdots| = \prod_f \int \frac{d^3p_f}{(2\pi)^3} \frac{\phi_f(\vec{p}_f)}{\sqrt{2E_f}} \,_{\text{out}}\langle\vec{p}_1\vec{p}_2\ldots| \ . \tag{12.4}$$

The probability P that the intial state $|\phi_{\mathcal{A}}\phi_{\mathcal{B}}\rangle_{\text{in}}$ evolves into the final state $_{\text{out}}\langle\phi_1\phi_2\ldots|$ is given by

$$P = |_{\text{out}}\langle\phi_1\phi_2\ldots|\phi_{\mathcal{A}}\phi_{\mathcal{B}}\rangle_{\text{in}}|^2 \,. \tag{12.5}$$

Thus we have to compute

$$_{\text{out}}\langle\vec{p}_1\vec{p}_2\ldots|\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle_{\text{in}} = \lim_{T\to\infty} \langle\vec{p}_1\vec{p}_2\ldots|e^{-iH(2T)}|\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle =: \langle\vec{p}_1\vec{p}_2\ldots|S|\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle \ , \tag{12.6}$$

where the last step defines the S-matrix. If there is no interaction $S = \mathbf{1}$ holds and thus the non-trivial scattering is captured by the T-matrix defined as $S = \mathbf{1} + iT$ (the factor i is convention). Computing $_{\text{out}}\langle\vec{p}_1\vec{p}_2...|iT|\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle_{\text{in}}$ one realizes that momentum conservation is garuanteed and therefore one defines the matrix $\mathcal{M}$ via

$$\langle\vec{p}_1\vec{p}_2\ldots|iT|\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle =: (2\pi)^4\delta^{(4)}(k_{\mathcal{A}}k_{\mathcal{B}} - \sum_f p_f) iM(k_{\mathcal{A}}, k_{\mathcal{B}} \to p_f) \ . \tag{12.7}$$

The relation bewteen σ and $\mathcal{M}$ is computed elsewhere [2] and is not repeated here. One finds for the differential cross section

$$d\sigma \ = \ \frac{(2\pi)^4\delta^{(4)}(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_f p_f)}{2E_{\mathcal{A}}2E_{\mathcal{B}}\,|\vec{v}_{\mathcal{A}} - \vec{v}_{\mathcal{B}}|} \left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right) |\mathcal{M}(k_{\mathcal{A}}, k_{\mathcal{B}} \to p_f)|^2 \ , \tag{12.8}$$

where $|\vec{v}_{\mathcal{A}} - \vec{v}_{\mathcal{B}}|$ denotes the relative velocity of the incoming beams in the laboratory frame and $p_{\mathcal{A}}, p_{\mathcal{B}}$ are the central values of the incoming momenta.

Later on we often consider the situation where only two particles are in the final state. In the center of mass frame (12.8) then simplifies to

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{2E_{\mathcal{A}}2E_{\mathcal{B}}}\frac{1}{|\vec{v}_{\mathcal{A}}-\vec{v}_{\mathcal{B}}|}\frac{|\vec{p}_1|}{(2\pi)^2 4E_{cm}}|\mathcal{M}(p_{\mathcal{A}},p_{\mathcal{B}}\to p_1,p_2)|^2. \tag{12.9}$$

Furthermore, if all four particles involved have the same mass, (12.9) further simplifies

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{cm}^2}. \tag{12.10}$$

For the decay rate Γ of a particle one finds

$$d\Gamma = \frac{(2\pi)^4\delta^{(4)}(p_{\mathcal{A}}+p_{\mathcal{B}}-\sum_f p_f)}{2m_{\mathcal{A}}}\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)|\mathcal{M}(m_{\mathcal{A}}\to\{p_f\})|^2\ , \tag{12.11}$$

The matrix $\mathcal{M}$ can be computed by the LSZ-formalism (the derivation will be given later) with the result

$$\langle\vec{p}_1\ldots\vec{p}_n|iT|\vec{p}_{\mathcal{A}}\vec{p}_{\mathcal{B}}\rangle = \lim_{T\to\infty(1-i\epsilon)}\left({}_0\langle\vec{p}_1\ldots\vec{p}_n|T\{e^{-i\int_{-T}^{T}dtH_I(t)}\}|\vec{p}_{\mathcal{A}}\vec{p}_{\mathcal{B}}\rangle_0\right)_{\substack{\text{fully connected,}\\ \text{amputated}}} \tag{12.12}$$

The notation "fully connected, amputated" refers to the contributing Feynman-diagrams. A Feynman-diagram is called amputated if an external point cannot be removed by cutting a single propagator. "Fully connected diagrams" are diagrams where all external points are connected to each other.

In order to evaluate ampltudes or Feynman diagrams we also need the expressions

$$\Phi^+(x)|p\rangle_0 = \int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{k}}}}e^{-ikx}a_{\vec{k}}(\sqrt{2E_{\vec{p}}}a^\dagger_{\vec{p}}|0\rangle) = e^{-ipx}|0\rangle\ , \tag{12.13}$$

where in the last step we used the commutator (3.12). Thus we conclude

$$\Phi_I(x)|p\rangle_0 = e^{-ipx}|0\rangle \qquad \text{and analogously} \qquad {}_0\langle p|\Phi_I(x) = \langle 0|e^{ipx} \tag{12.14}$$

As an example let us compute

$$_{\text{out}}\langle p_1p_2|p_{\mathcal{A}}p_{\mathcal{B}}\rangle_{\text{in}} = \langle p_1p_2|p_{\mathcal{A}}p_{\mathcal{B}}\rangle + \langle p_1p_2|iT|p_{\mathcal{A}}p_{\mathcal{B}}\rangle\ , \tag{12.15}$$

where the first term represents no scattering while the second term contains the interaction. Using (12.7) and (12.12) it evaluates at leading order to

$$\begin{aligned}
\langle p_1p_2|iT|p_{\mathcal{A}}p_{\mathcal{B}}\rangle &= \frac{-i\lambda}{4!}4!\int d^4z\langle p_1p_2|\Phi(z)\Phi(z)\,\Phi(z)\Phi(z)|p_Ap_B\rangle \\
&= -i\lambda\int d^4z e^{-i(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2)} \\
&= -i\lambda(2\pi)^4\delta^4(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2) \\
&= (2\pi)^4\delta^4(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2)i\mathcal{M}\ ,
\end{aligned} \tag{12.16}$$

which implies

$$\mathcal{M} = \lambda\ , \qquad \text{and} \qquad \left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2E_{CM}^2} = \frac{\lambda^2}{64\pi^2E_{CM}^2}\ . \tag{12.17}$$

13 Lecture 13: Feynman rules for QED

13.1 The Lagrangian of QED and its gauge invariance

The Lagragian for QED is given by

$$\mathcal{L} = \sum_{j=1}^{N} \bar{\Psi}_j \big(i\gamma^\mu D_\mu \Psi_j - m_j \Psi_j \big) - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \ , \tag{13.1}$$

where the covariant derivative $D_\mu \Psi$ is defined by

$$D_\mu \Psi_j := \partial_\mu \Psi_j + i q_j A_\mu \Psi_j \ . \tag{13.2}$$

A_μ is the photon field and the index j runs over all known elementary particles, i.e.

$$\begin{aligned} \text{leptons:} \quad & e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau \ , \\ \text{quarks:} \quad & u, d, c, s, t, b \ , \end{aligned} \tag{13.3}$$

and q_j is the electric charge of the respective field.

An alternative way to display the Lagrangian (13.1) reads

$$\mathcal{L} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{int}} \tag{13.4}$$

where

$$\begin{aligned} \mathcal{L}_{\text{Dirac}} &= \sum_{j=1}^{N} \bar{\Psi}_j \big(i\gamma^\mu \partial_\mu \Psi_j - m_j \Psi_j \big) \ , \\ \mathcal{L}_{\text{Maxwell}} &= -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \ , \\ \mathcal{L}_{\text{int}} &= A_\mu j^\mu \ , \qquad j^\mu = \sum_{j=1}^{N} \bar{\Psi}_j \gamma^\mu \Psi_j \ . \end{aligned} \tag{13.5}$$

Note that j^μ is a conserved Noether current satisfying $\partial_\mu j^\mu$.

The Lagrangian (13.1) is invariant under the gauge transformations

$$\begin{aligned} \Psi_j &\to \Psi'_j = e^{i q_j \alpha(x)} \Psi_j \ , \\ A_\mu &\to A'_\mu = A_\mu - \partial_\mu \alpha(x) \ . \end{aligned} \tag{13.6}$$

This is a generalization of the phase rotation discussed earlier in that the parameter of the transformation $\alpha(x)$ is an arbitrary real function of the space-time coordinates x. This situation is called a local symmtry or a gauge symmetry. Using (13.6) it is easy to check that

$$\begin{aligned} D_\mu \Psi_j &\to D_\mu \Psi'_j = e^{i q_j \alpha(x)} D_\mu \Psi_j \ , \\ F_{\mu\nu} &\to F'_{\mu\nu} = F_{\mu\nu} \ . \end{aligned} \tag{13.7}$$

The fact that $D_\mu \Psi_j$ transforms exactly like Ψ_j is the defining feature of a covariant derivative.

Finally, the Euler-Lagrange equations are given by

$$\big(i\gamma^\mu D_\mu - m_j \big) \Psi_j = 0 \ , \qquad \partial_\mu F^{\mu\nu} \ = j^\nu \ . \tag{13.8}$$

13.2 Quantization of the photon field

From (13.8) we immediatly find that the free field equation for the photon field reads

$$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu \partial_\mu A^\mu = 0 \ , \tag{13.9}$$

while the canonically conjugated momentum is $\pi^\mu = F^{\mu 0}$ (i.e. $\pi^0 = 0$). The quantization procedure for A_μ is complicated by the gauge invariance which renders one d.o.f. of A_μ unphysical. As a consequence the rules for canonical quatization cannot be applied naively in that a gauge has to be chosen. Common gauge choice are the Coulomb gauge $\vec{\nabla}\vec{A} = 0$, the temporal gauge $A^0 = 0$ or the Lorentz gauge $\partial_\mu A^\mu = 0$. Each gauge has its advantages and disadvantages; following [2] we continue in the Lorentz gauge which maintains Lorentz invariance.

In the Lorentz gauge (13.9) simplifies to

$$\Box A^\nu = 0 \ , \tag{13.10}$$

whose classical solutions we can give immediately as

$$A_\nu = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \sum_{r=0}^{3} \left(a^r_{\vec{k}} \epsilon^r_\nu(k) e^{-ikx} + a^{\dagger r}_{\vec{k}} \epsilon^{*r}_\nu(k) e^{ikx} \right) \ , \tag{13.11}$$

with $k_\mu k^\mu = 0$. ϵ^r_ν is a basis of polarization vectors which, due to $\partial_\mu A^\mu = 0$, are transversal and obey $k^\nu \epsilon^r_\nu = k^\nu \epsilon^{*r}_\nu = 0$. In scattering processes one imposes the additional boundfary condition that the initial and final photons are transversely polarized, i.e.

$$\epsilon_\nu = (0, \vec{\epsilon}) \ , \quad \text{with} \quad \vec{p} \cdot \vec{\epsilon} = 0 \ . \tag{13.12}$$

With these prerequisites one imposes the commutation relations

$$\begin{aligned} \left[a^r_{\vec{k}}, a^{\dagger r'}_{\vec{k}'}\right] &= (2\pi)^3 \delta^{rr'} \delta^{(3)}(\vec{k} - \vec{k}') \ , \\ \left[a^r_{\vec{k}}, \hat{a}^{r'}_{\vec{k}'}\right] &= 0 = \left[a^{\dagger r}_{\vec{k}}, a^{\dagger r'}_{\vec{k}'}\right]. \end{aligned} \tag{13.13}$$

As a consequence the propagator is given by

$$\langle 0 | T\{A_\mu(x) A_\nu(y)\} | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{k^2} \, e^{-ik(x-y)} \tag{13.14}$$

Before we continue let us note that the sign of $\langle 0 | T\{A_0(x) A_0(y)\} | 0 \rangle$ is "wrong". This is related to the unphysical gauge degree of freedom which, as we will see, decouples from physical amplitudes.

