Preliminary draft

Introduction to Supersymmetry and Supergravity

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ABSTRACT

These lecture give a basic introduction to supersymmetry and supergravity.

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1 The Lorentz Group and the Supersymmetry Algebra

1.1 Introduction

Supersymmetry is a symmetry between states or fields of different spin.[1] Supersymmetric field theories in four space-time dimension have the following properties:

- they exist,
- they have constraint quantum corrections and thus are simpler as quantum fields theories (QFT),
- they offer a solution of the naturalness problem of QFTs.

Furthermore, the supersymmetric Standard Model (SSM) suggests

- a candidate for dark matter,
- the unification of $SU(3) \times SU(2) \times U(1)$,
- that the coupling of the Standard Model (SM) to gravity is necessary.

In the course of these lecture we will derive these properties (and more).

1.2 The Lorentz Group

Let $x^\mu, \mu = 0, \ldots, 3$ be the coordinates of Minkowski space $M_{1,3}$ with metric

$$(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1) \ . \tag{1.1}$$

Lorentz Transformations are rotations in $M_{1,3}$ and thus correspond to the group $O(1,3)$

$$x^\mu \to x^{\mu\prime} = \Lambda^\mu_\nu x^\nu \ . \tag{1.2}$$

$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ is invariant for

$$\eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma} \ , \qquad \text{or in matrix form} \qquad \Lambda^T \eta \Lambda = \eta \ . \tag{1.3}$$

This generalizes the familiar orthogonal transformation $O^T O = \mathbf{1}$ of $O(4)$.

Λ depends on $4 \cdot 4 - (4 \cdot 4)_s = 16 - 10 = 6$ parameters. $\Lambda_R := \Lambda^i_j, i, j = 1, 2, 3$ satisfies $\Lambda_R^T \Lambda_R = \mathbf{1}$ corresponding to the $O(3)$ subgroup of three-dimensional space rotations. Λ_R depends on 3 rotation angles. $\Lambda_B := \Lambda^0_j$ corresponds to Lorentz boosts depending on 3 boost velocities.

[1] Textbooks of supersymmetry and supergravity include [1–5]. For review lectures see, for example, [7–9] .

One expands Λ infinitesimally near the identity as

$$\Lambda = \mathbf{1} - \tfrac{i}{2}\omega_{[\mu\nu]}L^{[\mu\nu]} + \dots \ , \tag{1.4}$$

where $\omega_{[\mu\nu]}$ are the 6 parameters of the transformation. The $L^{[\mu\nu]}$ are the generators of the Lie algebra $SO(1,3)$ and satisfy

$$[L^{\mu\nu}, L^{\rho\sigma}] = -i\big(\eta^{\nu\rho}L^{\mu\sigma} - \eta^{\mu\rho}L^{\nu\sigma} - \eta^{\nu\sigma}L^{\mu\rho} + \eta^{\mu\sigma}L^{\nu\rho}\big) \ . \tag{1.5}$$

1.3 The Poincare group

The Poincare group includes in addition the (constant) translations

$$x^\mu \rightarrow x^{\mu\prime} = \Lambda^\mu_\nu x^\nu + a^\mu \ , \tag{1.6}$$

generated by the momentum operator $P_\mu = -i\partial_\mu$. The algebra of the Lorentz generators (1.5) is augmented by

$$[P_\mu, P_\nu] = 0 \ , \qquad [P_\mu, L_{\nu\rho}] = i(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu) \ . \tag{1.7}$$

The Poincare group has two Casimir operators $P_\mu P^\mu$ and $W_\mu W^\mu$ where $W_\mu = \epsilon_{\mu\nu\rho\sigma}L^{\nu\rho}P^\sigma$ is the Pauli-Lubanski vector. Both commute with $P_\mu, L_{\mu\nu}$. Thus the representations can be characterized by the eigenvalues of P^2 and W^2.

1.4 Representations of the Poincare Group

Massive representations

For massive representation one has $P_\mu P^\mu = -m^2, W_\mu W^\mu = m^2 L^2$, where the physical requirement of a positive energy demands $m > 0$. One conveniently goes to the rest frame where $P_\mu = (-m, \vec{0})$ and $W_\mu = -2m(0, \vec{L})$. This choice is left invariant by $SO(3)$ known as the little group. The massive representation of the Poincare Group are thus labeled by m and s the eigenvalue of L^2, i.e. the spin of the $SO(3)$ representations.

Massless representations

For massless representation one has $P_\mu P^\mu = 0$ and conveniently goes to the frame $P_\mu = (-E, 0, 0, E)$. This is left invariant by the Poincare group in two dimensions. Its compact subgroup is the Abelian group $SO(2)$ with the helicity λ being the eigenvalues of the generator. The CPT-theorem of QFTs requires that the representations contain a pair of states corresponding to $\pm\lambda$.

Spinor representations of $SO(1,3)$

All $SO(n,m)$ groups also have spinor representations.[2] They are constructed from Dirac matrices γ^μ satisfying the Clifford/Dirac algebra[3]

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \ . \tag{1.8}$$

[2]They are two-valued in $SO(n,m)$ but single valued in the double cover denoted by $Spin(n,m)$.
[3]Here we use the somewhat unconventional convention of [5].

From the γ^μ one constructs the operators

$$S^{\mu\nu} := -\tfrac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] , \tag{1.9}$$

which satisfy (1.5) and thus are generator of (the spinor representations of) $SO(1,3)$.

The γ matrices are unique (up to equivalence transformations) and a convenient choice in the following is the chiral representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} , \qquad \text{where} \qquad \sigma^\mu = (-\mathbf{1}, \sigma^i) , \quad \bar\sigma^\mu = (-\mathbf{1}, -\sigma^i) . \tag{1.10}$$

Here σ^i are the Pauli matrices which satisfy $\sigma^i\sigma^j = \delta^{ij}\mathbf{1} + i\epsilon^{ijk}\sigma^k$. Inserted into (1.9) one finds

$$S^{\mu\nu} = i\begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar\sigma^{\mu\nu} \end{pmatrix} , \quad \text{where} \quad \sigma^{\mu\nu} = \tfrac14(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu) , \quad \bar\sigma^{\mu\nu} = \tfrac14(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu) . \tag{1.11}$$

For the boosts and rotations one has explicitly

$$S^{0i} = \tfrac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} , \qquad S^{ij} = \tfrac12\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix} . \tag{1.12}$$

Since they are block-diagonal the smallest spinor representation is the two-dimensional Weyl spinor. In the Van der Waerden notion one decomposes a four-component Dirac spinor Ψ_D as

$$\Psi_D = \begin{pmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix} , \qquad \alpha, \dot\alpha = 1, 2 , \tag{1.13}$$

where χ_α and $\bar\psi^{\dot\alpha}$ are two independent two-component complex Weyl spinors. The dotted and undotted spinors transform differently under the Lorentz group. Concretely one has

$$\begin{aligned} \delta\chi_\alpha &= \tfrac12\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha^\beta\chi_\beta = \tfrac12(\omega_{0i}\sigma^i + i\omega_{ij}\epsilon^{ijk}\sigma^k)\chi , \\ \delta\bar\psi^{\dot\alpha} &= \tfrac12\omega_{\mu\nu}(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\dot\beta}\bar\psi^{\dot\beta} = \tfrac12(-\omega_{0i}\sigma^i + i\omega_{ij}\epsilon^{ijk}\sigma^k)\bar\psi , \end{aligned} \tag{1.14}$$

where we used (1.11) and (1.12). These transformation laws are often referred to as $(\frac12, 0)$ and $(0, \frac12)$ respectively. Note that the two spinors transforms identically under the rotation subgroup while they transform with opposite sign under the boosts.

The spinor indices are raised and lowered using the Lorentz-invariant ϵ-tensor

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta , \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta , \tag{1.15}$$

where

$$\epsilon_{21} = -\epsilon_{12} = 1, \quad \epsilon_{11} = \epsilon_{22} = 0, \quad \epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = \delta_\alpha^\beta .$$

For dotted indices the analogous equations hold. One can check that σ^μ carries the indices $\sigma^\mu_{\alpha\dot\alpha}$ and $\bar\sigma^{\mu\alpha\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}$. Complex conjugation interchanges the two representations, i.e., $(\chi_\alpha)^* = \bar\chi_{\dot\alpha}$.

1.5 Supersymmetry Algebra

The supersymmetry algebra is an extension of the Poincare algebra. One augments the Poincare algebra by a fermionic generator Q_α which transforms as a Weyl spinor of the Lorentz group. Haag, Lopuszanski and Sohnius showed that the following algebra is the only extension compatibly with the requirements of a QFT [5, 6]

$$\begin{aligned} &\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}\, P_\mu\,, && \{Q_\alpha, Q_\beta\} = 0 = \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\}\,, \\ &[\overline{Q}_{\dot\alpha}, P_\mu] = 0 = [Q_\alpha, P_\mu]\,, && \\ &[Q_\alpha, L^{\mu\nu}] = \tfrac{1}{2}\,(\sigma^{\mu\nu})_\alpha^{\beta} Q_\beta\,, && [\overline{Q}_{\dot\alpha}, L^{\mu\nu}] = \tfrac{1}{2}\,(\bar\sigma^{\mu\nu})_{\dot\alpha}^{\dot\beta}\overline{Q}_{\dot\beta}\,. \end{aligned} \tag{1.16}$$

The only generalization we will discuss later on is the possibility of having N supersymmetric generators $Q^I_\alpha, I = 1, \ldots, N$ – a situation which is referred to as N-extended supersymmetry.

2 Representations of $N = 1$ Supersymmetry and the Chiral Multiplet

2.1 Representation of the Supersymmetry Algebra

Let us now discuss the representations of the supersymmetry algebra (1.16). Let us first show that for any finite-dimensional representations the number of bosonic states n_B and fermionic states n_F coincides and one has

$$\mathrm{Tr}\left((-)^{N_F}\right) = n_B - n_F = 0 \ . \tag{2.1}$$

Here the fermion number operator $(-)^{N_F}$ is defined by

$$(-)^{N_F}|B\rangle = |B\rangle \ , \qquad (-)^{N_F}|F\rangle = -|F\rangle \ , \tag{2.2}$$

where $|B\rangle$ ($|F\rangle$) denotes any bosonic (fermionic) state. Due to (2.2) and the fermionic nature of Q_α one has $(-)^{N_F}Q_\alpha = -Q_\alpha(-)^{N_F}$.

The cyclicity of the trace then implies

$$\mathrm{Tr}\left((-)^{N_F}\{Q_\alpha, \overline{Q}_{\dot\alpha}\}\right) = \mathrm{Tr}\left(-Q_\alpha(-)^{N_F}\overline{Q}_{\dot\alpha} + Q_\alpha(-)^{N_F}\overline{Q}_{\dot\alpha}\right) = 0 \ . \tag{2.3}$$

Inserting (1.16) yields

$$\mathrm{Tr}\left((-)^{N_F} 2\sigma^\mu_{\alpha\dot\beta}\, P_\mu\right) = 2\sigma^\mu_{\alpha\dot\beta}\, P_\mu \mathrm{Tr}\left((-)^{N_F}\right) = 0 \ , \tag{2.4}$$

where in the first step the trace was evaluated for fixed P_μ. This proves (2.1).

As for the Poincare group the representations (supermultiplets) of the algebra (1.16) are distinct for different values of the Casimir operator P^2.

Massive representations

For massive representations ($P^2 = -m^2$, $m > 0$) one again goes to the rest frame $P_\mu = (-m, 0, 0, 0)$ such that the superalgebra (1.16) becomes

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2m\delta_{\alpha\dot\beta} \ , \qquad \{Q_\alpha, Q_\beta\} = 0 = \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} \ . \tag{2.5}$$

Then one defines the operators

$$a_\alpha := \frac{1}{\sqrt{2m}} Q_\alpha \ , \qquad (a_\alpha)^\dagger := \frac{1}{\sqrt{2m}} \overline{Q}_{\dot\alpha} \tag{2.6}$$

such that (2.5) becomes

$$\{a_\alpha, (a_\beta)^\dagger\} = \delta_{\alpha\dot\beta} \qquad \{a_\alpha, a_\beta\} = 0 = \{a^\dagger_{\dot\alpha}, a^\dagger_{\dot\beta}\} \, . \tag{2.7}$$

This is the algebra of two fermionic harmonic oscillators and thus its representations can be constructed as in quantum mechanics. One defines a "ground state" (Clifford vacuum) $|0\rangle$ by the condition

$$a_\alpha|0\rangle = 0 \ , \tag{2.8}$$

and constructs the multiplet by acting with $a_\alpha^\dagger$

$$|0\rangle \ , \quad (a_\alpha)^\dagger|0\rangle \ , \quad (a_1)^\dagger(a_2)^\dagger|0\rangle \ . \tag{2.9}$$

By acting with the spin operator L^2 one determines that the first and the last state have spin $s=0$ while the two other states have $s=1/2$. We therefore have $n_B=n_F=2$ and this representation is called the chiral multiplet.

Other multiplets can be constructed in a similar way if one also assigns spin to the Clifford vacuum. In this case one finds the multiplet

$$|s\rangle \ , \quad (a_\alpha)^\dagger|s\rangle \ , \quad (a_1)^\dagger(a_2)^\dagger|s\rangle \ , \tag{2.10}$$

corresponding the spins $(s, s\pm\frac{1}{2}, s)$ and the multiplicities $2s+1, 2(s\pm\frac{1}{2})+1, 2s+1$. Thus altogether one has $n_B=n_F=4s+2$. The different multiplets are summarized in Table 2.1.

Spin	$\|0\rangle$	$\|\frac{1}{2}\rangle$	$\|1\rangle$	$\|\frac{3}{2}\rangle$
0	2	1		
$\frac{1}{2}$	1	2	1	
1		1	2	1
$\frac{3}{2}$			1	2
2				1
$n_B=n_F$	2	4	6	8
	chiral multiplet	vector multiplet	spin $\frac{3}{2}$ multiplet	spin 2 multiplet

Table 2.1: Massive $N=1$ multiplets.

Since P^2 commutes with Q it also is a Casimir operator of the supersymmetry algebra. Therefore all members of a supermultiplet have the same mass and in particular bosonic states are mass degenerate with fermionic states

$$m_{\rm B} = m_{\rm F} \quad \forall \text{ states} \ . \tag{2.11}$$

Hence, supersymmetry has to be broken, if realized in nature.

Massless representations

For massless representations one goes again to a light-like frame, $P_\mu=(-E,0,0,E)$. Inserted into (1.16) one obtains

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2E(-\sigma^0+\sigma^1)_{\alpha\dot\beta} = 2E\begin{pmatrix}1&0\\0&0\end{pmatrix} \ , \qquad \{Q_\alpha, Q_\beta\} = 0 = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} \ . \tag{2.12}$$

We see that the algebra is trivial for Q_2. Inserting

$$a := \frac{1}{\sqrt{2E}}\, Q_1 \ , \qquad a^\dagger := \frac{1}{\sqrt{2E}}\, \overline{Q}_1 \ , \tag{2.13}$$

into (17.12) one arrives at

$$\{a, a^\dagger\} = 1 \ , \qquad \{a, a\} = 0 = \{a^\dagger, a^\dagger\} \, , \tag{2.14}$$

which is the algebra of a single fermionic oscillator. In the massless case the representations are labeled by the helicity λ and a multiplet has only the two states

$$|\lambda\rangle \ , \qquad a^\dagger|\lambda\rangle \ , \tag{2.15}$$

corresponding to the helicities $\lambda, \lambda + \frac{1}{2}$. However, due to the CPT theorem of quantum field theories a massless particle with helicity corresponds to two states with helicities $\pm\lambda$. Therefore in quantum field theoretic applications one has to double the multiplets (2.15) appropriately. The relevant massless multiplets are summarized in Table 2.2.

λ	$\vert 0\rangle$	$\vert -\frac{1}{2}\rangle$	$\vert \frac{1}{2}\rangle$	$\vert -1\rangle$	$\vert 1\rangle$	$\vert -\frac{3}{2}\rangle$	$\vert \frac{3}{2}\rangle$	$\vert -2\rangle$
0	1	1						
$\pm\frac{1}{2}$	1	1	1	1				
± 1			1	1	1	1		
$\pm\frac{3}{2}$					1	1	1	1
± 2							1	1
$n_B = n_F$	2		2		2		2	
	chiral multiplet		vector multiplet		gravitino multiplet		graviton multiplet	

Table 2.2: The massless multiplets for $N = 1$.

2.2 The chiral multiplet in QFTs

The chiral multiplet has in the massive and massless case two states with spin/helicity zero and two states with spin/helicity 1/2. In a QFT this can be realized as a complex scalar $A(x)$ and a Weyl fermion $\chi_\alpha(x)$. However, with χ being complex it has initially (off-shell) four degrees of freedom (d.o.f.) and only after using the equation of motion (the Weyl equation) in carries two d.o.f. on-shell.

The next step is to find the supersymmetry transformation of the chiral multiplet. To this end we define

$$\delta_\xi := \xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}} \overline{Q}^{\dot{\alpha}} \, , \tag{2.16}$$

where the parameters of the transformation ξ_α are constant, complex anti-commuting Grassmann parameters obeying

$$\xi_\alpha \xi_\beta = -\xi_\beta \xi_\alpha \, . \tag{2.17}$$

The supersymmetry algebra (1.16) implies

$$[\delta_\eta, \delta_\xi] = -2i(\eta\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\eta})\partial_\mu \, . \tag{2.18}$$

One demands that (2.18) holds on all fields of a supermultiplet. For the chiral multiplet this is satisfied for

$$\delta_\xi A = \sqrt{2}\xi^\alpha \chi_\alpha \ , \qquad \delta_\xi \chi_\alpha \ = i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\partial_\mu A \ , \tag{2.19}$$

if the equation of motion $\bar{\sigma}^\mu \partial_\mu \chi = 0$ holds.

This set of transformation can be promoted to an off-shell realization by introducing an auxiliary complex scalar field $F(x)$ and the transformations

$$\begin{aligned} \delta_\xi A &= \sqrt{2}\xi\chi \ , \\ \delta_\xi \chi &= \sqrt{2}\xi F + i\sqrt{2}\sigma^\mu \bar{\xi}\partial_\mu A \ , \\ \delta_\xi F &= i\sqrt{2}\bar{\xi}\bar{\sigma}^\mu \partial_\mu \chi \ , \end{aligned} \tag{2.20}$$

which satisfy (2.18) without using any equation of motion. Note that $F = 0$ demands $\bar{\sigma}^\mu \partial_\mu \chi = 0$ and the transformation reduce to the previous case. Thus the off-shell chiral multiplet reads

$$\big(A(x), \chi_\alpha(x), F(x)\big) \ , \tag{2.21}$$

and has $n_B = n_F = 4$.

The supersymmetric Lagrangian for the kinetic terms of the chiral multiplet is found to be

$$\mathcal{L}_{\mathrm{kin}} = -\partial_\mu A \partial^\mu \bar{A} - i\bar{\chi}\sigma^\mu \partial_\mu \chi + F^i \bar{F}^i \ . \tag{2.22}$$

One can check $\delta_\xi \mathcal{L}_{\mathrm{kin}} = \partial_\mu j^\mu$ such that the action is invariant for appropriate boundary conditions of the fields. The equations of motion derived from $\mathcal{L}_{\mathrm{kin}}$ read

$$\Box A = 0 \ , \qquad \bar{\sigma}^\mu \partial_\mu \chi = 0 \ , \qquad F = 0 \ . \tag{2.23}$$

We see that the equations of motion is purely algebraic which is the characteristic feature of auxiliary fields in supersymmetric theories.

One can add mass terms as

$$\mathcal{L}_{\mathrm{m}} = -\tfrac{1}{2} m\big(\chi\chi + \bar{\chi}\bar{\chi} + 2AF + 2\bar{A}\bar{F}\big) \ . \tag{2.24}$$

$\mathcal{L}_{\mathrm{kin}} + \mathcal{L}_{\mathrm{m}}$ now have the equations of motion

$$\Box A + m\bar{F} = 0 \ , \qquad \bar{\sigma}^\mu \partial_\mu \chi + m\bar{\chi} = 0 \ , \qquad F + m\bar{A} = 0 \ . \tag{2.25}$$

Again the equation of motion for F is algebraic and thus can be inserted into the first equation yielding the familiar Klein-Gordon equation $(\Box - m^2)A = 0$.

Finally, the most renormalizable Lagrangian for n_c chiral multiplets reads

$$\begin{aligned} \mathcal{L} = &- \partial_\mu A^i \partial^\mu \bar{A}^i - i\bar{\chi}^i \bar{\sigma}^\mu \partial_\mu \chi^i + F^i \bar{F}^i \\ &- \tfrac{1}{2} W_{ij}\chi^i\chi^j - \tfrac{1}{2}\bar{W}_{ij}\bar{\chi}^i\bar{\chi}^j + F^i W_i + \bar{F}^i \bar{W}_i \ , \end{aligned} \tag{2.26}$$

where $i, j = 1, \ldots, n_c$. W_i and W_{ij} in (2.26) are the first and second derivatives of the superpotential $W(A)$, which is a holomorphic function of the fields A^i, and in renormalizable theories constrained to be at most cubic

$$\begin{aligned} W(A) &= \tfrac{1}{2} m_{ij} A^i A^j + \tfrac{1}{3} Y_{ijk} A^i A^j A^k \ , \\ W_i &\equiv \frac{\partial W}{\partial A^i} = m_{ij} A^j + Y_{ijk} A^j A^k \ , \\ W_{ij} &\equiv \frac{\partial^2 W}{\partial A^i \partial A^j} = m_{ij} + 2Y_{ijk} A^k \ . \end{aligned} \tag{2.27}$$

m_{ij} is the mass matrix while Y_{ijk} are the Yukawa couplings.[4] Eliminating the auxiliary fields F^i by

$$\frac{\delta \mathcal{L}}{\delta \bar{F}^i} = F^i + \bar{W}^i = 0\,, \tag{2.28}$$

and inserted back into (2.26) yields

$$\begin{aligned} \mathcal{L} = &- \partial_\mu A^i \partial^\mu \bar{A}^i - i\bar{\chi}^i \bar{\sigma}^\mu \partial_\mu \chi^i + F^i \bar{F}^i \\ &- \tfrac{1}{2} W_{ij} \chi^i \chi^j - \tfrac{1}{2} \bar{W}_{ij} \bar{\chi}^i \bar{\chi}^j - V(A, \bar{A})\ , \end{aligned} \tag{2.29}$$

where V is the scalar potential given by

$$V(A, \bar{A}) = F_i \bar{F}_i = W_i \bar{W}_i\,. \tag{2.30}$$

[4]Of course both couplings are constrained by any symmetry (e.g. gauge symmetry) the theory under consideration might have.

3 Super Yang-Mills Theories

3.1 The massless vector multiplet

In Section 2.1 (Table 2.2) we saw that the massless vector multiplet contains the states $|\lambda = \pm 1\rangle, |\lambda = \pm 1\rangle$. In a QFT they correspond to a gauge boson $v_\mu(x)$ and a Weyl fermion λ_α termed gaugino. Off-shell the gauge boson has $n_B = 3$ while the gaugino has again $n_F = 4$. Therefore we expect a real scalar auxiliary field $D(x)$ to complete the off-shell vector multiplet.

In general v_μ carries the adjoint representation of the gauge group G, i.e., $v_\mu = v_\mu^a T^a$ where T^a are the generators of G obeying

$$[T^a, T^b] = i f^{abc} T^c \ , \quad \mathrm{Tr}\,(T^a T^b) = k\,\delta^{ab} \ , \ k > 0 \ , \quad a = 1, \ldots, n_v = \dim(ad(G)) \ . \tag{3.1}$$

The generators of G commute with the supersymmetry generators, i.e. $[T^a, Q] = 0$, so that all members of any supermultiplet carry the same representation of G.

The supersymmetry transformation of the off-shell vector multiplet $\big(v_\mu^a(x), \lambda_\alpha^a(x), D^a(x)\big)$ read [5]

$$\begin{aligned}
\delta_\xi v_\mu^a &= -i\bar\lambda^a \bar\sigma^\mu \xi + i\bar\xi \bar\sigma^\mu \lambda^a \,, \\
\delta_\xi \lambda^a &= i\xi D^a + \sigma^{\mu\nu}\xi F_{\mu\nu}^a \,, \\
\delta_\xi D^a &= -\xi\sigma^\mu D_\mu \bar\lambda^a - (D_\mu \lambda^a)\,\bar\sigma^\mu \bar\xi \,,
\end{aligned} \tag{3.2}$$

where g is the gauge coupling and

$$D_\mu \lambda^a = \partial_m \lambda^a - g f^{abc} V_m^b \lambda^c \,, \qquad F_{\mu\nu}^a \ = \partial_\mu v_\nu - \partial_\nu v_\mu - g f^{abc} v_\mu^b v_\nu^c \ . \tag{3.3}$$

The supersymmetric Lagrangian is given by

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}^a F^{\mu\nu\,a} - i\bar\lambda^a \bar{D\!\!\!/} \lambda^a + \tfrac{1}{2} D^a D^a \ , \tag{3.4}$$

where we abbreviate $\bar{D\!\!\!/} \equiv \bar\sigma^\mu D_\mu$. The equation of motion for D^a is again the algebraic equation $D^a = 0$.

For G Abelian it is supersymmetric to add a Fayet-Iliopoulos (FI) term

$$\mathcal{L}_{FI} = \xi_{FI} D \ , \tag{3.5}$$

such that the equation of motion for D in this case becomes $D = -\xi_{FI}$.

