Problem 4.1

Pure $N = 1$ supersymmetric gauge theories (i.e. without any matter) have a β-function which can be written to all orders in perturbation theory as

$$\mu \frac{dg}{d\mu} = \frac{1}{16\pi^2} \frac{bg^3}{(1 + \frac{b}{24\pi^2} g^2)} , \tag{1}$$

where b is the 1-loop coefficient.

a) Show that

$$g^{-2}(\mu) = g^{-2}(M) + \frac{b}{8\pi^2} \ln \frac{M}{\mu} - \frac{b}{24\pi^2} \ln g^{-2}(\mu)$$

solves the RG-equation (1).

b) Show that

$$\Lambda = \mu \, g^{-\frac{2}{3}}(\mu) \exp[\tfrac{8\pi^2}{b} g^{-2}(\mu)]$$

is an RG-invariant scale, i.e. $\frac{d\Lambda}{d\mu} = 0$ holds.

c) Using a) show that

$$\Lambda = M \exp[\tfrac{8\pi^2}{b} g^{-2}(M)] \ .$$

Problem 4.2

In General Relativity one defines the covariant derivative of a vector V_μ as

$$\nabla_\mu V_\nu := \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho \ .$$

For a vector $V_a = e^\mu_a V_\mu$ one has

$$\nabla_\mu V_a := \partial_\mu V_a - \omega_{\mu a}{}^b V_b \ ,$$

where $\omega_{\mu ab} = -\omega_{\mu ba}$.

a) Derive an expression for $\nabla_\mu e^a_\nu$ using the consistency of the two expressions.

Using additionally $\nabla_\mu g_{\nu\rho} = 0$ show

$$\omega_{\mu\nu\rho} = \tfrac{1}{2}(C_{\mu\nu\rho} - C_{\nu\rho\mu} + C_{\mu\rho\nu}) \ ,$$

where $\omega_{\mu\nu\rho} := e^a_\rho e^b_\nu \omega_{\mu ba}, \ C_{\nu\mu\rho} := (\partial_\nu e^a_\mu - \partial_\mu e^a_\nu)\, \eta_{ac} e^c_\rho$.

b) Compute the transformation law of $\omega_{\mu a}{}^{b}$ under local Lorentz transformations.

Problem 4.3

a) Compute all Christoffel symbols and all components of the Riemann curvature tensor for a Kähler manifold with the metric $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$.

b) Show

$$[D_i, \bar{D}_{\bar{j}}] v_k = R_{i\bar{j}k}{}^{l} v_l \ ,$$

where $D_{\bar{j}} v_k = \partial_{\bar{j}} v_k$, $D_i v_k = \partial_i v_k - \Gamma^l_{ik} v_l$.

c) Show that the Ricci tensor is given by

$$R_{i\bar{j}} = \partial_i \partial_{\bar{j}} \ln \det(G_{l\bar{k}}) \ .$$

Problem 4.4

Consider a supergravity with $n_c + 1$ chiral multiplets $T, \phi^i, i = 1, \ldots, n_c$ and a Kähler potential

$$K = -3 \ln Y \ , \qquad \text{where} \qquad Y \equiv (T + \bar{T} - \phi^i \delta_{i\bar{j}} \bar{\phi}^{\bar{j}}) \ , \quad \kappa = 1 \ .$$

a) Compute all components of the metric.

b) Show

$$K_I G^{I\bar{J}} K_{\bar{J}} = 3 \ ,$$

where I runs over all $n_c + 1$ chiral fields.

Hint: For $G^{I\bar{J}}$ use the Ansatz

$$G^{I\bar{J}} = \frac{Y}{3} \begin{pmatrix} X & Z\phi^i \\ Z\bar{\phi}^{\bar{j}} & \delta^{i\bar{j}} \end{pmatrix} \ ,$$

and determine X, Z.

c) Compute V in this theory for $W = \text{constant}$.