13.3 The Feynman rules for QED

Let us give the Feynman rules directly in momentum space. One has

Note that the $\bullet$ indicates where the diagram continues. The arrow on the fermion lines indicates particle number flow and not the flow of the momentum.

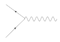
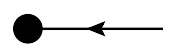
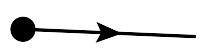
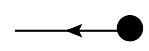
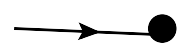
fermion propagator	$\Psi\bar{\Psi}$		$\frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2}$
photon propagator	$A_\mu A_\nu$		$\frac{-i\eta_{\mu\nu}}{k^2}$
vertex	$A_\mu \bar{\Psi}\Psi$		$-iq\gamma^\mu$
incoming fermion	$\Psi\|p, s\rangle$		$u^s(p)$
incoming anti-fermion	$\bar{\Psi}\|p, s\rangle$		$\bar{v}^s(p)$
outgoing fermion	$\langle s, p\|\bar{\Psi}$		$\bar{u}^s(p)$
outgoing anti-fermion	$\langle s, p\|\Psi$		$v^s(p)$
incoming photon	$A_\mu\|k, r\rangle$		$\epsilon^r_\mu(k)$
outgoing photon	$\langle r, k\|A_\mu$		$\epsilon^{*r}_\mu(k)$

Table 13.0: Feynman rules for QED

14 Lecture 14: $e^+e^- \to \mu^+\mu^-$

In this lecture we consider e^+e^--annihilation into a $\mu^+\mu^-$ – pair. The relevant Feynman-diagram at lowest order is given in fig. 14.

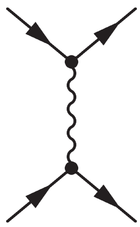


Figure 14.1: Feynman diagram for $e^+e^- \to \mu^+\mu^-$.

The corresponding $\mathcal{M}$-matrix is given by

$$i\mathcal{M} = (-ie)^2 \big(\bar{u}^r(k)\gamma^\mu v^{r'}(k')\big)_{(\mu)} \frac{-i\eta_{\mu\nu}}{(p+p')^2} \big(\bar{v}^{s'}(p')\gamma^\nu u^s(p)\big)_{(e)} \ , \tag{14.1}$$

which arises from the Wick-contraction

$$\langle \vec{k}, r; \vec{k}', r'| \int d^4x \bar{\psi}_{(\mu)}(x)\gamma^\mu \psi_{(\mu)}(x) A_\mu(x) \int d^4y \bar{\psi}_{(e)}(y)\gamma^\nu \psi_{(e)}(y) A_\nu(y) |\vec{p}, s; \vec{p}', s'\rangle \tag{14.2}$$

Note that the spins of the incoming and outgoing particles are fixed in (14.1). In order to compute $|\mathcal{M}|^2$ we need

$$(\bar{v}\gamma^\mu u)^* = (\bar{v}\gamma^\mu u)^\dagger = u^\dagger \gamma^{\mu\dagger}\gamma^0 v = \bar{u}\gamma^\mu v \ , \tag{14.3}$$

where in the last step we used (5.11). Together with (14.1) this yields

$$|\mathcal{M}|^2 = \frac{e^4}{(p+p')^4}\big(\bar{u}^r(k)\gamma^\mu v^{r'}(k')\big)_{(\mu)}\big(\bar{v}^{s'}(p')\gamma_\mu u^s(p)\big)_{(e)}\big(\bar{v}^{r'}(k')\gamma^\rho u^r(k)\big)_{(\mu)}\big(\bar{u}^s(p)\gamma_\rho v^{s'}(p')\big)_{(e)} \tag{14.4}$$

This expression can be further simplified if one sums over the external spins which experimentally corresponds to incoming and outgoing beams which are unpolarized. Using

$$\sum_s u_a^s(p)\,\bar{u}_b^s(p) = (\gamma^\mu p_\mu + m)_{ab}\ , \qquad \sum_s v_a^s(p)\,\bar{v}_b^s(p) = (\gamma^\mu p_\mu - m)_{ab}\ , \tag{14.5}$$

which were shown in problem 4.1 and the notation $\not{p} \equiv \gamma^\mu p_\mu$, we arrive at

$$\tfrac{1}{4}|\mathcal{M}|^2 = \frac{e^4}{4(p+p')^4}\mathrm{Tr}[(\gamma^\mu(\not{k}' - m_\mu)\gamma^\nu(\not{k} + m_\mu)]\mathrm{Tr}[(\gamma_\mu(\not{p} + m_e)\gamma_\nu(\not{p}' - m_e)] \tag{14.6}$$

To simplify this expression further one uses the following properties of the γ-matrices[10]

$$\begin{aligned} \mathrm{Tr}(\text{odd number of } \gamma^\mu) &= 0\ , && (14.7)\\ \mathrm{Tr}(\gamma^\mu\gamma^\nu) &= 4\eta^{\mu\nu}\ , && (14.8)\\ \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu} - \eta^{\mu\rho}\eta^{\nu\sigma})\ . && (14.9) \end{aligned}$$

Inserted into (14.6) together with the property $m_\mu \gg m_e \approx 0$ we arrive at

$$\frac{1}{4}\sum_{\text{Spin}}|M|^2 = \frac{8e^4}{(p+p')^4}\big((p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_{(\mu)}^2 p\cdot p'\big)\ . \tag{14.10}$$

We can further simplify this expression by going to the center-of-mass (CM) frame. Let us assume that the electrons come in along the z-axis, i.e. we choose $p = (E,0,0,E)$, $p' = (E,0,0,-E)$ with $p^2 = m_{(e)}^2 \approx 0$. The μ-pair is scattered in an angle θ relative to the z-axis, i.e. $k = (E,\vec{k})$, $k' = (E,-\vec{k})$ with $k^2 = k'^2 = m_{(\mu)}^2$ (which implies $|\vec{k}'| = |\vec{k}| = \sqrt{E^2 - m_{(\mu)}^2}$ and $\vec{k}\cdot\vec{e}_z = \vec{k}'\cdot\vec{e}_z = |\vec{k}|\cos\theta$). This further implies

$$\begin{aligned} &p\cdot p' = 2E^2\ , \qquad p\cdot k = p'\cdot k' = E^2 - E|\vec{k}|\cos\theta\ ,\\ &p\cdot k' = p'\cdot k = E^2 + E|\vec{k}|\cos\theta\ , \qquad (p+p')^2 = 4E^2 \end{aligned} \tag{14.11}$$

Inserted into (14.10) we arrive at

$$\begin{aligned} \tfrac{1}{4}\sum_{\text{Spin}}|M|^2 &= \frac{8e^4}{16E^4}\Big(E^2(E - \vec{k}\cos\theta)^2 + E^2(E + \vec{k}\cos\theta)^2 + 2m_{(\mu)}^2E^2\Big)\\ &= e^4\left(\Big(1 + \frac{m_{(\mu)}^2}{E^2}\Big) + \Big(1 - \frac{m_{(\mu)}^2}{E^2}\Big)\cos^2\theta\right)\ . \end{aligned} \tag{14.12}$$

[10] The first identity was shown in problem 6.3, the two latter in the problem 2 of the first exam.

Inserted into (12.9) we obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\vec{k}|}{4E_A E_B |\vec{v}_A - \vec{v}_B|} \cdot \frac{1}{16\pi^2 E_{CM}} |M(pp' \to kk')|^2. \tag{14.13}$$

Using $|\vec{v}_A - \vec{v}_B| \approx 2c$, $E_A = E_B = E = \frac{E_{CM}}{2}$ and $\alpha = \frac{e^2}{4\pi}$ yields

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{CM} &= \frac{|\vec{k}|}{32\pi^2 E_{CM}^3} \cdot \left(\tfrac{1}{4}\sum_{\text{Spin}} |M|^2\right) \\ &= \frac{\alpha^2}{4E^2}\sqrt{1 - \frac{m_{(\mu)}^2}{E^2}} \left(\left(1 + \frac{m_{(\mu)}^2}{E^2}\right) + \left(1 - \frac{m_{(\mu)}^2}{E^2}\right)\cos^2\theta\right) .\end{aligned} \tag{14.14}$$

The total cross section is obtained by integration

$$\begin{aligned}\sigma = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right) &= \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \left(\frac{d\sigma}{d\Omega}\right) \\ &= \frac{4\pi\alpha^2}{3E_{\text{CM}}^2}\sqrt{1 - \frac{m_{(\mu)}^2}{E^2}} \left(1 + \frac{m_{(\mu)}^2}{2E^2}\right)\end{aligned} \tag{14.15}$$

In the high energy limit $E \to \infty$ or $\frac{m_{(\mu)}}{E} << 1$ one obtains

$$\sigma \to R \equiv \frac{4\pi\alpha^2}{3E_{\text{CM}}^2} \tag{14.16}$$

Let us close this section with some further remarks. First of all the derived theoretical result is confirmed experimentally [2]. It can also be used to dertermine $m(\mu)$ or $m(\tau)$ for the analogous process $e^+e^- \to \tau^+\tau^-$. For the process $e^+e^- \to$ hadrons one replaces $\mu^+\mu^-$ by a quark pair. This changes on the one hand the charges (2/3 for u,c,t and $-1/3$ for d,s,b) and adds an overall factor of 3 for the three colors. Thus in the high energy limit one has

$$\sigma(e^+e^- \to \text{Hadronen}) \to 3R \sum_{f\in\{u,d,s,c,b,t\}} \left(\frac{q_f}{e}\right)^2 . \tag{14.17}$$

15 Lecture 15: Compton scattering $e^-\gamma \to e^-\gamma$

In this lecture we discuss aspects of Compton scattering. The leading order Feynman diagrams are depicted in fig. 15.1.

