Problem 1.1

a) Rotations in $\mathbb{R}^3$ are characterized by an orthogonal 3×3 rotation matrix D satisfying $D^T D = \mathbf{1}$. Determine D for a rotation in the $x-y$-plane in terms of a rotation angle α_1 such that $\det D = 1$ and $D(\alpha_1 = 0) = \mathbf{1}$ holds. How does D look for rotations in the $x-z$-and $y-z$-plane? Which form does a generic rotation matrix D have?

b) Expand D to first order in the rotation angles $\alpha_j, j = 1, 2, 3$, write it in the form

$$D = \mathbf{1} + i \sum_{j=i}^{3} \alpha_j T^j + \mathcal{O}(\alpha^2)$$

and determine the T^j.

c) Compute $[T^i, T^j]$.

d) Lorentz transformations are characterized by a 4×4 matrix Λ which satisfies $\Lambda^T \eta \Lambda = \eta$ where η is the pseudo-Euclidean metric tensor. Show that rotations in $\mathbb{R}^3$ are a special case of Lorentz transformations.

e) Show that

$$\Lambda = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a Lorentz transformation.

f) Show that Lorentz transformations satisfy $\det \Lambda = \pm 1$.

Problem 1.2

A second rank tensor $T^{\mu\nu}$ is defined by its transformation law

$$T^{\mu\nu} \to T^{\mu\nu\prime} = \sum_{\rho\sigma} \Lambda^\mu_\rho \Lambda^\nu_\sigma T^{\rho\sigma}$$

a) Check that $\sum_{\mu\nu\kappa\theta} T^{\mu\nu} T^{\kappa\theta} \eta_{\mu\kappa} \eta_{\nu\theta}$ is a Lorentz scalar.

b) Determine the transformation law of $T_{\mu\nu} := \sum_{\rho\sigma} \eta_{\mu\rho} \eta_{\nu\sigma} T^{\rho\sigma}$.

c) The field strength $F_{\mu\nu}$ is defined as $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ for a four-vector A_μ. Determine the transformation law of $F_{\mu\nu}$.

d) The ϵ-tensor is defined as

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{for even permutations of 0123} \\ -1 & \text{for odd permutations of 0123} \\ 0 & \text{otherwise} \end{cases}$$

It transforms as

$$\epsilon^{\mu\nu\rho\sigma} \to \epsilon^{\mu\nu\rho\sigma\prime} = \sum_{\mu'\nu'\rho'\sigma'} \Lambda^{\mu}_{\mu'} \Lambda^{\nu}_{\nu'} \Lambda^{\rho}_{\rho'} \Lambda^{\sigma}_{\sigma'} \epsilon^{\mu'\nu'\rho'\sigma'} \ .$$

Show that $\epsilon^{\mu\nu\rho\sigma}$ is invariant for $\det \Lambda = 1$.

Problem 1.3

Consider a complex scalar field ϕ with Lagrangian

$$\mathcal{L} = \sum_\mu \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi \phi^* \ .$$

a) Decompose ϕ into its real and imaginary part as $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ and determine $\mathcal{L}$ in terms of ϕ_1 and ϕ_2.

b) Compute the Euler-Lagrange equations, the conjugate momenta $\pi_{1,2}$ and the Hamiltonian density $\mathcal{H}$.

c) Derive the Euler-Lagrange equations, the conjugate momenta π and the Hamiltonian density $\mathcal{H}$ directly for the complex ϕ without decomposing into real and imaginary part.

Problem 1.4

Consider the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} \sum_{\mu\nu} F_{\mu\nu} F^{\mu\nu} - \sum_\mu A_\mu j^\mu$$

with $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$, $F^{\mu\nu} := \sum_{\rho\sigma} \eta^{\mu\rho}\eta^{\nu\sigma} F_{\rho\sigma}$ and A_μ, j^μ real.

a) Compute the Euler-Lagrange equations for A_μ.

b) Express $\sum_{\mu\nu} F_{\mu\nu}F^{\mu\nu}$ and the Euler-Lagrange equations for A_μ in terms of $E^i = F^{i0}$ and $\sum_{k=1}^{3} \epsilon^{ijk} B^k = -F^{ij}$ and show that the inhomogeneous Maxwell equations appear.

c) Show that the homogeneous Maxwell equations in terms of $F_{\mu\nu}$ are given by the four equations

$$\sum_{\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \, \partial^\nu F^{\rho\sigma} = 0 \ .$$