Problem 2.1

Consider a complex scalar field ϕ with Lagrangian

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi \phi^* \; .$$

a) Show that $\phi \to \phi' = e^{i\alpha}\phi$ is a symmetry of $\mathcal{L}$ for α constant. Compute the associated Noether current $\jmath^\mu$ and Noether charge Q. Show that

$$Q = i \int d^3x (\pi\phi - \phi^* \pi^*) \; ,$$

holds and confirm the conservation laws for j^μ und Q.

b) Compute the energy-momentum-tensor T^μ_ν and check $\partial_\mu T^\mu_\nu = 0$.

c) Show $T^{00} = \mathcal{H}$ and $T^{0i} = -(\pi \partial_i \phi + \pi^* \partial_i \phi^*)$.

Problem 2.2

The solution of the Klein-Gordon equation can be expressed as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip\cdot x^\mu} + a^\dagger_{\vec{p}}\, e^{ip\cdot x} \right)\Big|_{p^0 = E_{\vec{p}}} \; ,$$

with the conjugated momentum given by

$$\pi(x) = -i \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{E_{\vec{p}}}}{2} \left(a_{\vec{p}}\, e^{-ip\cdot x} - a^\dagger_{\vec{p}}\, e^{ip\cdot x} \right)\Big|_{p^0 = E_{\vec{p}}} \; .$$

a) Invert these relation and derive expressions for $a_{\vec{p}}, a^\dagger_{\vec{p}}$ in terms of ϕ and π.

Hint: $\int_{-\infty}^{\infty} d^3x e^{i\vec{x}\cdot(\vec{p}-\vec{p}')} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')$.

b) Show

$$\partial_t a_{\vec{p}} = \partial_t a^\dagger_{\vec{p}} = 0 \; .$$

Hint: Use partial integration and assume that ϕ vanishes at spatial infinty.

Problem 2.3

Show for a real scalar field ϕ

a)

$$H = \tfrac{1}{2}\int d^3x\big(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\big) = \int \frac{d^3p}{(2\pi)^3}\, E_{\vec{p}}\left(a^\dagger_{\vec{p}} a_{\vec{p}} + \tfrac{1}{2}[a_{\vec{p}}, a^\dagger_{\vec{p}}]\right) ,$$

b)

$$P^i = \int T^{0i} d^3x = -\int \pi\partial_i\phi d^3x = \int \frac{d^3p}{(2\pi)^3}\, p^i\left(a^\dagger_{\vec{p}} a_{\vec{p}} + \tfrac{1}{2}[a_{\vec{p}}, a^\dagger_{\vec{p}}]\right) .$$

Problem 2.4

Sow that the operators ϕ, π satisfy the Heisenberg equations

$$i\partial_t\phi = [\phi, H] , \qquad i\partial_t\pi = [\pi, H] .$$

Hint: In order to show the second equation first derive $[\pi, H] = i(\Delta - m^2)\phi$.