3.2 Coupling to matter

Let us add n_c chiral multiplets $(A^i, \chi^i, F^i), i = 1, \ldots, n_c = \dim(\mathbf{r})$ in some representation $\mathbf{r}$ of G. In this case the renormalizable supersymmetric Lagrangian reads

$$\begin{aligned}
\mathcal{L} = &- \tfrac{1}{4} F_{\mu\nu}^a F^{\mu\nu\,a} - i\bar\lambda^a \bar{D\!\!\!/} \lambda^a + \tfrac{1}{2} D^a D^a - D_\mu A^i D^\mu \bar A^i - i\bar\chi^i \bar{D\!\!\!/} \chi^i + F^i \bar F^i \\
&+ i\sqrt{2}\, g \left(\bar A^i T_{ij}^a \chi^j \lambda^a - A^i T_{ij}^a \bar\lambda^a \bar\chi^j\right) + g D^a \bar A^i T_{ij}^a A^j \\
&- \tfrac{1}{2} W_{ij} \chi^i \chi^j - \tfrac{1}{2} \bar W_{ij} \bar\chi^i \bar\chi^j + F^i W_i + \bar F^i \bar W_i \ ,
\end{aligned} \tag{3.6}$$

where W_i and W_{ij} are defined in (2.27) and the covariant derivatives are defined as

$$D_\mu A^i = \partial_\mu A^i + igv_\mu^a T_j^{ai} A^j \,, \qquad D_\mu \chi^i = \partial_\mu \chi^i + igv_\mu^a T_j^{ai} \chi^j \,. \tag{3.7}$$

$\mathcal{L}$ is invariant under the combined supersymmetry transformations

$$\begin{aligned} \delta_\xi A^i &= \sqrt{2}\xi\chi^i \;, \\ \delta_\xi \chi^i &= \sqrt{2}\xi F^i + i\sqrt{2}\sigma^\mu \bar{\xi} D_\mu A^i \;, \\ \delta_\xi F^i &= i\sqrt{2}\bar{\xi}^i \bar{\sigma}^\mu D_\mu \chi^i \;, \\ \delta_\xi v_\mu^a &= -i\bar{\lambda}^a \bar{\sigma}^\mu \xi + i\bar{\xi}\bar{\sigma}^\mu \lambda^a \,, \\ \delta_\xi \lambda^a &= i\xi D^a + \sigma^{\mu\nu}\xi F_{\mu\nu}^a \,, \\ \delta_\xi D^a &= -\xi\sigma^\mu D_\mu \bar{\lambda}^a - (D_\mu \lambda^a)\,\bar{\sigma}^\mu \bar{\xi} \,. \end{aligned} \tag{3.8}$$

The additional terms compared to (2.20) and (3.2) are enforced by gauge invariance.

The auxiliary fields F^i, D^a can be eliminated by their algebraic equations of motions

$$\frac{\delta\mathcal{L}}{\delta D^a} = D^a + g\bar{A}^i T_{ij}^a A^j = 0 \;, \qquad \frac{\delta\mathcal{L}}{\delta \bar{F}^i} = F^i + \bar{W}^i = 0 \,. \tag{3.9}$$

Inserted into the Lagrangian (3.6) then yields

$$\begin{aligned} \mathcal{L} = &- \tfrac{1}{4} F_{\mu\nu}^a F^{\mu\nu\,a} - i\bar{\lambda}^a \not{D} \lambda^a - D_\mu A^i D^\mu \bar{A}^i - i\bar{\chi}^i \not{D} \chi^i \\ &+ i\sqrt{2}\, g \left(\bar{A}^i T_{ij}^a \chi^j \lambda^a - A^i T_{ij}^a \bar{\lambda}^a \bar{\chi}^j \right) - \tfrac{1}{2} W_{ij} \chi^i \chi^j - \tfrac{1}{2} \bar{W}_{ij} \bar{\chi}^i \bar{\chi}^j - V(A, \bar{A}) \,, \end{aligned} \tag{3.10}$$

where V is the scalar potential given by

$$V(A,\bar{A}) = W_i \bar{W}_i + \tfrac{1}{2} g^2 \left(\bar{A}^i T_{ij}^a A^j \right) \left(\bar{A}^k T_{kl}^a A^l \right) = F_i \bar{F}_i + \tfrac{1}{2} D^a D^a \,. \tag{3.11}$$

Before we continue let us make the following remarks:

- V is positive semi-definite $V \geq 0$.
- V is not the most general scalar potential, i.e. there is no independent $\lambda(A\bar{A})^2$ coupling. Instead the quartic scalar couplings arise from Y^2 in the F-term or g^2 in the D-term. In the SSM this properties leads to a light Higgs boson.
- V depends only on g, m_{ij}, Y_{ijk} with no additional new parameters being introduced.
- There is a "new" Yukawa coupling proportional to $g\bar{A}\chi\lambda$.

3.3 Mass sum rules and the supertrace

In this section we compute the mass matrices of the various fields and derive a sum rule which will be useful later. In particular we will cover the case where the scalar fields A^i

have a non-trivial background value $\langle A^i\rangle \neq 0$. Let us start with the masses of the Weyl spinors χ and λ. They arise from the following terms of the Lagrangian (3.10)

$$\mathcal{L}_{M_{1/2}} = -\tfrac{1}{2}W_{ij}\chi^i\chi^j + \tfrac{1}{2}\bar{W}_{ij}\bar{\chi}^i\bar{\chi}^j + i\sqrt{2}g\left(\bar{A}^i T^a_{ij}\chi^j\lambda^a - \bar{\lambda}^a T^a_{ij}A^i\bar{\chi}^j\right) \tag{3.12}$$

These terms can be arranged in matrix form

$$\mathcal{L}_{M_{1/2}} = -\tfrac{1}{2}\left(\chi^i, \lambda^a\right) M_{1/2}\begin{pmatrix}\chi^j\\ \lambda^b\end{pmatrix} + \text{h.c.}\,, \tag{3.13}$$

for

$$M_{1/2} = \left.\begin{pmatrix} W_{ij} & i\sqrt{2}\partial_i D^a \\ i\sqrt{2}\partial_j D^b & 0\end{pmatrix}\right|_{\min(V)}\,, \tag{3.14}$$

where $\partial_i D^a = -g\bar{A}^j T^a_{ji}$. Similarly

$$\bar{M}_{1/2} = \left.\begin{pmatrix} \bar{W}_{ij} & -i\sqrt{2}\bar{\partial}_i D^a \\ -i\sqrt{2}\bar{\partial}_j D^b & 0\end{pmatrix}\right|_{\min(V)}\,. \tag{3.15}$$

Note that for $\langle A^i\rangle = 0$ only $W_{ij} = m_{ij}$ survives in $M_{1/2}$. For later use we compute

$$\operatorname{Tr} M_{1/2}\bar{M}_{1/2} = \left.\left(W_{ij}\bar{W}_{ji} + 4\partial_i D^a\bar{\partial}_i D^a\right)\right|_{\min(V)}\,. \tag{3.16}$$

In order to determine the scalar mass matrix we need to consider the second derivatives of V. From (3.11) we find

$$\begin{aligned}\partial_j V &= W_{ij}\bar{W}_i + (\partial_j D^a)D^a\,,\\ \partial_j\partial_k V &= W_{ijk}\bar{W}_i + (\partial_j D^a)(\partial_k D^a)\\ \partial_j\bar{\partial}_k V &= W_{ij}\bar{W}_{ik} + (\partial_j\bar{\partial}_k D^a)D^a + (\partial_j D^a)(\bar{\partial}_k D^a)\,,\end{aligned} \tag{3.17}$$

where

$$\begin{aligned} D^a = -g\bar{A}^i T^a_{ij}A^j - \xi_{FI}\delta^{aU(1)}\,, \qquad \partial_j D^a = -g\bar{A}^i T^a_{ij}\,,\\ \bar{\partial}_i D^a = -gT^a_{ij}A^j\,, \qquad \partial_j\bar{\partial}_k D^a = -gT^a_{kj}\,.\end{aligned} \tag{3.18}$$

The scalar masses can also be written in matrix form

$$V = \tfrac{1}{2}\left(\bar{A}^i, A^j\right) M_0^2\begin{pmatrix}A^k\\ \bar{A}^l\end{pmatrix} \tag{3.19}$$

for

$$M_0^2 = \left.\begin{pmatrix}\bar{\partial}_i\partial_k V & \bar{\partial}_i\bar{\partial}_l V\\ \partial_j\partial_k V & \partial_j\bar{\partial}_l V\end{pmatrix}\right|_{\min(V)}\,. \tag{3.20}$$

Note that for $\langle A^i\rangle = 0$ M_0^2 is block diagonal with m^2_{ij} appearing in the diagonal. The trace is

$$\operatorname{Tr} M_0^2 = 2\left.\left(W_{ij}\bar{W}_{ji} + (\partial_i\bar{\partial}_i D^a)D^a + (\partial_i D^a)(\bar{\partial}_i D^a)\right)\right|_{\min}\,. \tag{3.21}$$

Finally, the mass matrix of the gauge bosons arises from

$$\mathcal{L}_{M_1} = -D_\mu\bar{A}^i D^\mu A^i = -\tfrac{1}{2}M^2_{ab}v^a_\mu v^{b\,\mu} + \ldots\,, \tag{3.22}$$

with

$$M^2_{ab} = 2g^2 \bar{A}^j T^a_{jl} T^b_{lk} A^k = 2(\partial_l D^a)(\bar{\partial}_l D^b) \ , \tag{3.23}$$

where we used

$$D_\mu A^i = \partial_\mu A^i + igv^a_\mu T^a_{ij} A^j \ , \qquad D_\mu \bar{A}^i = \partial_\mu A^i - igv^a_\mu T^{aT}_{ij} \bar{A}^j \ . \tag{3.24}$$

Note that for $\langle A^i \rangle = 0$ all gauge bosons are massless.

One defines the <u>supertrace</u> of the mass matrices by

$$\mathrm{Str} M^2 := \sum_{s=0}^{1} (-)^{2s} (2s+1) \mathrm{Tr}\, M^2_s \ . \tag{3.25}$$

For the case at hand we find from (3.16), (3.21), (3.23)

$$\begin{aligned} \mathrm{Str} M^2 =& \mathrm{Tr}\, M_0^2 - 2\mathrm{Tr}\, M_{1/2} + 3\mathrm{Tr}\, M_1^2 \\ =& 2(W_{ij}\bar{W}_{ji} + (\partial_i \bar{\partial}_i D^a) D^a + (\partial_i D^a)(\bar{\partial}_i D^a)) \\ & - 2(W_{ij}\bar{W}_{ji} + 4\partial_i D^a \bar{\partial}_i D^a) + 6(\partial_i D^a)(\bar{\partial}_i D^a) \\ =& 2(\partial_i \bar{\partial}_i D^a) D^a = -2g\,(\mathrm{Tr}\, T^a)\, D^a \ . \end{aligned} \tag{3.26}$$

For a non-Abelian gauge group the generators are traceless while for an Abelian ($U(1)$) gauge group the trace is proportional to the sum of the $U(1)$ charges q. Thus we have altogether

$$\mathrm{Str} M^2 = -2g\,(\mathrm{Tr}\, T^a)\, D^a = \begin{cases} 0 & \text{for non-Abelian } G \\ -2g\,(\sum q)\, D^{U(1)} & \text{for } G = U(1) \end{cases} \ . \tag{3.27}$$

However, for $\sum q \neq 0$ the theory has a gravitational anomaly and thus cannot be coupled to gravity.

4 Superspace and the Chiral Multiplet

4.1 Basic set-up

The coordinates of superspace are $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ where $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$ are Grassmann coordinates which satisfy

$$\theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha = -\tfrac{1}{2}\epsilon^{\alpha\beta}\theta^2 \ , \qquad \theta^\alpha\theta^\beta\theta^\gamma = 0 \ . \tag{4.1}$$

Superfields are function on superspace and due to (4.1) have an expansion

$$\begin{aligned} f(x,\theta,\bar\theta) =& f(x) + \theta^\alpha \chi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\phi^{\dot\alpha}(x) + \theta^2 m(x) + \bar\theta^2 n(x) + \theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} v_\mu \\ &+ \theta^2 \bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x) + \bar\theta^2\theta^\alpha\psi_\alpha(x) + \theta^2\bar\theta^2 d(x) \ . \end{aligned} \tag{4.2}$$

We see that the following ordinary complex fields are combined in a superfield

$$\begin{aligned} s = 0 &: f(x), m(x), n(x), d(x) \ , && n_B = 8 \\ s = \tfrac{1}{2} &: \chi_\alpha(x), \bar\phi^{\dot\alpha}(x), \bar\lambda^{\dot\alpha}(x), \psi_\alpha(x) \ , && n_F = 8 \\ s = 1 &: v_\mu \ , && n_B = 8 \end{aligned} \tag{4.3}$$

Note that due to (4.1) sums and products of superfields are again a superfield

$$f_1(x,\theta,\bar\theta) + f_2(x,\theta,\bar\theta) = f_3(x,\theta,\bar\theta) \ , \qquad f_1(x,\theta,\bar\theta) f_2(x,\theta,\bar\theta) = f_4(x,\theta,\bar\theta) \ . \tag{4.4}$$

In this formalism supersymmetry transformations are translations in superspace. Recall that a finite translation in Minkowski space is generated by

$$G(a) := e^{i(-a^\mu P_\mu)} \ . \tag{4.5}$$

The generalization in superspace is defined to be

$$G(a,\bar\eta,\eta) := e^{i(-a^\mu P_\mu + \eta Q + \bar\eta\overline{Q})} \ . \tag{4.6}$$

The product of two transformation can be computed with help of the Hausdorff-formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}$

$$G(b,\xi,\bar\xi) G(a,\eta,\bar\eta) = G(a+b-i(\xi\sigma\bar\eta - \eta\sigma\bar\xi), \xi+\eta, \bar\xi+\bar\eta) \tag{4.7}$$

By acting infinitesimally on a superfield one determines $Q, \overline{Q}$ as differential operators

$$\begin{aligned} G(0,\xi,\bar\xi) f(x,\theta,\bar\theta) &= (1 + i\xi Q + \bar\xi\overline{Q}) f + \mathcal{O}(\xi^2) = f(x - i(\xi\sigma\bar\eta - \eta\sigma\bar\xi), \theta+\xi, \bar\theta+\bar\xi) \\ &= f(x,\theta,\bar\theta) - i(\xi\sigma^\mu\bar\eta - \eta\sigma^\mu\bar\xi)\partial_\mu f + +\xi^\alpha\partial_\alpha f + \bar\xi_{\dot\alpha}\partial^{\dot\alpha} f + \mathcal{O}(\xi^2) \ , \end{aligned} \tag{4.8}$$

where

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha} = -\epsilon_{\alpha\beta}\partial^\beta \ . \tag{4.9}$$

From (4.8) one finds a representation for $Q, \overline{Q}$ in terms of differential operators

$$Q_\alpha = \partial_\alpha - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \ , \qquad \overline{Q}_{\dot\alpha} = -\partial_{\dot\alpha} + i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \ , \tag{4.10}$$

and checks

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu \ , \qquad \{Q_\alpha, Q_\beta\} = 0 = \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} \, , \tag{4.11}$$

Note that for left multiplication that we used above the sign of P_μ changed. For right multiplication one finds the representation

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \ , \qquad \bar D_{\dot\alpha} = -\partial_{\dot\alpha} - i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \ , \tag{4.12}$$

which satisfy

$$\{D_\alpha, \bar D_{\dot\beta}\} = -2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu \ , \qquad \{D_\alpha, D_\beta\} = 0 = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} \, . \tag{4.13}$$

Whichever representation one uses the respective "other" differential operators represent covariant derivatives on superspace as they satisfy

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \overline{Q}_\beta\} = \{\bar D_{\dot\alpha}, Q_\beta\} = \{\bar D_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0 \ . \tag{4.14}$$

Supersymmetry transformations can be systematically computed by

$$\begin{aligned} \delta_\xi f(x,\theta,\bar\theta) =& \delta_\xi f(x) + \theta^\alpha \delta_\xi \chi_\alpha(x) + \bar\theta_{\dot\alpha}\delta_\xi \bar\phi^{\dot\alpha}(x) + \ldots + \theta^2\bar\theta^2\delta_\xi d(x) \\ =& (\xi Q + \overline{\xi Q}) f(x,\theta,\bar\theta) \end{aligned} \tag{4.15}$$

In particular one finds that the highest component $d(x)$ of any superfield always transforms as a total divergence.

4.2 Chiral Multiplet

We already observed that a general superfield $f(x,\theta,\bar\theta)$ has $n_B = n_F = 16$ which is too large for the multiplets we have constructed earlier. One can reduce the number of degrees of freedom by imposing algebraic supersymmetric constraints. For a chiral multiplet this constraint reads

$$\bar D_{\dot\alpha}\Phi = 0 = D_\alpha\bar\Phi \ . \tag{4.16}$$

They are supersymmetric since D anticommutes with Q. Furthermore, the solution of this constraint in terms of the components of $f(x,\theta,\bar\theta)$ are the algebraic equations

$$\phi = \psi = n = 0 \ , \qquad v_\mu = i\partial_\mu f \ , \qquad \lambda_{\dot\alpha} = -\tfrac{i}{2}\partial_\mu\chi^\beta\sigma^\mu_{\beta\dot\alpha} \ , \qquad d = \tfrac14\Box f \ . \tag{4.17}$$

Or if one renames $f = A, \chi \to \sqrt2\chi, m = F$

$$\begin{aligned} \Phi(x,\theta,\bar\theta) =& A(x) + \sqrt2\theta\chi(x) + \theta^2F(x) + \theta\sigma^\mu\bar\theta\partial_\mu A(x) \\ & - \tfrac{i}{\sqrt2}\theta^2\partial_\mu\chi(x)\sigma^\mu\bar\theta + \tfrac14\theta^2\bar\theta^2\Box A(x) \ . \end{aligned} \tag{4.18}$$

The field redefinition $y^\mu := x^\mu + i\theta\sigma^\mu\bar\theta$ removes the $\bar\theta$ dependence and yields

$$\Phi(y,\theta) = A(y) + \sqrt2\theta\chi(y) + \theta^2F(y) \ . \tag{4.19}$$

Now one can work out the supersymmetry transformation law

$$\begin{aligned}
\delta A &= (\xi Q + \bar{\xi}\bar{Q})\Phi\big|_{\theta=\bar{\theta}=0} = \ldots = \sqrt{2}\xi\chi \\
\delta\chi &= \tfrac{1}{\sqrt{2}}\,(\xi Q + \bar{\xi}\bar{Q})\Phi\big|_{\theta} = \ldots = \sqrt{2}\xi F + i\sqrt{2}\sigma^\mu\bar{\xi}\partial_\mu A\ , \\
\delta_\xi F &= (\xi Q + \bar{\xi}\bar{Q})\Phi\big|_{\theta^2} = \ldots = i\sqrt{2}\bar{\xi}\bar{\sigma}^\mu\partial_\mu\chi\ ,
\end{aligned} \tag{4.20}$$

which indeed coincides with (2.20).

The supersymmetric action is constructed by choosing appropriate highest components of superfields or rather products of superfields. Note that due to (4.16)

$$\bar{D}_{\dot\alpha}\Phi^n = n\Phi^{n-1}\bar{D}_{\dot\alpha}\Phi = 0\ , \qquad \bar{D}_{\dot\alpha}W(\Phi) = \frac{\partial W}{\partial\phi}\bar{D}_{\dot\alpha}\Phi = 0\ . \tag{4.21}$$

Thus the θ^2 component of W transforms as a total divergence. One finds

$$W(A + \sqrt{2}\theta\chi + \theta^2 F)\Big|_{\theta^2} = \partial W\big|_{\theta=\bar{\theta}=0} F + \tfrac{1}{2}\,\partial^2 W\big|_{\theta=\bar{\theta}=0}\chi\chi \tag{4.22}$$

or for n_c chiral multiplets $\Phi^i, i = 1, \ldots, n_c$

$$W(\Phi^i)\big|_{\theta^2} = W_i(A)F^i + \tfrac{1}{2}W_{ij}(A)\chi^i\chi^j\ , \tag{4.23}$$

where $W_i(A), W_{ij}(A)$ are defined in (2.27).

The kinetic terms arise from $\Phi\bar{\Phi}$ which is not chiral and thus one has to take the $\theta^2\bar{\theta}^2$ component

$$\Phi\bar{\Phi}\big|_{\theta^2\bar{\theta}^2} = -\partial_\mu A\partial^\mu\bar{A} + F\bar{F} - i\bar{\chi}\bar{\partial\!\!\!/}\chi\ . \tag{4.24}$$

Thus altogether we have

$$\mathcal{L} = \Phi\bar{\Phi}\big|_{\theta^2\bar{\theta}^2} + W(\Phi^i)\big|_{\theta^2} + \bar{W}(\bar{\Phi}^i)\big|_{\bar{\theta}^2}\ . \tag{4.25}$$

4.3 Berezin integration

There is an alternative way to display this result. One defines an integral for Grassmann variables by

$$\int d\theta = 0\ , \qquad \int \theta d\theta = 1\ , \tag{4.26}$$

such that for $f(\theta) = f(A + \theta\chi)$ one finds

$$\int f(\theta)d\theta = \chi\ , \qquad \int f(\theta)\theta d\theta = A\ . \tag{4.27}$$

This can be generalized to $\theta^\alpha, \bar{\theta}^{\dot\alpha}$ by defining the measures

$$d^2\theta := -\tfrac{1}{4}d\theta^\alpha d\theta^\beta\epsilon_{\alpha\beta}\ , \qquad d^2\bar{\theta} := -\tfrac{1}{4}d\theta_{\dot\alpha}d\theta_{\dot\beta}\epsilon^{\dot\alpha\dot\beta}\ , \qquad d^4\theta := d^2\theta d^2\bar{\theta}\ . \tag{4.28}$$

with

$$\int \theta^2 d^2\theta = 1 = \int \bar{\theta}^2 d^2\bar{\theta} = \int \theta^2\bar{\theta}^2 d^2\theta d^2\bar{\theta}\ . \tag{4.29}$$

In this notation the Lagrangian (4.25) reads

$$\mathcal{L} = \int \Phi\bar{\Phi} d^4\theta + \int \; W(\Phi^i) d^2\theta + \int \bar{W}(\bar{\Phi}^i) d^2\bar{\theta} \; . \tag{4.30}$$

Finally, we can discuss possible non-renormalizable generalizations. In this case W is not constrained to be cubic and $\Phi\bar{\Phi}$ is replaced by a Kähler potential $K(\Phi, \bar{\Phi})$ with the action

$$\mathcal{L} = \int K(\Phi^i, \bar{\Phi}^i) d^4\theta + \int W(\Phi^i) d^2\theta + \int \bar{W}(\bar{\Phi}^i) d^2\bar{\theta} \; . \tag{4.31}$$

Note that K is not uniquely defined but only up to Kähler transformations as for $K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + f(\Phi^i) + \bar{f}(\bar{\Phi}^i)$ one has

$$\int K(\Phi^i, \bar{\Phi}^i) d^4\theta \to \int K(\Phi^i, \bar{\Phi}^i) + \int f(\Phi^i) d^4\theta + \int \bar{f}(\bar{\Phi}^i) d^4\bar{\theta} = \int K(\Phi^i, \bar{\Phi}^i) \; , \tag{4.32}$$

where we used $\int f(\Phi^i) d^4\theta = 0 = \int \bar{f}(\bar{\Phi}^i) d^4\theta = 0$.

4.4 R-symmetry

The supersymmetry algebra (1.16) has an $U(1)$ automorphism (called R-symmetry) which transforms Q as

$$Q \to Q' = e^{-i\alpha} Q \; , \qquad \overline{Q} \to \overline{Q}' = e^{i\alpha} \overline{Q} \; , \qquad \alpha \in \mathbb{R} \; . \tag{4.33}$$

This implies that the members of supermultiplet transform differently and one has the R-charges for the chiral multiplet

$$R(\Phi) = R(A) = q \; , \qquad R(\chi) = q - 1 \; , \qquad R(F) = q - 2 \; , \qquad R(\theta) = 1 \; . \tag{4.34}$$

The kinetic terms are automatically invariant but the interactions might break this symmetry. $R(\theta) = 1$ implies $R(d^2\theta) = -2, R(d^4\theta) = 0$ and thus any $K(\Phi\bar{\Phi})$ is invariant but one needs $R(W) = 2$ which indeed constrains the interactions.

5 The Vector Multiplet in Superspace and non-Renormalization Theorems

5.1 The Vector Multiplet in Superspace

In the previous lecture we discussed the general superfield $f(x, \theta, \bar{\theta})$ in (5.1) which has $n_B = n_F = 16$. The vector multiplet V satisfies the constraint $V = V^\dagger$, has $n_B = n_F = 8$ and a θ-expansion

$$\begin{aligned} f(x,\theta,\bar{\theta}) =& f(x) + i\theta^\alpha \chi_\alpha(x) - i\bar{\theta}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}(x) + \tfrac{i}{2}\theta^2 m(x) - \tfrac{i}{2}\bar{\theta}^2\bar{m}(x) - \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} v_\mu \\ &+ i\theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}(x) - i\bar{\theta}^2\theta^\alpha\lambda_\alpha(x) + \tfrac{1}{2}\theta^2\bar{\theta}^2 d(x) \ , \end{aligned} \tag{5.1}$$

where the convention compared to (5.1) was slightly changed for later convenience. The bosonic fields of V are the real f, d, v_μ and the complex m while the fermions are χ, λ.

