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Background Fluxes in Type II String Compactifications

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Abstract

In this thesis we study some recently proposed generalized compactifications which are known as compactifications with background fluxes. We start from type II theories in ten dimensions and show that such compactifications lead to massive supergravities and thus they preserve some supersymmetry. As a byproduct we find a way to reconcile symplectic invariance with gauged supergravities by coupling two-form fields to both electric and magnetic field strengths. Furthermore we study mirror symmetry in such compactifications and we show that this holds naturally in the case of RR fluxes. For the case of NS-NS fluxes we propose some generalized Calabi–Yau manifolds which are termed half-flat manifolds with $SU(3)$ structure and we show that by performing the KK compactification on such manifolds one indeed obtains the mirror of the NS-NS three-form fluxes.

Zusammenfassung

In dieser Arbeit betrachten wir allgemeinere Kompaktifizierungen, wie sie kürzlich vorgeschlagen wurden, nämlich Kompaktifizierungen mit Hintergrundflüssen. Ausgehend von Typ II String Theorien in zehn Dimensionen zeigen wir, dass solche Kompaktifizierungen zu massiven Supergravitationstheorien führen und somit einen Teil der Supersymmetrie erhalten. Nebenbei finden wir eine Möglichkeit symplektische Invarianz mit geeichter Supergravitation in Einklang zu bringen, indem wir die 2-Form-Felder sowohl mit elektrischen als auch mit magnetischen Feldstärken koppeln. Außerdem betrachten wir Mirror-Symmetrie für solche Kompaktifizierungen, und wir zeigen, dass diese Symmetrie auf natürliche Weise im Falle der RR Flüsse erfüllt ist. Für den Fall von NS-NS Flüssen schlagen wir eine Verallgemeinerung der Calabi-Yau Mannigfaltigkeiten vor, sogenannte Half-Flat Mannigfaltigkeiten mit SU(3) als Strukturgruppe, und wir zeigen, dass man durch KK-Kompaktifizierung auf solchen Mannigfaltigkeiten wirklich das Spiegelbild mit NS-NS 3-Form Flüssen erhält.

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Contents

Chapter 1

Introduction

1.1 High energy physics: An overview

There is one fascinating question which people have asked since ancient times: *What are the elementary building blocks of the world we live in?* Despite the oldness of this question and of the progress which was done in understanding the structure of the matter a definite answer is missing so far. All the experimental results seem to point towards the fact that everything we can see is made out of quarks and leptons and their interactions are governed by the *Standard Model of Particle Physics.* However there are good reasons to believe that this is not the end of the story.

The standard model is a gauge theory based on the group $SU(3)_C \times SU(2)_W \times U(1)_Y$ with matter content (quarks and leptons) which falls into three distinct families (for a review see [1]). It can explain any fact in particle physics up to energies of order 100 GeV and its predictions have been confirmed by numerous experiments. Despite its great success there are definitely things which deserve an explanation. First of all it contains 19 free parameters which reduce its predictive power. Moreover in order to reproduce the experimental data one has to fine tune some of these parameters[1] or create arbitrary large hierarchies between parameters of the same type facts which do not have a satisfactory explanation. There are also other questions like *Why precisely three families?* or *How is the gauge group chosen?* which make one think that there should be something *beyond the Standard Model.* There were several attempts to find an extension of the standard model, but this proved very difficult because of the strict constraints imposed by the several precision tests. Moreover beside these 'theoretical' arguments no experiment up to the present energies could indicate that there is indeed some physics beyond the standard model and what the nature of this physics is.[2]

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[1] It is well known that the renormalization group drives the mass of scalar particles (the Higgs boson in this case) to the highest scale in the theory. Thus, if the standard model is supposed to be true even at higher energies one has tune this mass with a high precision in order to keep it at the level of the weak scale.

[2] Strictly speaking the recently discovered neutrino masses are part of the physics beyond the standard model. However in this case there are simple mechanisms which could account for small neutrino masses

One of the ideas for extending the Standard Model was to implement a symmetry, known as *supersymmetry*, between bosons and fermions. There are many attractive features of the supersymmetric standard models one of the most important being that a natural protection of the Higgs mass appears and so there is no need to fine-tune this parameter anymore. The price to be payed is on the other hand also big. As the particles discovered up to now do not come in super-multiplets in order to make this symmetry work one needs to introduce *superpartners* for all the known particles and thus (at least) double the number of elementary particles. The number of parameters blows up (in the minimal model being bigger than 100) and several other 'ad-hoc' mechanisms have to be introduced in order to avoid severe problems like the proton decay.

Another appealing extension of the standard model are the *Grand Unified Theories* (GUT) [1]. The main idea behind such models is that provided nothing new happens between the weak and the GUT scale the Standard Model descends from an unified theory which has a bigger gauge group such as $SU(5)$ or $SO(10)$. Due to their large amount of symmetry these models are very predictive. In particular it means that the evolution with the energy scale of the three coupling constants (strong, electro-magnetic and weak) of the standard model due to the renormalization group equations should be in such a way that the three curves meet at a point which gives the GUT scale. It is interesting to note that even if the standard model fails to satisfy this minimal requirement the three couplings come really closed together at an energy scale of order 10^{15} GeV. In the minimal supersymmetric extension of the standard model the meeting of the coupling constants does indeed occur at a slightly higher scale, namely 10^{16} GeV. This could indicate that that if the GUTs describe the physics beyond the standard model, supersymmetry might also play an important role. It is interesting to note that some of the quantum numbers of the standard model particles (like the $U(1)_Y$ charges) have a natural explanation if one regards them as descending from GUT multiplets. Moreover in the context of grand unified theories one can easily generate tiny neutrino masses.

Despite the nice features of the grand unified theories one cannot neglect the problems which they generate. In particular one of the most serious problems is the fact that the proton is unstable even at tree level. Thus the only suppression of this effect comes from the high scale of these models and the generic life time for the proton in such models is about to be reached by the present experiments. There are also theoretical problems which one encounters in such models like the fact that 'nothing new' happens over a range of 10^{15} orders of magnitude or the doublet triplet splitting problem.

To make a long story short no truly viable extension of the standard model is known at present and it appears that a more complicated physics is hidden beyond the standard model picture. There is another aspect which we have totally neglected until now which seems to point in the same direction: gravity.

At a classical level the gravitational interactions are described with high accuracy by Einstein's theory of general relativity (see for example [2]). However there are again reasons to suspect that there is something else behind. First of all the theory of general relativity is a classical theory which we can only believe if the gravitational fields are

without drastic changes in the structure of the standard model.

