**Problem 4.1**

The solution of the Dirac equation can be given as

$$\psi^+ = u(p)\, e^{-ip\cdot x} \qquad \text{where} \qquad u(p) = \frac{1}{N} \begin{pmatrix} (p\cdot\sigma + m)\,\xi^s \\ (p\cdot\bar\sigma + m)\,\xi^s \end{pmatrix} ,$$

and

$$\psi^- = v(p)\, e^{ip\cdot x} \qquad \text{where} \qquad v(p) = \frac{1}{N} \begin{pmatrix} (p\cdot\sigma + m)\,\xi^s \\ -(p\cdot\bar\sigma + m)\,\xi^s \end{pmatrix} ,$$

with

$$p^0 = E_{\vec p}\ , \qquad N = \sqrt{2(E_{\vec p} + m)}\ , \qquad \xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

a) Check the following relations

$$\begin{aligned} \bar u^r_a(p)\, u^s_a(p) &= 2m\delta^{rs}\ , \qquad \bar v^r_a(p)\, v^s_a(p) = -2m\delta^{rs}\ , \qquad \bar u^r_a(p)\, v^s_a(p) = 0\ , \\ u^{r\dagger}_a(p)\, u^s_a(p) &= 2p^0\delta^{rs}\ , \qquad v^{r\dagger}_a(p)\, v^s_a(p) = 2p^0\delta^{rs}\ , \end{aligned}$$

b) Show

$$\sum_{s=1}^{2} u^s_a(p)\, \bar u^s_b(p) = \gamma^\mu_{ab} p_\mu + m\delta_{ab}\ , \qquad \sum_{s=1}^{2} v^s_a(p)\, \bar v^s_b(p) = \gamma^\mu_{ab} p_\mu - m\delta_{ab}\ .$$

c) Show

$$\bar u(p')\, \gamma^\mu\, u(p) = \frac{1}{2m}\, \bar u(p') \Big( p'^\mu + p^\mu + 2iS^{\mu\nu}(p'-p)_\nu \Big) u(p)\ ,$$

where $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

*Hint*: Use $(\gamma^\mu p_\mu - m)u(p) = 0 = \bar u(p')(\gamma^\mu p'_\mu - m)$.

**Problem 4.2**

a) Show that $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ satisfies

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left( \eta^{\nu\rho} S^{\mu\sigma} - \eta^{\nu\sigma} S^{\mu\rho} - \eta^{\mu\rho} S^{\nu\sigma} + \eta^{\mu\sigma} S^{\nu\rho} \right) .$$

b) Determine the Lorentz transformation of

$$j^\mu := \bar\psi\, \gamma^\mu\, \psi\ , \qquad \text{and} \qquad A^{\mu\nu} := \tfrac{1}{2}\, \bar\psi\, [\gamma^\mu, \gamma^\nu]\, \psi\ .$$

## Problem 4.3

A quantized Dirac field $\psi(x)$ is given in terms of raising and lowering operators by

$$\begin{aligned}\psi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\sum_{s=1}^{2}\left(a^s_{\vec p}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\vec p}v^s(p)e^{ip\cdot x}\right)_{p^0=E_{\vec p}} ,\\ \bar\psi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\sum_{s=1}^{2}\left(b^s_{\vec p}\bar v^s(p)e^{-ip\cdot x} + a^{s\dagger}_{\vec p}\bar u^s(p)e^{ip\cdot x}\right)_{p^0=E_{\vec p}} ,\end{aligned}$$

with

$$\{a^s_{\vec p}, a^{r\dagger}_{\vec q}\} = \{b^s_{\vec p}, b^{r\dagger}_{\vec q}\} = (2\pi)^3\delta(\vec p - \vec q)\delta^{rs} ,$$

a) Show

$$\{\psi_a(\vec x, t), \psi^\dagger(\vec y, t)\} = \delta(\vec x - \vec y)\delta_{ab} .$$

*Hint*: Use the result from problem 4.2.b).

b) The Hamiltonian for a Dirac field is given by

$$H(\psi) = i\int d^3x\, \bar\psi\gamma^0\partial_0\psi .$$

Express $H$ in terms of raising and lowering operators and show

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec p}\sum_s \left(a^{s\dagger}_{\vec p}a^s_{\vec p} + b^{s\dagger}_{\vec p}b^s_{\vec p}\right) + c(\psi) ,$$

Compute $c(\psi)$.

c) Repeat the computation for the Noether charge $Q$ and show

$$Q = \int d^3x\, \psi^\dagger\psi = \int \frac{d^3p}{(2\pi)^3}\sum_s \left(a^{s\dagger}_{\vec p}a^s_{\vec p} - b^{s\dagger}_{\vec p}b^s_{\vec p}\right) + c' .$$

d) Add to $H(\psi)$ an appropriate $H(\phi^i)$ ($\phi^i$ being a set of $N$ scalar fields) such that

$$c(\psi) + c(\phi^i) = 0 .$$

Why is this not a solution to the problem of the cosmological constant?