Since the massless vector has a gauge invariance we need to implement this at the level of superfields. We will see that the right transformation (for the Abelian case) is

$$V \to V' = V + \Lambda + \bar{\Lambda} \ , \tag{5.2}$$

where Λ is a chiral multiplet (i.e. $\bar{D}_{\dot{\alpha}}\Lambda = 0 = D_\alpha\bar{\Lambda}$) . Let us denote the component of Λ by (Λ, ψ, F) and one really computes

$$\begin{aligned} V + \Lambda + \bar{\Lambda} =& f + (\Lambda + \bar{\Lambda}) + \theta(i\chi + \sqrt{2}\psi) - \bar{\theta}(i\bar{\chi} - \sqrt{2}\bar{\psi}) \\ &+ \tfrac{1}{2}\theta^2(im + 2F) + \tfrac{1}{2}\bar{\theta}^2(-\bar{m} + 2\bar{F}) - \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}(v_\mu - i\partial_\mu(\Lambda - \bar{\Lambda}) \\ &+ i\theta^2\bar{\theta}(\bar{\lambda} + \tfrac{1}{\sqrt{2}}\bar{\sigma}^\mu\partial_\mu\psi - i\bar{\theta}^2\bar{\theta}(\lambda - \tfrac{1}{\sqrt{2}}\sigma^\mu\partial_\mu\bar{\psi} + \tfrac{1}{2}\theta^2\bar{\theta}^2(d + \tfrac{1}{4}\Box(\Lambda + \bar{\Lambda})) \ , \end{aligned} \tag{5.3}$$

This shows that f, χ, m and the longitudinal component of v_μ are $n_B = n_F = 4$ gauge degrees of freedom. Finally one performs the field redefinition

$$\lambda \to \lambda + \tfrac{i}{2}\sigma^\mu\partial_\mu\bar{\chi} \ , \qquad d \to D + \tfrac{1}{2}\Box f \tag{5.4}$$

such that the gauge transformation become

$$\delta v_\mu = -i\partial_\mu(\Lambda - \bar{\Lambda}) \ , \qquad \delta\lambda = 0 \ , \qquad \delta D = 0 \ . \tag{5.5}$$

Thus V has the physical components (v_μ, λ, D). The supersymmetry transformation we gave already in (3.2) and the constructions via superfields yields the same transformations.

In the (non-supersymmetric) Wess-Zumino (WZ) gauge $f = \chi = m = 0$ one has

$$\begin{aligned} V =& -\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} v_\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\theta^2\bar{\theta}\lambda + \tfrac{1}{2}\theta^2\bar{\theta}^2 D \ , \\ V^2 =& -\tfrac{1}{2}\theta^2\bar{\theta}^2 v_\mu v^\mu \ , \\ V^3 =& 0 \end{aligned} \tag{5.6}$$

The gauge invariant field strength is

$$W_\alpha := -\tfrac{1}{4}\bar{D}^2 D_\alpha V \ , \qquad \bar{W}_{\dot{\alpha}} := -\tfrac{1}{4}D^2\bar{D}_{\dot{\alpha}} V \ , \tag{5.7}$$

and one checks that under the gauge transformation (5.2)

$$W'_\alpha = W_\alpha - \tfrac{1}{4}\bar{D}^2 D_\alpha(\Lambda + \bar{\Lambda}) - \tfrac{1}{4}D^2\bar{D}_{\dot\alpha}(\Lambda + \bar{\Lambda}) = W_\alpha \tag{5.8}$$

due to the chiral property of Λ. One also has

$$\bar{D}_{\dot\alpha} W_\beta = 0 = D_\beta \bar{W}_{\dot\alpha} \ , \tag{5.9}$$

and an expansion

$$W_\alpha = -i\lambda_\alpha + (\delta_\alpha^\beta D - \tfrac{i}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha^\beta F_{\mu\nu})\theta_\beta + \theta^2\sigma^\mu\partial_\mu\bar{\lambda} \ . \tag{5.10}$$

The Lagrangian in terms of superfields is

$$\begin{aligned}\mathcal{L} &= W_\alpha W^\alpha\big|_{\theta^2} + \bar{W}^{\dot\alpha}\bar{W}_{\dot\alpha}\big|_{\bar\theta^2} = \int W_\alpha W^\alpha d^2\theta + \int \bar{W}^{\dot\alpha}\bar{W}_{\dot\alpha} d^2\bar{\theta} \\ &= -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu\, a} - \tfrac{i}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - i\bar{\lambda}^a \not{D}\lambda^a + \tfrac{1}{2}D^a D^a \ ,\end{aligned} \tag{5.11}$$

where compared to (3.4) we also added the topological term $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$.

The non-Abelian generalization assigns the adjoint representation to V, i.e. $V = V^a T^a$ and a field strength

$$W_\alpha := -\tfrac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V \ , \tag{5.12}$$

with a gauge transformation

$$e^V \to e^{V'} = e^{-i\bar{\Lambda}} e^V e^{i\Lambda} \ , \qquad W_\alpha \to W'_\alpha = e^{-i\Lambda} W_\alpha^{i\Lambda} \tag{5.13}$$

The coupling to chiral multiplets is achieved by changing the kinetic term to

$$\bar{\Phi}\Phi \to \bar{\Phi} e^V \Phi \ , \tag{5.14}$$

with gauge invariance

$$\Phi \to \Phi' = e^{-i\Lambda}\Phi \ , \qquad \bar{\Phi} \to \bar{\Phi} e^{i\bar{\Lambda}} \ . \tag{5.15}$$

The non-Abelian Lagrangian then reads

$$\mathcal{L} = \int \mathrm{Tr}\, W_\alpha W^\alpha d^2\theta + \int \mathrm{Tr}\, \bar{W}^{\dot\alpha}\bar{W}_{\dot\alpha} d^2\bar{\theta} + \int \bar{\Phi} e^V \Phi d^4\theta + \int W(\Phi) d^2\theta + \int \bar{W}(\bar{\Phi}) d^2\bar{\theta} \ . \tag{5.16}$$

5.2 Quantization and non-Renormalization Theorems

There are basically two ways to quantize $N = 1$ supersymmetric field theories:

1. The standard (perturbative) quantization in terms of component fields with $s = 0, 1/2, 1$ (as done in a standard QFT course). This procedure is usually followed today. It was shown that a supersymmetric regulator exists and thus supersymmetry has no anomaly. Therefore the quantum corrections have to preserve supersymmetry and as a consequence they are "simpler" than in a non-supersymmetric field theories (in that for example some set of Feynman diagrams vanish). This approach has the disadvantage that supersymmetry is not manifest and has to be checked/ensured at the end of the computation.

2. One can also quantize the theory in superspace with superfields and develop a supergraph formalism [5,10]. In this case the supersymmetry is manifest throughout but the formalism is more complicated.

In ordinary QFTs one has wave function renormalization and renormalization of the couplings m, Y, g. In $N = 1$ supersymmetric theories one finds instead:

- Wave function renormalization at all orders in perturbation theory (which can be viewed as correction to K).
- $m, Y, W(A)$ is not renormalized in perturbation theory but only non-perturbatively.
- $g, f(A)$ is only renormalized at one-loop but not beyond in perturbation theory.

Two independent proofs have been given:

1. Using superfields and supergraphs [10].
2. Using a specific background field methods and the holomorphicity of the superpotential [11, 12].

The idea of the second method is to view all couplings as background values of chiral superfields. So for example the superpotential is viewed as a function $W(\Phi, m, Y)$ which cannot depend on $\bar{m}, \bar{Y}$. As an example consider a chiral superfield Φ with interaction

$$W_{\text{tree}} = m\Phi^2 + Y\Phi^3 \tag{5.17}$$

The theory has a $U(1) \times U(1)_R$ symmetry with the charge assignment

$$\begin{aligned} q(W) &= 0 \ , \qquad q(\Phi) = 1 \ , \qquad q(m) = -2 \ , \qquad q(Y) = -3 \ , \\ q_R(W) &= 2 \ , \qquad q_R(\Phi) = 1 \ , \qquad q_R(m) = 0 \ , \qquad q_R(Y) = -1 \ , \end{aligned} \tag{5.18}$$

where q denotes the charge of $U(1)$ and q_R denotes the charge of $U(1)_R$. Since the quantum corrected W has to respect the symmetry one concludes

$$W_{\text{qc}} = m\Phi^2 f(t) \ , \qquad \text{for} \qquad t = \frac{Y\Phi}{m} \ . \tag{5.19}$$

At weak coupling $Y \to 0$ one needs $W_{\text{qc}} \to W_{\text{tree}}$ and concludes $f = 1 + t$. The same property has to hold in the limit $Y \to 0, m \to 0$ but Y/m arbitrary. Thus $f = 1 + t$ has to hold for arbitrary t and one concludes that W_{tree} is exact. For the gauge coupling one introduces a chiral field S with $\langle S \rangle = g^{-2} + i\frac{\theta}{8\pi^2}$ and $f_{\text{tree}} = S$. Note that ImS plays the role of an axion with a Peccei-Quinn symmetry $S \to S + i\gamma, \gamma \in \mathbb{R}$. This symmetry or equivalently the correct dependence of the action on the θ-angle imposes

$$f_{\text{qc}} = S + \text{const.} \ , \tag{5.20}$$

with no other polynomial S-dependence allowed.

6 The supersymmetric Standard Model

The basic idea of the supersymmetric Standard Model (SSM) is to promote each field of the Standard Model (SM) to an appropriate supermultiplet. In particular the quarks, leptons and Higgs reside in chiral multiplets while the gauge bosons are members of vector multiplets. Since the gauge generators commute with the Qs, the supermultiplets have to carry the same representations as their SM-components. The gauge group of the SM is $G = SU(3) \times SU(2) \times U(1)_Y$ which is spontaneously broken to $G = SU(3) \times U(1)_{\text{em}}$.

6.1 The Spectrum

The spectrum of the SSM is summarized in table 6.1.

	SM fields	$SU(3)\times SU(2)\times U(1)_Y$	$U(1)_{\text{em}}$	supermultiplet	F	B
quarks	$q_L^I = \begin{pmatrix} u_L^I \\ d_L^I \end{pmatrix}$	$(3,2,\frac{1}{6})$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$	$Q_L^I = \begin{pmatrix} U_L^I \\ D_L^I \end{pmatrix}$	q_L^I	$\tilde{q}_L^I$
	u_R^I	$(\bar{3}, 1, -\frac{2}{3})$	$-\frac{2}{3}$	U_R^I	u_R^I	$\tilde{u}_R^I$
	d_R^I	$(\bar{3}, 1, -\frac{1}{3})$	$-\frac{1}{3}$	D_R^I	d_R^I	$\tilde{d}_R^I$
leptons	$l_L^I = \begin{pmatrix} \nu_L^I \\ e_L^I \end{pmatrix}$	$(1,2,-\frac{1}{2})$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$L_L^I = \begin{pmatrix} N_L^I \\ E_L^I \end{pmatrix}$	l_L^I	$\tilde{l}_L^I$
	e_R^I	$(1,1,1)$	1	E_R^I	e_R^I	$\tilde{e}_R^I$
	ν_R^I	$(1,1,0)$	0	N_R^I	ν_R^I	$\tilde{\nu}_R^I$
Higgs	$\begin{pmatrix} h_u^+ \\ h_u^0 \end{pmatrix}$	$(1,2,\frac{1}{2})$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}$	$\begin{pmatrix} \tilde{h}_u^+ \\ \tilde{h}_u^0 \end{pmatrix}$	$\begin{pmatrix} h_u^+ \\ h_u^0 \end{pmatrix}$
		$(1,2,-\frac{1}{2})$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}$	$\begin{pmatrix} \tilde{h}_d^0 \\ \tilde{h}_d^- \end{pmatrix}$	$\begin{pmatrix} h_d^0 \\ h_d^- \end{pmatrix}$
gauge	G	$(8,1,0)$	0	G	$\tilde{G}$	G
bosons	W	$(1,3,0)$	$(0, \pm 1)$	W	$\tilde{W}$	W
	B	$(1,1,0)$	0	B	$\tilde{B}$	B

Table 6.1: Particle content of the supersymmetric Standard Model. The column below 'F' ('B') denotes the fermionic (bosonic) content of the model. The index $I = 1, 2, 3$ labels the three families of the SM.

Before we turn to the Lagrangian let us note that two Higgs doublets (i.e. an extended Higgs sector) are necessary. This is imposed on the theory by supersymmetry as gauge invariance of the superpotential otherwise can not be achieved. Alternatively, the absence of a gauge anomaly leads to the same conclusion as the Higgs multiplets contain two new chiral fermions which have to be in vector-like representations of the gauge group.

Let us also summarize the new fields in the spectrum. For $s = 0$ these are the squarks $\tilde{q}, \tilde{u}, \tilde{d}$ and the sleptons $\tilde{l}, \tilde{e}, \tilde{\nu}$. For $s = 1/2$ these are the Higgsinos $\tilde{h}_u, \tilde{h}_d$ and the gauginos $\tilde{G}, \tilde{W}, \tilde{B}$. They will often be regrouped into the four neutralinos $\tilde{h}_u^0, \tilde{h}_d^0, \tilde{\gamma}^0, \tilde{Z}$ (where $\tilde{\gamma}^0, \tilde{Z}$ are called photino and Zino) and the four charginos $\tilde{h}_u^+, \tilde{h}_d^-, \tilde{W}^\pm$.

6.2 The Lagrangian

The Lagrangian for the supersymmetric Standard Model has to be of the form (3.10) with gauge group $G = SU(3) \times SU(2) \times U(1)_Y$. This specifies the covariant derivatives in (3.3) and (3.7) appropriately. The superpotential (2.27) has to be chosen such that the Lagrangian of the non-supersymmetric Standard Model is contained. This is achieved by

$$W = \sum_{I,J=1}^{3} \Big((Y_u)_{IJ} h_u \tilde{q}_L^I \tilde{u}_R^J + (Y_d)_{IJ} h_d \tilde{q}_L^I \tilde{d}_R^J + (Y_l)_{IJ} h_d \tilde{l}_L^I \tilde{l}_R^I + m_{IJ} \tilde{\nu}_R^I \tilde{\nu}_R^J \Big) + \mu\, h_u h_d \ , \quad (6.1)$$

where $(Y_u)_{IJ}, (Y_d)_{IJ}, (Y_l)_{IJ}$ are the measured Yukawa couplings of the SM, μ a Higgs-mass parameter and m_{IJ} a possible mixing matrix of the right handed neutrinos. Now we see more explicitly that a $h_u \bar{h}_u$ Higgs mass term as in the SM is incompatible with the holomorphicity of W. This forces the presence of a second Higgs doublet h_d in the complex conjugate representation of $SU(2) \times U(1)$.

From Table 6.1 we see that in terms of quantum numbers there is no distinction between the chiral superfields L_L and H_d. This in turn leads to additional gauge invariant couplings which are possible in W. These are

$$\Delta W = a h_u \tilde{l}_L + b\, \tilde{l}_L \tilde{q}_L \tilde{d}_R + c\, \tilde{d}_R \tilde{d}_R \tilde{u}_R + d\, \tilde{l}_L \tilde{l}_L \tilde{e}_R \,, \quad (6.2)$$

which, however, violate baryon or lepton number conservation and thus easily lead to unacceptable physical consequences (for example fast proton decay). Such couplings can be excluded by imposing a discrete R-parity. Particles of the Standard Model (including both Higgs doublets) are assigned R-charge 1 while all new supersymmetric particles are assigned R-charge -1. This eliminates all terms in (6.2) while the superpotential given in (6.1) is left invariant. An immediate consequence of this additional symmetry is the fact that the lightest supersymmetric particle (often denoted by the 'LSP') is necessarily stable and thus a candidate for WIMP dark matter. However, one should stress that R-parity is not a phenomenological necessity. Viable models with broken R-parity can be constructed and they also can have some phenomenological appeal.

Another extension of the SSM (often called the NMSSM) adds an additional singlet chiral multiplet S with couplings

$$W_{NMSSM} = \tfrac{1}{2}\mu_S S^2 + \tfrac{1}{6} Y_S S^3 + \lambda_S S h_u h_d + W_{SSM} \ . \quad (6.3)$$

7 Spontaneous Supersymmetry Breaking

7.1 Order parameters of supersymmetry breaking

Recall that in a theory with a spontaneously broken symmetry the action of the theory is invariant under the symmetry transformation but its ground state or background is not. Here we consider backgrounds which preserve four-dimensional Lorentz invariance and minimize the potential V. In supersymmetric theories we have generically

$$\langle \delta\text{fermion}\rangle \sim \langle \text{boson}\rangle\ , \qquad\qquad \langle \delta\text{boson}\rangle \sim \langle \text{fermion}\rangle = 0\ , \tag{7.1}$$

where the second transformation always vanishes in a Lorentz-invariant background. Therefore we see that the Lorentz-scalar part of $\langle \delta\text{fermion}\rangle$ is the order parameter of supersymmetry breaking. For super Yang-Mills theories we have

$$\langle \delta\chi^i_\alpha\rangle = \sqrt{2}\xi_\alpha\langle F^i\rangle\ , \qquad \langle \delta\lambda^a_\alpha\rangle = i\xi_\alpha\langle D^a\rangle\ , \tag{7.2}$$

where all additional terms vanish in a Lorentz-invariant background. We see that we can have spontaneous supersymmetry breaking if and only if

$$\langle F^i\rangle \neq 0 \quad (F-\text{term breaking})\ , \quad \text{and/or} \quad \langle D^a\rangle \neq 0 \quad (D-\text{term breaking})\ , \tag{7.3}$$

i.e. $\langle F^i\rangle$ and $\langle D^a\rangle$ are the order parameters of supersymmetry breaking in that non-vanishing F- or D-terms signal spontaneous supersymmetry breaking.

Let us determine the minimum of the scalar potential (3.11)

$$V = F_i\bar{F}_i + \tfrac{1}{2}D^aD^a \geq 0\ . \tag{7.4}$$

Its first derivative reads

$$\partial_j V = F_i\partial_j\bar{F}_i + (\partial_j D^a)D^a = \bar{W}_i W_{ij} + (\partial_j D^a)D^a = 0\ . \tag{7.5}$$

We immediately see that the minimum of V is at

$$\langle F_i\rangle = \langle \bar{F}_i\rangle = \langle D^a\rangle = \langle V\rangle = 0\ . \tag{7.6}$$

Conversely, $\langle V\rangle = 0$ implies that supersymmetry is unbroken while $\langle V\rangle \neq 0$ implies that supersymmetry is broken.

7.2 Goldstone's theorem for supersymmetry

Goldstone's theorem implies that any spontaneously broken global symmetry leads to a massless state in the spectrum. This also holds for supersymmetry where the broken generator is a Weyl spinor and thus there has to be an massless Goldstone fermion.

We already computed the mass matrix $M_{1/2}$ of the fermions in (3.15) for arbitrary F- and D-terms. Or in other words in the derivation of $M_{1/2}$ we did not assume that supersymmetry is unbroken. Therefore we have to show that $M_{1/2}$ always has a zero eigenvalue corresponding to the Goldstone fermion. We do so by identifying the corresponding null vector. Consider

$$M_{1/2}\begin{pmatrix}\bar{W}_j\\ \frac{-i}{\sqrt{2}}D^a\end{pmatrix}=\begin{pmatrix}W_{ij}\bar{W}_j+(\partial_j D^a)D^a\\ i\sqrt{2}(\partial_j D^b)\bar{W}_j\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\ , \tag{7.7}$$

where in the first equation we used (3.15). In the second equation the upper component vanishes due to (7.5) while the lower component vanishes due to gauge invariance of W. Gauge invariance indeed implies

$$\delta W=W_i\delta A^i=i\alpha^a W_i\,(T^a)^i_j\,A^j=i\alpha^a W_i\partial_{\bar{i}}D^a=0\,. \tag{7.8}$$

This proves Goldstones theorem for supersymmetry. Phenomenologically, however, the presence of a massless Goldstone fermion poses a problem for the SSM as no massless fermion has been observed yet. This already hints at the super Higgs effect where the Goldstone fermion is “eaten” by the gauge field of local supersymmetry, the gravitino.

7.3 Models for spontaneous supersymmetry breaking

Let us now discuss models for spontaneous supersymmetry breaking. The idea is to add fields to the spectrum with couplings such that supersymmetry is spontaneously broken. Concretely one needs to forbid solutions with $\langle F_i\rangle=\langle D^a\rangle=0$ which is surprisingly difficult to arrange. Let us start with F-term breaking.

7.3.1 F-term breaking

In the O'Raifeartaigh model [13] one introduces three chiral superfields Φ_0,Φ_1,Φ_2 and the following superpotential:

$$W=\lambda A_0+mA_1A_2+YA_0A_1^2\ ,\qquad m^2>2\lambda Y\ . \tag{7.9}$$

The algebraic equations for the F-terms are:

$$\begin{aligned}F_0&=\frac{\partial W}{\partial A^0}=\lambda+YA_1^2\,,\\ F_1&=\frac{\partial W}{\partial A^1}=mA_2+2YA_0A_1\,,\\ F_2&=\frac{\partial W}{\partial A^2}=mA_1\,.\end{aligned} \tag{7.10}$$

$\langle F_0\rangle=0=\langle F_2\rangle$ has no solution and thus supersymmetry must be broken.

The scalar potential reads

$$V=\left|\lambda+YA_1^2\right|^2+|mA_2+2YA_0A_1|^2+|mA_1|^2\ . \tag{7.11}$$

It is minimized by $\langle A_1\rangle=0=\langle A_2\rangle$, $\langle A_0\rangle$ arbitrary, such that $\langle F_1\rangle=0=\langle F_2\rangle$ and $\langle F_0\rangle\neq 0$. The mass spectrum of the 6 real bosons and the 3 Weyl fermions is found to be

$$\begin{aligned}\text{bosons}:&\quad(0,0,m^2,m^2,m^2\pm 2Y\lambda)\,,\\ \text{fermions}:&\quad(0,m,m)\ .\end{aligned} \tag{7.12}$$

We observe a mass splitting of the boson-fermion mass degeneracy but the sum rule (3.27) still holds. For the case at hand we find

$$\mathrm{Tr}\, M_0^2 = 4m^2 \ , \qquad \mathrm{Tr}\, M_{1/2}^2 = 2m^2 \ , \tag{7.13}$$

such that $\mathrm{Str}M^2 = 0$. This is no coincidence as we derived the sum rule (3.27) without any assumption about $\langle F_i \rangle$ or $\langle D^a \rangle$. Phenomenologically the sum rule (3.27) is problematic for the supersymmetric Standard Model. Since none of the supersymmetric partners has been observed yet and they must be heavier than the particles of the Standard Model. Close inspection of (3.27) shows that this cannot be arranged within a spontaneously broken supersymmetric Standard Model. Nevertheless let us continue and discuss D-term breaking.

7.3.2 D-term breaking

We already discussed the possibility of adding a Fayet-Iliopoulos term to the supersymmetry Lagrangian for any $U(1)$ factor in the gauge group. Let us therefore consider a $U(1)$ vector multiplet and one chiral multiplet with vanishing $W = 0$ but the additional FI coupling (3.5). In this case the D-term and the potential read

$$D = -(g\bar{A}A + \xi_{FI}) \ , \qquad V = \tfrac{1}{2}D^2 = \tfrac{1}{2}\left(g\bar{A}A + \xi_{FI}\right)^2 \ . \tag{7.14}$$

We need to distinguish the cases $g\xi_{FI} < 0$ and $g\xi_{FI} > 0$. For $g\xi_{FI} < 0$ the minimum is at $\langle \bar{A}A \rangle = -\xi_{FI}/g$ with $\langle D \rangle = 0 = \langle V \rangle$. Thus the $U(1)$ gauge symmetry is spontaneously broken but supersymmetry is intact. For $g\xi_{FI} > 0$ the condition $\langle D \rangle = 0$ has no solution. The minimum is at $\langle A \rangle = 0$ with $\langle V \rangle = \xi_{FI}^2/2$, $\langle D \rangle = -\xi_{FI}$. In this case the $U(1)$ is unbroken but supersymmetry is broken. Thus the vector multiplet remains massless, the chiral fermion remains massless as $W = 0$ and only A receives a mass

$$m_A^2 = \langle \partial_A \partial_{\bar{A}} V \rangle = -2\xi_{FI}g \ . \tag{7.15}$$

The sum rule (3.27) is again satisfied as in this case

$$\mathrm{Str}M^2 = m_A^2 = -2gD \ . \tag{7.16}$$

As a second example let us consider $U(1)$ gauge theory with FI-term and two massive chiral multiplets $\Phi_\pm$ carrying opposite $U(1)$ charge $q = \pm 1$. The superpotential, F- and D-terms read

$$W = mA_+A_- \ , \quad \bar{F}_\pm = -W_\pm = -mA_\mp \ , \quad D = -(g\bar{A}_+A_+ - \bar{A}_-A_-) - \xi_{FI} \ . \tag{7.17}$$

The potential thus is

$$\begin{aligned} V &= |F_+|^2 + |F_-|^2 + \tfrac{1}{2}D^2 \\ &= \tfrac{1}{2}\xi_{FI}^2 + (m^2 + \xi_{FI}g)|A_+|^2 + (m^2 - \xi_{FI}g)|A_-|^2 + \tfrac{1}{2}g^2(|A_+|^2 - |A_-|^2)^2 \ . \end{aligned} \tag{7.18}$$

We need to distinguish $m^2 > \xi_{FI}g$ and $m^2 < \xi_{FI}g$. In the first case the minimum is at

$$m^2 > \xi_{FI}g : \quad \langle A_+ \rangle = \langle A_- \rangle = \langle F_+ \rangle = \langle F_- \rangle = 0 \ , \quad \langle D \rangle = \xi_{FI} \ , \quad \langle V \rangle = \tfrac{1}{2}\xi_{FI}^2 \ . \tag{7.19}$$

Thus supersymmetry is broken but the gauge symmetry is intact. One checks again that the sum rule (3.27) holds.

For $m^2 < \xi_{FI} g\ A_-$ has a negative mass^2 and thus the $U(1)$ is broken. One finds for generic values of the couplings

$$\langle A_+ \rangle \neq 0 \ , \quad \langle V \rangle \neq 0 \ , \tag{7.20}$$

and thus both the gauge symmetry and supersymmetry is broken. By tuning $m \to 0$ one can arrange $\langle A_+ \rangle \neq 0, \langle V \rangle = 0$ and thus broken gauge symmetry with supersymmetry intact.

To summarize, the lesson of this section is that spontaneously broken supersymmetry run into phenomenological difficulties. The only way out is an explicit breaking of (global) supersymmetry which we discuss next.