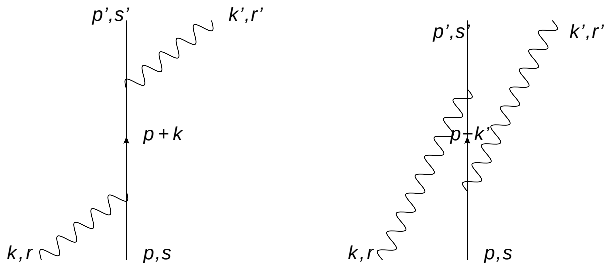


Figure 15.1: Feyman diagrams for Compton scattering

The corresponding $\mathcal{M}$-matrix reads

$$\begin{aligned} i\mathcal{M} = -ie^2\Big(&\gamma^\mu \epsilon^{*r'}_\mu(k')\bar{u}^{s'}(p')\frac{(\not{p}+\not{k})+m}{(p+k)^2-m^2}\,u^s(p)\gamma^\nu\epsilon^r_\nu(k) \\ &+\gamma^\nu\epsilon^r_\nu(k)\bar{u}^{s'}(p')\frac{(\not{p}-\not{k}')+m}{(p-k')^2-m^2}\,u^s(p)\gamma^\mu\epsilon^{r'}_\mu(k')\Big) \end{aligned} \tag{15.1}$$

This can be further simplified due to the properties

$$\begin{aligned} &p^2 = p'^2 = m^2\ , \qquad k^2 = k'^2 = 0\ , \\ \Rightarrow\quad &(p+k)^2 - m^2 = 2p\cdot k\ , \qquad (p-k')^2 - m^2 = -2p\cdot k'\ , \end{aligned} \tag{15.2}$$

and

$$(\not{p}+m)\gamma^\mu u^s(p) = 2p^\mu u^s(p) - \gamma^\mu(\not{p}-m)u^s(p) = 2p^\mu u^s(p)\ , \tag{15.3}$$

where in the last step we used (5.29) and (5.30). Inserted into (15.1) we arrive at

$$i\mathcal{M} = -ie^2\epsilon^{*r'}_\mu(k')\epsilon^r_\nu(k)\bar{u}^{s'}(p')\Big(\frac{(\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu)}{2pk} + \frac{(\gamma^\nu\not{k}'\gamma^\mu) - 2\gamma^\nu p^\mu)}{2pk'}\Big)u^s(p) \tag{15.4}$$

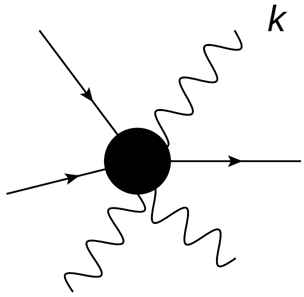
This expression can be further simplified by computing $|\mathcal{M}|^2$ and summing over spins and photon polarisations. Therefore let us pause and determine the sum over photon polarisation.

15.1 Sum over photon polarisation

Consinder an arbitrary Feyman diagram of the type depicted in 15.1.

The details do not matter but let us focus on one outgoing photon line with momentum k. The corresponding $\mathcal{M}$-matrix reads

$$i\mathcal{M} = i\epsilon^{*r}_\mu(k)\mathcal{M}^\mu \quad\Rightarrow\quad \sum_r |\epsilon^{*r}_\mu(k)\mathcal{M}^\mu|^2 = \sum_r \epsilon^{*r}_\mu(k)\epsilon^r_\nu(k)\mathcal{M}^\mu\mathcal{M}^{*\nu}\ . \tag{15.5}$$



Here $\mathcal{M}^\mu$ represents the "rest" of the diagram and due to the $A_\mu J^\mu$ coupling necessarily has the structure

$$\mathcal{M}^\mu = \int d^4x e^{ikx} \langle f | j^\mu(x) \ldots | i \rangle \ . \tag{15.6}$$

Since j^μ is a conserved Noether current we can derive the Ward identity

$$k_\mu \mathcal{M}^\mu = -i \int d^4x (\partial_\mu e^{ikx}) \langle f | j^\mu(x) \ldots | i \rangle = i \int d^4x e^{ikx} \langle f | \partial_\mu j^\mu(x) \ldots | i \rangle = 0 \ , \tag{15.7}$$

where in the second step we used partial integration. Without loss of generality we can choose $k^\mu = (k, 0, 0, k)$ and transverse polarisations $\epsilon^1_\mu = (0, 1, 0, 0), \epsilon^2_\mu = (0, 0, 1, 0)$. In this basis (15.7) implies $\mathcal{M}^0 = \mathcal{M}^3$ and thus

$$\sum_r |\epsilon^{*r}_\mu(k) \mathcal{M}^\mu|^2 = |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 = -|\mathcal{M}^0|^2 + |\mathcal{M}^1|^2 |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2 = -\eta_{\mu\nu} \mathcal{M}^\mu \mathcal{M}^{*\nu} \ . \tag{15.8}$$

Comparing with (15.5) we conclude

$$\sum_r \epsilon^{*r}_\mu \epsilon^r_\nu = -\eta_{\mu\nu} + k_\mu X_\nu + k_\nu X_\mu \ , \tag{15.9}$$

where X_ν is anarbitrary four-vector. Another way of saying this is that in physical amplitudes we can replace

$$\sum_r \epsilon^{*r}_\mu \epsilon^r_\nu \to -\eta_{\mu\nu} \ . \tag{15.10}$$

15.2 Unpolarized Compton scattering

Let us return to the Compton scattering for unpolarized beams. However, in the follwing we will not give the details of the computation but merely state the results.[11] Using (14.5) and (15.10) we compute from (15.4)

$$\begin{aligned} \tfrac{1}{4} \sum_{r,r',s,s'} |\mathcal{M}|^2 &= \tfrac{1}{4} e^4 \, \mathrm{tr} \Big(\big(\frac{(\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu)}{2pk} + \frac{(\gamma^\nu \not{k}' \gamma^\mu) - 2\gamma^\nu p^\mu)}{2pk'} \big) \big(\not{p} + m \big) \cdot \\ &\quad \cdot \big(\frac{(\gamma_\nu \not{k} \gamma_\mu + 2\gamma_\mu p_\nu)}{2pk} + \frac{(\gamma_\mu \not{k}' \gamma^\nu) - 2\gamma_\nu p_\mu)}{2pk'} \big) \big(\not{p}' + m \big) \Big) \qquad (15.11) \\ &= 2e^4 \Big(m^4 \big(\frac{1}{pk} + \frac{1}{pk'} \big)^2 + 2m^2 \big(\frac{1}{pk} - \frac{1}{pk'} \big) + \big(\frac{pk'}{pk} - \frac{pk}{pk'} \big) \Big) \ . \end{aligned}$$

[11] Some details can be found in [2].

The partial cross section is conveniently given in the lab-frame where

$$k = (\omega, 0, 0, \omega)\ , \quad p = (m, \vec{0})\ , \quad k' = (\omega', \omega' \sin\theta, 0, \omega' \cos\theta)\ , \quad p = (E', \vec{p}')\ , \tag{15.12}$$

where θ is again the scattering angle. One then finds the Klein-Nishina formula

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right)\ , \quad \text{where} \quad \omega' = \frac{\omega}{1 + \frac{\omega}{m}(1 - \cos\theta)}\ . \tag{15.13}$$

In the limit $\omega \to 0$ one has $\omega'/\omega \to 1$ and

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(1 + \cos^2\theta\right)\ , \tag{15.14}$$

which results in the total Thomson cross section

$$\sigma_{\text{tot}} = \frac{8\pi\alpha^2}{3m^2}\ . \tag{15.15}$$

16 Lecture 16: Dimensional regularization

In this lecture we compute the leading order correction to the fermion propagator. This amounts to evaluating the diagram in fig. 16.1. Truncating the two external fermion lines and using the momentum space Feynamn rules one arrives at

$$-i\Sigma_2(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2} \gamma_\mu \frac{-i}{(p-k)^2-\mu^2} \; , \tag{16.1}$$

where a IR regulator μ is included for later convenience. As a next step one simplifies the denominator using Feynman integrals.

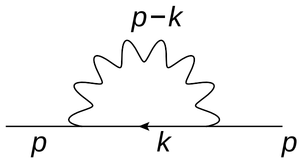


Figure 16.1: Leading order correction for the fermion propagator

An expression $1/AB$ can be given an integral representation of the form

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{[xA+(1-x)B]^2} = \int_0^1 dx \int_0^1 dy \, \frac{\delta(x+y-1)}{[xA+yB]^2} \; , \tag{16.2}$$

which can be proven by elementary integration. It is a special case of the more general representation

$$\frac{1}{A_1 \ldots A_n} = \int_0^1 dx_1 \ldots dx_n \, \delta\Big(\sum_{i=1}^n x_i - 1\Big) \frac{(n-1)!}{[x_1A_1 + \ldots + x_nA_n]^n} \; , \tag{16.3}$$

which is proven for example by induction.

Applied to Σ_2 one derives

$$\begin{aligned} \frac{1}{[k^2-m^2][(p-k)^2-\mu^2]} &= \int_0^1 dx \, \frac{1}{[k^2-(1-x)2pk+(1-x)(p^2-\mu^2)-xm^2]^2} \\ &= \int_0^1 dx \, \frac{1}{[l^2-\Delta]^2} \; , \end{aligned} \tag{16.4}$$

where

$$l = k-(1-x)p \; , \qquad \Delta = xm^2 + (1-x)\mu^2 + x(1-x)p^2 \; . \tag{16.5}$$

Using $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$ and $\gamma^\mu\gamma_\mu = 4$ we have altogether

$$-i\Sigma_2(p) = -e^2 \int \frac{d^4k}{(2\pi)^4} \int_o^1 dx \, \frac{(4m-2\not{k})}{[l^2-\Delta]^2} \; . \tag{16.6}$$

Changing the integration variables to l we arrive at

$$-i\Sigma_2(p) = -e^2 \int_o^1 dx \, \big(4m - 2(1-x)\not{p}\big) \int \frac{d^4l}{(2\pi)^4} \frac{1}{[l^2-\Delta]^2} \; , \tag{16.7}$$

where the term linear in l vanishes when integrated over a symmetric domain. From this expression we see that Σ_2 is not well defined. The l-integral does not exist in that it is logarithmically divergent for large l, i.e. in the UV.

In order to proceed the integral has to be "regulated". For the case at hand a simple momentum cut-off would do but for later use we introduce the idea of 't Hooft and Veltman to regulate such integrals by going away from four space-time dimensions. One first Wick rotates to Euclidean momenta[12]

$$l^0 \to i l_E^0 \ , \quad \vec{l} \to \vec{l}_E \ , \qquad \Rightarrow \qquad l^2 \to -l_E^2 \ , \quad d^4 l \to i d^4 l_E \ . \tag{16.8}$$

Now one evaluates the integral by considering it as an analytic function of the space-time dimension d. That is one replaces

$$I_1 = i \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{[l_E^2 + \Delta]^2} = \lim_{d\to 4} I_1(d) = i \lim_{d\to 4} \int \frac{d^d l_E}{(2\pi)^d} \frac{1}{[l_E^2 + \Delta]^2} \ . \tag{16.9}$$

Choosing spherical coordinates with a measure $d^d l_E = l_E^{d-1} dl d\Omega_d$ and an area element $d\Omega_d$ of the d-dimensional unit sphere normalized with

$$\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \ , \qquad \Gamma(\alpha) := \int_0^\infty dz\, z^{\alpha-1} e^{-z} \ , \tag{16.10}$$

one arrives at

$$I_1(d) = \frac{i}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \int_0^\infty dl_E \frac{l_E^{d-1}}{[l_E^2 + \Delta]^2} \ . \tag{16.11}$$

Substituting $y = \Delta[l_E^2 + \Delta]^{-1}$ into I_1 and using the integral representation

$$\int_0^1 dy y^{\alpha-1} (1-y)^{\beta-1} \ = \ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \ , \tag{16.12}$$

one obtains

$$I_1(d) = \frac{i}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \ \Delta^{\frac{d}{2}-2} \ . \tag{16.13}$$

$\Gamma(\alpha)$ has isolated poles at $\alpha = 0, -1, -2, \ldots$ and as a consequence I_1 has poles at $d = 4, 6, 8, \ldots$. The pole at $d = 4$ precisely corresponds to the logarithmic divergence we already observed earlier. However in $d = 4 - \epsilon$ dimension the integral I_1 is finite and thus regulated by going away from $d = 4$. The behavior of the Γ-function near the pole is

$$\Gamma(2 - \frac{d}{2}) \ = \ \Gamma(\frac{\epsilon}{2}) \ = \ \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \ . \tag{16.14}$$

This implies

$$\begin{aligned} I_1 = \lim_{\epsilon\to 0} I_1(\epsilon) = i \lim_{\epsilon \to 0} \frac{\Gamma(\frac{\epsilon}{2}) \Delta^{\frac{\epsilon}{2}}}{(4\pi)^{2-\frac{\epsilon}{2}}} &= \frac{i}{(4\pi)^2} \lim_{\epsilon\to 0} \Gamma(\frac{\epsilon}{2}) e^{-\frac{\epsilon}{2} \ln(\Delta/4\pi)} \\ &= \frac{i}{(4\pi)^2} \lim_{\epsilon\to 0} \Big(\frac{2}{\epsilon} - \gamma - \ln(\Delta/4\pi) + \mathcal{O}(\epsilon) \Big) \ . \end{aligned} \tag{16.15}$$

[12]The rotation is done such that no pole is encountered in the complex l^0-plane.