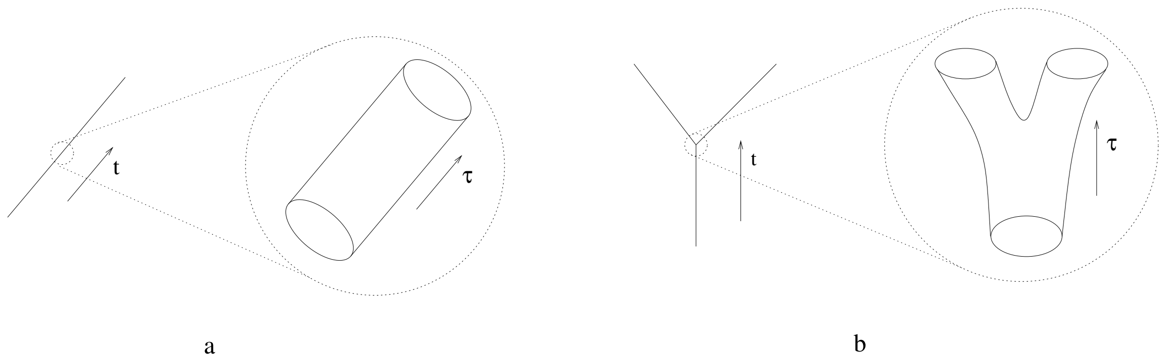

Figure 1.1: a) Particle vs. (closed) string propagation. b) Particle vs. string interaction

not too strong. This is definitely the case in most of the applications of particle physics, astrophysics, or even cosmology. Nevertheless in order to describe strong gravitational fields (like near black holes, or at very high energies) one would really need a quantum description of gravity and such a theory has not been constructed until now. Moreover we would also like to have unified description of all interactions in nature and for this one again needs a quantum version of gravity.

It is also worth mentioning that the picture we have about the physics at 100 GeV where the gravitational interactions decouple from the standard model features one of the biggest fine tuning problem ever encountered. Even if gravity plays no role in the standard model the converse is not true. The reason is that the vacuum expectation value of the Higgs field which gives masses to all the particles in the standard model is also the source of a vacuum energy which curves the space. As a result the naive cosmological constant is 60 orders of magnitude bigger than the observed value and in order to agree with experiments one has to add by hand a negative cosmological constant which is tuned with the precision 1 into 10^{60} (see for example [3]).

All the arguments we have presented up to now seem to indicate that a 'bottom-up' approach is difficult to adopt in this case and one would need a more drastic change in the way of thinking. One of the most promising candidates at the moment is the theory of (super)strings. Strings are one dimensional extended objects which sweep out a two-dimensional surface, known as world sheet, in a D dimensional space which is also called the target space. depending on their boundary conditions the string can be closed or open. The action which describes the movement of the string is a (super)conformal two dimensional non-linear sigma-model whose only free parameter is the string tension which is commonly denoted by $1/\sqrt{\alpha'}$. In the limit $\alpha' \to 0$ the string tension becomes infinite and thus the string shrinks to a point. In this limit string theory reduces to an ordinary field theory. The difference between string and field theory can be visualized as in figure 1.1.

Consistency of string theory at quantum level imposes very strict constraints on the theory itself. In particular the absence of the Weyl anomaly implies the existence of a critical dimension which is the dimension of the space in which the string can consistently propagate. For the purely bosonic string the critical dimension is 26 while for the superstring it turns out to be 10. Moreover the quantum conformal invariance requires that the geometry of the target space has to be Ricci flat.

This is a very surprising result as up to now the space-time was chosen by hand while now it seems to be imposed by the consistency of the theory we consider. The drawback of this fact is that in neither of the cases discussed above the space time is four dimensional as it appears to be according to our observations. This will turn out to be one of the major difficulties in trying to make contact with the real world and we will discuss this subject at large as it is one of the central themes of this work.

Starting with such a theory the particles in the D-dimensional space-time can be thought as excitations of the string. As the only scale in the theory is $1/\sqrt{\alpha'}$ which is supposed to be large the masses of these modes will typically be of this order. Thus the only interesting excitations for low energy physics are the massless ones. However it turns out that for the case of the bosonic string the lowest mass state is in fact a tachyon (it has negative mass square) which reveals some inconsistency of the theory. For the superstrings this is not the case and the first excited states correspond to massless particles. Thus from now on we will only concentrate on the superstrings and we implicitly assume that the space-time is ten-dimensional.

One of the major observations was that among the excitations of the string one usually finds a state with spin 2 which was immediately identified to the graviton. Thus one hopes that string theory can be a consistent description of quantum gravity.

It appears that in ten dimensions there are five consistent string theories which are denoted as: type IIA, type IIB, heterotic $E_8 \times E_8$, heterotic $SO(32)$ and type I. They are all supersymmetric and beside the spin two state which we mentioned above, there are several other massless excitations which assemble themselves into representations of the supersymmetry algebra in ten dimensions. Moreover it turns out that the low energy description of any superstring theory (i.e. $\alpha' \to 0$ limit) is the corresponding supergravity in ten dimensions. Thus, it appears natural to consider string theory as a candidate for a theory which in the low energy limit can reproduce the theories we know in four dimensions: the standard model and the theory of general relativity. In the next section we will present a couple of ideas in this direction.

1.2 Lower dimensional models

In the previous section we argued that string theory is one of the most natural candidates for a unified quantum theory of all interactions. It is indeed very attractive to study it as it has only one free parameter which is the string scale $\sqrt{\alpha'}$ and thus it can be a very predictive theory. As we have seen that gravity is automatically included in a low energy approximation it is natural to relate the string scale to the Planck scale. This means that it will be difficult to see stringy effects in the future experiments and thus the best one can do at the moment is to find an appropriate limit in which the string theory reduces to the known theories in four dimensions, namely the standard model and general relativity. However this is not a straightforward exercise and in the following we will sketch the main ideas of obtaining phenomenologically interesting models starting from a ten-dimensional string theory.

First of all even though it seems that string theory is a very restrictive theory very few things are known about its vacuum structure apart from the fact that the space in which it moves should be a ten dimensional Ricci flat manifold. This is in fact one of the biggest problems which are faced nowadays in string theory which is known as the vacuum selection problem.

Of course what one would expect is that the vacuum exhibits a splitting of the ten dimensional coordinates into four which constitute the world we observe and some hidden six dimensions. The structure of these six hidden coordinates is very poorly understood and at the moment there are roughly two main ideas: one where it is assumed that the six extra dimensions are compact and small and escaped our observations until now and the second which states that we are in fact living on a four dimensional hyper-plane (D-brane) embedded in ten dimensions and again the extra dimensions have not been observed until now. The first scenario is termed of Kaluza–Klein compactifications and the second braneworld models. In the following we give a brief description of each of them.

1. Kaluza-Klein reductions.

Kaluza-Klein (KK) compactifications is one of the main topics of this thesis. The idea is based on the observation made by Kaluza in 1920 that pure gravity in five dimensions can be interpreted from a four dimensional point of view as gravity coupled to an electromagnetic field and a scalar, provided that the fifth dimension is compact and its size is taken to be small. We review this example in more detail in appendix D.1 and here we outline how this can help us to obtain four dimensional models.