8 Soft Supersymmetry Breaking

8.1 Excursion: The Hierarchy and Naturalness Problem

Before we continue let us briefly review the hierarchy and naturalness problem of QFTs.[5] Consider the following (non-supersymmetric) Lagrangian of a complex scalar A and a Weyl fermion χ

$$\begin{aligned}\mathcal{L} = &- \partial_\mu \bar{A}\partial^\mu A - i\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi - \tfrac{1}{2}\, m_f\,(\chi\chi + \bar{\chi}\bar{\chi}) \\ &- Y\,(A\chi\chi + \bar{A}\bar{\chi}\bar{\chi}) - m_b^2\,\bar{A}A - \lambda\,(\bar{A}A)^2 + \kappa\left(A\bar{A}^2 + \bar{A}A^2\right)\,.\end{aligned} \tag{8.1}$$

This Lagrangian is supersymmetric for $m_f = m_b, Y^2 = \lambda$ and $\kappa = mY$ and then has a superpotential $W = \frac{1}{2}mA^2 + \frac{1}{3}YA^3$. For $m_f = 0 = \kappa$ $\mathcal{L}$ has a chiral symmetry acting as

$$A \to e^{-2i\alpha}\,A\,, \qquad \chi \to e^{i\alpha}\,\chi\,. \tag{8.2}$$

This symmetry prohibits the generation of a fermion mass by quantum corrections. For $m_f \neq 0$ the fermion mass does receive radiative corrections, but all possible diagrams have to contain a mass insertion as can be seen from the one-loop diagram shown in Fig. 8.1. Since the propagator of the boson (upper dashed line in the diagram) is $\sim \frac{1}{k^2}$ while the propagator of the fermion (lower solid line) is $\sim \frac{1}{k}$ one obtains a mass correction which is proportional to m_f

$$\delta m_f \sim Y^2 m_f\, \ln\frac{m_f^2}{\Lambda^2}\,, \tag{8.3}$$

where Λ is the ultraviolet (UV) cutoff. Hence the mass of a chiral fermion does not receive large radiative corrections if the bare mass m_f is small. For that reason 't Hooft calls fermion masses "natural" – an extra symmetry appears when the mass is set to zero which in turn leads to a protection of the fermion mass by an approximate chiral symmetry [17].

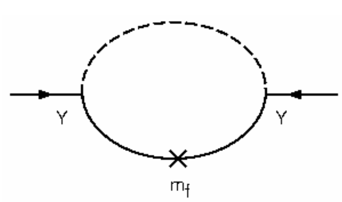

Figure 1: The one-loop correction to the fermion mass.

This state of affairs is different for scalar fields. The diagrams giving the one-loop corrections to m_b are shown in Fig. 8.1. Both diagrams are quadratically divergent but they have an opposite sign because in the second diagram fermions are running in the loop. One finds

$$\delta m_b^2 \sim (\lambda - \ Y^2)\,\Lambda^2\,. \tag{8.4}$$

Thus, in non-supersymmetric theories scalar fields receive large mass corrections (even if the bare mass is set to zero) and small scalar masses are "unnatural". They can only be

[5]The discussion of this section follows ref. [15].

arranged by delicately fine-tuning the bare mass and the couplings λ, Y. This problem becomes apparent in extensions of the Standard Model which apart from the weak scale M_Z do have a second larger scale, say $M_{\rm GUT}$ with $M_{\rm GUT} \gg M_Z$. In such theories the mass of the scalar boson is naturally of the order of the largest mass parameter in the theory. This discussion applies to the Higgs boson of the Standard Model and it is difficult to understand the smallness of M_Z and how it can be kept stable against quantum corrections whenever the Standard Model is the low energy limit of a theory with a large mass scale.

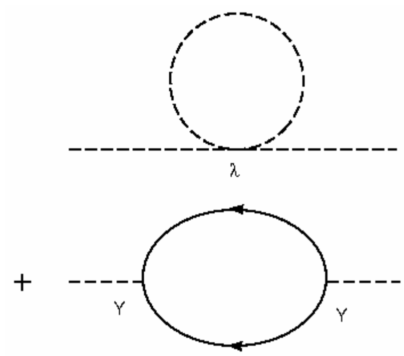

Figure 2: The one-loop corrections to the boson mass.

There are basically two different suggestions for 'solving' this problem. The first class of models assume that the Higgs boson of the Standard Model is not an elementary scalar, but rather a condensate of strongly interacting 'techni'- fermions. These theories are called "technicolor" theories. The second class of models are supersymmetric theories where the Higgs boson is elementary but the quadratic divergence in (8.4) exactly cancels due to the supersymmetric relation $Y^2 = \lambda$.

The cancellation of quadratic divergences is a general feature of supersymmetric quantum field theories and a consequence of the more general non-renormalization theorem which was discussed in Section 5.2. The 'taming' of the quantum corrections is one of the attractive features of supersymmetric quantum field theories. It leads (among other things) to the possibility of stabilizing the weak scale M_Z. In that sense supersymmetry solves the naturalness problem in that it allows for a small and stable weak scale without fine-tuning. However, supersymmetry does not solve the hierarchy problem in that it does not explain why the weak scale is small in the first place.

8.2 Soft Breaking of Supersymmetry

As we have seen in section 7 models with spontaneously broken supersymmetry are phenomenologically not acceptable. For example the mass formula (3.27), generally valid in such cases, forbids that all supersymmetric particles acquire masses large enough to make them invisible in present experiments. One way to overcome those difficulties is to allow explicit supersymmetry breaking.

We observed that the absence of quadratic divergences in supersymmetric theories stabilizes the Higgs mass and thus the weak scale. This 'attractive' feature of supersymmetric field theories can be maintained in theories with explicitly broken supersymmetry if the supersymmetry breaking terms are of a particular form. Such terms which break supersymmetry explicitly and generate no quadratic divergences are called "soft breaking terms".

One possibility to identify the soft breaking terms is to investigate the divergence structure of the effective potential [18]. Consider a quantum field theory of a scalar field A in the presence of an external source J. The generating functional for the Green's functions is given by

$$e^{-iE[J]} = \int \mathcal{D}A \exp\left[i\int d^4x(\mathcal{L}[A(x)] + J(x)A(x))\right] . \tag{8.5}$$

The effective action $\Gamma(A_{cl})$ is defined by the Legendre transformation

$$\Gamma(A_{cl}) = -E[J] - \int d^4x J(x) A_{cl}(x) , \tag{8.6}$$

where $A_{cl} = -\frac{\delta E[J]}{\delta J(x)}$. $\Gamma(A_{cl})$ can be expanded in powers of momentum; in position space this expansion takes the form

$$\Gamma(A_{cl}) = \int d^4x[-V_{eff}(A_{cl}) - \frac{1}{2}(\partial_m A_{cl})(\partial^m A_{cl})Z(A_{cl}) + \dots\ ] . \tag{8.7}$$

The term without derivatives is called the effective potential $V_{eff}(A_{cl})$. It can be calculated in a perturbation theory of $\hbar$:

$$V_{eff}(A_{cl}) = V^{(0)}(A_{cl}) + \hbar V^{(1)}(A_{cl}) + \dots \tag{8.8}$$

where $V^{(0)}(A_{cl})$ is the tree level and $V^{(1)}(A_{cl})$ the one-loop contribution. In a theory with scalars, fermions and vector bosons the one-loop contribution takes the form [19]

$$V^{(1)} \sim \int d^4k\, \mathrm{Str} \ln(k^2 + M^2) = \sum_s (-1)^{2s}(2s+1)\, \mathrm{Tr} \int d^4k\ \ln(k^2 + M_s^2) , \tag{8.9}$$

where M_s^2 is the matrix of second derivatives of $\mathcal{L}|_{k=0}$ at zero momentum for scalars ($s = 0$), fermions ($s = 1/2$) and vector bosons ($s = 1$).[6] The UV divergences of (8.9) can be displayed by expanding the integrand in powers of large k. This leads to

$$V^{(1)} \sim \mathrm{Str}\mathbf{1} \int \frac{d^4k}{(2\pi)^4} \ln k^2 + \mathrm{Str} M^2 \int \frac{d^4k}{(2\pi)^4} k^{-2} + \dots\ . \tag{8.10}$$

If a UV-cutoff Λ is introduced the first term in (8.10) is $\mathcal{O}(\Lambda^4 \ln \Lambda)$. Its coefficient $\mathrm{Str}\mathbf{1} = n_B - n_F$ vanishes in theories with a supersymmetric spectrum of particles. The second term in (8.10) is $\mathcal{O}(\Lambda^2)$ and determines the presence of quadratic divergences at one-loop level. Therefore quadratic divergences are absent if

$$\mathrm{Str} M^2 = 0\ . \tag{8.11}$$

[6] M_s^2 is not necessarily evaluated at the minimum of V_{eff}. Rather it is a function of the scalar fields in the theory. The mass matrix is obtained from M_s^2 by inserting the vacuum expectation values of the scalar fields.

More precisely, one can also tolerate $\mathrm{Str}M^2 = \text{const.}$ since this would correspond to a shift of the zero point energy which, without coupling to gravity, is undetermined. In theories with exact or spontaneously broken supersymmetry (8.11) is fulfilled whenever the trace-anomaly vanishes as we learned in (3.27).[7]

The soft supersymmetry breaking terms are defined as those non-supersymmetric terms that can be added to a supersymmetric Lagrangian without spoiling $\mathrm{Str}M^2 = \text{const.}$. One finds the following possibilities [18]:

- Holomorphic terms of the scalars proportional to A^2, A^3 and the corresponding complex conjugates.[8]
- Mass terms for the scalars proportional to $\bar{A}A$. (They only contribute a constant, field independent piece in $\mathrm{Str}M^2$).
- Gaugino mass terms.

Thus the most general Lagrangian with softly broken supersymmetry takes the form

$$\mathcal{L} = \mathcal{L}_{\rm susy} + \mathcal{L}_{\rm soft} \ , \tag{8.12}$$

where $\mathcal{L}_{\rm susy}$ is of the form (3.6) and

$$\mathcal{L}_{\rm soft} = -m^2_{ij} A^i \bar{A}^j - (B_{ij} A^i A^j + A_{ijk} A^i A^j A^k + \text{h.c.}) - \tfrac{1}{2}\, \tilde{m}_{ab} \lambda^a \lambda^b + \text{h.c.} \ . \tag{8.13}$$

m^2_{ij} and B_{ij} are mass matrices for the scalars, A_{ijk} are trilinear couplings (often called 'A-terms') and $\tilde{m}_{ab}$ is a mass matrix for the gauginos.

We see that many new parameters are introduced which are only constrained by gauge invariance. For the SSM (with R-parity) one has

$$\begin{aligned} \mathcal{L}_{\rm soft} \;\; = \;\; & -\Big((A_u)_{IJ} h_u \tilde{q}^I_L \tilde{u}^J_R + (A_d)_{IJ} h_d \tilde{q}^I_L \tilde{d}^J_R + (A_e)_{IJ} h_d \tilde{l}^I_L \tilde{e}^J_R + B h_u h_d + \text{ h.c. } \Big) \\ & - \sum_{\text{all scalars}} m^2_{ij} A^i \bar{A}^j - \Big(\tfrac{1}{2} \sum_{(a)=1}^{3} \tilde{m}_{(a)} (\lambda\lambda)_{(a)} + \text{ h.c. } \Big) \ , \end{aligned} \tag{8.14}$$

where the index (a) runs over the three factors in the SM gauge group. Obviously a huge number of new parameters is introduced via $\mathcal{L}_{\rm soft}$. The parameters of $\mathcal{L}_{\rm susy}$ are the Yukawa couplings Y and the parameter μ in the Higgs potential. The Yukawa couplings are determined experimentally already in the non-supersymmetric Standard Model. In the softly broken supersymmetric Standard Model the parameter space is enlarged by

$$\big(\mu, (a_u)_{IJ}, (a_d)_{IJ}, (a_e)_{IJ}, b, m^2_{ij}, \tilde{m}_{(a)}\big) \ . \tag{8.15}$$

Not all of these parameters can be arbitrary but quite a number of them are experimentally constrained.

[7] Indeed, theories with a non-vanishing D-term have been shown to produce a quadratic divergence at one-loop [20].

[8] Higher powers of A are forbidden since they generate quadratic divergences at the 2-loop level [18].

Within this much larger parameter space it is possible to overcome several of the problems encountered in the supersymmetric Standard Model. For example, the supersymmetric particles can now easily be heavy (due to the arbitrariness of the mass terms m^2_{ij}) and therefore out of reach of present experiments. Furthermore, the Higgs potential is changed and vacua with spontaneous electroweak symmetry breaking can be arranged.

However, the soft breaking terms introduce their own set of difficulties. For generic values of the parameters (8.15) the contribution to flavor-changing neutral currents is unacceptably large, additional (and forbidden) sources of CP-violation occur and finally the absence of vacua which break the $U(1)_{\rm em}$ and/or $SU(3)$ is no longer automatic. It is beyond the scope of these lectures to review all of these aspects in detail.

In the spirit of these lectures, the choice of specific soft parameters corresponds to a specific "model" or rather a specific UV-theory with a particular mediation mechanism. Ideally the soft terms should be computed in string theory.

One specific choice is to assume universal soft parameters at some high scale (e.g. $M_{\rm GUT}$)

$$\begin{aligned} & m^2_{ij} = m_0 \delta_{ij} \ , \qquad \tilde m_1 = \tilde m_2 = \tilde m_3 = \tilde m \ , \qquad B = B_0 \mu m_0 \ , \\ & A_u = A_0 Y_u \ , \qquad A_d = A_0 Y_d \ , \qquad A_l = A_0 Y_l \ . \end{aligned} \tag{8.16}$$

This choice, often called "msugra", has a parameter space

$$m_0 \ , \tilde m \ , B_0 \ , A_0 \ , \mu \ , \tag{8.17}$$

with two constraints

$$M^2_Z(m_0, \tilde m, B_0, A_0, \mu) = (91 GeV)^2 \ , \qquad M^2_{\rm Higgs}(m_0, \tilde m, B_0, A_0, \mu) = (126 GeV)^2 \ . \tag{8.18}$$

In the next section we discuss the solution of these constraints.

9 The Higgs sector in supersymmetric theories

Recall that in the SSM we necessarily have two Higgs doublets

$$h_u = \begin{pmatrix} h_u^+ \\ h_u^0 \end{pmatrix} , \quad h_d = \begin{pmatrix} h_d^0 \\ h_d^- \end{pmatrix} . \tag{9.1}$$

These are eight real degrees of freedom, two positively charged, two negatively charged and four neutral. One needs three Goldstone bosons to be eaten by $W^\pm, Z^0$ and thus there are five physical real scalars, three neutral denoted as h^0, H^0 (CP-even), A (CP-odd) and 2 charged Higgs H^+, H^-.

The Higgs potential has two contributions. The supersymmetric terms are computed from (3.11) by setting all scalars to zero but the two Higgs doublets. This implies

$$W = \mu\, h_u h_d \, , \qquad D^a_{SU(2)} = -\tfrac{1}{2} g_2 (\bar{h}_u \sigma^a h_u + \bar{h}_d \sigma^a h_d) \, , \qquad D_{U(1)} = -\tfrac{1}{2} g_1 (|h_u|^2 - |h_d|^2) \, , \tag{9.2}$$

where by abuse of notation $a = 1, 2, 3$ labels the adjoint of $SU(2)$ and $g_{1,2}$ are the gauge couplings of $U(1)_Y$ and $SU(2)$ respectively. Inserted into (3.11) one obtains[9]

$$\begin{aligned} V_{\text{susy}} =& |\mu|^2 \left(|h_u^+|^2 + |h_d^-|^2 + |h_u^0|^2 + |h_d^0|^2\right) + \tfrac{1}{2} g_2^2 \left|h_u^+ \bar{h}_d^0 + h_u^0 \bar{h}_d^-\right|^2 \\ &+ \tfrac{1}{8} \left(g_1^2 + g_2^2\right) \left(|h_u^0|^2 + |h_u^+|^2 - |h_d^0|^2 - |h_d^-|^2\right)^2 \, . \end{aligned} \tag{9.3}$$

In addition one has the soft terms

$$V_{\text{soft}} = m^2_{h_u} \left(|h_u^+|^2 + |h_u^0|^2\right) + m^2_{h_d} \left(|h_d^-|^2 + |h_d^0|^2\right) + \left(B(h_u^+ h_d^- - h_u^0 h_d^0) + \text{h.c.}\right) \, . \tag{9.4}$$

Note that no independent λh^4 coupling exits but instead $\lambda \sim g^2$. As we will see this is the origin for the light Higgs in supersymmetric theories. Furthermore, once supersymmetry is broken $V = V_{\text{susy}} + V_{\text{soft}}$ is no longer positive and electroweak symmetry breaking is possible.

In order to minimize the potential one can choose the $SU(2)$ gauge freedom to set one $SU(2)$ component to zero in the minimum, e.g.

$$\langle h_u \rangle = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_u \end{pmatrix} . \tag{9.5}$$

One also needs to check that the $U(1)_{\text{em}}$ is unbroken or in other words $\langle h_u^+ \rangle = \langle h_d^- \rangle = 0$ holds. For the SSM with universal soft terms one finds that indeed there is no breaking of $U(1)_{\text{em}}$ but in any generalizations one has check this condition. In the following we assume unbroken $U(1)_{\text{em}}$ and minimize V for $h_u^+ = h_d^- = 0$. In this case one obtains from (9.3) and (15.17) (we drop the superscript 0 on h_u, h_d for now)

$$V = \hat{m}_u^2 |h_u|^2 + \hat{m}_d^2 |h_d|^2 - B h_u h_d - \bar{B} \bar{h}_u \bar{h}_d + \tfrac{1}{8} \left(g_1^2 + g_2^2\right) \left(|h_u|^2 - |h_d|^2\right)^2 \, , \tag{9.6}$$

where

$$\hat{m}_u^2 = m_u^2 + |\mu|^2 \, , \qquad \hat{m}_d^2 = m_d^2 + |\mu|^2 \, . \tag{9.7}$$

(Note that $\hat{m}_u^2, \hat{m}_d^2$ are no longer positive as m_u^2, m_d^2 are arbitrary.) To simplify the analysis further one can choose phases of h_u, h_d such that B is real.

Let us now formulate conditions for the parameters such that an electroweak symmetry breaking minimum exist.

---

[9]Note that $|\mu|^2 \geq 0$ and thus no electroweak symmetry breaking is possible for V_{susy}.

1. V has to be bounded from below for h_u, h_d large. This is indeed the case due to the positive quartic term in (9.6). However, for $|h_u| = |h_d|$ one needs an additional condition which can be seen by rewriting V in terms of $h = h_r + ih_i$. One obtains

$$\begin{aligned} V(h_u = h_d) &= \left(\hat{m}_u^2 + \hat{m}_d^2\right)|h|^2 - B\left(h^2 + \bar{h}^2\right) \\ &= \left(\hat{m}_u^2 + \hat{m}_d^2 - 2B\right)h_r^2 + \left(\hat{m}_u^2 + \hat{m}_d^2 + 2B\right)h_i^2 \ , \end{aligned} \tag{9.8}$$

which imposes

$$\hat{m}_u^2 + \hat{m}_d^2 \geq 2|B| \ . \tag{9.9}$$

2. The existence of an electroweak breaking minimum requires $\det M^2 < 0$ where from (9.6) we infer in terms of real scalars

$$M^2 = \begin{pmatrix} \hat{m}_u^2 & 0 & -B & 0 \\ 0 & \hat{m}_u^2 & 0 & +B \\ -B & 0 & \hat{m}_d^2 & 0 \\ 0 & +B & 0 & \hat{m}_d^2 \end{pmatrix} , \qquad \det M^2 = \left(\hat{m}_u^2\hat{m}_d^2 + B^2\right)\left(\hat{m}_u^2\hat{m}_d^2 - B^2\right) \ . \tag{9.10}$$

Thus $\det M^2 < 0$ imposes

$$\hat{m}_u^2\hat{m}_d^2 < |B|^2 \ . \tag{9.11}$$

Note that $\hat{m}_u^2 = \hat{m}_d^2$ is <u>not</u> possible!

3. One determines v_u, v_d via the following two conditions at the $SU(2)\times U(1)$ breaking minimum:

$$\begin{aligned} \left.\frac{\partial V}{\partial h_u}\right|_{h_u=\frac{v_u}{\sqrt{2}},\, h_d=\frac{v_d}{\sqrt{2}}} &= \hat{m}_u^2 v_u - Bv_d + \tfrac{1}{8}\left(g_1^2 + g_2^2\right)\left(v_u^2 - v_d^2\right)v_u = 0 \ , \\ \left.\frac{\partial V}{\partial h_d}\right|_{h_u=\frac{v_u}{\sqrt{2}},\, h_d=\frac{v_d}{\sqrt{2}}} &= \hat{m}_u^2 v_d - Bv_u - \tfrac{1}{8}\left(g_1^2 + g_2^2\right)\left(v_u^2 - v_d^2\right)v_d = 0 \ . \end{aligned} \tag{9.12}$$

(We exclude $v_u = v_d = 0$ since by construction it is a local maximum.) One defines

$$v_u = v\sin\beta\,, \quad v_d = v\cos\beta\,, \tag{9.13}$$

such that

$$v_u^2 + v_d^2 = v^2 \ , \qquad \frac{v_u}{v_d} = \tan\beta \ , \qquad v_u^2 - v_d^2 = -v^2\cos 2\beta \tag{9.14}$$

and

$$M_Z^2 = \tfrac{1}{4}v^2\left(g_1^2 + g_2^2\right) \ , \qquad M_W^2 = \tfrac{1}{4}v^2 g_2^2 \ . \tag{9.15}$$

In this convention the two conditions of (9.12) turn into

$$\begin{aligned} \hat{m}_u^2 - B\cot\beta - \tfrac{1}{2}M_Z^2\cos 2\beta = 0 \ , \\ \hat{m}_d^2 - B\tan\beta + \tfrac{1}{2}M_Z^2\cos 2\beta = 0 \ . \end{aligned} \tag{9.16}$$

They determine (v_u, v_d) or equivalently $(M_Z, \tan\beta)$ in terms of the supersymmetric and soft parameters. One can rewrite them in yet another form

$$\begin{aligned} M_Z^2 &= \frac{2}{\tan^2\beta - 1}\left(\hat{m}_d^2 - \hat{m}_u^2 \tan^2\beta\right) = -2|\mu|^2 + \frac{2\,(m_d^2 - m_u^2\tan^2\beta)}{\tan^2\beta - 1} \ , \\ B &= \tfrac{1}{2}\sin 2\beta\left(\hat{m}_u^2 + \hat{m}_d^2\right) \ . \end{aligned} \tag{9.17}$$

The constraint $M_Z \simeq 91\,\mathrm{GeV}$ has to be imposed which eliminates one parameter in the Higgs sector. From (9.17) one also sees that without any fine-tuning the soft parameters should obey $m_{\mathrm{soft}} = \mathcal{O}(M_Z)$. However, in the next lecture we discuss that with the current LHC-date this seems to be ruled out.

As the next step we compute the Higgs masses or more precisely the eigenvalues of the mass matrix

$$M_{ij} = \left.\frac{\partial^2 V}{\partial\phi^i\partial\phi^j}\right|_{\mathrm{min}} \ , \tag{9.18}$$

where here ϕ^i are the 8 real components of the Higgs sector. M splits into a 4×4 matrix of neutral Higgses and a 4×4 matrix of charged Higgses. Apart from the zero-eigenvalues of the three Goldstone bosons $G^\pm, G^0$ one finds for the five remaining scalars $(H, h, A, H^\pm)$

$$\begin{aligned} m_{h,H}^2 &= \tfrac{1}{2}\left(m_A^2 + M_Z^2 \mp \sqrt{(m_A^2 + M_Z^2)^2 - 4m_A^2 M_Z^2\cos^2(2\beta)}\right) \ , \\ m_A^2 &= \hat{m}_u^2 + \hat{m}_d^2 \ , \\ m_{H^\pm}^2 &= m_A^2 + M_W^2 \ , \end{aligned} \tag{9.19}$$

Note that the square root in the first equation obeys

$$|m_A^2 - M_Z^2|^2 \le \sqrt{(m_A^2 + M_Z^2)^2 - 4m_A^2 M_Z^2\cos^2(2\beta)} \le (m_A^2 + M_Z^2)^2 \ . \tag{9.20}$$

Remarks:

1. The eigenvalues satisfy (independent of the scale of supersymmetry breaking):

$$m_{H^\pm} \ge M_W \ , \qquad m_H \ge M_Z \ , \qquad m_h \le \min(m_A, M_Z) \ . \tag{9.21}$$

   Since LHC measured $m_{Higgs} = 126$ GeV the third property is problematic and requires large quantum correction. To be close to $m_h \simeq M_Z$ one needs $\cos 2\beta \to 1$, i.e., $\beta \to 0, \pi$ or $\tan\beta \to 0, \infty$.

2. Let us also consider the limit $m_{\mathrm{soft}} \to \infty$. In this case one infers from (9.19) $m_A \to \infty, m_H \to \infty, m_{H^\pm} \to \infty$ and

$$m_h^2 \to \tfrac{1}{2}(m_A^2 + M_Z^2)\left(1 - \sqrt{1 - \frac{4M_Z^2 m_A^2\cos^2 2\beta}{(m_A^2 + M_Z^2)^2}}\right) \simeq M_Z^2\cos^2 2\beta \ , \tag{9.22}$$

   or in other words, one Higgs is always light in the MSSM!