In order to give a consistent formula for Σ_2 we need to reevaluate the numerate in d-dimensions. One insists that the Dirac algebra is unchanged with 4×4 γ-matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}_{4\times 4} \ , \qquad \mu, \nu = 0, \ldots, d-1 \ . \tag{16.16}$$

This in turn implies

$$\gamma^\mu \gamma_\mu = d\mathbf{1}_{4\times 4} \ , \qquad \gamma^\mu \gamma^\nu \gamma_\mu = (\epsilon - 2)\gamma^\nu \ . \tag{16.17}$$

Inserted into Σ_2 one obtains

$$\begin{aligned} -i\Sigma_2(p) &= -e^2 \int_o^1 dx \left((\epsilon - 2)(1-x)\not{p} + (4-\epsilon)m \right) I_1 \\ &= \frac{-ie^2}{(4\pi)^2} \lim_{\epsilon \to 0} \int_o^1 dx \left((\epsilon - 2)(1-x)\not{p} + (4-\epsilon)m \right) \left(\frac{2}{\epsilon} - \gamma - \ln(\Delta/4\pi) + \mathcal{O}(\epsilon) \right) \ . \end{aligned} \tag{16.18}$$

For the moment we leave this as an divergent expression and return to its interpretation when we discuss renormalization in section 20.

17 Lecture 17: Field strength renormalization

17.1 The Kallén-Lehmann-representation of the propagator

In the last lecture we found an UV-divergence in the leading order perturbative correction of the electron or rather fermion propagator $\langle\Omega|T\{\bar{\psi}(x)\psi(y)\}|\Omega\rangle$. A similar divergence occurs for the scalar propagator in a ϕ^4-theory, i.e. in $\langle\Omega|T\{\Phi(x)\Phi(y)\}|\Omega\rangle$. Therefore we need to reconsider the structure of these quantities independent of perturbation theory. For simplicity we will focus on $\langle\Omega|T\{\Phi(x)\Phi(y)\}|\Omega\rangle$ and then only give the result for $\langle\Omega|T\{\bar{\psi}(x)\psi(y)\}|\Omega\rangle$.

For the free theory $\langle 0|T\{\Phi(x)\Phi(y)\}|0\rangle$ gives the amplitude for a particle propagating from y to x. Now we want to understand its meaning in the interacting theory. Let us insert a complete set of eigenstates of H and $\vec{p}$. States with $\vec{p}\neq 0$ can be viewed as Lorentz transformation of states with $\vec{p}=0$, i.e.

$$1 = |\Omega\rangle\langle\Omega| + \sum_{\lambda_{\vec{0}}}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}(\lambda)}|\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|\ , \tag{17.1}$$

where the sum is over all zero-momentum states and $|\lambda_{\vec{p}}\rangle$ denotes all boasts of the zero-momentum states $|\lambda_{\vec{0}}\rangle$. (Note that $\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}(\lambda)}(...)$ is Lorentz invariant).

Choosing $x^0 > y^0$ we obtain

$$\begin{aligned}\langle\Omega|T\{\Phi(x)\Phi(y)\}|\Omega\rangle &= \langle\Omega|\Phi(x)|\Omega\rangle\langle\Omega|\Phi(y)|\Omega\rangle\\ &\quad+\sum_{\lambda_{\vec{0}}}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\langle\Omega|\Phi(x)|\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|\Phi(y)|\Omega\rangle.\end{aligned}\tag{17.2}$$

For simplicity let us consider theories where the vacuum expectation value of Φ vanishes, i.e. theories with $\langle\Omega|\Phi(x)|\Omega\rangle = 0$. Furthermore let us use the space-time translation operator $\hat{P}^\mu = (\hat{H},\hat{P}^i)$ to write $\Phi(x) = e^{i\hat{P}\cdot x}\Phi(0)e^{-i\hat{P}\cdot x}$. Using the invariance of the ground state $\langle\Omega|e^{i\hat{P}\cdot x} = \langle\Omega|$ we obtain

$$\begin{aligned}\langle\Omega|\Phi(x)|\lambda_{\vec{p}}\rangle &= \langle\Omega|\Phi(0)|\lambda_{\vec{p}}\rangle e^{-ipx}|_{p^0=E_{\vec{p}}}\\ &= \langle\Omega|U^{-1}U\Phi(0)U^{-1}U|\lambda_{\vec{p}}\rangle e^{-ipx}|_{p^0=E_{\vec{p}}}\\ &= \langle\Omega|\Phi(0)|\lambda_0\rangle e^{-ipx}|_{p^0=E_{\vec{p}}}\ ,\end{aligned}\tag{17.3}$$

where in the second line we inserted an operator U. One can choose U to be the unitary operator which represents Lorentz-boasts in the Fock-space and connects $U|\lambda_{\vec{p}}\rangle = |\lambda_0\rangle$. Since Φ is a scalar one also has $U\Phi(0)U^{-1} = \Phi(0)$ and these two properties then lead to the third line (17.3). Inserting (17.3) into (17.2) we obtain

$$\begin{aligned}\langle\Omega|\Phi(x)\Phi(y)|\Omega\rangle &= \sum_{\lambda_0}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\,e^{-ip(x-y)}|_{p^0=E_{\vec{p}}}\ |\langle\Omega|\Phi(0)|\lambda_0\rangle|^2\\ &= \sum_{\lambda_0}\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-m_\lambda^2+i\epsilon}\,e^{-ip(x-y)}|_{p^0=E_{\vec{p}}}\ |\langle\Omega|\Phi(0)|\lambda_0\rangle|^2.\end{aligned}\tag{17.4}$$

An analogous computation for $y^0 > x^0$ similarly yields

$$\langle\Omega|T\{\Phi(x)\Phi(y)\}|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\varrho(M^2)G_F(x-y,M^2), \tag{17.5}$$

which is called Kallén-Lehmann-representation of the propagator. The spectral density ϱ is given by

$$\begin{aligned}\varrho(M^2) &= \sum_{\lambda_0}(2\pi)\delta(M^2-m_\lambda^2)|\langle\Omega|\Phi(0)|\lambda_{\vec{p}}\rangle|^2 \\ &= 2\pi\delta(M^2-m^2)Z + \text{ possibly bound states} \\ &\qquad + \text{multiparticle excitations for } M^2 \geq 4m^2 \ .\end{aligned} \tag{17.6}$$

$Z = |\langle\Omega|\Phi(0)|\lambda_{\vec{p}}\rangle|^2$ is called field strength renormalization and m is the physical mass. It is important to note that m does not coincide with the so called bare parameter m_0 which appears in the Lagrangian.

It is also useful to consider the Fourier transform

$$\begin{aligned}\int d^4x e^{ipx}\langle\Omega|T\{\Phi(x)\Phi(0)\}|\Omega\rangle &= \int_0^\infty \frac{dM^2}{2\pi}\,\varrho(M^2)\,\frac{i}{p^2-M^2+i\epsilon} \\ &= \frac{iZ}{p^2-m^2+i\epsilon} + \int_{\approx 4m^2}^\infty \frac{dM^2}{2\pi}\,\varrho(M^2)\,\frac{i}{p^2-M^2+i\epsilon}\ .\end{aligned} \tag{17.7}$$

Comparison with the free theory

$$\int d^4x e^{ipx}\langle 0|T\{\Phi(x)\Phi(0)\}|0\rangle = \frac{i}{p^2-m_0^2+i\epsilon} \tag{17.8}$$

reveals the following differences:

(i) the factor Z which in the free theory becomes $Z = |\langle 0|\Phi(0)|p\rangle|^2 = 1$,

(ii) the difference between the physical mass m and the bare mass $m_0 \neq m$,

(iii) the existence of multi-particle states for $M^2 \geq 4m^2$.

For fermions one similarly determines

$$\int d^4x e^{ipx}\langle\Omega|T\{\Psi(x)\overline{\Psi}(y)\}|\Omega\rangle = \frac{iZ_2(\not{p}+m)}{p^2-m^2+i\epsilon} + \text{ multi-particle states }, \tag{17.9}$$

where

$$\langle\Omega|\Psi(0)|p,s\rangle = \sqrt{Z_2}u^s(p)\ . \tag{17.10}$$

17.2 Application: the electron self energy

Let us now apply these consideration to Σ_2 computed in (16.18).[13] We immediately observe that there is no simple pole at $p^2 = m_0^2$. This is an artefact of perturbation theory

[13]To be consistent with the notation of this section we should replace $m \to m_0$ in (16.18).

as can be seen as follows. Let us define one-particle irreducible Feynman diagrams (1PI) as diagrams which cannot be split by removing a single line. For the case at hand the 1PI diagrams are depicted in fig. 17.1. Let us denote this sum by $-i\Sigma(p)$ and the first diagram in the sum by $-i\Sigma_2(p)$.

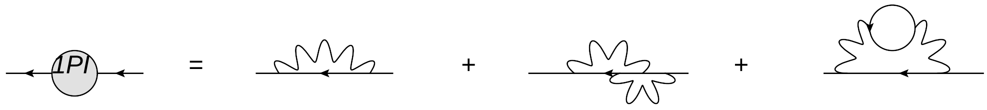


Figure 17.1: 1PI diagrams

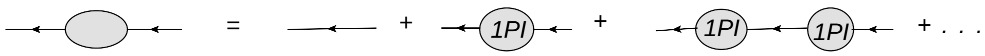


Figure 17.2: Diagramatic expression for exact fermion propagator

Let us further denote the full propagator $\int d^4x e^{ipx} \langle\Omega|T\{\Psi(x)\overline{\Psi}(y)\}|\Omega\rangle$ by a blob without any label. This full propagator can then by given by a sum of 1PI diagrams as depicted in fig. 17.2. This sum corresponds to

$$\frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}\big(-i\Sigma(p)\big)\frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}\big(-i\Sigma(p)\big)\frac{i}{\not{p}-m_0}\big(-i\Sigma(p)\big)\frac{i}{\not{p}-m_0} + \dots$$
$$= \frac{i}{\not{p}-m_0}\sum_{n=0}^{\infty}\left(\frac{\Sigma(p)}{\not{p}-m_0}\right)^n = \frac{i}{\not{p}-m_0-\Sigma} \ . \tag{17.11}$$

Now we see that there is a simple pole at

$$\big(\not{p}-m_0-\Sigma(p)\big)_{(\not{p}=m)} = 0 \ , \tag{17.12}$$

which defines the physical mass $m = m_0 + \Sigma(\not{p}=m)$. Taylor expansion around the pole yields

$$\Sigma = \Sigma(\not{p}=m) + \frac{d\Sigma}{d\not{p}}\Big|_{(\not{p}=m)}(\not{p}-m) + \mathcal{O}\big((\not{p}-m)^2\big) \ . \tag{17.13}$$

which implies

$$\big(\not{p}-m_0-\Sigma(p)\big) = (\not{p}-m)\Big(1-\frac{d\Sigma}{d\not{p}}\Big|_{(\not{p}=m)}\Big) + \mathcal{O}\big((\not{p}-m)^2\big) \ . \tag{17.14}$$

Comparing with (17.9) we conclude

$$Z_2^{-1} = 1 - \frac{d\Sigma}{d\not{p}}\Big|_{(\not{p}=m)} \ . \tag{17.15}$$

18 Lecture 18: Renormalization of the electric charge

In the previous lecture we discussed the corrections to the fermion propagator. Let us now repeat a similar analysis for the photon propagator. The exact photon propagator is diagramatically depicted in fig. 18.1. It can be expressed in terms of the 1 PI diagramms which we denote by $i\Pi^{\mu\nu}(q)$ and which are depicted in fig. 18.2. As before we denote by $i\Pi_2^{\mu\nu}(q)$ the leading contribution of the 1PI diagrams which is the first diagram of fig. 18.2.