The basic assumption is again to consider that some of the ten dimensions are actually small and compact and only four are extended and can be observed. Consequently one chooses the then dimensional space to be a direct product of a four dimensional one (which is often chosen to be just the flat Minkowski space) and some internal unknown manifold K_6

$$\mathcal{M}_{10} = \mathbb{R}^{1,3} \times K_6 \tag{1.1}$$

This is equivalent to choosing a background metric which is a direct product between a four dimensional Minkowski metric and some metric on the internal manifold

$$\hat{G}_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g^{int}_{mn} \end{pmatrix} , \tag{1.2}$$

where g_{mn} has to solve the Einstein equations for the particular background field configuration which is chosen. It turns out that the ten dimensional fields give rise in four dimensions to infinite towers of states whose masses are multiples of $1/R$, where R is the generic size of the extra dimensions. In a low energy approximation one only keeps the massless fields which correspond to harmonic forms on the internal manifold. Performing the integration over the internal space one obtains the low energy effective action which describes the truncated theory in four dimensions. One of the first questions which can be asked is how small the additional dimensions are. As the masses in the KK tower crucially depend on the size of the internal manifold it implies that the existence of extra dimensions is constrained by the four dimensional physics. In particular just from the fact that no massive states which fit in the KK pattern were observed one infers that the compactification radius has to be small enough so that the corresponding states are heavier than the present energies obtained in the laboratory. However from the weak to the Planck scale there is a broad spectrum of choices for the dimension of the internal manifold which is in principle not fixed. Note that in this approach the string scale $1/\sqrt{\alpha'}$ is in general fixed to be of the order of the Planck scale.

The setup we have just presented is in some sense very arbitrary as the existence of four large dimensions and six small, compact ones is imposed by hand while in string theory there is no mechanism which can fix this. In fact this seems to be a big puzzle as there is no way one can fix the geometry of the ten dimensional space.[3] The structure of these internal dimensions is again a delicate problem: beside the constraint on the ten dimensional manifold that it should be Ricci flat there is no other indication that string theory prefers any particular geometry for the internal directions. Usually one makes a choice for the background field configuration like in (1.2) according to the desired properties of the four dimensional theory.

2. Braneworlds.

Even though in the rest of this work we are going to concentrate on the 'traditional' approach of string compactifications let us address for completeness some other possibilities to obtain four dimensional models from string theory. After the discovery of D-branes (for a review see for example [5]) in 1995 a lot of work was devoted to the idea of realizing the standard model on a 3-brane which lives in the ten-dimensional bulk. Here we briefly review, without going into details, some of the ideas which are used in such scenarios.

First of all the branes are supersymmetric objects and in general they preserve half of the bulk supersymmetry. In this way one can find an alternative for reducing the large amounts of supersymmetry with which string theory comes in ten dimensions. Combining branes with internal manifolds of special holonomy one easily obtains models with $N = 1$ supersymmetries in four dimensions. Second of all the open strings which end on the branes give rise to gauge fields. Considering more branes which are coincident one obtains non-Abelian gauge groups which can even contain the standard model gauge group. Obviously this opens up a large variety of possibilities for obtaining viable models. For the case that the dimensions transverse to the brane are compact the string scale is no longer fixed but it dramatically changes with the volume of the extra dimensions. In particular one can obtain models where the string scale is of order of TeV which tell us that unlike the traditional approach the stringy effects could be around the corner

[3] In [4] it was argued that starting with ten small dimensions around which strings can wrap one naturally ends up with a configuration where four of the dimensions extend to infinity while the other six remain small.

waiting to be discovered in the future experiments.

Another interesting direction was revealed in [6, 7] where the extra dimensions are not compact but the geometry includes a warp factor which exponentially goes to zero away from the brane. In this way even if the extra dimensions extend to infinity the warp factor ensures that this can not be noticed from the point of view of the brane-observer and the volume of the extra dimension space is effectively finite.

Recently very attractive ideas have appeared which are based on the observation made in [8] that at brane intersections chiral fermions may appear. This opened up a whole set of possibilities for obtaining the standard model spectrum using such configurations.

1.3 Dualities and mirror symmetry

Let us now come back to the traditional approach, namely compactification of the extra dimensions. For phenomenological reasons the $E_8 \times E_8$ heterotic string appeared to be the most promising theory due to the fact that its gauge group can easily accommodate the standard model group or even GUT groups. In particular in the mid '80s there were several attempts to construct four-dimensional models by compactifications of the heterotic string which preserved one quarter of the original amount of supersymmetry, namely $N = 1$ in four dimensions [9]. This was done by choosing the internal manifold to be a Calabi–Yau space i.e. a manifold with $SU(3)$ holonomy. However a question still remains: *How do we choose precisely the heterotic string and what role do the other string theories play?* The answer seems to come from a surprising direction. It turns out that the vacua of these theories are related or in other words there are some duality maps which connect the different string theories among themselves (for a review see [10]). This is a very interesting feature which is specific to string theories and which has many applications.

Roughly speaking these dualities can be split into two major classes: perturbative and non-perturbative ones. The perturbative dualities relate two theories at small couplings while the non-perturbative ones relate the weak coupling regime of one theory to the strong coupling regime of the other. For example it is known that type IIA and type IIB theories compactified on a circle are equivalent. Moreover the same happens with the two heterotic strings. On the other hand type IIB theory exhibits a symmetry which exchanges the weakly coupled regime with the strongly coupled one.

A special role among the web of dualities is played by the so called mirror symmetry (for a review see [11, 12]). As this will be one of the main topics of this thesis let us shortly introduce the basic notions.

From a physical point of view mirror symmetry states that for any Calabi–Yau manifold Y there exist another Calabi–Yau manifold $\tilde{Y}$ such that type IIA theory compactified on Y is equivalent to type IIB compactified on $\tilde{Y}$. Even if at a first glance this does not seem to be very surprising the implications of the above statement are far reaching. In particular we will see later on the vector fields in type IIA compactification come in one to one correspondence with the harmonic $(1, 1)$ forms and thus they are counted by the elements of $(1,1)$ cohomology group whose dimensions is denoted by $h^{(1,1)}$. In type IIB theory on the other hand the vector fields are related to the cohomology group $H^{2,1}(Y)$ which has dimension $h^{(2,1)}$. Identifying the vector field sectors in the two theories leads to the first major implication of mirror symmetry that the odd and even cohomologies are exchanged. Consequently we have

$$h^{(1,1)}(Y) = h^{(2,1)}(\tilde{Y}) \; ; \qquad h^{(2,1)}(Y) = h^{(1,1)}(\tilde{Y}) \; . \tag{1.3}$$

In fact this is part of a much stronger statement namely that the moduli spaces of Kähler and complex structure deformations have to be interchanged.

From a mathematical point of view this is a highly non-trivial statement as it is not at all clear a priori that the Calabi–Yau manifolds should really come in mirror pairs which we stress again are manifolds with different topology. On the other hand if one interprets the Calabi–Yau compactifications as superconformal sigma models with Calabi–Yau target space the difference between the mirror manifolds is just a convention for a $U(1)$ current[4].

It is a remarkable fact that mirror symmetry can be used to compute quantities (like Yukawa couplings) which otherwise would have been inaccessible. Moreover as the equivalence of type II theories on mirror manifolds is supposed to hold at quantum level one can obtain in special cases fully corrected quantities on the mirror side. A typical example is the $N = 2$ prepotential which in type IIB theory can be computed exactly (i.e. it does not receive world sheet instanton corrections). In type IIA theory on the other hand the prepotential does receive corrections from the world sheet instantons and these corrections are in general difficult to compute. However using mirror symmetry one can obtain the fully corrected prepotential.

Assuming fully equivalence of the theories at the non-perturbative level one can obtain the most intuitive interpretation of mirror symmetry. It was argued in [13] that mirror manifolds can be viewed as T^3 fibers over a three-dimensional base and that mirror symmetry is nothing else but T-duality along the T^3 fibers. However for the purposes of this work we will only use a simpler picture of mirror symmetry. More precisely we work in the supergravity approximation which is valid in the limit of large volumes. In this limit the world sheet instanton corrections can be neglected[5] and we will reduce mirror symmetry to a map between different low energy effective actions.