In order to satisfy the Higgs mass measurement the classical relations (9.19) require large quantum correction. This is indeed possible as the quantum corrections to the quartic Higgs coupling are dominated by top/stop loops. One finds [21]

$$\delta m_h^2 = \frac{3g^2 m_t^4}{2\pi^2 M_W^2} \left(\log \frac{m_s^2}{m_t^2} + \frac{X_t^2}{m_s^2}\left(1 - \frac{X_t^2}{12 m_s^2}\right)\right) \ , \tag{9.23}$$

where

$$m_s^2 = m_{\tilde{t}_1}^2 m_{\tilde{t}_2}^2 \ , \qquad X_t = A_t - \mu \cot\beta \ . \tag{9.24}$$

Here $m_{\tilde{t}_{1,2}}$ are the two stop mass eigenvalues. The second terms is maximized for $X_t^2 = 6m_s^2$ which, as we will see in the next lecture, corresponds to maximal mixing between the two stops in the electroweak basis. In order to obtain $m_h = 126$ GeV one either needs to make m_s large which induces considerable fine-tuning in (9.17) or X_t maximal. Concretely one finds the bounds [22]

$$m_s \leq 1\text{TeV} \quad \text{for} \quad X_t \sim \sqrt{6} m_s \ , \qquad m_s \leq 3\text{TeV} \quad \text{for} \quad X_t \sim m_s \ . \tag{9.25}$$

In the first case the lighter stop can be in the range $200 - 400$ GeV [27].

10 Experimental signals of Supersymmetry

The experimental searches for supersymmetric particles impose additional constraints on the supersymmetric parameter space. First and foremost the direct lower bounds on the masses of the supersymmetric particles determined at the LHC exclude certain regions of the parameter space. The analysis at ATLAS and CMS is rather involved and in general strongly model/assumption dependent. The best bounds obtained at the LHC are for strongly interacting particles, i.e. gluinos and squarks but due to the complexity of the analysis we cannot adequately review them here. For summaries and reviews see, for example, [23–26].

The translations of experimental bounds into the supersymmetric parameter space is complicated by the fact that the states which are listed in table 6.1 are interaction eigenstates, but not necessarily mass eigenstates. The only exception are the gluinos $\tilde{g}$ and the mass bounds directly translate into bounds on $\tilde{m}_3$. Let us start with the squarks and sleptons.

10.1 Squarks and slepton masses

The mass matrices in of the quarks $\tilde{q}_L^I, \tilde{u}_R^I, \tilde{d}_R^I$ and slepton $\tilde{l}_L^I, \tilde{e}_R^I$ appear in the Lagrangian in the following form

$$\mathcal{L} = -\mathbf{U} M_U^2 \mathbf{U}^\dagger - \mathbf{D} M_D^2 \mathbf{D}^\dagger - \mathbf{E} M_E^2 \mathbf{E}^\dagger \ , \tag{10.1}$$

where we abbreviate

$$\mathbf{U} \equiv (\tilde{u}_L^I, \bar{\tilde{u}}_R^I) \ , \qquad \mathbf{D} \equiv (\tilde{d}_L^I, \bar{\tilde{d}}_R^I) \ , \qquad \mathbf{E} \equiv (\tilde{e}_L^I, \bar{\tilde{e}}_R^I) \ . \tag{10.2}$$

M^2 are 6×6 matrices which, if one ignores the intergenerational mixing, are composed of three 2×2 blocks of the form

$$M_U^2 = \begin{pmatrix} m_{\tilde{u}_L}^2 + m_u^2 + L_u & m_u X_u^* \\ m_u X_u & m_{\tilde{u}_R}^2 + m_u^2 + R_u \end{pmatrix} \ , \tag{10.3}$$

where $m_{\tilde{u}_{L,R}}$ are the soft masses, m_u is the mass of the up-quarks and

$$L_u = (\tfrac{1}{2} - e_u \sin^2\theta_W) M_Z^2 \cos^2\beta \ , \quad R_u = e_u \sin^2\theta_W M_Z^2 \cos^2\beta \ , \quad X_u = A_u - \mu^* \cot\beta \ . \tag{10.4}$$

Similar formuli for $M_{D,L}$ can be found in [1, 23].

For the squarks of the first two generations the off-diagonal terms are negligible and in a large region of parameter space one finds the "generic" bound

$$m_{\tilde{q}} \gtrsim 800\,\text{GeV} \ . \tag{10.5}$$

For the stops the off-diagonal term in (10.3) can be significant and lead to large mass splitting between the two mass eigenstates. Therefore the lighter stop can be currently as light as $200 - 400$ GeV [27]. For the selectron and smuon one finds in a large region of parameter space

$$m_{\tilde{e},\tilde{\mu}} \gtrsim 275\,\text{GeV} \ . \tag{10.6}$$

10.2 Gluinos, Neutralinos and Charginos

As we already said the gluinos $\tilde{g}$ are already in the mass eigenbasis and in a large region of parameter space LHC obtained the limit

$$m_{\tilde{g}} \gtrsim 1\,\text{TeV} . \tag{10.7}$$

The three Winos $\tilde{W}^{\pm}, \tilde{W}^3$, the $\tilde{B}$ and the four Higgsinos $\tilde{h}^0_{u,d}, \tilde{h}^-_d, \tilde{h}^+_u$ combine into a four-vector of neutral Weyl fermions consisting of $\mathbf{N} \equiv (\tilde{B}, \tilde{W}^3, \tilde{h}^0_u, \tilde{h}^0_d)$ and two pairs of charged Weyl fermions $\mathbf{C}^- \equiv (\tilde{W}^-, \tilde{h}^-_d)$, $\mathbf{C}^+ \equiv (\tilde{W}^+, \tilde{h}^+_u)$ with the following set of mass matrices

$$\mathcal{L}_{\text{fmass}} = -\ \mathbf{C}^- M_C (\mathbf{C}^+)^T \ -\ \tfrac{1}{2}\mathbf{N} M_N \mathbf{N}^T + h.c. \ , \tag{10.8}$$

where

$$M_C = \begin{pmatrix} \tilde{m}_2 & -\frac{i}{\sqrt{2}} g_2 v_u \\ -\frac{i}{\sqrt{2}} g_2 v_d & \mu \end{pmatrix} \ , \tag{10.9}$$

and

$$M_N = \begin{pmatrix} \tilde{m}_1 & 0 & \frac{i}{2} g_1 v_u & -\frac{i}{2} g_1 v_d \\ 0 & \tilde{m}_2 & -\frac{i}{2} g_2 v_u & \frac{i}{2} g_2 v_d \\ \frac{i}{2} g_1 v_u & -\frac{i}{2} g_2 v_u & 0 & \mu \\ -\frac{i}{2} g_1 v_d & \frac{i}{2} g_2 v_d & \mu & 0 \end{pmatrix} \ . \tag{10.10}$$

Thus, the physical mass eigenstates of M_C and M_N are parameter dependent linear combinations of the corresponding interaction eigenstates and they are termed *charginos* and *neutralinos*, respectively. In the limit $\tilde{m}, \mu \gg M_W$ these matrices are diagonal.

The current LHC limit for the charginos is

$$m_{\chi^{\pm}} \gtrsim 330\,\text{GeV} . \tag{10.11}$$

while there is no limit on the lightest neutralino and for the next-to-lightest one has the limit

$$m_{\chi^2} \gtrsim 330\,\text{GeV} . \tag{10.12}$$

10.3 Flavor and CP-violation

Rare decays are other possible processes where new physics could surface. The smallness of flavor-changing-neutral-currents is a prominent feature of the SM as they can only be induced by loop effects. For example $b \to s\gamma$ has been observed at a rate consistent with the SM. In supersymmetric theories additional loop-diagrams exist and contribute but due to supersymmetry they all cancel. However, for softly broken theories they contribute a generically large (and unacceptable) piece. This is due to the fact that the soft scalar masses given in (8.13) need not have any properties in flavor space. Only if they are approximately universal their contribution to rare decays can be within the observed bounds. This in turn imposes strong constraints on the structure of the soft terms.

In a similar spirit the soft terms generically introduce additional CP-violating phases which are strongly constrained by the measurements in the K-system, the B-system and by the bounds on the electric dipole moment of the neutron.

The measured anomalous magnetic moment of the μ deviates at 2-3σ from the SM prediction and could be a first sign of New Physics.

11 Supersymmetric Grand Unified Theories

11.1 Non-supersymmetric GUTs

The basic idea is to embed the gauge group of the SM $G_{\rm SM}$ into a simple GUT-group $G_{\rm GUT}$ with a breaking pattern

$$G_{\rm GUT} \stackrel{M_{\rm GUT}}{\longrightarrow} G_{\rm SM} \stackrel{M_Z}{\longrightarrow} SU(3)\times U(1)_{\rm em}\ . \tag{11.1}$$

Since $\mathrm{rk}(G_{\rm SM}) = 4$ one needs $\mathrm{rk}(G_{\rm GUT}) \geq 4$.

The minimal example is $G_{\rm GUT} = SU(5)$ [29] where the generators of $G_{\rm SM}$ in the fundamental **5** representation are chosen as follows:

$$\begin{aligned}
SU(3): &\qquad T^{\hat a} = \begin{pmatrix} T^{\hat a} & 0 \\ 0 & 0\end{pmatrix}\ , \qquad \hat a = 1,\ldots,8\ ,\\
SU(2): &\qquad T^{a} = \begin{pmatrix} 0 & 0 \\ 0 & T^{a}\end{pmatrix}\ , \qquad a = 9,10,11\ ,\\
U(1): &\qquad T^{12} = \tfrac{1}{\sqrt{15}}\,\mathrm{diag}(-1,-1,-1,\tfrac32,\tfrac32)\ .
\end{aligned} \tag{11.2}$$

T^{12} is chosen such that

$$[T^{12}, T^{\hat a}] = 0 = [T^{12}, T^{a}]\ , \qquad \mathrm{Tr}\, T^{12} = 0\ , \qquad \mathrm{Tr}_{\,\mathbf r}(T^A T^B) = I(\mathbf r)\,\delta^{AB}\ , \tag{11.3}$$

with indices $I(\mathbf 5) = I(\bar{\mathbf 5}) = 1/2$. The remaining twelve generators have appropriate off-diagonal entries

$$T^A = \tfrac12 \begin{pmatrix} 0 & * \\ * & 0\end{pmatrix}\ . \tag{11.4}$$

A field $\chi^i, i = 1,\ldots,5$ in the **5** of $SU(5)$ transforms according to

$$\delta\chi^i = -i\alpha^A (T^A)^i_j \chi^j\ , \qquad A = 1,\ldots,24\ . \tag{11.5}$$

Together with (11.2) this determines the transformation law of the components of χ under $G_{\rm SM}$. One finds

$$\mathbf 5 \to (3,1,q_3)\oplus(1,2,q_2)\ , \qquad \bar{\mathbf 5} \to (\bar 3,1,q_{\bar 3})\oplus(1,\bar 2,q_{\bar 2})\ . \tag{11.6}$$

The charges are fixed by the consistency relations (e.g. for the $\bar{\mathbf 5}$)

$$\tfrac{-1}{\sqrt{15}}\, g_{\rm GUT} = q_{\bar 3}\, g_1\ , \qquad \tfrac{3}{2\sqrt{15}}\, g_{\rm GUT} = q_{\bar 2}\, g_1\ , \tag{11.7}$$

which determines for $q_{\bar 2} = 1/2$

$$g_1 = \sqrt{\tfrac35}\, g_{\rm GUT}\ , \qquad q_{\bar 3} = -\tfrac13\ . \tag{11.8}$$

Therefore we can identify $\bar{\mathbf 5} \to d_R \oplus \bar l_L$.

For a field $\psi^{[ij]}$ in the **10** representation one finds the decomposition

$$\mathbf{10} \ \longrightarrow\ (\bar 3, 1, -\tfrac23)\oplus(3,2,\tfrac16)\oplus(1,1,1)\ , \tag{11.9}$$

which thus can be identified with the SM fermions u_R, q_L and e_R, respectively. We see that one family of SM fermions fit precisely into a $\mathbf{\bar{5}}+\mathbf{10}$ of $SU(5)$ with an "explanation" of the fractional quark charges.

The adjoint **24** representations decomposes as

$$\mathbf{24} \longrightarrow (8,1,0) \oplus (1,3,0) \oplus (1,1,0) \oplus (3,2,\tfrac{5}{6}) \oplus (\bar{3},\bar{2},-\tfrac{5}{6}) \ , \tag{11.10}$$

where, for vector fields, the first three entries are identified with the gauge bosons of the SM while the last 12 states are additional, new gauge bosons which have to get a mass when G_{GUT} is spontaneously broken. The Higgs Σ which achieves this breaking also transforms in the **24** and one arranges the potential such that the singlet in the decomposition (11.10) gets a VEV

$$\langle \Sigma \rangle = \tfrac{v_\Sigma}{\sqrt{2}}\, T^{12} \ . \tag{11.11}$$

The mass matrix of the new gauge bosons then is

$$m^2_{AB} = \tfrac{1}{2} g^2 v^2 [T^A, T^{12}][T^B, T^{12}] \ . \tag{11.12}$$

The new gauge bosons do couple to the quarks and leptons and thus new processes are allowed. The most spectacular is proton decay (for example in the channel $p \to \pi_0 + e^+$) with a life-time

$$\tau_p \sim \frac{M^4_{\mathrm{GUT}}}{\alpha_5^2 m_p^5} \ . \tag{11.13}$$

Via this decay M_{GUT} is in principle observable! The current limit (super Kamiokande) at about $\tau > 5 \cdot 10^{33}$ years implies $M_{\mathrm{GUT}} \geq 10^{15}$ GeV.

The electroweak symmetry is broken by introducing a second Higgs H^i in the **5** representation. From (11.6) we see that the doublet in the decomposition can be identified with the SM Higgs but the triplet has to be heavy. Altogether one needs

$$m_{\mathbf{3}} \gg M_Z \ , \qquad \langle H_{\mathbf{2}} \rangle \ll \langle \Sigma \rangle \ , \qquad \langle H_{\mathbf{3}} \rangle = 0 \ . \tag{11.14}$$

The first property is known as the "doublet-triplet" splitting problem while the second property is the hierarchy problem.

Altogether the $SU(5)$ invariant Lagrangian reads

$$\begin{aligned} \mathcal{L} = & -\tfrac{1}{4} F^A_{\mu\nu} F^{\mu\nu\,A} - i\bar{\chi}^i \not{D} \chi^i - i\bar{\psi}^{[ij]} \not{D} \psi^{[ij]} - \tfrac{1}{2} D_\mu \Sigma D^\mu \Sigma - D_\mu \bar{H}^i D^\mu H^i \\ & - Y_{dIJ} \bar{H}^i \chi^{jI} \psi^J_{[ij]} - Y_{uIJ} \epsilon^{ijklm} H_i \psi^I_{[jk]} \psi^J_{[lm]} - V(\Sigma, H) \end{aligned} \tag{11.15}$$

We see that only two Yukawa couplings are possible which lead to the problematic mass relation $m_d = m_l$.

11.2 Unification of the gauge couplings

At M_{GUT} a GUT theory has to obey

$$g_3(M_{\mathrm{GUT}}) = g_s(M_{\mathrm{GUT}}) = \sqrt{5/3}\, g_1(M_{\mathrm{GUT}}) = g_{\mathrm{GUT}} \ . \tag{11.16}$$

At a scale μ below $M_{\rm GUT}$ (i.e. $\mu < M_{\rm GUT}$) one has

$$g_{(a)}^{-2}(\mu) = g^{-2}(M_{\rm GUT}) + \frac{b_{(a)}}{8\pi^2} \ln \frac{M_{\rm GUT}}{\mu} + \Delta_{(a)} \ , \qquad (a) = 1, 2, 3 \ , \tag{11.17}$$

where

$$b_{(a)} = -\tfrac{11}{3} I(ad) + \sum_{\mathbf{r}} \left(\tfrac{2}{3} n_{\rm WF}(\mathbf{r}) I(\mathbf{r}) + \tfrac{1}{6} n_{\rm S}(\mathbf{r}) I(\mathbf{r}) \right) \ , \tag{11.18}$$

is the coefficient of the one-loop β-function and $\Delta_{(a)}$ are the threshold corrections which arise from integrating out heavy $\mathcal{O}(M_{\rm GUT})$ modes. For $SU(N)$ one has $I(N) = I(\bar{N}) = 1/2, I(ad) = N$ and hence

$$\begin{aligned} b_3 &= -11 + \tfrac{4}{3} N_F = -7 \ , \\ b_2 &= -\tfrac{22}{3} + \tfrac{4}{3} N_F + \tfrac{1}{6} N_H = -\tfrac{19}{6} \ , \\ b_1 &= \tfrac{20}{9} N_F + \tfrac{1}{6} N_H = \tfrac{41}{6} \ , \end{aligned} \tag{11.19}$$

where in the second step we used $N_F = 3, N_H = 1$. For the measured gauge couplings at M_Z and the running in the SM the gauge couplings do not obey (11.16) while in the supersymmetric SM they do. Therefore let us now turn to supersymmetric GUTs.

11.3 Supersymmetric GUTs

As in the SSM in super GUTs the gauge bosons are promoted to vector multiplets while the matter fermions reside in chiral multiplets. Concretely for an $SU(5)$ super GUT one has the spectrum:

1 vector multiplet	in	$\mathbf{24}$	$V^A, A = 1, \ldots, 24,$
N_F chiral multiplet	in	$\bar{\mathbf{5}} \oplus \mathbf{10}$	$\chi^{iI}, \psi^{[ij]J}, i, j = 1, \ldots, 5,\ I, J = 1, \ldots, N_F,$
2 chiral Higgs	in	$\mathbf{5} \oplus \bar{\mathbf{5}}$	$H_u, H_d,$
1 chiral Higgs	in	$\mathbf{24}$	$\Sigma.$

The superpotential is given by

$$W = \tfrac{1}{2} M \mathrm{Tr}\, \Sigma^2 + \tfrac{1}{3} \mathrm{Tr}\, \Sigma^3 + \mu H_u H_d + \kappa H_u \Sigma H_d - Y_{dIJ} \chi^{iI} \psi^J_{ij} H^j_d - Y_{uIJ} \epsilon^{ijklm} \psi^I_{ij} \psi^J_{kl} H_{um} \tag{11.20}$$

For supersymmetric theories the β-function coefficient (11.18) changes into

$$b_{(a)} = -3 I(ad) + \sum_{\mathbf{r}} n_c(\mathbf{r}) I(\mathbf{r}) \ , \tag{11.21}$$

which for the SSM results in

$$\begin{aligned} b_3 &= -9 + 2N_F = -3 \ , \\ b_2 &= -6 + 2N_F + \tfrac{1}{2} N_H = -1 \ , \\ b_1 &= \tfrac{10}{2} N_F + \tfrac{1}{2} N_H = 16 \ . \end{aligned} \tag{11.22}$$

Now unification of couplings constants holds for weak scale supersymmetry with a scale for supersymmetry breaking $M_{\rm susy} = \mathcal{O}(1TeV)$.

Further properties of supersymmetric GUTs are:

- There is a hierarchy problem, i.e no explanation why $M_Z, M_{\rm susy} \ll M_{\rm GUT}$. Supersymmetry renders this hierarchy stable but it does not explain it in the first place.

- The fermionic mass relations $m_d = m_l$ remain problematic and can only be improved by making the Higgs sector (a lot) more complicated. Today a popular way out is to combine a GUT theory with the appearance of Kaluza-Klein like extra dimensions and have $G_{\rm GUT}$ broken at the Kaluza-Klein scale.

- Generalization to other GUT-groups $SO(10), E_6, \ldots$ have the attractive feature of giving a right-handed neutrino with a see-saw mechanism to generate neutrino masses. For example the **16** of $SO(10)$ decomposes under $SU(5)$ according to

$$\mathbf{16} \to \mathbf{10} \oplus \bar{\mathbf{5}} \oplus \mathbf{1}\ , \tag{11.23}$$

with the singlet being identified with the right-handed neutrino. In this case all SM fermions including the right-handed neutrino transform in one irreducible representation $G_{\rm GUT}$. Within the see-saw mechanism the GUT-scale also is related to neutrino masses via

$$m_\nu \sim \frac{M_Z^2}{M_{\rm GUT}}\ . \tag{11.24}$$

12 $N = 1$ Supergravity

12.1 General Relativity

Let us first recall a few facts about General Relativity. It can be viewed as a (semi-) classical field theory for a spin 2 field, the metric $g_{\mu\nu}(x)$ which is a symmetric tensor field on an arbitrary (pseudo-) Riemannian manifold. Its Lagrangian is given by

$$\mathcal{L} = -\tfrac{1}{2\kappa^2}\sqrt{-g}\big(R + \Lambda\big) + \mathcal{L}_{\rm mat}\,, \tag{12.1}$$

where $\kappa^2 = 8\pi M_{Pl}^{-2}$, $g = \det g_{\mu\nu}$, R is the Ricci-scalar, Λ is the cosmological constant and $\mathcal{L}_{\rm mat}$ contains the couplings to matter and gauge fields. The equations of motion derived from the action are the Einstein equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\big(R + \Lambda\big) = \kappa T_{\mu\nu}\ , \tag{12.2}$$

where $R_{\mu\nu}$ is the Ricci tensor while $T_{\mu\nu}$ is the energy-momentum tensor defined as

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\partial\mathcal{L}_{\rm matter}}{\partial g_{\mu\nu}}\ . \tag{12.3}$$

The matter couplings summarized in $\mathcal{L}_{\rm mat}$ are obtained from the corresponding flat-space version by replacing $\eta^{\mu\nu} \to g^{\mu\nu}$ and multiplication by $\sqrt{-g}$. For a scalar field it reads

$$\mathcal{L}_{\rm mat} = -\sqrt{-g}\, g^{\mu\nu}\partial_\mu A \partial_\nu A\ , \tag{12.4}$$

for a gauge field one has

$$\mathcal{L}_{\rm mat} = -\tfrac{1}{4}\sqrt{-g}\, g^{\mu\nu}g^{\kappa\rho}F_{\mu\kappa}F_{\nu\rho} \tag{12.5}$$

In order to couple fermions on needs the vierbein formalism where one defines the 4×4 matrix, the vierbein, $e_\mu^a(x)$ by

$$g_{\mu\nu}(x) = e_\mu^a(x)\,\eta_{ab}\,e_\nu^b(x)\,,\quad \mu,\nu = 0,\ldots,3\ ,\quad a,b = 0,\ldots,3\ . \tag{12.6}$$

At each space-time point x^μ it erects a local Lorentz-frame. The inverse vierbeins are defined by

$$e_\mu^a e_a^\nu = \delta_\mu^\nu\ ,\qquad e_b^\mu e_\mu^a = \delta_b^a\ . \tag{12.7}$$

With the help of the vierbein one can give the Weyl action for a spin-1/2 fermion χ as

$$\mathcal{L} = -i\sqrt{-g}\,\bar\chi\bar\sigma^a e_a^\mu D_\mu\chi\,, \tag{12.8}$$

where σ^a are the Pauli matrices as defined in (1.10). The covariant derivative is given by

$$D_\mu\chi = \partial_\mu\chi + \omega_{\mu ab}\sigma^{ab}\chi \tag{12.9}$$

where $\omega = \omega(e,\partial e)$ is the spin connection and σ^{ab} is defined in (1.11).

With the help of the vierbein one defines for a vector field v_μ

$$v_a := e_a^\mu v_\mu\ , \tag{12.10}$$

and the covariant derivatives

$$
\begin{aligned}
D_\mu v_\nu &= \partial_\mu v_\nu - \Gamma^\rho_{\mu\nu} v_\rho \ , \qquad D_\mu v^\nu = \partial_\mu v^\nu + v^\rho \Gamma^\nu_{\mu\rho} \ , \\
D_\mu v_a &= \partial_\mu v_a - \omega_{\mu a}{}^b v_b \ , \qquad D_\mu v^a = \partial_\mu v^a + v^b \omega_{\mu b}{}^a \ ,
\end{aligned}
\tag{12.11}
$$

where Γ is the Christoffel connection. Imposing $D_\mu g_{\nu\rho} = 0$ expresses $\Gamma = \Gamma(g, \partial g)$. Similarly, Imposing $D_\mu e^a_\nu = 0$ expresses $\omega = \omega(e, \partial e)$ and give the relation

$$
\Gamma^\rho_{\mu\nu} e^a_\rho = \partial_\mu e^a_\nu + e^b_\nu \omega_{\mu b}{}^a \ . \tag{12.12}
$$

The action (12.1) has two sets of invariances. Firstly there are the general coordinate transformations

$$
x^\mu \to x'^\mu = x^\mu - a^\mu(x) \ , \tag{12.13}
$$

which leads to the infinitesimal transformations of vector fields v_μ

$$
\delta v_\mu = -a^\rho \partial_\rho v_\mu - (\partial_\mu a^\rho) v_\rho \ . \tag{12.14}
$$

In addition there are local Lorentz transformations for vector fields which are defined in the local tangent space and which carry a-type indices

$$
\delta v^a = v^b L_b{}^a(x) \ , \qquad \delta v_a = -L_a{}^b(x)\, v_b \ . \tag{12.15}
$$

Thus the vierbein itself transforms accordingly as

$$
\delta e^a_\mu = -a^\rho \partial_\rho e^a_\mu - (\partial_\mu a^\rho)\, e^a_\rho + e^b_\mu L_b{}^a \ , \tag{12.16}
$$

while ω transforms as

$$
\delta \omega_{\mu a}{}^b = -a^\rho \partial_\rho \omega_{\mu a}{}^b - (\partial_\mu a^\rho)\, \omega_{\rho a}{}^b + \omega_{\mu a}{}^c L_c{}^b - L_a{}^c \omega_{\mu c}{}^b - \partial_\mu L_a{}^b \ , \tag{12.17}
$$

12.2 $N = 1$ Supergravity

In order to couple the supersymmetric theories we discussed so far to General Relativity the dynamical metric $g_{\mu\nu}$ has to be part of a supermultiplet. Furthermore, since the supersymmetry algebra enlarges the Lorentz- and Poincare group it is necessary to promote the supersymmetry transformations to a local symmetry, i.e. the supersymmetry parameters become space-time dependent

$$
\xi_\alpha \to \xi_\alpha(x) \, . \tag{12.18}
$$

As for any local gauge symmetry one needs to introduce an appropriate gauge field whose spin is one unit higher than than that of the parameter of the transformation. For supersymmetry we thus need a field $\psi_{\mu\alpha}$, called gravitino, with spin $s = 3/2$. This can be explicitly seen by considering the local supersymmetry transformation laws. For the scalar in a chiral multiplet one has

$$
\delta_\xi A = \sqrt{2} \xi^\alpha(x) \chi_\alpha \, , \tag{12.19}
$$

and thus

$$\partial_\mu\left(\delta_\xi A\right) = \sqrt{2}\,\xi^\alpha \partial_\mu \chi_\alpha + \sqrt{2}\,\left(\partial_\mu \xi^\alpha\right)\chi_\alpha\,. \tag{12.20}$$

In order to maintain supersymmetry one needs a gauge field with an inhomogeneous transformation law

$$\delta_\xi \psi_{\mu\alpha} = -\,\partial_\mu \xi_\alpha + \dots\,, \tag{12.21}$$

which has to be included in the covariant derivative

$$\hat{D}_\mu A = \partial_\mu A + \chi_{\mu\alpha}\psi^\alpha + \dots\,. \tag{12.22}$$

The gravitino $\psi_{\mu\alpha}$ can be part of two distinct supermultiplets. Recall from Table 2.2 that the two multiplets are

1. the gravitino multiplet with helicities $(\lambda = \pm 1) \oplus (\lambda = \pm\frac{3}{2})$
2. the gravity multiplet with helicities $(\lambda = \pm\frac{3}{2}) \oplus (\lambda = \pm 2)$.