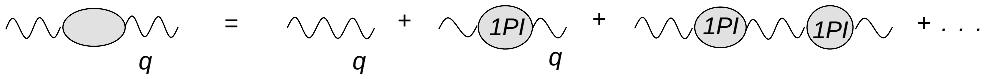


Figure 18.1: Diagramatic expression for the exact photon propagator

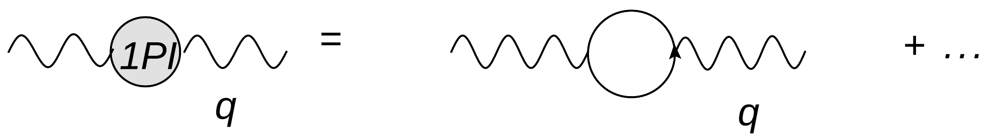


Figure 18.2: 1PI diagrams

18.1 General properties

Before computing $i\Pi_2^{\mu\nu}(q)$ we can constrain the structure of the exact expression $i\Pi^{\mu\nu}(q)$. The Ward identity $q_\mu \Pi^{\mu\nu} = 0$ implies

$$\Pi^{\mu\nu} = \left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2)\ , \tag{18.1}$$

where Π has to be regular at $q = 0$. Furthermore, we can write the expansion of fig. 18.1 as

$$\begin{aligned} &\frac{-i\eta_{\mu\nu}}{q^2} + \frac{-i\eta_{\mu\nu}}{q^2}\big(i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)\big)\frac{-i\eta_{\mu\nu}}{q^2} + \dots \\ =&\frac{-i\eta_{\mu\nu}}{q^2} + \frac{-i\eta_{\mu\rho}}{q^2}\Delta^\rho_\nu\Pi(q^2) + \frac{-i\eta_{\mu\rho}}{q^2}\Delta^\rho_\sigma\Delta^\sigma_\nu\Pi^2(q^2) + \dots\ , \end{aligned} \tag{18.2}$$

where

$$\Delta^\rho_\nu = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\big(\frac{-i\eta_{\mu\nu}}{q^2}\big) = \big(\delta^\rho_\nu - \frac{q^\rho q_\nu}{q^2}\big)\ . \tag{18.3}$$

Δ^ρ_ν is a projection operator in that it satisfies

$$\Delta^\rho_\sigma\Delta^\sigma_\nu = \Delta^\rho_\nu\ . \tag{18.4}$$

Inserted into (18.2) one obtains

$$\frac{-i\eta_{\mu\nu}}{q^2} + \frac{-i\eta_{\mu\rho}}{q^2}\Delta^\rho_\nu\big(\Pi + \Pi^2 + \dots\big) = \frac{-i}{q^2(1-\Pi)}\big(\eta_{\mu\nu} - q_\mu q_\nu\big) - i\frac{q_\mu q_\nu}{q^4}\ , \tag{18.5}$$

where used the geometric series. The last term in (18.5) vanishes in amplitudes due to the Ward identity and thus we have

$$\frac{-i}{q^2(1-\Pi)}\big(\eta_{\mu\nu} - q_\mu q_\nu\big) = \frac{-iZ_3\eta_{\mu\nu}}{q^2} + \dots \ , \tag{18.6}$$

where

$$Z_3 = \frac{1}{1-\Pi(0)} \tag{18.7}$$

is the first term in the Taylor expansion of the exact photon propagator and Z_3 the residue of the pole at $q = 0$.

18.2 Computation of $\Pi_2^{\mu\nu}$

Let us now compute the first diagram in fig. 18.2. Using momentum-space Feynman rules one obtains

$$i\Pi_2^{\mu\nu} = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\Big(\gamma^\mu \frac{i(\not k + m)}{k^2 - m^2}\gamma^\nu \frac{i(\not k + \not q + m)}{(k+q)^2 - m^2}\Big) \ . \tag{18.8}$$

Using properties of the γ-matrices the numerator N can be simplified as[14]

$$N = -4e^2\big(k^\mu(k+q)^\nu + (k+q)^\mu k^\nu - \eta^{\mu\nu}(k(k+q) - m^2)\big) \ . \tag{18.9}$$

Expressed in terms of Feynman integrals the denominator D reads

$$\begin{aligned} D = \frac{1}{k^2-m^2}\frac{1}{(k+q)^2-m^2} &= \int_0^1 dx\, \frac{1}{[(1-x)(k^2-m^2) + x((k+q)^2 - m^2)]^2} \\ &= \int_0^1 dx\, \frac{1}{(l^2-\Delta)^2} \ , \end{aligned} \tag{18.10}$$

where

$$l = k + xq \ , \qquad \Delta = m^2 - x(1-x)q^2 \ . \tag{18.11}$$

Inserting (18.9) and (18.10) into (18.8) and dropping terms linear in l one arrives at

$$i\Pi_2^{\mu\nu} = -4e^2 \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \frac{(2l^\mu l^\nu - \eta^{\nu\mu}l^2 - 2x(1-x)q^\mu q^\nu + \eta^{\nu\mu}(m^2 + x(1-x)q^2)}{(l^2-\Delta)^2} \ . \tag{18.12}$$

This integral appears to be quadratically divergent in the UV and has to dimensionally regulated. Following the same steps as before we Wick rotate $l^0 \to il_E^0$, go to d-dimension and use that under the integral $l^\mu l^\nu = \frac{1}{d}\eta^{\mu\nu}l^2$ holds to obtain

$$i\Pi_2^{\mu\nu} = -4ie^2 \int_0^1 dx \int \frac{d^dl_E}{(2\pi)^d} \frac{\eta^{\nu\mu}l_E^2(1-2/d) - 2x(1-x)q^\mu q^\nu + \eta^{\nu\mu}(m^2 + x(1-x)q^2)}{(l_E^2+\Delta)^2} \ . \tag{18.13}$$

[14] Note that the expression also holds in d-dimensions.

Before we proceed note that regulated the divergent integral by a simple momentum cut-off Λ we would get a quadratic divergence

$$\int \frac{d^4l}{(2\pi)^4} \frac{\eta^{\nu\mu} l^2}{(l_E^2+\Delta)^2} \sim \Lambda^2 \eta^{\nu\mu} \ , \tag{18.14}$$

which violates the Ward identity in that $q_\mu \Pi_2^{\mu\nu} \neq 0$. As we will see shortly this does not happen in dimensionally regularization which is one of the reason for ist usefullness.

In problem 9.2 we showed

$$\begin{aligned} \int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2+\Delta)^2} &= \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{1}{\Delta^{1-d/2}} \frac{\Gamma(1-d/2)}{\Gamma(2)} \\ &= \frac{1}{(4\pi)^{d/2}} \frac{d}{2(1-d/2)} \frac{\Delta}{\Delta^{2-d/2}} \Gamma(2-d/2) \ , \end{aligned} \tag{18.15}$$

where in the second step we used $\Gamma(u+1) = u\Gamma(u)$. In problem 9.2 we also showed

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2+\Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)} \ . \tag{18.16}$$

Inserting both integrals into (18.13) we arrive at

$$\begin{aligned} i\Pi_2^{\mu\nu} &= -4ie^2 \lim_{d\to 4} \int_0^1 dx \, \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} \big(\eta^{\mu\nu}(m^2 + x(1-x)q^2 - \Delta) - 2x(1-x)q^\mu q^\nu\big) \\ &= -4ie^2 \big(\eta^{\mu\nu} q^2 - q^\mu q^\nu\big) \lim_{d\to 4} \int_0^1 dx \, x(1-x) \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} \ , \end{aligned} \tag{18.17}$$

where in the second step we merely used (18.11). We see that, as promised, $\Pi_2^{\mu\nu}$ is proportional to the projection operator and thus the Ward identity is fullfilled. The quadratic divergence cancelled and the final expression is only logarithmically divergent. Comparing with (18.1) we conclude

$$\Pi_2 = -\frac{2\alpha}{\pi} \lim_{d\to 4} \int_0^1 dx \, x(1-x) \frac{\Gamma(2-d/2)}{(4\pi)^{d/2-2}} \frac{1}{\Delta^{2-d/2}} \ , \tag{18.18}$$

which indeed has no pole at $q = 0$.

19 Lecture 19: The electron vertex

19.1 General considerations

So far we considered electron scattering at lowest order (see fig.19.1). In this lecture we compute a first set of loop corrections to this process.

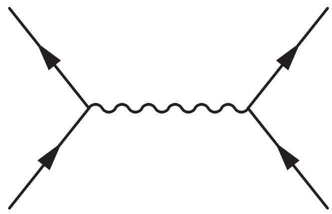


Figure 19.1: electron scattering at lowest order

We insist that the initial and final state are unchanged and thus the loops occur "inside" the diagramm. One set of corrections – termed vertex corrections – are depicted in fig.19.2.

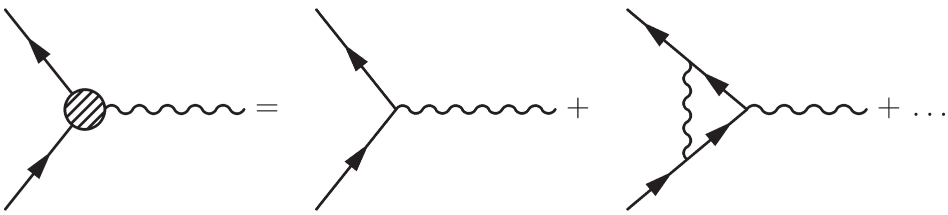


Figure 19.2: Leading correction to the electron vertex

In the notation we amputate the photon line and define the vertex correction as

$$i\mathcal{M} = -ie\overline{u}(p')\Gamma^{\mu}(p,p')u(p)\ldots \tag{19.1}$$

where $-ie\Gamma^{\mu}(p,p')$ stands for the sum of all vertex diagrams, i.e. for the diagrams given in fig.19.2. We also see that at lowest order $\Gamma^{\mu} = \gamma^{\mu}$ holds. The corrections can be constrained by the following considerations:

1. Lorentz invariance constrains the corrections to be of the form

$$\Gamma^{\mu} = \gamma^{\mu}A + (p'^{\mu} + p^{\mu})B + (p'^{\mu} - p^{\mu})C \ , \tag{19.2}$$

   where A, B, C can only depend on the Lorentz-scalars m, e or $q^2 = (p-p')^2 = 2m - 2p\cdot p'$.[15]

2. Current conservation $\partial_{\mu}j^{\mu} = 0$ implies the Ward-identity

$$q_{\mu}\Gamma^{\mu} = 0 \ . \tag{19.3}$$

[15]Dependence on $\not{p}$ und $\not{p}'$ can be neglected since we can always use $\overline{u}(p')\not{p}' = \overline{u}(p')m$ and $\not{p}u(p) = mu(p)$. Thus the only non-trivial scalar is q^2.