1.4 Topics and organization of the thesis

As we have seen in section 1.2 nowadays there exist several ideas to obtain models which can be interesting from a phenomenological point of view by starting from string theory.

[4]One should bare in mind that the above argument can not be a rigorous proof as it is not clear that any sigma model has the geometric interpretation of a Calabi–Yau compactification.

[5]This kind of corrections come with a weight e^{-t} where t is a generic Kähler modulus which gives the size of the Calabi–Yau manifold and thus going to a point in the moduli space where the vacuum expectation value of t is large enough one can neglect such contributions.

Nevertheless in this thesis we will only concentrate only on the Kaluza–Klein approach. Extended supersymmetry in four dimensions is inconsistent with observations and thus one needs to break in the compactification some of the supersymmetry which is present in ten dimensions. The way to do this is to consider manifolds with special holonomy. As we are only going to study six dimensional internal manifolds Calabi–Yau spaces are the most natural candidates as they preserve only one quarter of the total amount of supersymmetry present in ten dimensions.

It is well known that Calabi–Yau compactifications on the other hand generically lead to a series of problems. First of all as we will see later, a large number of moduli (scalar fields which are flat directions of the potential) appear in the four-dimensional effective action. These moduli can have arbitrary vacuum expectation values and this reduces dramatically the predictive power of string theory. Furthermore one would also need a mechanism to further break the residual supersymmetry. Generically if no potential is present this can not happen. This will motivate the first part of this work as we will argue that allowing for some more general compactifications a potential for some of the moduli can be generated. In these generalizations fluxes of some p-form field strengths through internal cycles are generated and that is why we will denote these compactifications as compactifications in the presence of background fluxes.

Thus in the first part of this work we show how to perform the Calabi–Yau compactification when background fluxes are turned on. We explicitly derive the low energy effective action for several cases and show whenever it is possible that such compactifications lead to known gauged supergravities in four dimensions.

The second main topic of this work is mirror symmetry. As we have argued in the previous section there is a precise relation which maps the low energy effective action of type IIA into the one of type IIB. The question on which we focus in this second part is whether mirror symmetry still holds when fluxes are turned on. We will see that in some cases mirror symmetry requires the presence of some ‘generalized’ Calabi–Yau manifolds.

The structure of the thesis is as follows. Chapter 2 is intended to be an introduction to the main features of Calabi–Yau compactifications of type II theories. We start by giving a short description of $N = 2$ supersymmetry in ten (section 2.1.1) and four dimensions (section 2.2). We also introduce the two type II theories in ten dimensions in sections 2.1.2 and 2.1.3 and some notions about Calabi–Yau spaces and their moduli space in section 2.3. Then in sections 2.3.3/2.3.4 we derive the low energy effective actions of type IIA/IIB compactified to four dimensions on Calabi–Yau three-folds. We end this discussion by showing that as expected from mirror symmetry the low energy effective actions of type IIA and type IIB supergravities compactified on mirror manifolds coincide. We close the chapter with a short discussion which motivates the further work.

In chapter 3 we start the study of compactification with fluxes. We consider the type IIA theory and we focus on the derivation of the low energy effective actions and on the differences which appear compared to the usual compactifications from section 2.3.3. We split the discussion in two parts: NS-NS fluxes in section 3.2 and RR fluxes in section 3.3. We explicitly compute the scalar potentials and the new couplings which are generated in this way and compare these results with what one expects to find from gauged supergravity.

In the next chapter we concentrate on mirror symmetry. We perform the compactification of type IIB theory with fluxes and show that for the case of RR fluxes (section 4.1) one precisely obtains a low energy effective action which is mirror to the one obtained in type IIA with RR fluxes. For the NS-NS fluxes we conclude that a more general configuration has to be considered in order to recover mirror symmetry.

In the last chapter of this work we present some ideas of who one can generalize the setup from chapters 3 and 4 in order to reconcile mirror symmetry with the presence of NS-NS fluxes. It will turn out that different geometries have to be considered and what we will focus on will be the so called half-flat manifolds with $SU(3)$ structure. In order to understand better such spaces and see how the mirror of the NS fluxes can appear we will first give a short overview of manifolds with $SU(3)$ structure and then concentrate on the particular subclass of half-flat manifolds in section 5.1. In the following section, 5.2 we describe how to perform the KK reduction on such spaces. In particular in sections 5.2.2 and 5.2.3 we derive the low energy effective actions of type IIA and type IIB theories compactified on half-flat manifolds and show that these are indeed the configurations mirror to the corresponding compactifications with NS fluxes found in sections 3.2 and 4.2. However it appears that only half of the NS fluxes can be reproduced in this way and in the last section 5.3 we present some arguments about how to further generalize the half-flat spaces in order to accommodate the mirror of all NS fluxes which can be turned on.

The conclusions of this work are presented in chapter 6.

In order to ease the understanding of the above-mentioned subjects we assembled at the end a couple of appendices. First in appendix A we record the conventions we use throughout this thesis. In appendix B we give a short overview of the $N = 2$ supergravities in four dimensions and then in B.2 we describe the related special geometries which arise from the moduli spaces of Calabi–Yau manifolds. In appendix C we give a short introduction into the mathematical machinery which we need in chapter 5. We also compute the Ricci scalar of half-flat manifolds which we use in the derivation of the scalar potentials in this chapter. The main features of KK reductions are presented in appendix D including the dualization of different p-form fields in four dimensions.

The contents of this thesis are based on the research performed in the period 2000–2003 and which was published in some scientific articles. In particular chapters 3, 4 are entirely based on [14], though some features can also be encountered in a previous publication [15]. The second part of this thesis, chapter 5, is based on the work performed in [16, 17].

Chapter 2

Calabi–Yau compactifications of type II theories

In this chapter we give a short review of the Calabi–Yau compactifications of type II string theories. We start by recording the structure of the type II theories in ten dimensions in section 2.1 and then we present in section 2.3 the main features of their Calabi–Yau compactifications. We end this chapter with a discussion of the results obtained which will motivate the developments in the following chapters.

2.1 Type II theories in ten dimensions

2.1.1 Supersymmetry in ten dimensions

As we have pointed out in the introduction all consistent string theories in ten dimensions are supersymmetric and their low energy effective actions are the corresponding supergravities. As we are going to study these theories in quite some detail let us first start with some basic notions about supersymmetry/supergravity in ten dimensions. The physical states in the theory are given by (massless) representations of the supersymmetry algebra (we do not take into account the central charges)

$$\{Q, \bar{Q}\} = 2P^M \cdot \Gamma_M \ , \qquad M = 0, \dots, 9 \ , \tag{2.1}$$

where Q and $\bar{Q}$ are the supercharges, P denotes the momentum and Γ are gamma matrices. The minimal amount of supersymmetry in D dimensions is given by the dimension of the smallest irreducible spinor representation of the Lorentz group $SO(1, D-1)$. In ten dimensions one can define spinors which obey both Majorana and Weyl constraints [18,19] and such a spinor has 16 real components. Consequently the minimal amount of supersymmetry in ten dimensions which we denote by $N = 1$ has 16 *real supercharges*.