In an interacting field theory the equations of motion for the gravitino are only consistent if the theory has a local symmetry. This selects the gravity multiplet as the only multiplet where the gravitino can be consistently incorporated.

The supersymmetry transformations are found to be

$$\delta_\xi e^a_\mu = i\left(\psi_\mu \sigma^a \bar{\xi} - \xi \sigma^a \bar{\psi}_\mu\right)\,, \qquad \delta_\xi \psi^\alpha_\mu = -D_\mu \xi^\alpha\,. \tag{12.23}$$

The Lagrangian for the gravity multiplet reads

$$\mathcal{L} = -\tfrac{1}{2\kappa^2}\sqrt{-g}R + \tfrac{1}{2}\sqrt{-g}\epsilon^{\kappa\rho\mu\nu}\left(\bar{\psi}_\kappa \bar{\sigma}_\rho D_\mu \psi_\nu - \psi_\kappa \sigma_\rho D_\mu \bar{\psi}_\nu\right)\,, \tag{12.24}$$

where $\sigma_\rho = e^a_\rho \sigma_a$. The second term in (12.24) is called the Rarita-Schwinger field strength.

The $N = 1$ gravity multiplet also has an off-shell representation with six auxiliary fields, a real b_μ and a complex M. Altogether the off-shell multiplet is

$$\left(g_{\mu\nu}, \psi_{\mu\,\alpha}, b_\mu, M\right)\,, \quad \text{with d.o.f.}: \quad (6, 12, 4, 2)\,. \tag{12.25}$$

The auxiliary fields are added to (12.24) by

$$\Delta\mathcal{L} = \sqrt{-g}\left(M\bar{M} + b^\mu b_\mu\right) \tag{12.26}$$

13 Coupling of $N = 1$ Supergravity to matter

In this lecture we discuss the couplings of chiral and vector multiplets to $N = 1$ supergravity. As in theories including gravity renormalizability is no longer a necessary condition let first recall a few facts about non-linear σ-models.

13.1 Excursion: non-linear σ-model

The renormalizable Lagrangian for n scalar fields

$$\mathcal{L} = -\delta_{ij}\partial_\mu A^i \partial^\mu A^j \,, \quad i = 1, \ldots, n \,, \tag{13.1}$$

can be generalized as

$$\mathcal{L} = -G_{ij}(A)\,\partial_\mu A^i \partial^\mu A^j \,, \tag{13.2}$$

where $G_{ij}(A)$ is a symmetric, positive and invertible matrix depending on A^i. A theory with the Lagrangian (13.2) is called non-linear σ-model which, due to the A dependence of G_{ij} is non-renormalizable.

The scalar fields A^i can be interpreted as coordinate of an n-dimensional Riemannian target space $\mathcal{M}$ and G_{ij} as its metric. Indeed an arbitrary field redefinition $A^i \to A'^i(A^i)$ implies

$$\partial_\mu A^i \to \partial_\mu A'^i = \frac{\partial A'^i}{\partial A^j}\,\partial_\mu A^j \,. \tag{13.3}$$

$\mathcal{L}$ is invariant if G_{ij} transforms inversely, i.e.,

$$G_{ij} \to G'_{ij} = \frac{\partial A^l}{\partial A'^i}\frac{\partial A^k}{\partial A'^j}\,G_{lk} \,, \tag{13.4}$$

which is precisely the transformation of the metric on $\mathcal{M}$. The scalar fields can thus be viewed as the map

$$A^i(x): \quad M_4 \to \mathcal{M} \,, \tag{13.5}$$

where M_4 is the Minkowski space and $\mathcal{M}$ a Riemannian target space.

Let us also recall that the metric has an expansion in Riemann normal coordinates as

$$G_{ij} = \delta_{ij} + R_{ijkl}(A = 0)A^k A^l + \ldots \,, \tag{13.6}$$

where R_{ijkl} is the curvature tensor on $\mathcal{M}$. R_{ijkl} has mass dimension -2 which is another way to see the non-renormalizability of the theory. For complex scalar fields $\mathcal{M}$ is a complex manifold.

13.2 Couplings of neutral chiral multiplet

The couplings of chiral multiplets to $N = 1$ supergravity has the following purely bosonic terms

$$\mathcal{L} = -\tfrac{e}{2\kappa^2}R - G_{i\bar{j}}(A, \bar{A})\partial_\mu A^i \partial^\mu \bar{A}^{\bar{j}} - V(A, \bar{A}) + \ldots \,, \tag{13.7}$$

where the $\ldots$ denote fermionic terms. This $\mathcal{L}$ is supersymmetric if and only if

1. $\mathcal{M}$ is Kähler manifold, i.e. the metric satisfies

$$G_{ij} = \frac{\partial}{\partial A^i}\frac{\partial}{\partial \bar{A}^{\bar{j}}} K(A, \bar{A}) \,, \tag{13.8}$$

   where K is called Kähler potential.

2. The scalar potential V is given by

$$V = e^{\kappa^2 K}\left(D_i W G^{i\bar{j}} D_{\bar{j}}\bar{W} - 3\kappa^2 |W|^2\right) \tag{13.9}$$

   where

$$D_i W := \frac{\partial W}{\partial A^i} + \kappa^2 \left(\frac{\partial K}{\partial A^i}\right) W \tag{13.10}$$

   is a Kähler covariant derivative. Since we do not insist on renormalizability W has no longer to be cubic.

Before we proceed let us note that for Kähler manifolds the Christoffel connection and the curvature tensor enjoy special properties. The only non-vanishing Christoffel symbols are

$$\Gamma^k_{ij} = G^{k\bar{l}}\partial_i G_{\bar{l}j} \ , \qquad \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = G^{\bar{k}l}\bar{\partial}_{\bar{i}} G_{l\bar{j}} \ . \tag{13.11}$$

The only non-vanishing curvature tensor component is

$$R_{i\bar{j}k\bar{l}} = G_{m\bar{l}}\,\partial_{\bar{j}}\Gamma^m_{ik} \ . \tag{13.12}$$

Remarks:

- In the flat or global limit $\kappa^2 \to 0$, i.e., the limit of decoupling gravity, we need to distinguish two cases.
  - (i) An intermediate scale M exists with $M_Z \ll M \ll M_{\rm Pl}$.
    In this case one has

$$\begin{aligned}\lim_{\kappa\to 0} G_{i\bar{j}}(A, \bar{A}, \kappa, M) &= G^g_{i\bar{j}}(A, \bar{A}, M) \ , \\ \lim_{\kappa\to 0} V(A, \bar{A}, \kappa, M) &= \partial_i W G^{g\,i\bar{j}}\partial_{\bar{j}}\bar{W} \ ,\end{aligned} \tag{13.13}$$

    and W is not restricted to be cubic. In this case the flat limit is a non-renormalizable field theory with an intermediate scale M.
  - (ii) No intermediate scale M exists.
    In this case one has

$$\begin{aligned}\lim_{\kappa\to 0} G_{i\bar{j}}(A, \bar{A}, \kappa) &= \delta_{i\bar{j}} \ , \\ \lim_{\kappa\to 0} V(A, \bar{A}, \kappa) &= \partial_i W \delta^{i\bar{j}}\partial_{\bar{j}}\bar{W} \ ,\end{aligned} \tag{13.14}$$

    and W is cubic.

- The Lagrangian (13.7) has a Kähler invariance under which the couplings transform accordingly

$$K \to K + F(A) + F(\bar{A}) \ , \quad W \to W e^{-\kappa^2 F} \ , \tag{13.15}$$

which leave the metric $G_{i\bar{j}}$ and the potential V invariant. The fermions transform according to

$$\psi_\mu \to \psi_\mu \, e^{-\frac{i}{2}\mathrm{Im}F} \ , \qquad \chi \to \chi \, e^{\frac{i}{2}\mathrm{Im}F} \ , \qquad \lambda \to \lambda \, e^{\frac{i}{2}\mathrm{Im}F} \ . \tag{13.16}$$

This invariance is also reflected in the covariant derivatives of the fermions

$$D_\mu f = \partial_\mu f \pm A_\mu f + \dots \ , \qquad \text{where} \qquad A_\mu = \tfrac{1}{4}(K_i \partial_\mu A^i - K_{\bar{i}} \partial_\mu A^{\bar{i}}) \tag{13.17}$$

is the Kähler connection.

- Frequently K and W are combined into the combination $G = K + \ln |W|^2$ and in terms of G the potential takes the form

$$V = e^G \big(G_i G^{i\bar{j}} G_{\bar{j}} - 3 \big) \ . \tag{13.18}$$

However, the definition of G is problematic for $\langle W \rangle = 0$.

13.3 Coupling to vector multiplets – gauged supergravity

In order to couple vector multiplets the metric $G_{i\bar{j}}$ needs to have isometries which then will be gauged. The isometries are generated by $\dim(ad(G))$ Killing vectors $X^a, \bar{X}^a$ which satisfy

$$\big[X^a, X^b\big] = -f^{abc} X^c \ , \qquad \big[\bar{X}^a, \bar{X}^b\big] = -f^{abc} \bar{X}^c \ , \qquad \big[X^a, \bar{X}^b\big] = 0 \ . \tag{13.19}$$

On a complex manifold $\mathcal{M}$ they can be expanded as

$$X^a = X^{a\,i} \frac{\partial}{\partial A^i} \, , \qquad \bar{X}^a = \bar{X}^{a\,\bar{\imath}} \frac{\partial}{\partial \bar{A}^{\bar{\imath}}} \, , \tag{13.20}$$

such that the coordinates on $\mathcal{M}$, i.e. the scalar fields A^i transform as

$$\delta A^i = \alpha^a(x) X^{a\,i}(A) \, , \qquad \delta \bar{A}^{\bar{\imath}} = \alpha^a(x) \bar{X}^{a\,\bar{\imath}}(\bar{A}) \, , \qquad a = 1, \dots, \dim(ad(G)) \ , \tag{13.21}$$

where α^a is the gauge parameter. Consistency requires that the $X^{a\,i}(A)$ are holomorphic functions of the A^i and we will see shortly that this also results from the solution of the Killing equation.

Demanding $\delta_\alpha G_{i\bar{j}} = 0$ results in the Killing equations

$$\nabla_i \bar{X}^a_j + \nabla_j \bar{X}^a_i = 0 = \nabla_i X^a_{\bar{j}} + \bar{\nabla}_{\bar{j}} \bar{X}^a_i \ , \tag{13.22}$$

where

$$\bar{X}^a_j(A, \bar{A}) := G_{j\bar{\imath}}(A, \bar{A}) \, \bar{X}^{\bar{\imath}}(\bar{A}) \ , \qquad X^a_{\bar{j}}(A, \bar{A}) := G_{\bar{j}i}(A, \bar{A}) \, X^i(A) \ . \tag{13.23}$$

The first equation is solved by an (anti-)holomorphic Killing vector field $\bar{X}^{\bar{\imath}} = \bar{X}^{\bar{\imath}}(\bar{A})$. The solution of the second equation locally reads

$$X^a_{\bar{\jmath}} = G_{\bar{\jmath}l}X^{a\,l} = -i\frac{\partial}{\partial \bar{A}^{\bar{\jmath}}}D^a\,, \qquad \bar{X}^a_i = G_{i\bar{l}}\bar{X}^{\bar{l}a} = -i\frac{\partial}{\partial A^i}D^a\ , \tag{13.24}$$

where the D^a are real Killing prepotentials. They are unique up to Fayet-Iliopoulos integration constants. The relation (13.24) can also be inverted leading to

$$D^a = -i\bar{X}^{a\bar{\jmath}}\partial_{\bar{\jmath}}K = iX^{aj}\partial_j K\ . \tag{13.25}$$

Let us check the flat limit. In this case we indeed have

$$K = \delta_{i\bar{\jmath}}A^i\bar{A}^{\bar{\jmath}}\ , \qquad G_{i\bar{\jmath}} = \delta_{i\bar{\jmath}}\ , \qquad X^{a\,i} = iT^{a\,i}{}_j A^j\,, \qquad D^a = -\bar{A}T^{a\,i}{}_j A^j\,. \tag{13.26}$$

Let us now give the bosonic term for chiral and vector multiplets coupled to supergravity. They read

$$\begin{aligned} \mathcal{L} =& \tfrac{e}{2\kappa^2}\,R - \tfrac{1}{4}\,\mathrm{Re}f_{ab}(A)F^a_{\mu\nu}F^{b\,\mu\nu} + \tfrac{1}{4}\,\mathrm{Im}f_{ab}(A)F^a\widetilde{F}^b \\ & - G_{i\bar{\jmath}}D_\mu A^i D^\mu \bar{A}^{\bar{\jmath}} - V(A,\bar{A}) + \text{ fermionic terms}\,, \end{aligned} \tag{13.27}$$

where

$$\begin{aligned} D_\mu A^i &= \partial_\mu A^i - v^a_\mu X^{i\,a} \\ V &= e^{\kappa^2 K}\left[(D_i W)G^{i\bar{\jmath}}D_{\bar{\jmath}}\bar{W} - 3\kappa^2|W|^2\right] + \tfrac{1}{2}\,\mathrm{Re}f^{-1}_{ab}D^a D^b\,. \end{aligned} \tag{13.28}$$

$f_{ab}(A)$ is the holomorphic gauge kinetic function with its real part being the matrix of inverse gauge couplings and its imaginary part being the matrix of θ-angles

$$\mathrm{Re}f_{ab} = g^{-2}_{ab}\,, \qquad \mathrm{Im}f_{ab} = \frac{\theta_{ab}}{8\pi^2}\ . \tag{13.29}$$

We see that altogether $\mathcal{L}$ is determined by 3 functions $K(A,\bar{A}), W(A), f(A)$.

14 Quantum corrections in $N = 1$ supergravity

We already discussed the issue of quantum corrections in section 5.2. In supergravity the holomorphic coupling functions $W(A), f(A)$ have unchanged renormalization properties. That is, W is not renormalized in perturbation theory while f receives a perturbative correction only at one-loop. Thus

$$W = W_0 + W_{\text{np}} \ , \qquad f = f_0 + f_1 + f_{\text{np}} \ , \tag{14.1}$$

where the index 0(1) indicates tree (one-loop) level and "np" stands for non-perturbative corrections. K on the other hand is renormalized at every loop order. However, there is a caveat in that these NRT only hold for Wilsonian couplings while the physical couplings generically are corrected at any order. For example the gauge couplings obey [30]

$$g^{-2}(p) \ = \mathrm{Re} f \ + \ \tfrac{b}{16\pi^2} \log \frac{M_{Pl}^2}{p^2} \ + \ \tfrac{c}{16\pi^2} K + \ \tfrac{T(ad)}{8\pi^2} \log g^{-2}(p) \ - \sum_{\mathbf{r}} \tfrac{T(\mathbf{r})}{8\pi^2} \log \det Z^{(\mathbf{r})}(p) \ , \tag{14.2}$$

where p is the renormalization scale and $Z(p)$ the wave-function renormalization of the charged fields. The numerical coefficients in this formula are as follows:

$$\mathrm{Tr}_{\mathbf{r}}(T^a T^b) = T(\mathbf{r})\delta^{ab} \ , \quad b = \sum_r n_r T(\mathbf{r}) - 3T(ad) \ , \quad c = \sum_{\mathbf{r}} n_{\mathbf{r}} T(\mathbf{r}) - T(ad) \ . \tag{14.3}$$

where $n_{\mathbf{r}}$ is the number of massless charged matter fields which transform in the representation r and ad denotes the adjoint representation. Note that $g^{-2}(p)$ is no longer harmonic in that

$$\partial_i \bar{\partial}_{\bar{j}} g^{-2}(p) \neq 0 \ , \tag{14.4}$$

but the "failure" is entirely determined by the light modes. The right hand side of (14.4) is often called the holomorphic anomaly.

From (14.2) we also see that Kähler invariance is anomalous at one-loop which, however, can be cured by assigning the following transformation law to f

$$f \to f - \tfrac{c}{8\pi^2} F \ . \tag{14.5}$$

15 Spontaneous supersymmetry breaking in supergravity

15.1 Generalities

As in section 7 the order parameters of spontaneous supersymmetry breaking are the scalar parts of the fermionic supersymmetry transformations. In supergravity they are given by

$$\delta_\xi \chi^i \sim F^i \xi \ , \qquad \delta_\xi \lambda^a \sim g D^a \xi \ , \qquad \delta_\xi \psi_\mu \sim D_\mu \xi + i e^{\frac{1}{2}\kappa^2 K} W \sigma_\mu \xi \ , \tag{15.1}$$

where $F^i \sim e^{\frac{1}{2}\kappa^2 K} G^{i\bar{j}} \bar{D}_{\bar{j}} \bar{W}$. We see that, as before, $\langle F^i \rangle$ and $\langle D^a \rangle$ are the order parameters of supersymmetry breaking.[10]

For $\langle F^i \rangle = \langle D^a \rangle = 0$, the potential evaluated at the minimum is

$$\langle V \rangle = -3\kappa^2 \langle e^{\kappa^2 K} |W|^2 \rangle \leq 0 \ . \tag{15.2}$$

$\langle V \rangle$ plays the role of a cosmological constant and for $\langle W \rangle = \langle V \rangle = 0$ one has a Minkowski background M_4. For $\langle W \rangle \neq 0$ follows $\langle V \rangle < 0$, i.e. one has an AdS_4-background. Note that a dS-background is incompatible with unbroken supersymmetry.

For $\langle F^i \rangle \neq 0$ supersymmetry is broken. In M_4 the gravitino mass $m_{3/2}$ is given by

$$m_{3/2}^2 = \kappa^4 \langle e^{\kappa^2 K} |W|^2 \rangle \ , \tag{15.3}$$

and the F terms are related via

$$\langle F^i \bar{F}^{\bar{i}} G_{i\bar{j}} \rangle = 3\kappa^{-2} m_{3/2}^2 \ . \tag{15.4}$$

In terms of $G = K + \ln |W|^2$ the bosonic mass matrices read

$$\begin{aligned} M_{i\bar{j}}^2 &= \langle (D_i G_k \bar{D}_{\bar{j}} G^k - R_{i\bar{j}k\bar{l}} G^k G^{\bar{l}} + G_{i\bar{j}}) e^G \rangle \ , \\ M_{ij}^2 &= \langle (G^k D_i D_j G_k + D_i G_j + D_j G_i) e^G \rangle \ . \end{aligned} \tag{15.5}$$

The fermionic mass matrix reads

$$m_{ij} = \langle (D_i G_j + \tfrac{1}{3} G_i G_j) \rangle m_{3/2} \ , \tag{15.6}$$

and one can show that the Goldstone fermion is "eaten" by the gravitino.

The sum rule (3.27) is modified and reads

$$\mathrm{Str} M^2 \equiv \sum_{J=0}^{3/2} (-)^{2J} (2J+1) \mathrm{Tr}\, M_J^2 = 2(n_c - 1)\, m_{3/2}^2 - 2\langle R_{i\bar{j}} G^i G^{\bar{j}} \rangle m_{3/2}^2 \ . \tag{15.7}$$

Now it is phenomenologically viable due to the contribution of the massive gravitino.

In AdS_4 similar formulas exist but they are more complicated as the cosmological explicitly contributes.

[10] For $\langle F^i \rangle = \langle D^a \rangle = 0$ one can always find $\langle \delta_\xi \psi_\mu \rangle = 0$ which determines a Minkowski or AdS-background.

15.2 The Polonyi model

After these generalities let us come to a concrete realization of supersymmetry breaking. As in the global case the basic idea is to add a "hidden sector" which is responsible for the supersymmetry breaking. That is one adds

$$W = W_{\text{MSSM}} + W_{\text{hidden}} \tag{15.8}$$

such that

$$\langle \bar{F}^{\bar{\imath}} \rangle = \langle e^{\kappa K/2} G^{\bar{\imath}j} D_j W_{\text{hidden}} \rangle \neq 0 \ . \tag{15.9}$$

The simplest concrete W is the Polonyi model where one singlet ϕ is added with the following couplings

$$\begin{aligned} W &= M_s^2 \left(\phi + \beta\right) \ , \qquad M_s, \beta \in \mathbb{R} \ , \\ K &= \phi\bar{\phi} + K_{\text{MSSM}} \ , \quad G_{\phi\bar{\phi}} = \partial_\phi \partial_{\bar{\phi}} K = 1 \ . \end{aligned} \tag{15.10}$$

Computing

$$D_\phi W = \partial_\phi W + \kappa^2 K_\phi W = m^2 + \kappa^2 \bar{\phi} m_s^2 \left(\phi + \beta\right) \ , \tag{15.11}$$

one see that $D_\phi W = 0$ has no solution for $\kappa\beta < 2$. Minimizing V and tuning $\langle V \rangle = 0$ by choosing β appropriately one finds

$$\begin{aligned} \kappa\beta = \pm(2 - \sqrt{3}) \ , \qquad \langle \phi \rangle = \pm(\sqrt{3} - 1) \ , \\ \langle D_\phi W \rangle = \sqrt{3} m_s^2 e^{\left(2-\sqrt{3}\right)} \ , \qquad \langle W \rangle = \pm \kappa^{-1} m_s^2 \ . \end{aligned} \tag{15.12}$$

15.3 Generic gravity mediation

In this section we want to identify the effect of supersymmetry breaking in the observable (MSSM) sector. We distinguish the observable charged matter fields Q^I from neutral (hidden) scalars T^i and assume $\langle Q^I \rangle = 0$. Then we expand their Kähler potential in a power series in Q^I as

$$K \ = \ \kappa^{-2} \hat{K}(T, \bar{T}) \ + \ Z_{\bar{I}J}(T, \bar{T})\, \bar{Q}^{\bar{I}} Q^J \ + \ \left(\tfrac{1}{2} H_{IJ}(T, \bar{T})\, Q^I Q^J \ + \ \text{c.c.}\right) + \ \cdots \ , \tag{15.13}$$

where we neglect terms of order $\mathcal{O}(Q^3)$. In this notation the superpotential is given by

$$W(T, Q) = W_{\text{obs}}(T, Q) + W_{\text{hidden}}(T) \ , \tag{15.14}$$

with

$$W_{\text{obs}}(T, Q) = \tfrac{1}{2} m_{IJ}(T)\, Q^I Q^J + \tfrac{1}{3} Y_{IJL}(T)\, Q^I Q^J Q^K \ + \ \cdots \tag{15.15}$$

For $W_{\text{hidden}}(T)$ we make the following assumption:

1. some $\langle F^i \rangle \neq 0$,
2. all $\langle T^i \rangle$ fixed,
3. $\langle V \rangle = 0$,

4. $m_{3/2} \ll M_{Pl}$.