Eq. (19.3) implies

$$q_\mu \Gamma^\mu = (p_\mu - p'_\mu)\gamma^\mu A + \underbrace{(p_\mu - p'_\mu)(p'^\mu + p^\mu)}_{p^2 - p'^2 = 0} B + \underbrace{(p_\mu - p'_\mu)(p'^\mu - p^\mu)}_{-(p_\mu - p'_\mu)^2} C = 0\ . \tag{19.4}$$

The first term drops out of any amplitude since

$$\overline{u}(p')(\not p - \not p')u(p) = \overline{u}(p')u(p)(m-m) = 0\ , \tag{19.5}$$

as a consequence of $\overline{u}(p')(\not p' - m) = (\not p - m)u(p) = 0$. Thus in (19.4) only the last term proportional to C survives and therefore $C = 0$ has to hold. Finally, one can use the Gordon-identity (proven in problem 10.1)

$$\overline{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\,\overline{u}(p')\big((p'^\mu + p^\mu) + 2iS^{\mu\nu}q_\nu\big)u(p)\ , \tag{19.6}$$

where $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. Eliminating the $(p+p')$ dependence in (19.2) via the Gordon identity we have altogether the structure

$$\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + \tfrac{i}{m}\,S^{\mu\nu}q_\nu F_2(q^2)\ . \tag{19.7}$$

F_1 and F_2 are called form factors which at lowest order are given by $F_1 = 1, F_2 = 0$.

19.2 Leading order results

Let us now consider the first corrections $\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu$ given by the diagram of fig. 19.3. It corresponds to

$$\begin{aligned} &- ie\overline{u}(p')\delta\Gamma^\mu(p, p')u(p) = \\ &\int \frac{d^4k}{(2\pi)^4}\overline{u}(p')(-ie\gamma^\nu)\frac{-i\eta_{\nu\rho}}{(k-p)^2}\frac{i(\not k' + m)}{k'^2 - m^2}(-ie\gamma^\mu)\frac{i(\not k + m)}{k^2 - m^2}(-ie\gamma^\rho)u(p)\ , \end{aligned} \tag{19.8}$$

where $k' = k + q, q = p' - p$. Note that also this correction is logarithmically divergent in the UV.

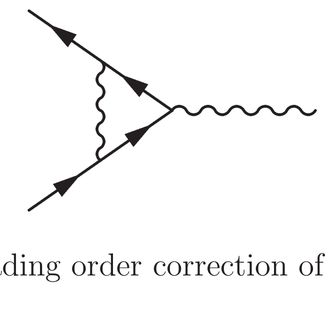


Figure 19.3: Leading order correction of electron vertex

Expressions of the type given in (19.8) we have already computed a few times in the previous lecture. Therefore we only state the result here and refer for more details to the literature [2]. Using the identies for γ-matrices derived in problem 9.3 the numerator can be simplified as

$$\overline{u}(p')\big(-2\not k\gamma^\mu\not k' + \epsilon\not k'\gamma^\mu\not k + 4m(k^\mu + k'^\mu) - m\epsilon(\gamma^\mu\not k + \not k'\gamma^\mu) + (\epsilon - 2)^2m^2\gamma^\mu\big)u(p)\ , \tag{19.9}$$

where we give the results straight away in $4-\epsilon$ dimensions. The denominator is again rewritten in terms of Feynman integrals as

$$\frac{1}{[(k-p)^2-\mu^2][k'^2-m^2][k^2-m^2]} = \int_0^1 dxdydz\,\delta(x+y+z-1)\,\frac{2}{D^3} \tag{19.10}$$

where

$$D = x[k^2-m^2]+y[k'^2-m^2]+z[(k-p)^2-\mu^2] = l^2-\Delta \tag{19.11}$$

with

$$l = k+4yq-zp\;, \qquad \Delta = -xyq^2+(1-z)^2m^2+z\mu^2\;. \tag{19.12}$$

The next steps are well known by now:

1. change the integration variables $k \to l$,
2. go to d-dimensions,
3. Wick rotate,
4. express the d-dimensional integral in terms of Γ-function according to problem 9.2.

In addition we can use the Gordon identity (19.6) to write the fianl result in the form (19.7). One finds

$$\begin{aligned} \delta F_1 &= \frac{\alpha}{2\pi}\int_0^1 dxdydz\,\delta(x+y+z-1)\Big(\frac{(2-\epsilon)^2}{2(4\pi)^{d/2-2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \\ &\qquad + \frac{\Gamma(3-d/2)}{\Delta^{3-d/2}}\big(q^2[(1-x)(1-y)-\epsilon xy]+m^2[2(1-4z+z^2)+\epsilon(1-z)^2]\big)\Big)\;, \\ \delta F_2 &= \frac{\alpha}{2\pi}\int_0^1 dxdydz\,\delta(x+y+z-1)\Big(\frac{2m^2z(1-z)}{m^2(1-z)^2-xyq^2}\Big)\;, \end{aligned} \tag{19.13}$$

where $\alpha = \frac{e^2}{4\pi}$ and $\Delta = (1-z)^2m^2 - yxq^2 + z\mu^2$. Note that δF_1 is UV divergent while δF_2 is finite.

20 Lecture 20: Renormalization of QED

20.1 Superficial degree of divergence of a Feynman diagram

In this section we introduce a quantity D which represents the superficial degree of divergence of a Feynman diagram. As we will see it is a useful guide in classifying the divergent diagrams in QED. Let us first introduce some notation:

$$\begin{aligned}
N_e &= \text{number of external fermion lines} \\
N_\gamma &= \text{number of external photon lines} \\
P_e &= \text{number of fermion propagators} \\
P_\gamma &= \text{number of photon propagators} \\
V &= \text{number of vertices} \\
L &= \text{number of loops}
\end{aligned}$$

As an example consider the diagram in fig. 20.1. It has

$$N_e = 4\ , \quad N_\gamma = 0\ , \quad P_e = 6\ , \quad P_\gamma = 5\ , \quad V = 10\ , \quad L = 4\ . \tag{20.1}$$

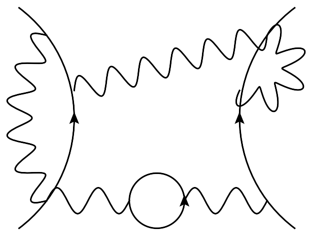


Figure 20.1:

Now we define the superficial degree of divergence D as

$$\begin{aligned}
D &= \text{power of } k\text{'s in the numerator} - \text{power of } k\text{'s in the denominator} \\
&= 4L - P_e - 2P_\gamma\ ,
\end{aligned} \tag{20.2}$$

which holds since each loop adds a d^4k while a fermion progators adds a $\not{k}^{-1}$ and a photon propagator a k^{-2}.

The UV-divergence can now estimated to be

$$\begin{aligned}
\Lambda^D \quad &\text{for} \quad D > 0\ , \\
\ln\Lambda \quad &\text{for} \quad D = 0\ , \\
\text{none} \quad &\text{for} \quad D < 0\ .
\end{aligned} \tag{20.3}$$

This counting is useful but often too naive. For example, the first diagram in fig. 19.2 has $D = 0$ but is finite. Similarly, the diagram 18.2 has $D = 4 - 2 = 2$ but is only ln-divergent and the first diagram of fig. 17.1 has $D = 4 - 1 - 2 = 1$ and is also only ln-divergent. Nevertheless D is a usefull quantity since for QED it can be expressed entirely in terms of the number of external lines. This holds since the quatities defined above enjoy further relations. First of all one has

$$L = P_e + P_\gamma - V + 1 \ , \tag{20.4}$$

which holds since each propagator has a momentum integral but each vertex has a δ-function. (The $+1$ expresses the overall momentum conservation.) For the example of fig. 20.1 one indeed has $L = 8 + 5 - 10 + 1 = 4$.

A second relation is

$$V = 2P_\gamma + N_\gamma = \tfrac{1}{2}(2P_e + N_e) \ , \tag{20.5}$$

which holds since out of each vertex comes one γ-line and two e-lines. The factor of two accounts for the fact that a photon or electron propagator always connects two vertices while an external line does not. For the example of fig. 20.1 one has indeed $V = 2 \cdot 5 = 10 = \frac{1}{2}(2 \cdot 8 + 4)$. As a consequence of (20.4) and (20.5) one derives

$$D = 4L - P_e - 2P_\gamma = 2P_\gamma - P_e - 2N_e + 4 = 4 - N_\gamma - \tfrac{3}{2}N_e \ . \tag{20.6}$$

The last expression is useful as it expresses D solely in terms of the external legs and furthermore it shows that only diagrams with a small number of external legs can have a UV divergence i.e. $D \geq 0$.

20.2 Application to QED

As a consequence of (20.6) the superficially divergent diagrams can simply be listed in fig.20.2. The diagram a) is the vacuum energy which in theories without gravity is unobservable. The diagramms b) and d) vanish due to Furry's theorem which states that one cannot have an odd number of currents j^μ in an amplitude. To show that one inserts the unitary charge conjugation operators C and uses $Cj^\mu C^\dagger = -j^\mu$ together with the $C|\Omega\rangle = |\Omega\rangle$ to arrive at

$$\langle\Omega|j^{\mu_1}\ldots j^{\mu_n}|\Omega\rangle = \langle\Omega|CC^\dagger j^{\mu_1}CC^\dagger\ldots CC^\dagger j^{\mu_n}|\Omega\rangle = (-1)^n\langle\Omega|j^{\mu_1}\ldots j^{\mu_n}|\Omega\rangle \ , \tag{20.7}$$

which thus vanishes for n odd. The diagram f) vanishes as it does not respect fermion number. All other diagrams we look at in slightly more detail now.

The diagram g) can be Taylor-expanded around $p = 0$ to obtain

$$A_0 + A_1\not{p} + A_2p^2 + \ldots \ , \tag{20.8}$$

where in the coefficient A_n terms like $\frac{d^n}{d\not{p}^n}\frac{1}{(\not{k}+\not{p}-m})$ appear. As a consequence D is lower for higher terms in the expansion (20.8). Thus we conclude

$$A_0 \sim \Lambda \quad (D=1) \ , \qquad A_1 \sim \ln\Lambda \quad (D=0) \ , \qquad A_{n\geq 2}\text{finite} \ . \tag{20.9}$$

However in the chiral limit $m \to 0$ one can show $A_0 \to 0$ and hence one finds

$$A_0 \sim m\ln\Lambda \ , \tag{20.10}$$

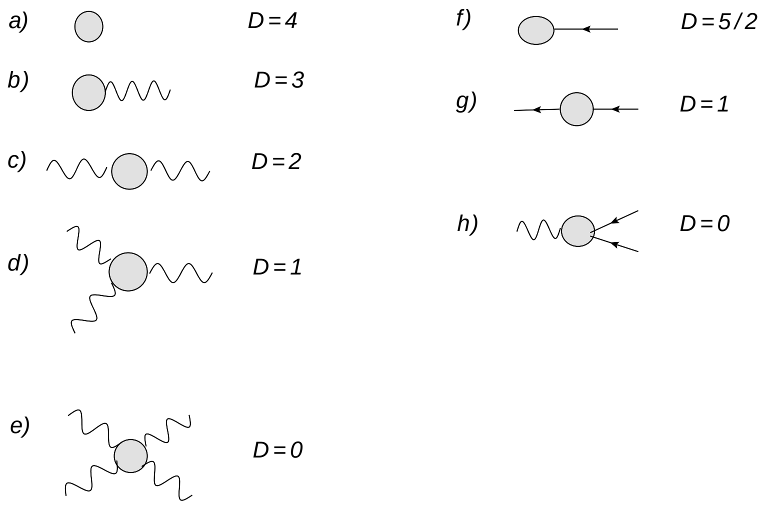


Figure 20.2: Diagrams with $D \geq 0$.

i.e. only a ln-divergence occurs.