Let us now see how one can construct massless representations of the supersymmetry algebra (2.1). It is a well known fact that for massless states the supersymmetry algebra implies the vanishing of half of the supercharges [20, 21]. Thus the massless states are constructed using only eight supercharges which form a spinor representation of the little group $SO(8)$. The simplest representation is given by the the vector (super)multiplet which consists of a vector field A_M and a spinor λ^+. It is not hard to check that the number of bosonic and fermionic degrees of freedom coincide. Knowing that the physical states are counted by the little group one sees that a vector in ten dimensions has 8 degrees of freedom which is precisely the same as the degrees of freedom of an $SO(8)$ spinor. In order to construct higher spin representations one considers tensor products of the vector multiplet representation with other $SO(8)$ non-trivial representations [20, 22] and the result can be summarized in table 2.1. $\hat{g}_{MN}$ denotes the metric, $\Psi^{\pm}_M$ are grav-

Multiplet	Bosons	Fermions
vector	$\hat{A}_M$	$\lambda^{\pm}$
graviton	$\hat{g}_{MN},\ \hat{B}_{MN},\ \hat{\phi}$	$\Psi^+_M,\ \lambda^+$
gravitino	$l,\ \hat{C}_{MN},\ \hat{A}^*_4$	$\Psi^+_M,\ \lambda^+$
gravitino	$\hat{A}_M,\ \hat{C}_{MNP}$	$\Psi^-_M,\ \lambda^-$

Table 2.1: $N = 1$ supermultiplets in ten dimensions.

itinos, $\lambda^{\pm}$ are spinors, $\hat{\phi}$ is the dilaton, l is a scalar, $\hat{A}, \hat{B}, \hat{C}$ are totally antisymmetric tensors of various degrees.[1] The $\pm$ superscripts denote the chirality of the spinors and $*$ denotes the fact that $\hat{A}_4$ has a self-dual field strength. Note that there are two inequivalent gravitino representations which differ in the bosonic content and the spinors have opposite chiralities. As we will see in a while this leads to two inequivalent theories one can construct in ten dimensions. In what follows we are going to be interested in extended supersymmetry in ten dimensions, namely $N = 2$ which is equivalent to 32 real supercharges. The massless representations in this case are straightforward to determine by putting together the representations found above. Before we list these representations let us make few comments on $N = 2$ supersymmetry in ten dimensions. The lowest spin representation necessarily contains a spin 2 particle – the graviton. Thus in such theories there are no vector multiplets and the graviton multiplet incorporates all particles with spin less or equal to 2. Moreover since it is not known how to consistently couple to gravity particles with spin higher than 2, the amount of 32 supercharges is the maximum supersymmetry one can deal with.

As mentioned above the two inequivalent gravitino representations found in table 2.1 lead to two different $N = 2$ multiplets and consequently to two different $N = 2$ theories in ten dimensions. They are known as type IIA and type IIB theories.

Type IIA theory is non-chiral in the sense that one chooses the gravitino multiplet which has opposite chirality to the gravitino sitting in the graviton multiplet. Thus the massless spectrum in this case comprises the metric $\hat{g}_{MN}$ the two-form $\hat{B}_2$, the dilaton $\hat{\phi}$, a one and a three-form $\hat{A}_1$ and $\hat{C}_3$ while the fermionic components are given by two

[1] Hats are used in order to be consistent with the later conventions that hatted quantities live in ten dimensions.

gravitini $\Psi_M^{\pm}$ and two spin 1/2 fields $\lambda^{\pm}$.

Type IIB theory is known as the chiral type II theory and its massless modes are the metric $\hat{g}_{MN}$ the two-form $\hat{B}_2$, the dilaton $\hat{\phi}$, and zero, two and four-forms l, $\hat{C}_2$ and $\hat{A}_4$ two gravtini Ψ_M^+ and two spin 1/2 fields λ^+.

In what follows we will neglect the fermions and look only at the bosonic modes. We will however keep in mind that the theories are in fact supersymmetric and that the fermionic parts can be obtained using supersymmetry. Before we move on let us make one more comment. In type II string theories the massless bosonic fields described above arise from two different sectors known as Neveu-Schwarz–Neveu-Schwarz (NS-NS) and Ramond-Ramond (RR) respectively. The (bosonic) fields in the graviton multiplet in table 2.1 come from the NS-NS sector while the ones in the gravitino multiplets appear from the RR sectors. Thus the NS-NS sectors of the two type II theories are identical while the distinction comes only from the RR sectors. Type IIA contains odd form fields $\hat{A}_1$ and $\hat{C}_3$ with even form field strengths $\hat{F}_2$ and $\hat{F}_4$, while in type IIB one encounters even form fields l, $\hat{C}_2$, $\hat{A}_4$ with odd form field strengths $\mathrm{d}l$, $\hat{F}_3$, $\hat{F}_5$.

In the next subsections we give a brief review of the dynamics of the two type II theories in ten dimensions whose spectra we described above. Before starting let us make a couple of remarks regarding our conventions. We use a units system where the Planck constant, the speed of light and the Newton constant are taken to be equal to the unity. Throughout the thesis we use differential form notation which we record in appendix A. As we mainly compactify theories from ten to four dimensions we distinguish the fields by using hatted symbols for the fields in ten dimensions. The Hodge star operator will be encountered in ten, six and four dimensions. However we do not introduce different notations for these cases as it should be quite clear from the context on what spaces this operator acts.

2.1.2 (Massive) Type IIA theory

For the case of the type IIA theory, the bosonic action is given by [21][2]

$$\begin{aligned} S_{10} &= \int \Big[e^{-2\hat{\phi}} \Big(-\frac{1}{2}\hat{R} * \mathbf{1} + 2\mathrm{d}\hat{\phi} \wedge *\mathrm{d}\hat{\phi} - \frac{1}{4}\hat{H}_3 \wedge *\hat{H}_3 \Big) \\ &\qquad - \frac{1}{2}\Big(\hat{F}_2 \wedge *\hat{F}_2 + \hat{F}_4 \wedge *\hat{F}_4 \Big) + \mathcal{L}_{top} \Big] , \end{aligned} \tag{2.2}$$

where the field strengths are defined as

$$\hat{F}_2 = \mathrm{d}\hat{A}_1 \,, \quad \hat{F}_4 = \mathrm{d}\hat{C}_3 - \hat{B}_2 \wedge \mathrm{d}\hat{A}_1 \,, \quad \hat{H}_3 = \mathrm{d}\hat{B}_2 \,, \tag{2.3}$$

and the topological terms read

$$\mathcal{L}_{top} = -\frac{1}{2}\Big[\hat{B}_2 \wedge \mathrm{d}\hat{C}_3 \wedge \mathrm{d}\hat{C}_3 - (\hat{B}_2)^2 \wedge \mathrm{d}\hat{C}_3 \wedge \mathrm{d}\hat{A}_1 + \frac{1}{3}(\hat{B}_2)^3 \wedge \mathrm{d}\hat{A}_1 \wedge \mathrm{d}\hat{A}_1 \Big] . \tag{2.4}$$

[2]This supergravity was first constructed in [23].