With these assumption one can compute the leading order effect in the limit $M_{\rm Pl} \to \infty$ with $m_{3/2}$ fixed. One finds that the (canonically normalized) gaugino masses are given by

$$\tilde{m} \;=\; \tfrac{1}{2} F^i \partial_i \log g^{-2} + \tfrac{1}{16\pi^2}\, b\, m_{3/2}\,, \tag{15.16}$$

where the second term is know as a contribution from <u>anomaly mediation</u> [32]. The potential reads

$$\begin{aligned} V \;&=\tfrac{1}{4} g^2 \big(\bar{Q}^{\bar{I}} Z_{\bar{I}J} T^a Q^J\big)^2 \;+\; \partial_I \hat{W} Z^{I\bar{J}} \bar{\partial}_{\bar{J}} \hat{\bar{W}} \\ &\quad+\; m^2_{I\bar{J}} Q^I \bar{Q}^{\bar{J}} \;+\; \big(\tfrac{1}{3} A_{IJL} Q^I Q^J Q^L \;+\; \tfrac{1}{2} B_{IJ} Q^I Q^J \;+\; \text{c.c.}\big)\,. \end{aligned} \tag{15.17}$$

The first line is the scalar potential of an effective theory with unbroken rigid supersymmetry while the second line is comprised of the soft supersymmetry-breaking terms. $\hat{W}$ is given by

$$\hat{W}(Q) \;=\; \tfrac{1}{2} \hat{m}_{IJ} Q^I Q^J \;+\; \tfrac{1}{3} \hat{Y}_{IJL} Q^I Q^J Q^L, \tag{15.18}$$

where

$$\begin{aligned} \hat{m}_{IJ} &:=\; e^{\hat{K}/2} m_{IJ} \;+ m_{3/2} H_{IJ} \;-\; \bar{F}^{\bar{j}} \bar{\partial}_{\bar{j}} H_{IJ}\,, \\ \hat{Y}_{IJL} &:=\; e^{\hat{K}/2}\, Y_{IJL}\,. \end{aligned} \tag{15.19}$$

The coefficients of the soft terms in the second line of (15.17) are as follows:

$$\begin{aligned} m^2_{I\bar{J}} \;&=\; m^2_{3/2} Z_{I\bar{J}} \;-\; F^i \bar{F}^{\bar{j}} R_{i\bar{j}I\bar{J}}\,, \\ A_{IJL} \;&=\; F^i D_i \hat{Y}_{IJL}\,, \\ B_{IJ} \;&=\; F^i D_i \hat{m}_{IJ} \;-\; m_{3/2} \hat{m}_{IJ}\,, \end{aligned} \tag{15.20}$$

where

$$\begin{aligned} R_{i\bar{j}I\bar{J}} \;&=\; \partial_i \bar{\partial}_{\bar{j}} Z_{I\bar{J}} \;-\; \Gamma^N_{iI} Z_{N\bar{L}} \bar{\Gamma}^{\bar{L}}_{\bar{j}\bar{J}}\,, \qquad \Gamma^N_{iI} \;=\; Z^{N\bar{J}} \partial_i Z_{I\bar{J}}\,, \\ D_i \hat{Y}_{IJL} \;&=\; \partial_i \hat{Y}_{IJL} \;+\; \tfrac{1}{2} \hat{K}_i \hat{Y}_{IJL} \;-\; \Gamma^N_{i(I} \hat{Y}_{JL)N}\,, \\ D_i \hat{m}_{IJ} \;&=\; \partial_i \hat{m}_{IJ} \;+\; \tfrac{1}{2} \hat{K}_i\, \hat{m}_{IJ} \;-\; \Gamma^N_{i(I}\, \hat{m}_{J)N}\,. \end{aligned} \tag{15.21}$$

Notice that all quantities appearing in eqs. (15.16), (15.19) and (15.20) are covariant with respect to the supersymmetric reparametrization of matter and moduli fields as well as covariant under Kähler transformations.

According to eq. (15.20), $m^2_{\bar{I}J} \sim m^2_{3/2}$, $A_{IJL} \sim m_{3/2} \hat{Y}_{IJL}$, and $B_{IJ} \sim m_{3/2} \hat{m}_{IJ}$; nevertheless, the soft terms are generally *not universal*, i.e. $A_{IJL} \neq \text{const} \cdot m_{3/2} \hat{Y}_{IJL}$ and $m^2_{I\bar{J}} \neq \text{const} \cdot m^2_{3/2} Z_{I\bar{J}}$, even at the tree level. In the context of the MSSM, this non-universality means that the absence of flavor-changing neutral currents is not an automatic feature of supergravity but a non-trivial constraint that has to be satisfied by a fully realistic theory.

Phenomenological viability of the MSSM imposes yet another requirement: The supersymmetric mass term μ for the two Higgs doublets should be comparable in magnitude with the non-supersymmetric mass terms. Equation (15.19) displays m_{IJ} and H_{IJ} as two independent sources of $\hat{m}_{IJ}$. The contribution of a non-vanishing H_{IJ} to $\hat{m}$ is automatically of order $m_{3/2}$, without any fine-tuning. This fact is known as the Giudice-Masiero mechanism [33].

16 Gauge mediation & gaugino condensation

In the previous lecture we discussed spontaneous breaking of supergravity and its effects in the observable sector. In the Polonyi model and its generalization the breaking is communicated by gravitational and Planck-sized scalar interactions. One of the problems is that generically the resulting soft terms are not flavor blind and induce dangerously large FCNC. As an alternative mediation by gauge interactions has been proposed.

16.1 Gauge mediation

The basic idea of gauge mediation is to use the gauge interactions as the messenger of supersymmetry breaking. The flavor blindness of the gauge interactions then automatically ensures the smallness of FCNC.

The "prototype model" contains a singlet N (or an O'Raifeartaigh sector) which is responsible for supersymmetry breaking with a non-vanishing $\langle F_N \rangle \neq 0$. In addition there are SM-charged heavy messengers M_L, M_R that mediate supersymmetry breaking to the MSSM. (One often assumes $M_L \sim \mathbf{5}$, $M_R \sim \bar{\mathbf{5}}$ of $SU(5)$.) The superpotential reads

$$W = W_{\text{obs}} + \lambda N M_L M_R + W_{\text{hid}}(N) \ , \tag{16.1}$$

and one imposes the following additional assumptions:

1. $W_{\text{hid}}(N)$ is such that $\langle F_N \rangle \neq 0$.
2. λ is such that $\lambda\langle N \rangle \gg M_Z$ and thus the $M_{L,R}$ are heavy.
3. $\langle F_N \rangle < |\lambda\langle N \rangle|^2$ which ensures the stability of the potential. This can be seen by computing the eigenvalues of the mass matrix of the scalars which reads

$$\left(\bar{M}_L,\, M_R\right) \begin{pmatrix} |\lambda\langle N \rangle|^2 & \lambda\langle F_N \rangle \\ \lambda\langle F_N \rangle & |\lambda\langle N \rangle|^2 \end{pmatrix} \begin{pmatrix} M_L \\ \bar{M}_R \end{pmatrix} \ . \tag{16.2}$$

Supersymmetry breaking is mediated to observable sector solely via loops which can be found, for example, in [1]. In the limit $\langle F \rangle \ll \lambda\langle N \rangle$ one finds the gaugino masses are generated at one-loop and given by

$$\tilde{m}_a = \frac{\alpha_a}{4\pi}\frac{\langle F_N \rangle}{\langle N \rangle} \ . \tag{16.3}$$

The scalar masses are generated at two-loops and read

$$m^2 = 2\left(\frac{\langle F_N \rangle}{\langle N \rangle}\right)^2 \sum_a C^a \left(\frac{\alpha_a}{4\pi}\right)^2 \tag{16.4}$$

Remarks:

- m^2 is automatically flavor blind (universal).
- To have $m^2 > \mathcal{O}(100\,\text{GeV} - 1\,\text{TeV})$ one needs $\Lambda \equiv \frac{F_N}{\langle N \rangle} > \mathcal{O}(30\,\text{TeV})$

- In these scenarios the gravitino is very light and the LSP. For example, for $\langle F_N \rangle = \mathcal{O}(10^5\,\text{GeV})^2$ one has $m_{3/2} \simeq \frac{\langle F_N \rangle^2}{M_{\text{Pl}}} = \mathcal{O}(10^{-9}\,\text{GeV}) = \mathcal{O}(1\,\text{eV})$. It is then important to determine the next-to-lightest-supersymmetric-particle (NLSP) as its decay can provide characteristic signatures.

- If μ is forbidden by some symmetry in W_{obs} it can be generated at $\mathcal{O}(m_{3/2})$ by the Guidice-Masiero mechanism in gravity mediated scenarios as discussed in the previous lecture. In gauge mediated scenarios μ can be generated by a loop effect and arranged to be of order $\mathcal{O}(m)$. However, at the same time $B_\mu \sim \mu\Lambda$ then holds which via (9.17) implies a severe fine-tuning. This is called the B_μ/μ-problem.

- A similar problem occurs for the A-terms which is called the A/m_H problem. The A-term is generated by a loop effect and thus generically $\mathcal{O}(m)$ or below. However, the observed values of the Higgs suggests large A-terms as discussed in lecture 9.

16.2 Gaugino Condensation

Let us now turn to the issue of how the hierarchy $m_{3/2} \ll M_{\text{Pl}}$ can be generated. The prime (and simplest) example is gaugino condensation where the hidden sector is assumed to be an asymptotically free non-Abelian gauge theory which is weakly coupled at M_{Pl}. It becomes strong at the condensation scale Λ_c defined by

$$g^{-2}(\Lambda_c) - \tfrac{T(ad)}{8\pi^2} \log g^{-2}(\Lambda_c) = 0\,. \tag{16.5}$$

Via (14.2) this implies

$$|\Lambda_c| = p\, e^{\frac{8\pi^2}{bg^2(p)}} g^{-\frac{2}{3}}(p) = M_{\text{Pl}} e^{\frac{8\pi^2}{b}\text{Re}f} e^{\frac{1}{6}\kappa^2 K} \ll M_{\text{Pl}}\,. \tag{16.6}$$

Note that Λ_c is an RG-invariant scale as it satisfies $\frac{d\Lambda_c}{dp} = 0$. At that scale the gauginos of the hidden sector condense and one estimates for the condensate

$$|\langle \lambda^\alpha \lambda_\alpha \rangle| = e^{\frac{1}{2}\kappa^2 K} |\langle W^\alpha W_\alpha \rangle| \sim e^{\frac{1}{2}\kappa^2 K} |\Lambda_c^3|\,. \tag{16.7}$$

The phase of $|\langle W^\alpha W_\alpha \rangle|$ can be determined from the transformation law (13.16) and (14.5). Altogether we then find

$$\langle W^\alpha W_\alpha \rangle \sim \Lambda_c^3 = M_{\text{Pl}}^3 e^{\frac{24\pi^2}{b} f}\,. \tag{16.8}$$

This value for the condensate can be obtained from the Veneziano-Yankielowicz superpotential

$$W = \tfrac{1}{4} U f + \tfrac{1}{32\pi^2} \left(T(ad) \ln \frac{U}{M_{\text{Pl}}} + \text{const.} \right)\,, \tag{16.9}$$

where $U = W^\alpha W_\alpha$. The supersymmetric minimum (which satisfies $\frac{\partial W}{\partial U} = 0$) is found at $U \sim \Lambda_c^3$. Inserted back into (16.9) yields

$$W \sim \Lambda_c^3 = M_{\text{Pl}}^3 e^{\frac{24\pi^2}{b} f}\,. \tag{16.10}$$

The problem is that for field-independent f supersymmetry is intact and only a field-dependent $f(A)$ can possibly break supersymmetry. We will return to this issue later on.

17 N-extended Supersymmetries

Now we consider the general case of N supercharges $Q^I_\alpha,\ I=1,\ldots,N$. In this case the superalgebra reads

$$\{Q^I_\alpha,\overline{Q}^J_{\dot\beta}\}=2\sigma^\mu_{\alpha\dot\beta}\,P_\mu\delta^{IJ}\,,\quad \{Q^I_\alpha,Q^J_\beta\}=\epsilon_{\alpha\beta}Z^{IJ}\,,\quad \{\overline{Q}_{\dot\alpha},\overline{Q}_{\dot\beta}\}=\epsilon_{\dot\alpha\dot\beta}\bar{Z}^{IJ}\,, \tag{17.1}$$

where the commutations relations for each of the Q^I with the generators of the Poincare group $L_{\mu\nu},P_\mu$ are as in (1.16). The Jacobi-identity requires that Z commutes with all generators $[Z,Q]=[Z,P]=[Z,L]=0$ and thus these are (Lorentz-invariant) central charges of the algebra.

Furthermore (17.1) is left invariant by an $U(N)$ automorphism which rotates the charges according to

$$\begin{aligned} Q^I\to Q'^I=Q^J\,U_J{}^I\,,\qquad \overline{Q}^I\to\overline{Q}'^I=U^{\dagger I}{}_J\,\overline{Q}^{\ J}\,,\\ Z^{IJ}\to Z'^{IJ}=U^I{}_K\,Z^{KL}\,U_L{}^J\,,\end{aligned} \tag{17.2}$$

where $UU^\dagger=\mathbb{1}$. One can use this freedom to bring Z into "normal-form", i.e., into 2×2 antisymmetric block-matrices leaving $N/2$ physical real central charges:[11]

$$Z^{IJ}=\begin{pmatrix} 0 & -Z_1 & & & \\ Z_1 & 0 & & & \\ & & 0 & -Z_2 & \\ & & Z_2 & 0 & \\ & & & & \ddots \end{pmatrix}\,. \tag{17.3}$$

17.1 Representations of extended supersymmetry

Let discuss the representations for $N=2$ in slightly more detail and then just give the result for other values of N. The construction is completely analogous to the construction given in lecture 2.

For massive representation with $P_\mu=(-m,0,0,0)$, the $N=2$ superalgebra becomes

$$\{Q^I_\alpha,\overline{Q}^J_{\dot\beta}\}=2m\delta_{\alpha\dot\beta}\delta^{IJ}\,,\qquad \{Q^I_\alpha,Q^J_\beta\}=2\epsilon_{\alpha\beta}\epsilon^{IJ}Z\,,\qquad \{\overline{Q}^I_{\dot\alpha},\overline{Q}^J_{\dot\beta}\}=2\epsilon_{\dot\alpha\dot\beta}\epsilon^{IJ}\bar{Z}\,. \tag{17.4}$$

One defines

$$a_\alpha=\frac{1}{\sqrt{2}}(Q^1_\alpha+\epsilon_{\alpha\beta}(Q^2_\beta)^\dagger)\,,\qquad b_\alpha=\frac{1}{\sqrt{2}}(Q^1_\alpha-\epsilon_{\alpha\beta}(Q^2_\beta)^\dagger)\,, \tag{17.5}$$

and obtains from the non-trivial commutators (17.4)

$$\{a_\alpha,a^\dagger_\beta\}=2\delta_{\alpha\beta}(m+Z)\,,\qquad \{b_\alpha,b^\dagger_\beta\}=2\delta_{\alpha\beta}(m-Z)\,, \tag{17.6}$$

with all others vanishing. Positivity of the quantum mechanical Hilbert space requires

$$m\geq Z\,. \tag{17.7}$$

[11] For N odd there is a single zero in the bottom right corner.

This constraint is known as the Bogolmoni-Prasad-Sommerfield (BPS) bound. From (17.6) we see that for $m > Z$ there are $4\,(= 2N)$ fermionic creation operators $a_\alpha^\dagger, b_\beta^\dagger$. Let us combine these four operators as $A^\dagger := (a_\alpha^\dagger, b_\beta^\dagger)$ and construct the representations by acting with $A^\dagger$ on the spin-s Clifford vacuum $|s\rangle$ which is annihilated by A, i.e.

$$A|s\rangle = 0\ . \tag{17.8}$$

For $|0\rangle$ the states with there multiplicities are:

states	multiplicity
$\lvert 0\rangle$	1
$A\lvert 0\rangle$	$\binom{4}{1} = 4$
$AA\lvert 0\rangle$	$\binom{4}{2} = 6$
$AAA\lvert 0\rangle$	$\binom{4}{3} = 4$
$AAAA\lvert 0\rangle$	$\binom{4}{4} = 1$
	16

Computing the associated spins one finds that this multiplet is a massive vector multiplet with the $1 \times [s = 1] + 4 \times [s = \frac{1}{2}] + 5 \times [s = 0]$. Thus it has $n_{\mathrm{B}} = n_{\mathrm{F}} = 8$.

For general N and vacuum $|s\rangle$ the total number of states of a multiplet is given by

$$n = (2s+1) \sum_{k=0}^{2N} \binom{2N}{k} = 2^{2N}(2s+1)\,, \tag{17.9}$$

where $(2s+1)$ is the multiplicity of $|s\rangle$. The number of bosonic and fermionic states therefore is

$$n_{\mathrm{B}} = 2^{2N-1}(2s+1) = n_{\mathrm{F}}\,. \tag{17.10}$$

The different spins occurring in the multiplet are $(s + \frac{N}{2}, \ldots, s - \frac{N}{2})$.

Let us now turn to the situation where the mass m saturates the BPS bound in (17.6), i.e., $m = Z$. In this case the N fermionic creation operators $b_\alpha^\dagger$ decouple and we are left only with the $a_\alpha^\dagger$ or in other words with an "$N/2$ situation". The number of states in a multiplet is only half, i.e., $n = 2^N(2s+1)$. Also in this situation there is a massive vector multiplet which is constructed from $|\frac{1}{2}\rangle$. It contains the states

$$|\tfrac{1}{2}\rangle\ , \quad a^\dagger|\tfrac{1}{2}\rangle\ , \quad a^\dagger a^\dagger|\tfrac{1}{2}\rangle\ , \tag{17.11}$$

corresponding to $[s = 1] + 2 \times [s = \frac{1}{2}] + [s = 0]$ and has $n_{\mathrm{B}} = n_{\mathrm{F}} = 4$. Thus in $N = 2$ supersymmetry there are two distinct vector multiplets. A "short" BPS multiplet with a total of 8 states and a "long" non-BPS multiplet with a total of 16 states.

For $N > 2$ there can be $N/2$ distinct central charges $Z_i, i = 1, \ldots, N/2$ and the multiplet structure depends on how many BPS bounds are saturated. In the generic case one has $m > Z_i, \forall\, i$. Then one can have the situation that $r < N/2$ BPS bounds are saturated, i.e., $m = Z_i, \forall\, i = 1, \ldots, r$. Finally, all BPS charges might be saturated $m = Z_i, \forall\, i$. The importance of the BPS bound comes the fact that it only depends on the algebra and therefore is expected to hold after including quantum corrections.

Finally let us turn to massless representations where again a light-like frame, $P_\mu = (-E, 0, 0, E)$, is chosen. The superalgebra becomes

$$\begin{aligned} \{Q^I_\alpha, \overline{Q}^J_{\dot\beta}\} &= 2E(-\sigma^0 + \sigma^1)_{\alpha\dot\beta}\, \delta^{IJ} = 2E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \delta^{IJ} , \\ \{Q^I_\alpha, Q^J_\beta\} &= 0 = \{\overline{Q}^I_{\dot\alpha}, \overline{Q}^J_{\dot\beta}\} \end{aligned} \tag{17.12}$$

Thus we have the same situation as in the BPS case with all charges saturated, namely N fermionic creation operators $(Q^J_1)^\dagger$. For multiplets which are in accord with the CPT theorem we thus have for the number of states in a multiplet

$$n = 2^N \times \begin{cases} 1 & \text{if the multiplet is CPT complete} \\ 2 & \text{if the CPT conjugate has to be added} \end{cases} \tag{17.13}$$

The massless multiplets for $N = 2, 4, 8$ are given in Tables 17.1,17.1,17.1, respectively. We see that for $N \geq 4$ no matter multiplets exists and for $N = 8$ there is a unique

λ	$\lvert -\frac{1}{2}\rangle$	$\lvert 0\rangle$	$\lvert -1\rangle$	$\lvert \frac{1}{2}\rangle$	$\lvert -\frac{3}{2}\rangle$	$\lvert 1\rangle$	$\lvert -2\rangle$
0	2	1	1				
$\pm\frac{1}{2}$	1+1	2	2	1	1		
± 1		1	1	2	2	1	1
$\pm\frac{3}{2}$				1	1	2	2
± 2						1	1
	half-hyper-multiplet (CPT compl.)	vector multiplet		gravitino multiplet		graviton multiplet	

Table 17.1: Massless multiplets for $N = 2$

λ	$\lvert -1\rangle$	$\lvert -\frac{1}{2}\rangle$	$\lvert -\frac{3}{2}\rangle$	$\lvert 0\rangle$	$\lvert -2\rangle$
0	6	4	4		
$\pm\frac{1}{2}$	4+4	6+1	1+6	4	4
± 1	1+1	4	4	6	6
$\pm\frac{3}{2}$		1	1	4	4
± 2				1	1
	vector multiplet (CPT compl.)	gravitino multiplet		graviton multiplet	

Table 17.2: Massless multiplets for $N = 4$

massless multiplet incorporating all helicities $\lambda = 0, \ldots, \pm 2$. For $N > 8$ one necessarily has states with $|\lambda| > 2$ in the spectrum which is believed to be inconsistent in a Minkowski background. Therefore one confines the attention to $N \leq 8$.

λ	$\vert -2\rangle$
0	70
$\pm\frac{1}{2}$	56
± 1	28
$\pm\frac{3}{2}$	8
± 2	1

Table 17.3: Massless multiplet for $N = 8$

17.2 The $N = 4$ action for massless vector multiplets

From Table 17.1 we see that an $N = 4$ vector multiplet contains a vector v_μ and six real scalars $\phi^i, i = 1, \dots, 6$ as bosonic components. Their Lagrangian is given by

$$\mathcal{L} = -\,\tfrac{1}{4g^2} F^a_{\mu\nu} F^{b\mu\nu} - D_\mu \phi^{ia} D^\mu \phi^{ia} - V(\phi)\ , \qquad (17.14)$$

where $a = 1, \dots, n_v$ and the potential takes the form

$$V \sim \sum_{ij} \mathrm{Tr}\,[\phi^i, \phi^j]^2\ , \quad \text{for} \quad \phi^i \equiv \phi^{ia} T^a\ . \qquad (17.15)$$

Remarks:

- The σ-model metric for the scalars is flat and the gauge kinetic function is constant.
- It is a conformal theory with a vanishing β function to all orders.
- It is a finite theory.

18 Seiberg-Witten theory

18.1 The $N = 2$ action for massless vector multiplets

The massless vector multiplet consists in one vector v_μ, two fermions λ^I_α $(I = 1, 2)$ and a complex scalars z. For n_v vector multiplets we use the notation $\left(v^a_\mu, \lambda^{aI}_\alpha, z^a\right)$ with $a = 1, \ldots, n_v$. All members of the multiplet transform in the adjoint representation of some gauge group G. In terms of $N = 1$ multiplets, we have the decomposition:

$$\left(v^a_\mu, \lambda^{aI}_\alpha, z^a\right) \to \left(v^a_\mu, \lambda^{a1}_\mu\right) \oplus \left(\lambda^{a2}_\alpha, z^a\right) \ , \tag{18.1}$$

where the first multiplet is the $N = 1$ vector multiplet while the second is a $N = 1$ chiral multiplet. The bosonic Lagrangian is

$$\begin{aligned}\mathcal{L} = &- (\mathrm{Im}F)_{ab}(z, \bar{z})\, F^a_{\mu\nu} F^{b\mu\nu} - \tfrac{1}{2}(\mathrm{Re}F)_{ab}(z, \bar{z})\, F^a_{\mu\nu} F^b_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \\ &- G_{a\bar{b}}(z, \bar{z}) D_\mu z^a D^\mu \bar{z}^b - V(z, \bar{z}) \ ,\end{aligned} \tag{18.2}$$

where due to supersymmetry the couplings are now interrelated. In particular the gauge kinetic function F_{ab} and the σ-model metric $G_{a\bar{b}}$ are both expressed in terms of one holomorphic prepotential $F(z)$.[12] Concretely, $G_{a\bar{b}}$ is again Kähler but with a specific Kähler potential

$$K = i\left(\bar{F}_a z^a - F_a \bar{z}^a\right) \ , \qquad F_a = \partial_a F(z) \tag{18.3}$$

such that

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K = 2\mathrm{Im}F_{ab} \ . \tag{18.4}$$

Manifolds with this property have been termed <u>rigid special Kähler manifolds</u>.

The gauge kinetic functions are also determined by the second derivative of F according to

$$F_{ab} = \partial_a \partial_b F(z) \ . \tag{18.5}$$

Note that the physical requirement of properly propagating fields imposes $\mathrm{Im}\left(F_{ab}\right) > 0$.

The scalars z^a transform in the adjoint representation of G and thus the covariant derivatives are given by

$$D_\mu z^a = \partial_\mu z^a - v^b_\mu X^{ab}(z) \ . \tag{18.6}$$

As in $N = 1$ the Killing vectors X^{ab} are expressed in terms of Killing prepotentials P^a according to

$$X^a_{\bar{b}}(z, \bar{z}) = G_{\bar{b}c} X^{ca}(z) = -i\partial_{\bar{b}} P^a(z, \bar{z}) \ , \qquad \bar{X}^a_b(z, \bar{z}) = G_{b\bar{c}} \bar{X}^{\bar{c}a}(\bar{z}) = i\partial_b P^a(z, \bar{z}) \ . \tag{18.7}$$

In $N = 2$ no superpotential is possible and the potential is entirely determined by the Killing vectors

$$V = G_{a\bar{b}} X^a_{\bar{c}} \bar{X}^{\bar{b}}_d \bar{z}^{\bar{c}} z^d \ . \tag{18.8}$$

For renormalizable theories one finds

$$F = \tfrac{i}{4} z^a z^a \ , \qquad K = \delta_{a\bar{b}} z^a \bar{z}^{\bar{b}} \ , \qquad G_{a\bar{b}} = \delta_{a\bar{b}} \ , \qquad X^{ab} = i f^{abc} z^c \ . \tag{18.9}$$

[12]Note that compared to $N = 1$ the notation changed as the gauge kinetic function is now called F_{ab} and the role of real and imaginary part have been interchanged.

Thus the potential is quartic and reads

$$V \;\sim\; \delta_{a\bar{d}} f^{abc} f^{def} z^b \bar{z}^c z^e \bar{z}^f \;\sim\; \mathrm{Tr}\,[z, \bar{z}]^2 \geq 0\ , \tag{18.10}$$

where in the second step we defined $z = z^a T^a$, $\bar{z} = \bar{z}^a T^a$.

Due to the semi-positivity of V its minimum is at $\langle V \rangle = 0$, which occurs at the origin of field space $\langle z^a \rangle = 0$. In fact, there is a moduli space of solutions spanned by the directions which point along the Cartan subalgebra of G. For $z = z^{\hat{a}} T^{\hat{a}}$ where $T^{\hat{a}}$ are the generators of the Cartan subalgebra which obey $[T^a, T^{\hat{a}}] = 0$, the potential remains zero but the gauge group is broken $G \to U(1)^r$. This moduli space is called the Coulomb branch of the theory.

As an example consider $G = SU(2)$. For $\langle z^{1,2,3} \rangle = 0$ the $SU(2)$ is unbroken while for $\langle z^3 \rangle \neq 0$ one has $SU(2) \to U(1)$. The $W^{\pm}$ gauge bosons gain a BPS-mass $m_{W^{\pm}} \sim \langle z^3 \rangle$.