The diagram h) we already computed at leading order and did indeed see the ln-divergence. Finally, the diagrams c) and e) are less divergent that esitimated by D. We already argued argued in lecture 18 that due to the Ward identity $\Pi^{\mu\nu} = (\eta^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2)$. As a consequence Π can only be ln-divergent. The same argument implies that the diagram e) has to be finite. Thus we are left with three only ln-divergent diagrams which are the diagrams c), g) and h). In the next lecture we will see that these divergence can be absorbed by redefining the fields and coupling.

21 Lecture 21: Renormalized pertubation theory of QED

In this lecture we discuss how the UV-divergence of QED can be absorbed in a redefinition of fields and couplings. This systematic procedure is sometimes called renormalized perturbation theory.

So far we started from the 'bare' Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\Psi} \left(i\gamma^\mu \partial_\mu - m_0 \right) \Psi - e_0 \overline{\Psi} \gamma^\mu \Psi A_\mu \ , \tag{21.1}$$

with the 'bare' parameters m_0, e_0. In lecture 17 we argued that the fermion propagator has a single pole at $\not{p} = m$ where m is the physical or renormalized mass with a residue Z_2 (cf. (17.9)). Similarly we argued in lecture 18 that the photon propagator has a single pole at $q^2 = 0$ with a residue Z_3.

Z_2 and Z_3 can be absorbed in a new set of <u>renormalized</u> field variables $\Psi_r, A_{r\mu}$ by the redefinition

$$\Psi = \sqrt{Z_2}\, \Psi_r \ , \qquad A_\mu = \sqrt{Z_3}\, A_{r\mu} \ . \tag{21.2}$$

Inserted into (21.1) one obtains

$$\mathcal{L} = -\tfrac{1}{4} Z_3 F_{r\mu\nu} F_r^{\mu\nu} + Z_2 \overline{\Psi}_r \left(i\gamma^\mu \partial_\mu - m_0 \right) \Psi_r - e_0 Z_2 \sqrt{Z_3}\, \Psi_r \gamma^\mu \Psi_r A_{r\mu}. \tag{21.3}$$

In addition one defines <u>physical</u> or <u>renormalized</u> parameters m, e as

$$Z_2 m_0 = m + \delta_m \ , \qquad e_0 Z_2 \sqrt{Z_3} = e Z_1 \ , \tag{21.4}$$

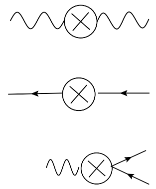
and splits

$$Z_{1,2,3} = 1 + \delta_{1,2,3} \ . \tag{21.5}$$

This split is at the moment arbitrary but as we will see shortly it will be defined such that the UV-divergence disappears from the physical quantities. Inserted into (21.3) we arrive at

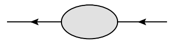
$$\begin{aligned} \mathcal{L} = & -\tfrac{1}{4} F_{r\mu\nu} F_r^{\mu\nu} + \overline{\Psi}_r \left(i\gamma^\mu \partial_\mu - m \right) \Psi_r - e \overline{\Psi}_r \gamma^\mu \Psi_r A_{r\mu} \\ & - \frac{1}{4} \delta_3 F_{r\mu\nu} F_r^{\mu\nu} + \overline{\Psi}_r \left(i\delta_2 \gamma^\mu \partial_\mu - \delta_m \right) \Psi_r - e\delta_1 \overline{\Psi}_r \gamma^\mu \Psi_r A_{r\mu} \ . \end{aligned} \tag{21.6}$$

The prescription now is to do perturbation in e (instead of e_0) with the redefined Lagrangian (21.6) instead of (21.1). For this we need new Feynman rules. The first line in (21.6) leads to the exact same Feynman rules as given in table 13.0. However the second line leads to new vertices which are called <u>counterterms</u> and which are given by

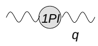


$$\begin{aligned} &= -i(\eta^{\mu\nu} q^2 - q^\mu q^\nu)\delta_3 \ , \\ &= i(\not{p}\delta_2 - \delta_m) \ , \\ &= -ie\gamma^\mu \delta_1 \ . \end{aligned} \tag{21.7}$$

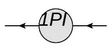
Now one defines the split given in (21.4) and (21.5) by the requirement that the renormalized fields $\Psi_r, A_{r\mu}$ have propagators like a free field with the renormalized mass m being the position of the pole. In other words

$$\int d^4x e^{ipx} \langle \Omega | T\{\Psi_r(x)\bar{\Psi}_r(0)\} | \Omega \rangle \quad = \quad$$  $$\quad = \quad \frac{i}{\not{p} - m} + \dots \qquad (21.8)$$

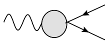
Now we need to redo the perturbation theory for the Lagrangian given in (21.6). This leads to same diagrams as before plus new diagrams involving the counterterms. One thus has



$$= \quad i\Pi^{\mu\nu} = i(\eta^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2) \ ,$$

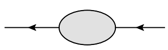


$$= \quad -i\Sigma(\not{p}) \ , \qquad (21.9)$$


$$= \quad -ie\Gamma^\mu(p,p') \ ,$$

where in this notation now all contribution of counterterms are included.

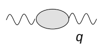
In lecture 17 we already computed the full fermion propagator in terms of the 1PI contribution Σ with the result


$$= \quad \frac{i}{\not{p} - m - \Sigma} = \frac{i}{\not{p} - m - \Sigma(\not{p} = m) - \frac{d\Sigma}{d\not{p}}\big|_{\not{p}=m}(\not{p} - m) + \dots} \qquad (21.10)$$

Comparing to (21.8) we obtain the first two <u>renormalization conditions</u>

$$(i) \quad \Sigma(\not{p} = m) = 0 \ ,$$
$$(ii)) \quad \frac{d\Sigma}{d\not{p}}\big|_{\not{p}=m} = 0 \ , \qquad (21.11)$$

which fix the physical mass m and the residue of the electron-propagator. Similarly we computed in lecture18

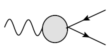


$$= \quad -\frac{i}{q^2}\frac{1}{1 - \Pi(0)} + \dots \qquad (21.12)$$

This implies the third renormalization condition

$$(iii) \quad \Pi(q^2 = 0) = 0 \ , \qquad (21.13)$$

which fixes the γ-propagator. Finally in lecture 19 we computed


$$= \quad -ie\Gamma^\mu(p,p') \ . \qquad (21.14)$$

In order to fix the electric charge e we demand

$$(iv) \quad \Gamma^\mu(p' - p = 0) = \gamma^\mu \ , \qquad (21.15)$$

which is the forth renormalization condition. Thus we managed to derive four conditions for the four counterterms $\delta_{1,2,3,m}$.

Let us see how this works explicitly at one-loop. The one-loop contribution to Σ was computed in lecture 16 and termed Σ_2. Now we need to add the contribution of the second counterterm in (21.7). Inserted into (21.11) we obtain

$$\begin{aligned} (i) \quad & -i\Sigma_2(\not{p}=m) + i(m\delta_2 - \delta_m) = 0 \ , \\ (ii) \quad & -i\frac{d\Sigma}{d\not{p}}\Big|_{\not{p}=m} + i\delta_2 = 0 \ , \end{aligned} \tag{21.16}$$

The one-loop contribution of $\Pi^{\mu\nu}$ was computed in lecture 18 and termed $\Pi_2^{\mu\nu}$. Here we need to add the first counterterm in (21.7) and inserted it into (21.13) we obtain

$$\Pi_2(0) - \delta_3 = 0 \ . \tag{21.17}$$

Finally, Γ^μ was computed in lecture 19. Now we need to add the contribution of the third counterterm in (21.7) and inserted it into (21.15) we obtain

$$F_1(p'-p=0) + \delta_1 = 1 \ . \tag{21.18}$$

Using the explicit expressions computed previously one sees that all UV-divergent contributions are absorbed into the counterterms $\delta_{1,2,3,m}$. This means that the physical parameters are finite (and small) while the bare parameters carry the infinities.

Physically this implies that a theory depends on the distance scale it is looked at or rather being probed at. Due to vacuum polarization where virtual e^+e^- pairs screen a bare electric charge the physical charge is measured at a different value. Such scale dependent couplings have indeed been measured and will be subject to further studies in the second part of this course.

22 Lecture 22: IR-divergences

In this section we discuss IR-divergences but for a more exhaustive treatment we have to refer the reader to the literature (see for example [2]).

In lecture 19 we computed the leading order correction to the electron vertex. The result for the two form-factors $F_{1,2}$ was given in (19.13). In the previous lecture we learned that in order to properly treat the UV-divergence we have to add δ_1 to F_1 which is determined by (21.18). So altogether we have

$$F_1 = 1 + \delta F_1(q^2) + \delta_1 = 1 + \delta F_1(q^2) - \delta F_1(0) \ . \tag{22.1}$$

In this lecture we study a completely different divergence of the Feynman diagram 19.3 which occurs when the momentum of the photon in the loop vanishes. This divergence is called an IR divergence and can be seen in the expression (19.13) in the limit $z \to 1, x \to 0, y \to 0, \mu \to 0$ such that $\Delta \to 0$. The parameter μ was introduced as a small mass of the internal photon precisely to regulate this divergence. More precisely we substituted

$$\frac{1}{(k-p)^2} \to \frac{1}{(k-p)^2 - \mu^2} \qquad \text{corresponding to} \qquad \Delta_{Photon} \to \Delta_{Photon} + z\mu^2 \ . \tag{22.2}$$

In the following we only keep the leading singular terms to arrive at

$$\delta\bar{F}_1(q) \equiv \delta F_1(q) - \delta F_1(0) \simeq \frac{\alpha}{2\pi}\int_0^1 dz \int_0^{1-z} dy \left[\frac{q^2 - 2m^2}{\Delta(q)} - \frac{2m^2}{\Delta(0)}\right] \ , \tag{22.3}$$

where $\Delta(q) = (1-z)^2 m^2 + \mu^2 - q^2 y(1-z-y)$. Substituting

$$z = 1 - w \ , \quad y = (1-z)\xi = w\xi \ , \quad dz = -dw \ , \quad dy = w d\xi \ , \quad w \in [0,1] \ , \quad \xi \in [0,1] \tag{22.4}$$

we obtain

$$\begin{aligned}\delta\bar{F}_1 &\simeq \frac{\alpha}{2\pi}\int_0^1 d\xi \, \tfrac{1}{2}\int_0^1 d(w^2)\left[\frac{q^2 - 2m^2}{(m^2 - q^2\xi(1-\xi))w^2 + \mu^2} - \frac{-2m^2}{m^2 w^2 + \mu^2}\right] \\ &= \frac{\alpha}{4\pi}\int_0^1 d\xi \, \frac{q^2 - 2m^2}{m^2 - q^2\xi(1-\xi)} \ln\left[\frac{m^2 - q^2\xi(1-\xi) + \mu^2}{\mu^2}\right] + 2\ln\left[\frac{m^2+\mu^2}{\mu^2}\right] \ .\end{aligned}$$