The action above can be easily seen to be invariant under the Abelian gauge transformations

$$\begin{aligned} \delta\hat{A}_1 &= \mathrm{d}\lambda\ , & \delta\hat{C}_3 &= \mathrm{d}\Sigma_2, \\ \delta\hat{B}_2 &= \mathrm{d}\Lambda_1\ , & \hat{\delta}C_3 &= \Lambda_1\wedge\mathrm{d}\hat{A}_1\ . \end{aligned} \tag{2.5}$$

It turns out that in ten dimensions one can find a whole family of non-chiral $N = 2$ theories parameterized by a constant parameter m [24]. The action for this theory can again be written as in (2.2) with the only modifications that a cosmological constant term has to be added $-\frac{1}{2}m^2 * 1$ and now the field strengths are given by

$$\hat{F}_2 = \mathrm{d}\hat{A}_1 + m\hat{B}_2\ , \quad \hat{F}_4 = \mathrm{d}\hat{C}_3 - \hat{B}_2\wedge\mathrm{d}\hat{A}_1 - \frac{m}{2}(\hat{B}_2)^2\ , \quad \hat{H}_3 = \mathrm{d}\hat{B}_2\ , \tag{2.6}$$

while the topological terms read

$$\begin{aligned} \mathcal{L}_{top} &= -\frac{1}{2}\Big[\hat{B}_2\wedge\mathrm{d}\hat{C}_3\wedge\mathrm{d}\hat{C}_3 - (\hat{B}_2)^2\wedge\mathrm{d}\hat{C}_3\wedge\mathrm{d}\hat{A}_1 + \frac{1}{3}(\hat{B}_2)^3\wedge\mathrm{d}\hat{A}_1\wedge\mathrm{d}\hat{A}_1 \\ &\qquad -\frac{m}{3}(\hat{B}_2)^3\wedge\mathrm{d}\hat{C}_3 + \frac{m}{4}(\hat{B}_2)^4\wedge\mathrm{d}\hat{A}_1 + \frac{m^2}{20}(\hat{B}_2)^5\Big]\ . \end{aligned} \tag{2.7}$$

In this case the gauge transformations (2.5) take the form

$$\begin{aligned} \delta\hat{A}_1 &= \mathrm{d}\lambda\ , & \delta\hat{C}_3 &= \mathrm{d}\Sigma_2\ , \\ \delta\hat{B}_2 &= \mathrm{d}\Lambda_1\ , & \delta\hat{C}_3 &= \Lambda_1\wedge\mathrm{d}\hat{A}_1\ , \quad \delta\hat{A}_1 = -m\Lambda_1\ . \end{aligned} \tag{2.8}$$

We should notice that the field A_1 has a transformation similar to the one of a Goldstone boson and thus can be gauged away; in this gauge the two-form field B_2 effectively becomes massive. As a last remark we notice that in the limit $m \to 0$ the massive type IIA action reduces to (2.2).

2.1.3 Type IIB theory

Let us now come to the type IIB theory. In this case there is one subtlety which one has to take into account when writing an action, namely the fact that the four-form field $\hat{A}_4$ has self-dual field strength. If one would naively try to write down a kinetic term for this field like $\hat{F}_5\wedge *\hat{F}_5$ with $\hat{F}_5$ being the field strength of $\hat{A}_4$ imposing the self-duality condition on $\hat{F}_5$ namely $\hat{F}_5 = *\hat{F}_5$ the kinetic term above vanishes identically. The fact that a covariant action can not be written in such cases was noticed long time ago [25], but nevertheless the theory is perfectly well defined by the equations of motion the fields have to satisfy. More recently a covariant action for the type IIB theory was constructed [26], but we will content ourselves to write an action from which the self-duality does not follow, but it rather has to be imposed by hand in order to obtain the correct equations of motion. In the string frame this action has the form [21]

$$\begin{aligned} S_{\mathrm{IIB}} &= \int e^{-2\hat{\phi}}\left(-\frac{1}{2}\hat{R} * \mathbf{1} + 2\mathrm{d}\hat{\phi}\wedge *\mathrm{d}\hat{\phi} - \frac{1}{4}\hat{H}_3\wedge *\hat{H}_3\right) \\ &\qquad -\frac{1}{2}\left(\mathrm{d}l\wedge *\mathrm{d}l + \hat{F}_3\wedge *\hat{F}_3 \ + \frac{1}{2}\hat{F}_5\wedge *\hat{F}_5\right) - \frac{1}{2}\hat{A}_4\wedge\hat{H}_3\wedge\mathrm{d}\hat{C}_2\ , \end{aligned} \tag{2.9}$$

where the field strengths are defined as

$$\hat{H}_3 = \mathrm{d}\hat{B}_2 \ , \qquad \hat{F}_3 = \mathrm{d}\hat{C}_2 - l\mathrm{d}\hat{B}_2 \, , \qquad \hat{F}_5 = \mathrm{d}\hat{A}_4 - \frac{1}{2}\hat{C}_2 \wedge \mathrm{d}\hat{B}_2 + \frac{1}{2}\hat{B}_2 \wedge \mathrm{d}\hat{C}_2 \ . \quad (2.10)$$

As in the case of type IIA theory the action above is invariant under the Abelian gauge transformations

$$\begin{aligned} &\delta A_4 = \mathrm{d}\Sigma_3 \ , \\ &\delta B_2 = \mathrm{d}\Lambda_1 \ , \qquad \delta A_4 = -\frac{1}{2}\Lambda_1 \wedge \mathrm{d}C_2 \ , \\ &\delta C_2 = \mathrm{d}\Lambda_1' \ , \qquad \delta A_4 = -\frac{1}{2}\Lambda_1' \wedge \mathrm{d}B_2 \ . \end{aligned} \quad (2.11)$$

Besides (2.11) type IIB supergravity features an $SL(2,\mathbb{R})$ which rotates the two two-forms $(\hat{B}_2,\ \hat{C}_2)$ and the two scalars $(\hat{\phi},\ l)$ into one another. In order two make this symmetry manifest one goes to the Einstein frame and introduces the complex scalar $\tau = l + ie^{-\phi}$ and the complex two-form $G_3 = \mathrm{d}C_3 + \tau H_3$. With these redefinitions the action reads [21, 27][3]

$$\begin{aligned} S_{IIB}^{E} \ = \ & \frac{1}{2}\int \left(-R * 1 - \frac{\mathrm{d}\tau \wedge *\mathrm{d}\bar{\tau}}{2(\tau_2)^2} - \frac{1}{2}\,\frac{G_3 \wedge *\bar{G}_3}{\tau_2} - \frac{1}{2}F_5 \wedge *F_5\right) \\ & -\frac{1}{8}\int \frac{A_4 \wedge G_3 \wedge \bar{G}_3}{\tau_2} \ , \end{aligned} \quad (2.12)$$

where τ_2 denotes the imaginary part of the scalar τ. It is easy to see that in this form the action is invariant under and the following transformations

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \ , \qquad G_3 \to G_3' = \frac{G_3}{c\tau + d} \ . \quad (2.13)$$

where $ad - bc = 1$. Note that under the above transformations the fields C_2 and B_2 transform as

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix}' = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \ . \quad (2.14)$$

2.2 $N = 2$ supersymmetry in four dimensions

Anticipating that the theories which we obtain in four dimensions are $N = 2$ supersymmetric we present in this section some basic notions about supersymmetry in four dimensions.