18.2 Quantum Corrections

The one loop corrections to the gauge coupling reads

$$g^{-2}(\mu) = g_0^{-2}(\Lambda_{UV}) + \frac{b}{8\pi^2} \ln \frac{\Lambda_{UV}}{\mu}\ , \tag{18.11}$$

where g_0 is the bare coupling defined at some UV-scale Λ_{UV}. If G is asymptotically free there also is an IR-scale Λ_{IR} where the gauge coupling becomes infinite. For $SU(2)$ and $\langle z^3 \rangle > \Lambda_{IR}$ the logarithmic running stops and the gauge coupling stays constant below $\langle z^3 \rangle$. For large $\langle z^3 \rangle$ one has a classical $U(1)$ theory at all scales while for small $\langle z^3 \rangle$ classically a gauge enhancement to $SU(2)$ occurs. The question Seiberg and Witten addressed is to what extent this perturbative picture holds non-perturbatively [34].[13]

Concretely they determined the prepotential F exactly, i.e. including all non-perturbative corrections. The generic form of F was known to be of the form [35]

$$F(a) = \tfrac{1}{2}\tau_0 a^2 + \tfrac{i}{\pi}\, a^2 \ln \frac{a^2}{\Lambda_{UV}^2} + \frac{a}{2\pi i} \sum_{l=1}^{\infty} c_l \left(\frac{\Lambda_{UV}}{a}\right)^{4l}\ , \tag{18.12}$$

where for simplicity one denotes $a = z^3$. The first two terms are perturbative but due to the non-renormalization theorem there is no further perturbative correction and only a sum of non-perturbative contributions. Note that the perturbative part of F is not single valued due to the $\ln a^2$ term.

The holomorphic gauge coupling is defined as

$$\tau(a) = \tfrac{1}{\pi}\theta(a) + 8\pi i g^{-2}(a) = \frac{\partial^2 F}{\partial a^2} = \tau_0 + \tfrac{2i}{\pi} \ln \frac{a^2}{\Lambda^2} + \ldots\ . \tag{18.13}$$

Since $\mathrm{Im}\tau$ also determines the σ-model metric we need to have $\mathrm{Im}\tau > 0$. However since $\mathrm{Im}\tau$ is harmonic it can no minimum unless it is constant and thus turns negative somewhere on the moduli space. This in turn implies that $\tau(a)$ is only locally and for large a well defined but the global description of the moduli space should be different.

[13] For a review of Seiberg-Witten theory see [36].

On the other hand the physics properties of a theory should not depend on a specific parameterization.

The resolution of this apparent paradox is that only the equation of motion have to be well defined while the action might not be. For the case at hand it turns out that for small a the theory is better described in terms of a dual gauge theory. Let us therefore pause and discuss the electric-magnetic duality.

18.3 Electric-magnetic duality

For a $U(1)$ gauge theory the e.o.m. and the Bianchi identity reads

$$\partial^\mu F_{\mu\nu} = 0 \ , \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \ . \tag{18.14}$$

In terms of the dual field strength $\tilde{F}^{\mu\nu} = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ one has

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu \tilde{F}_{\rho\sigma} = 0 \ , \qquad \partial^\mu \tilde{F}_{\mu\nu} = 0 \ , \tag{18.15}$$

i.e. e.o.m. and B.I. are interchanged. For field dependent gauge couplings one has

$$\partial^\mu(g^{-2}(a)F_{\mu\nu} + \tfrac{i}{8\pi^2}\theta(a)\tilde{F}_{\mu\nu}) = 0 \ , \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \ . \tag{18.16}$$

It is convenient to define the self-dual and anti self-dual combinations

$$F^\pm_{\mu\nu} := \tfrac{1}{2}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}) \ , \tag{18.17}$$

and

$$G^-_{\mu\nu} = \tau F^-_{\mu\nu} \ , \qquad G^+_{\mu\nu} = \bar{\tau} F^+_{\mu\nu} \ . \tag{18.18}$$

In terms of these quantities e.o.m. and B.I. read

$$\partial^\mu \mathrm{Im} F^-_{\mu\nu} = 0 \ , \qquad \partial^\mu \mathrm{Im} G^-_{\mu\nu} = 0 \ . \tag{18.19}$$

In terms of these quantities the electromagnetic duality can be expressed as a $SL(2,\mathbb{R})$ transformation

$$\begin{pmatrix} G^-_{\mu\nu} \\ F^-_{\mu\nu} \end{pmatrix} \to \begin{pmatrix} G'^-_{\mu\nu} \\ F'^-_{\mu\nu} \end{pmatrix} = S \begin{pmatrix} G^-_{\mu\nu} \\ F^-_{\mu\nu} \end{pmatrix} \ , \qquad \tau \to \tau' = \frac{a\tau + b}{c\tau + d} \ , \tag{18.20}$$

where

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \qquad ad - bc = 1 \ , \quad a,b,c,d \in \mathbb{R} \ . \tag{18.21}$$

At the same time one needs to transform

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} a'_D \\ a' \end{pmatrix} = S \begin{pmatrix} a_D \\ a \end{pmatrix} \ , \qquad \text{where} \qquad a_D := \frac{\partial F}{\partial a} \ . \tag{18.22}$$

18.4 The Seiberg-Witten solution

For the case at hand we have for the perturbative terms

$$\begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} \frac{2i}{\pi}\sqrt{u}\ln\frac{u}{\Lambda^2} \\ \sqrt{u} \end{pmatrix} , \tag{18.23}$$

where $u = a^2$. The transformation $u \to u' = e^{2\pi i}u$ induces

$$\begin{pmatrix} a'_D \\ a' \end{pmatrix} = M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix} , \qquad \text{where} \qquad M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} . \tag{18.24}$$

Seiberg and Witten suggested that a global description of the moduli space exists with two singularities at $u = \pm\Lambda^2$ where magnetically charged states (a monopole and a dyon) become massless and a perturbative description in terms of the dual gauge theory exits.[14] Consistency requires that the mondromy matrices M obey

$$M_{+\Lambda^2} M_{-\Lambda^2} = M_\infty . \tag{18.25}$$

For a dyon of magnetic charge g and electric charge q the monodromy matrix is

$$M^{(g,q)} = \begin{pmatrix} 1+qg & q^2 \\ -g^2 & 1-gq \end{pmatrix} . \tag{18.26}$$

One can check that (18.25) is satisfied for a monopole of charge $(1, 0)$ and a dyon of charge $(1, -2)$.

The next step is find $a(u), a_D(u)$ that display the required monodomies. This is a version of the Riemann-Hilbert problem and there are two basic strategies:

1. determine $a(u), a_D(u)$ as a solution of a singular so called Picard-Fuchs differential equation.
2. express $a(u), a_D(u)$ as period integrals of an auxiliary spectral surface.

Seiberg and Witten chose the second route and due to the $SL(2)$ considered a torus as the auxiliary spectral surface. One finds

$$\tau(u) = \frac{\omega_D}{\omega} , \qquad \omega_D = \frac{\partial a_D}{\partial u} = \oint_\beta \omega , \qquad \omega = \frac{\partial a}{\partial u} = \oint_\alpha \omega , \tag{18.27}$$

where ω is a certain one-form on the torus and (α, β) are the two cycles of the torus. Further discussions about the solution are beyond the scope of these lecture and we refer to the literature [34, 36].

[14] The Seiberg-Witten proposal was later on verfied by explicit instanton computations. See [36] for a list of references.

19 $N = 1$ SQCD and Seiberg-Duality

19.1 Preliminaries

In this section we consider an $N = 1$ supersymmetric version of QCD or more precisely a supersymmetric gauge theory with gauge group $G = SU(N_c)$ and N_f quark flavours denoted by $Q^I, \tilde{Q}^I, I = 1, \ldots, N_f$. The Q^I transforms in the fundamental $\mathbf{N_c}$ of $SU(N_c)$ while $\tilde{Q}^I$ transforms in the anti-fundamental $\bar{\mathbf{N}}_\mathbf{c}$ of $SU(N_c)$ [12, 37, 38]. (The gauge indices are suppressed throughout.) For $K = Q^I \bar{Q}^I + \tilde{Q}^I \bar{\tilde{Q}}^I$ and $W = 0$ the theory has the flavour symmetry $G_f = SU(N_f)_L \times SU(N_f)_R \times U(1)_A \times U(1)_B \times U(1)_R$ with the following charges

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_B$	$U(1)_R$
Q	$\mathbf{N_f}$	1	1	1	$R(Q)$
$\tilde{Q}$	1	$\bar{\mathbf{N}}_\mathbf{f}$	1	1	$R(\tilde{Q})$

(19.1)

The $U(1)_A$ has an anomaly $\sim N_f(q_Q + q_{\tilde{Q}}) = 2N_f$ while the $U(1)_B$ is anomaly free. For arbitrary R the $U(1)_R$ has an anomaly $\sim N_f(R(Q)+R(\tilde{Q})-2)-N_c$ since the N_c gauginos carry R-charge $R(\lambda) = -1$ and $R(\chi_Q) = R(Q) - 1$. The $U(1)_R$ can be chosen anomaly free by assigning the charges

$$R(Q) = R(\tilde{Q}) = 1 - \frac{N_c}{N_f} \ . \tag{19.2}$$

The goal is to learn something about this theory at low energies when the coupling is strong.

19.2 $0 \leq N_f < N_c$

First of all recall that in section 16.2 we saw that for $N_f = 0$ gaugino condensation $\langle \lambda\lambda \rangle \neq 0$ led to the non-perturbative $W_{\text{eff}} = c\Lambda^3$. For $N_f > 0$ also the squarks can condense and form "mesons" $\langle Q^I \cdot \tilde{Q}^J \rangle$. W_{eff} has to be a singlet of the flavour symmetry and should carry R-charge $R(W) = -2$. Since $\langle Q^I \cdot \tilde{Q}^J \rangle$ transforms in the $(\mathbf{N_f}, \bar{\mathbf{N}}_\mathbf{f})$ of $SU(N_f)_L \times SU(N_f)_R$ the superpotential has to be of the form

$$W = c\Lambda^3 \left(\frac{\det(Q\tilde{Q})}{\Lambda^{2N_f}} \right)^{\alpha} \ . \tag{19.3}$$

The requirement $R(W) = -2$ then determines $\alpha = (N_f - N_c)^{-1}$ and thus

$$W = c \left(\frac{\Lambda^{3N_c - N_f}}{\det(Q\tilde{Q})} \right)^{\frac{1}{(N_c - N_f)}} \ . \tag{19.4}$$

As we can see the theory has no stable vacuum.

19.3 $N_f \geq N_c$

In this case also gauge invariant baryons exist which are of the form

$$B^{[I_1 \dots I_{N_c}]} = Q^{I_1} \cdots Q^{I_{N_c}} \ , \qquad \tilde{B}^{[I_1 \dots I_{N_c}]} = \tilde{Q}^{I_1} \cdots \tilde{Q}^{I_{N_c}} \ , \tag{19.5}$$

where the gauge indices are contracted with $\epsilon_{i_1 \dots i_{N_c}}$.

19.3.1 $N_f = N_c$

In this case there is a classical constraint

$$\det(Q^I \cdot \tilde{Q}^J) - B\tilde{B} = 0 \ . \tag{19.6}$$

It is modified due to quantum corrections according to

$$\det(Q^I \cdot \tilde{Q}^J) - B\tilde{B} = \Lambda^{2N_c} \ , \tag{19.7}$$

but not non-perturbative superpotential is generated. The quantum constraint is derived by giving one of the quark flavours a mass via a tree level superpotential $W_{\text{tree}} = m_{IJ} Q^I \tilde{Q}^J$ and integrating out the heavy fields. Below the mass scale one then has a theory with $N_f = N_c - 1$ flavours and should recover the superpotentail (19.4). This indeed requires (19.7) [12, 37, 38].

19.3.2 $N_f = N_c + 1$

In this case one defines

$$B_J = \epsilon_{JI_1 \dots I_{N_c}} B^{I_1 \dots I_{N_c}} \ , \qquad \tilde{B}_J = \epsilon_{JI_1 \dots I_{N_c}} \tilde{B}^{I_1 \dots I_{N_c}} \ , \tag{19.8}$$

and has a classical constraint

$$\det(Q^I \cdot \tilde{Q}^J)(Q^K \tilde{Q}^L)^{-1} - B_K \tilde{B}_L = 0 \ , \qquad (Q^K \tilde{Q}^L)\tilde{B}_L = B_K (Q^K \tilde{Q}^L) = 0 \ , \tag{19.9}$$

which however stays unchanged quantum mechanically. (This can be shown using the same methods.)

19.3.3 $\frac{3}{2} N_c < N_f < 3N_c$

In this case it is argued that the gauge coupling does not grow arbitraryly large in the IR but rather reaches a fixed point of the RG-equation corresponding to an interacting superconformal field theory. For $N = 1$ pure gauge theories the exact β-function reads

$$\beta(g) = \mu \frac{dg}{d\mu} = \frac{1}{16\pi^2} \frac{bg^3}{1 - \frac{I(ad)g^2}{8\pi^2}} \ , \qquad b = -3I(ad) + \sum_{\mathbf{r}} n_c(\mathbf{r}) I(\mathbf{r}) \ . \tag{19.10}$$

For the case at hand we have $I(ad) = N_c, I(N_c) = I(\bar{N}_c) = \frac{1}{2}$ and thus $b = -3N_c + N_f$. In the presence of chiral matter (19.10) is modified according to

$$\begin{aligned} \beta(g) &= -\frac{g^3}{16\pi^2} \frac{3N_c - N_f + N_f \gamma(g^2)}{1 - \frac{N_c g^2}{8\pi^2}} \ , \\ \gamma(g^2) &= -\frac{g^2}{8\pi^2} \frac{N_c^2 - 1}{N_c} + \mathcal{O}(g^4) \ , \end{aligned} \tag{19.11}$$

where $\gamma(g^2)$ is the anomalous dimension. Expanding β in a power series in g one sees that the two-loop correction has the other sign and thus a fixed point might exist. It can be established in the limit $N_c, N_f \to \infty$ with $N_c g^2$ fixed and $\frac{N_f}{N_c} = 3 - \epsilon$ fixed when

$$g_*^2 N_c = \tfrac{8\pi^2}{3}\epsilon + \mathcal{O}(\epsilon^2) \ . \tag{19.12}$$

In [38] it is argued that this fixed point exists for all $\frac{3}{2}N_c < N_f < 3N_c$ and thus an interacting superconformal field theory is the IR-limit.

The same paper also suggested that there exists an equivalent "magnetic" description with a magnetic gauge group $G = SU(N_f - N_c)$, N_f magnetic quarks $q^I, \tilde{q}^J$, additional singlets M^{IJ} and a superpotential

$$W = \mu^{-1} M_{IJ} q^I \tilde{q}^J \ . \tag{19.13}$$

The scale μ relates the dynamical scales of the electric and magnetic theory $\Lambda, \tilde{\Lambda}$ via

$$\Lambda^{3N_c - N_f} \tilde{\Lambda}^{3(N_c - N_f) - N_f} = (-)^{N_f - N_c} \mu^{N_f} \ . \tag{19.14}$$

Under the "electric-magnetic duality" the M_{IJ} are mapped to the condensates $\langle Q^I \cdot \tilde{Q}^J \rangle$ and the baryons B to corresponding magnetic baryons b.

The proposed duality satisfies many consistency checks which can be found in [12, 38].

19.3.4 $N_c + 2 < N_f < \frac{3}{2}N_c$

For the magnetic theory one compute the one-loop coefficient of the β-function to be $b_{mag} = -3(N_f - N_c) + N_f = 3N_c - 2N_f$. Therefore for $N_f > \frac{3}{2}N_c$ the magnetic theory is asymptotically free while for $N_f < \frac{3}{2}N_c$ the magnetic theory is IR free. Thus the window $N_c + 2 < N_f < \frac{3}{2}N_c$ is best described by the IR free (weakly coupled) magnetic theory.

19.3.5 $N_f > 3N_c$

In this case the electric theory is not asymptotically free but IR free.

20 $N = 2$ Supergravity

The massless multiplets for $N = 2$ are given in Table 17.1. The gravitational multiplet contains the metric g_{mn}, two gravitini $\psi_m^{1,2}$ and a vector v_m^0 called the graviphoton. The vector multiplet contains a vector v_m, two gaugini $\lambda^{1,2}$ and a complex scalar z. For n_v vector multiplets we use the notation $(v_m^a, \lambda^{a1,2}, z^a)$ with $a = 1, \ldots, n_v$.[15] A hypermultiplet is buildt from two half-hypermultiplets. For n_H hypermultiplets we use the notation: (χ^i, q^u), where χ^i, $i = 1, \ldots, 2n_H$, are the fermions and the $q^u, u = 1, \ldots, 4n_H$ are $4n_H$ real scalars.

The bosonic Lagrangian in reads

$$\begin{aligned}\mathcal{L} = & -\tfrac{1}{2}R - \mathrm{Im}\,(\mathcal{N})_{AB}\, F^A_{\mu\nu}F^{B\mu\nu} - \tfrac{i}{2}\mathrm{Re}\,(\mathcal{N})_{AB}\, F^A_{\mu\nu}F^B_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \\ & - G_{a\bar{b}}(z,\bar{z})D_\mu z^a D^\mu \bar{z}^b - h_{uv}(q)D_\mu q^u D^\mu q^v - V(z,\bar{z},q) \ ,\end{aligned} \tag{20.1}$$

where $A = 0, \ldots, n_v$. The scalar field space is locally the product

$$M = M^{2n_v}_{v,SK} \times M^{4n_h}_{h,QK} \ , \tag{20.2}$$

where $M^{2n_v}_{v,SK}$ is a $2n_v$-dimensional special Kähler manifold while $M^{4n_h}_{h,QK}$ is a $4n_h$-dimensional quaternionic-Kähler manifold. Let us discuss both geometries in turn [3, 39].

20.1 Special Kähler geometry

A special Kähler manifold is a Kähler manifold where the Kähler potential is of the specific form

$$K = -\ln i\left(\bar{Z}^A F_A(Z) - Z^A \bar{F}_A(\bar{Z})\right) \ , \tag{20.3}$$

with

$$F_A := \frac{\partial F}{\partial Z^A} \qquad \text{and} \qquad Z^A F_A = 2F \ , \tag{20.4}$$

i.e. $F(Z)$ is homogeneous of degree 2 in the coordinates Z^A. The physical scalar fields z^a are defined as the projective coordinates $z^a = \frac{Z^a}{Z^0}$. Using the homogeneity of $F(Z)$ we can define a $\mathcal{F}(z^a)$ via $F(Z^A) = i(Z^0)^2\mathcal{F}(z^a)$. In terms of $\mathcal{F}$ the Kähler potential reads

$$K = -\ln\left(2(\mathcal{F} + \bar{\mathcal{F}}) - \left(\mathcal{F}_a - \bar{\mathcal{F}}_a\right)(z^a - \bar{z}^a)\right) - \ln|Z^0|^2 \ , \tag{20.5}$$

where the last terms can be removed by a Kähler transformations. Indeed, the rescalings $Z^A \to Z^A e^{-f(z)}, F_A \to F_A e^{-f(z)}$ induce a Kähler transformation of the form $K \to K + f(z) + \bar{f}(z)$ and can be used to set $Z^0 = 1$. The choice of coordinates $Z^A = (1, z^a)$ are called special coordinates.

There also is again invariant symplectic form of K given by

$$K = -\ln i\left(V^\dagger \Omega V\right) \ , \qquad \text{with} \qquad \Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \ , \qquad V = \begin{pmatrix} F_A \\ Z^B \end{pmatrix} \ . \tag{20.6}$$

[15] In terms of $N = 1$ multiplets, we have the decomposition: $\left(v_m^a, \lambda^{a1}\right) \oplus \left(\lambda^{a2}, z^a\right)$, where the first one is the vector multiplet of $N = 1$ and the second the chiral multiplet $N = 1$.

The symplectic section V transforms according to

$$V \to V' = SV \ , \tag{20.7}$$

with S being an element of $Sp(2n_v + 2, \mathbb{R})$ obeying

$$S^{\mathrm{T}}\Omega S = \Omega \ . \tag{20.8}$$

The gauge kinetic matrix $\mathcal{N}$ is given by:

$$\mathcal{N}_{AB} = \bar{F}_{AB} - \frac{(\mathrm{Im}F)_{AC}\, Z^C\, (\mathrm{Im}F)_{BD} Z^D}{Z^C\, (\mathrm{Im})_{CD}\, Z^D} \ , \tag{20.9}$$

where the second term is due to the graviphoton. Finally, the covariant derivatives read

$$D_\mu z^a = \partial_\mu z^a - v_\mu^B\, X^{Ba}(z) \ , \tag{20.10}$$

where the holomorphic Killing vectors $X^{Ba}(z)$ can again be expressed in terms of Killing prepotential P_0^B by

$$X_{\bar{a}}^B = G_{\bar{a}b} X^{Bb} = i\partial_{\bar{a}} P_0^B \ . \tag{20.11}$$

Finally, if one decouples gravity the geometry reduces to the geometry discussed in Section 18.

20.2 Quaternionic-Kähler geometry

From Table 17.1 we see that the half-hypermultiplet is an irreducible CPT-complete representation of the $N = 2$ algebra. However, in terms of field-theoretic representation it has two problems:

1. The two $\lambda = 0$ states have to be a doublet of $SU(2)_R$ yet at the same time they are real.
2. Since the generators of any gauge group G commute with the supercharges the states with $\lambda = \pm 1/2$ have to carry the same charge. Again for a complex fermion this is not possible and thus the half-hypermultiplet could at best we neutral.

The way out is to combine two half-hypermultiplets with opposite charges into one hypermultiplet. This multiplet then contains two Weyl fermions (equivalent to one Dirac-fermion) and four real scalars. This in turn immediately implies that any $N = 2$ matter representation is non-chiral or in other words the SM cannot be straightforwardly embedded into $N = 2$ supersymmetric theories.

For n_h hypermultiplets one has $4n_h$ real scalars $q^u, u = 1, \ldots, 4n_h$ which span the $4n_h$-dimensional target space $M_{h,QK}^{4n_h}$. It is not a Kähler manifold bur rather a quaternionic-Kähler manifold. This means that it admits three almost complex structures $(J^x)_u^v, x = 1, 2, 3$, which satisfy[16]

$$J^x J^y = -\delta^{xy}\mathbf{1} + i\epsilon^{xyz} J^z \ , \tag{20.12}$$

[16]An almost complex structure is called a complex structure if in addition the Niejenhuis tensor $N(J)$ vanishes. In that case the manifold is complex and complex coordinates exist globally. For further details see [3].

and the metric h_{uv} is Hermitian with respect to all three of them

$$(J^x)_u^v h_{vw}(J^x)_s^w = h_{us} \ . \tag{20.13}$$

They are also covariantly constant with respect to an $SU(2)$ connection ω

$$\nabla_w (J^x)_u^v + \epsilon^{xyz}\omega_w^y (J^z)_u^v = 0 \ . \tag{20.14}$$

For each J^x there is an associated Kähler two-form K^x with coefficients $K^x_{uv} = h_{uw}(J^x)_v^w$. They obey

$$dK^x + \epsilon^{xyz} w^y \wedge K^z = 0 \ . \tag{20.15}$$

The q^u can be charged with respect to an Abelian or non-Abelian gauge group. This requires the couplings to vector multiplet via the covariant derivatives

$$D_\mu q^u = \partial_\mu q^u - v_\mu^A X_A^u(q) \ , \tag{20.16}$$

where the Killing vectors $X_A^u(q)$ can be expressed in terms of Killing prepotential P_A^x by

$$X_A^u K_{uv}^x = -D_v P_A^x = -(\partial_\nu P_A^x + \epsilon^{xyz} w_v^y P_A^z) \ . \tag{20.17}$$

Altogether the potential is given by

$$V = e^K \Big(G_{a\bar{b}} X_A^a \bar{X}_B^{\bar{b}} Z^A \bar{Z}^B + 4 h_{uv} X_A^u X_B^v Z^A \bar{Z}^B + \big(G^{a\bar{b}} (\partial_a Z^A)(\bar{\partial}_{\bar{b}} \bar{Z}^B) - 3 Z^A \bar{Z}^B \big) P_A^x P_B^x \Big) \ . \tag{20.18}$$

Before we continue let us mention one caveat. The situation discussed here only features multiplets which are charged with respect to electric gauge bosons but not their magnetic duals. In string theory it is sometimes convenient to go to a different symplectic basis and includes magnetic charges. This can be done via the embedding tensor formalism [40].

Decoupling gravity in the hypermultiplet sector reduces the target space geometry from quaternionic-Kähler to hyper-Kähler. Hyper-Kähler manifold are Kähler manifolds with three complex structures J^x which obey (20.12) and are covariantly constant $\nabla J^x = 0$. As a consequence the associated Kähler two-forms are closed, i.e. $dK^x = 0$. Hyper-Kähler manifold are Ricci-flat while quaternionic-Kähler manifold are Einstein manifolds.

20.3 Partial supersymmetry breaking

From the algebra (17.1) one infers that in the rest-frame where $P_\mu = (-H, \vec{0})$ one has

$$H = \tfrac{1}{4} Q^I \cdot \bar{Q}^I \qquad \forall I \ . \tag{20.19}$$

Therefore if one supercharge, say Q^1, is unbroken one has

$$Q^1|0\rangle = 0 \quad \Rightarrow \quad H|0\rangle = 0 \ . \tag{20.20}$$

This in turn implies

$$\tfrac{1}{4} Q^2 \cdot \bar{Q}^2 |0\rangle = 00 \quad \Rightarrow \quad Q^2|0\rangle = 0 \tag{20.21}$$

since $Q^2 \cdot \bar{Q}^2$ is a positive operator. This line of argument is one way to state the no-go theorem of Refs. [41, 42] that $N = 2$ supersymmetry is either preserved or completely broken but partial breaking is not possible. The subtlety in this argument was found in [43, 44] in that for a broken symmetry the charges do not properly exist and one has to discuss the associated current algebra. It was found that in the current a magnetic FI-term is possible and can spontaneously break $N = 2 \rightarrow N = 1$. In $N = 2$ supergravity the analysis has been performed in [44–47].

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