In the limit $\mu \to 0$ the details of the numerator inside the ln do not matter and we can take it effectively out of the integral to obtain

$$\begin{aligned}\delta\bar{F}_1 &= -\frac{\alpha}{2\pi} f_{IR}(q^2) \ln\left[\frac{\text{“}-q^2 \text{ or } m^2\text{”}}{\mu^2}\right] \ , \\ \text{with} \qquad f_{IR}(q^2) &\equiv \int_0^1 d\xi \left(\frac{m^2 - q^2/2}{m^2 - q^2\xi(1-\xi)}\right) - 1 \ .\end{aligned} \tag{22.5}$$

For later use let us compute

$$\lim_{q^2 \to -\infty} f_{IR}(q^2) = -\frac{1}{2}\int_0^1 d\xi \frac{1}{m^2/q^2 - \xi(1-\xi)} \approx -\ln\left(-\frac{m^2}{q^2}\right) \tag{22.6}$$

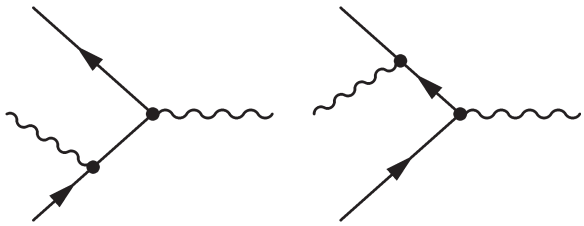


Figure 22.1: Feynman diagrams for Bremsstrahlung in leading order

In this limit the Sudakov double logs appear and we have

$$\delta\bar{F}_1(q^2) \approx -\frac{\alpha}{2\pi}\ln\left(-\frac{q^2}{m^2}\right)\ln\left(-\frac{q^2}{\mu^2}\right) \ . \qquad (22.7)$$

In the following it is argued that the IR divergence are canceled by soft photons in the final state (Bremsstrahlung) depicted in fig. 22. The contribution to the $i\mathcal{M}$ from these diagrams is

$$\begin{aligned}-i\mathcal{M} = &-\imath e\bar{u}(p')\left(\mathcal{M}_0(p',p-k)\frac{\imath(\not{p}-\not{k}+m)}{(p-k)^2-m^2}\gamma^\mu\varepsilon^*_\mu(k)\right)u(p)\\ &-\imath e\bar{u}(p')\left(\gamma^\mu\varepsilon^*_\mu(k)\frac{\imath(\not{p}'+\not{k}+m)}{(p'+k)^2-m^2}\mathcal{M}_0(p'+k,p)\right)u(p)\end{aligned} \qquad (22.8)$$

where $\mathcal{M}_0$ is the matrix element where the external photon line has been cut, i.e. exactly the diagram which has been computed in the last lectures. Assuming that the external photon is “soft” corresponds to the limit

$$|\vec{k}| \ll |\vec{p}' - \vec{p}| \qquad (22.9)$$

In this limit we also have

1. $\mathcal{M}_0(p',p-k) \approx \mathcal{M}_0(p',p) \approx \mathcal{M}_0(p'+k,p)$,
2. we can ignore $\not{k}$ in the numerator,
3. $(\not{p}+m)\gamma^\mu\varepsilon^*_\mu u(p) = 2p^\mu\varepsilon^*_\mu u(p)$
4. $\bar{u}(p')\gamma^\mu\varepsilon^*_\mu(\not{p}'+m) = \bar{u}(p')2p'^\mu\varepsilon^*_\mu$
5. $(p-k)^2 - m^2 = -2pk,\ (p'+k)^2 - m^2 = 2p'k$

Inserted in (22.8)) one obtains

$$i\mathcal{M} = -e\bar{u}(p')\mathcal{M}_0 u(p)\left(\frac{p'\varepsilon^*}{p'k} - \frac{p\varepsilon^*}{pk}\right)$$

and thus the differential cross section becomes

$$d\sigma(p\to p'+\gamma) = d\sigma(p\to p')\int_0^{|\vec{k}|_{max}}\frac{d^3k}{(2\pi)^3}\frac{1}{2k^0}\sum_\lambda e^2\left|\frac{p'\varepsilon^\lambda}{p'k} - \frac{p\varepsilon^\lambda}{pk}\right|^2 \qquad (22.10)$$

which is divergent for $k \to 0$.

The sum in (22.10) can be simplified as

$$\begin{aligned}\sum_\lambda \left| \left(\frac{p'}{p'k} - \frac{p}{pk} \right) \varepsilon^\lambda \right|^2 &= \left(\frac{p'^\mu}{p'k} - \frac{p^\mu}{pk} \right) \sum_\lambda \varepsilon^\lambda_\mu \varepsilon^\lambda_\nu \left(\frac{p'^\nu}{p'k} - \frac{p^\nu}{pk} \right) \\ &= \frac{2p'p}{(p'k)(pk)} - \frac{m^2}{(p'k)^2} - \frac{m^2}{(pk)^2} \\ &= \frac{1}{k^2} I(p, p', \cos(\Theta)) \ , \end{aligned} \tag{22.11}$$

where we used

$$\sum_\lambda \varepsilon^\lambda_\mu \varepsilon^\lambda_\nu = -\eta_{\mu\nu} \tag{22.12}$$

$I(p, p', \cos(\Theta))$ is an angle-dependent contribution whose precise form we do not need (see [2] for an explicit expression). Inserting (22.11) into (22.10) yields

$$\begin{aligned} d\sigma(p \to p' + \gamma) &\approx d\sigma(p \to p')_0 \frac{e^2}{4\pi^2} \int_\mu^{|\vec{q}|} dk \frac{k^2}{k^3} \int \frac{d\Omega_3}{4\pi} I(p, p', \cos(\Theta)) \\ &\approx d\sigma(p \to p')_0 \frac{\alpha}{\pi} \ln\left(-\frac{q^2}{\mu^2} \right) \int \frac{d\Omega_3}{4\pi} I(p, p', \cos(\Theta)) \\ &\overset{q^2 \to -\infty}{\longrightarrow} d\sigma(p \to p')_0 \frac{\alpha}{\pi} \ln\left(-\frac{q^2}{\mu^2} \right) \ln\left(-\frac{q^2}{m^2} \right) \ , \end{aligned} \tag{22.13}$$

where again the photon mass μ was introduced. We see that the same Sudakov double log appears albeit with opposite sign.

Now the claim ist that this cancellation persists for arbitrary q and to all orders (see [2]). Thus Feynman diagrams can well be IR-divergent but the physical observables are not.

23 Lecture 23: The LSZ-reduction formalism

In this lecture we sketch the proof for the earlier promised LSZ-relation. For $2 \to n$ body scattering it states

$$\int d^4x_1 \ldots d^4x_n dy_A dy_B e^{ip_1\cdot x_1} \ldots e^{ip_1\cdot x_n} e^{-ik_A\cdot y_A} e^{-ik_B\cdot y_B} \langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\phi(y_A)\phi(y_B)\}|\Omega\rangle$$

$$\underset{\substack{p_i \to E_{\vec{p}_i} \\ k_A \to E_{\vec{k}_A}}}{\sim} \Big(\prod_{i=1}^{n} \frac{i\sqrt{Z}}{p_i^2 - m^2}\Big)\Big(\prod_{A=1}^{2} \frac{i\sqrt{Z}}{k_A^2 - m^2}\Big) \langle p_1 \ldots p_n|S|k_A k_B\rangle_{\text{amputated}} \ , \tag{23.1}$$

which relates the time-ordered product of field operators to the S-matrix.

Let us sketch the proof in the following. Start with Fourier-transforming one leg and consider

$$A = \int d^4x_1 e^{ip_1\cdot x_1} \langle\Omega|T\{\phi(x_1)\ldots\}|\Omega\rangle \tag{23.2}$$

In order to take of the time ordering split the time integration into

$$\int dx^0 = \int_{T_+}^{\infty} dx^0 + \int_{T_-}^{T_+} dx^0 + \int_{-\infty}^{T_-} dx^0 \ , \tag{23.3}$$

and choose $T_+ \gg x^0_{i\neq 1} \gg T_-$. In the first integral x^0 is the latest time and one can insert a complete set of states.

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_{\lambda_{\vec{0}}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2E_{\vec{q}}(\lambda)} |\lambda_{\vec{q}}\rangle\langle\lambda_{\vec{q}}| \ , \tag{23.4}$$

where the sum is over all zero-momentum states and $|\lambda_{\vec{q}}\rangle$ denotes all boasts of the zero-momentum states $|\lambda_{\vec{0}}\rangle$. Using

$$\langle\Omega|\phi(x)|\Omega\rangle = 0 \ , \qquad \langle\Omega|\phi|\lambda_{\vec{q}}\rangle = \langle\Omega|\phi(0)|\lambda_0\rangle e^{-iqx}|_{q^0=E_{\vec{q}}} \ , \tag{23.5}$$

one obtains

$$\begin{aligned} A = \sum_{\lambda_0} \int_{T_+}^{\infty} dx^0 \int \frac{d^3q}{(2\pi)^3} \frac{1}{2E_{\vec{q}}(\lambda)} e^{ip^0x^0} e^{-iE_{\vec{q}}x^0} \langle\Omega|\phi(0)|\lambda_0\rangle \times \\ \int d^3x_1 e^{i(\vec{p}-\vec{q})\cdot x} \langle\lambda_{\vec{q}}|T\{\phi(x_2)\ldots\}|\Omega\rangle \\ = \sum_{\lambda_0} \frac{1}{2E_{\vec{p}}} \Big(\int_{T_+}^{\infty} dx^0 e^{i(p^0-E_{\vec{p}})\cdot x^0}\Big) \langle\Omega|\phi(0)|\lambda_0\rangle\langle\lambda_{\vec{p}}|T\{\phi(x_2)\ldots\}|\Omega\rangle \ . \end{aligned} \tag{23.6}$$

For the integral we have

$$\int_{T_+}^{\infty} dx^0 e^{i(p^0-E_{\vec{p}})\cdot x^0} = \frac{e^{i(p^0-E_{\vec{p}})\cdot x^0}}{i(p^0 - E_{\vec{p}})}\Bigg|_{T_+}^{\infty} \ , \tag{23.7}$$

which has a pole at $p^0 = E_{\vec{p}}$. Therefore we obtain

$$A \overset{p^0 \to E_{\vec{p}}}{\sim} \frac{i\sqrt{Z}}{p^2 - m^2} \langle \lambda_{\vec{p}} | T\{\phi(x_2) \ldots\} | \Omega \rangle \ . \tag{23.8}$$

We can repeat the same analysis for region III with the result

$$A \overset{p^0 \to -E_{\vec{p}}}{\sim} \frac{i\sqrt{Z}}{p^2 - m^2} \langle \Omega | T\{\phi(x_2) \ldots\} | \lambda_{-\vec{p}} \rangle \ . \tag{23.9}$$

In region II there is no pole since the integral is taken over a bounded region.

One can now also repeat the analysis for all particles using sharply concentrated wave packets around x_i in order to separate them in space. For details we refer to [2].

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