[3] In fact one has to redefine all the RR fields in (2.9) by a factor of $\sqrt{2}$ i.e. $A_p^{RR} \to 1/\sqrt{2}A_p^{RR}$ in order to obtain this action. However we work with the convention in (2.9) so that we obtain directly the correct normalization in four dimensions.

Minimal supersymmetry is denoted by $N = 1$ which is equivalent to saying that the theory is invariant under the action of four real supercharges. This is so because in four dimensions one can impose either the Weyl or the Majorana condition on spinors (but not both) which effectively reduces the number of degrees of freedom in a spinor to four. As in section 2.1.1 when we look for massless representations of the $N = 1$ supersymmetry algebra only half of the supercharges are non-vanishing. These two supercharges can be organized in 'creation' and 'annihilation' operators (i.e. raising and lowering the helicity). Thus the $N = 1$ super-multiplet contains only two states whose spin differ by 1/2. The physical interesting multiplets are listed in table 2.2.

Multiplet	Bosons	Fermions
chiral	Φ	ψ
vector	A_μ	λ
gravitino	A_μ	Ψ_μ
graviton	$g_{\mu\nu}$	$\Psi_\mu,$

Table 2.2: $N = 1$ supermultiplets in four dimensions. Φ denotes a complex scalar, $A_\mu,\ \mu = 0, \ldots, 3$ is a vector, $g_{\mu\nu}$ is the metric, $\psi,\ \lambda$ are spin 1/2 fields and Ψ_μ denotes the gravitino.

As in section 2.1.1 the $N = 2$ multiplets are obtained by combining various $N = 1$ multiplets. Putting together two chiral multiplets one obtains the so called hyper-multiplet. It contains two spinors and four real scalars and they form the matter part in $N = 2$ theories. A vector and a chiral multiplet give rise to an $N = 2$ vector multiplet[4] which consists of a vector, two gaugini (spin 1/2 fields) and one complex scalar. Finally the graviton and gravitino multiplet from table 2.2 give rise to the $N = 2$ gravity multiplet which consists of the graviton, two gravitini and a vector field known also as the graviphoton. These possibilities are summarized in table 2.3.

Multiplet	Bosons	Fermions
hyper-multiplet	$4 \times \varphi,$	$2 \times \psi$
vector	A_μ, Φ	$2 \times \lambda$
graviton	$g_{\mu\nu}, A^0_\mu$	$2 \times \Psi_\mu,$

Table 2.3: $N = 2$ supermultiplets in four dimensions. Note that we have used the symbol φ to denote the real scalars in contrast with Φ which denotes complex ones.

It is worth mentioning that in four dimensions these super-multiplets have (Poincaré) dual descriptions which will in many cases appear directly from the compactifications of higher dimensional theories. As a real scalar in four dimensions has a dual description in

[4] This is also known as the chiral $N = 2$ multiplet.

terms of an antisymmetric tensor field a hyper-multiplet can appear as a tensor multiplet whose bosonic degrees of freedom are given by $(B_{\mu\nu}, 3 \times \varphi)$. Furthermore one can also encounter double tensor multiplets $(B_{\mu\nu}, C_{\mu\nu}, 2 \times \varphi)$. Less usual are the vector-tensor multiplets $(A_\mu, B_{\mu\nu}, \varphi)$ where $B_{\mu\nu}$ is the dual description of one of the real scalars in the vector multiplet from table 2.3. In most cases we will dualize these additional multiplets to the ones in table 2.3 and we will refer to these multiplets as the standard spectrum of $N = 2$ theories. However we will encounter some cases where this dualization is not possible as some of the fields become massive and the dual description is less easy to formulate (for more details about dualizations see appendix D.2).

2.3 Calabi–Yau compactifications

Having introduced the main features of the type II theories in ten dimensions we can turn to study their compactification to four dimensions. As explained in the introduction Calabi–Yau threefolds are the most natural internal spaces to consider as they preserve minimal amount of supersymmetry. After a general introduction where we present in more detail the relation between the the compactification manifold and the supersymmetries which survive in four dimensions we highlight the main ideas encountered in deriving the low energy effective action of type II supergravities compactified on Calabi–Yau three-folds.

2.3.1 Calabi–Yau threefolds and supersymmetry

Calabi–Yau threefolds have originally appeared when trying to obtain $N = 1$ vacua of the heterotic string [9]. Such compactification were thought to play an important role in string phenomenology as the four-dimensional theories generically have minimal amount of supersymmetry, chiral fermions and non-Abelian gauge groups which are large enough to contain the standard model gauge group or even GUT groups. Our main interest though will be the study of the low energy effective theories ($N = 2$ supergravities) obtained by compactifying type II theories on Calabi–Yau manifolds and in particular the mirror symmetry which relates them in four dimensions. Such compactifications were first considered in [28, 29] and the low energy effective actions in four dimensions were first derived in [30–32].

Before we start the derivation of the effective actions in four dimensions let us describe how supersymmetry is directly related to the choice for the internal manifold. This short review is going to be essential for understanding the generalizations we consider in chapter 5.

We assume that the ten dimensional space splits as

$$\mathcal{M} = \mathbb{R}^{1,3} \times K_6 \ , \tag{2.15}$$

for some yet unknown internal six-dimensional manifold K_6. First of all in order to have a supersymmetric action in four dimensions one needs to reduce the ten dimensional supercharge (or equivalently the ten dimensional gravitino) to a four dimensional one. For this the minimal requirement would be that the internal manifold admits globally-defined nowhere-vanishing spinors which we denote by $\eta^I,\ I = 1,\dots,N$. The ten dimensional gravitino $\hat{\Psi}_M$ can be expanded in the internal spinors and one defines the four dimensional gravitini Ψ_μ^I as

$$\hat{\Psi}_\mu = \sum_{I=1}^{N} \Psi_\mu^I \otimes \eta^I \ . \tag{2.16}$$

From this schematic expansion one sees that the number of four dimensional gravitini (i.e. the number of supercharges) in four dimensions depends on the number of independent spinors on the internal manifold. In particular, in order to preserve the minimal amount of supersymmetry one needs one single spinor on the internal manifold which is globally defined and nowhere vanishing. This is equivalent to saying (for a six dimensional manifold) that the structure group[5] of the manifold has been reduced to $SU(3)$ and the spinor which is globally defined is a singlet under this $SU(3)$. (for a more detailed discussion see appendix C) For generic manifolds such spinors do not exist and thus compactifications on such spaces break all supersymmetries while special manifolds like the torus have the maximum number of independent spinors and so they preserve all supersymmetries.

A second issue related to supersymmetry is whether the ground state which we choose is a supersymmetric one. In case we ask for the four-dimensional theory to have a Minkowski vacuum (like in (2.15)) all the fields which transform non-trivially under the Lorentz group have to vanish in the background. In particular all the fermions vanish and in order for the vacuum to be supersymmetric we also need that the same happens with their supersymmetry variations. Consider the ten dimensional supersymmetry variations of the gravitinos which can schematically be written as (for the exact supersymmetry variations see for example [22, 23, 33])

$$\delta\psi_M = \nabla_M \epsilon + \sum_p (\Gamma \cdot \hat{F}_p)_M \epsilon + \dots \ ; \qquad M = 0,\dots,9 \ , \tag{2.17}$$

where the sum goes over the p-form field strengths in the theory, the dots indicate fermionic terms, while the term $(\Gamma\cdot F_p)_M$ denotes contractions of the p-form field strengths with gamma matrices of the form

$$(\Gamma \cdot \hat{F}_p)_M \sim \Gamma_{MN_1\dots N_p}(\hat{F}_p)^{N_1\dots N_p} + \Gamma_{N_1\dots N_{p-1}}(\hat{F}_p)^{N_1\dots N_{p-1}}{}_M \ . \tag{2.18}$$

Additionally there can be further factors of the dilaton, but as in this analysis we always keep it constant on the internal space these factors are not relevant for the future discussion. The simplest ground state consistent with four-dimensional Lorentz invariance is one where all the fields (except for the metric) are zero. In this background the field strengths $\hat{F}_p$ also vanish and the condition for supersymmetry $\delta\psi_M = 0$ becomes

$$\nabla_M \epsilon = 0 \ . \tag{2.19}$$

[5] Given a manifold and a metric one can introduce orthonormal frames at every point. The transitions between different such frames define the structure group which in general is a subgroup of $SO(n)$ where n is the dimension of the manifold. For a more precise definition see appenxid C.

Note that this is a ten dimensional equation and in order to obtain some constraints for the internal manifold one has to decompose it into its space-time and internal parts. The standard thing one does is to write the spinor ϵ as a direct product

$$\epsilon = \theta \otimes \eta \ , \tag{2.20}$$

were θ is a space-time spinor while η is an internal one. Since both θ and ϵ are physical spinors they have to be anticommuting and this implies that η has to be a commuting spinor. With this decomposition the supersymmetry preserving condition reads

$$\begin{aligned} \partial_\mu \theta = 0 \ , \qquad & \mu = 0, \ldots, 3 \ , \\ \nabla_m \eta = 0 \ , \qquad & m = 1, \ldots, 6 \ , \end{aligned} \tag{2.21}$$

where ∇_m denotes the covariant derivative on K_6 with respect to the Levi-Civita connection. The first equation does not constrain the space-time any further as Minkowski space admits constant spinors. The second relation on the other hand is highly non-trivial and imposes very restrictive conditions on the internal manifold. Note that in particular this relation means that η is globally defined[6] which implies that such a manifold automatically preserves some supersymmetry of the effective action. Manifolds with a covariantly constant spinor are known as Calabi–Yau manifolds and we will consider them in this section as compactification manifolds. Using (2.21) one can show [22] that such manifolds are complex Kähler and have $SU(3)$ holonomy. These properties together are in fact the usual definition of Calabi–Yau manifolds which one can find in the literature [34]. Equivalently, Calabi–Yau manifolds can be characterized by the existence of a Kähler form J and a unique holomorphic $(3,0)$ form Ω which are both covariantly constant. It was proven by Yau that Calabi–Yau manifolds admit a unique Ricci flat metric. This means that the background (2.15) with all other fields vanishing is a solution to the ten-dimensional Einstein equations. Moreover it is well known that backgrounds of the type (2.15) are consistent geometries in which strings can move [34].

Before we move on to consider compactifications of type II theories on Calabi–Yau manifolds, let us summarize the results of this section. In order that the four dimensional effective action is supersymmetric we need that the internal manifold admits a globally-defined nowhere-vanishing spinor or in other words its structure group reduces to $SU(3)$ or a subgroup thereof. Independently, if one asks for a supersymmetric ground state this spinor has to satisfy further constraints like for example (2.21) which came from the vanishing of the supersymmetry variations for the fermions in a certain background. For the analysis of this work this last requirement will not play a special role and we will see in the next chapters cases where the supersymmetry variations of the fermions do not vanish either because of of the presence of non-trivial fields in the background (fluxes) or because the relations (2.21) are no longer satisfied.

[6] One can define the spinor η by parallel transport as (2.21) guarantees that this is path independent.

2.3.2 Calabi–Yau manifolds and their moduli space

Even though no Calabi–Yau metric was explicitly constructed up to now and thus in some sense it is hard to picture such a manifold, their topology is quite well understood. For obtaining the low energy physics this is enough as the harmonic forms encode the information about the massless spectrum in four dimensions. Denoting by $h^{(p,q)}$ the Hodge numbers i.e. the dimension of the corresponding (p, q) cohomology group $H^{p,q}$, it is easy to see that the only non-trivial ones are given by [22]

$$\begin{aligned} &h^{(0,0)} = h^{(3,0)} = h^{(0,3)} = h^{(3,3)} = 1 \ , \\ &h^{(1,1)} = h^{(2,2)} \ , \qquad h^{(2,1)} = h^{(2,1)} \ , \end{aligned} \tag{2.22}$$

and thus the topology of such manifolds is characterized by two independent natural numbers[7] $h^{(1,1)}$ and $h^{(2,1)}$. Consequently one can write the Hodge diamond in the following way

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{(1,1)} & & 0 & \\ 1 & & h^{(2,1)} & & h^{(2,1)} & & 1 \\ & 0 & & h^{(1,1)} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \ . \tag{2.23}$$

The fact that this diamond is symmetric with respect to to a vertical axis which passes through the middle is due to complex conjugation while the symmetry with respect to the horizontal axis is due to Poincaré duality. As we have seen in the introduction there is one further symmetry with respect to a diagonal axis which exchanges $h^{(1,1)}$ and $h^{(2,1)}$ between the mirror manifolds.

In order to characterize the different cohomology groups introduced above we denote by $\omega_i, \ i = 1, \dots, h^{(1,1)}$ a basis for the $(1, 1)$ harmonic forms and by $\tilde{\omega}^i$ their duals which form a basis for the harmonic $(2, 2)$ forms. We choose these forms to be normalized as

$$\int_{Y_3} \omega_i \wedge \tilde{\omega}^j = \delta_i^j \ . \tag{2.24}$$

Similarly we can introduce a real basis for $H^3(Y)$ $(\alpha_A, \beta^A), \ A = (0, a) = 0, \dots, h^{(2,1)}$ which is normalized as

$$\begin{aligned} \int_{Y_3} \alpha_A \wedge \beta^B \;&=\; \delta_A^B \;=\; -\int_{Y_3} \beta^B \wedge \alpha_A \ , \qquad A, B = 0, \dots, h^{(2,1)} \ , \\ \int_{Y_3} \alpha_A \wedge \alpha_B \;&=\; \int_{Y_3} \beta^A \wedge \beta^B \;=\; 0 \ . \end{aligned} \tag{2.25}$$

[7] For Calabi–Yau manifolds $h^{(1,1)}$ has to be at least one because they are Kähler. $h^{(2,1)}$ on the other hand can also be zero, but in this case mirror symmetry is less obvious so we only consider cases for which $h^{(2,1)}$ is also non-vanishing.

Let us now see what the notion of moduli space means from a physical point of view. Compactifying a theory on a Calabi–Yau manifold Y implies that the ten dimensional space splits into a direct product $M_{10} = R^{3,1} \otimes Y$ where $R^{3,1}$ denotes the four dimensional Minkowski space. Formally this means to choose a background metric of the form

$$\hat{G}_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g^{0\,CY}_{mn} \end{pmatrix} , \tag{2.26}$$

where $\eta_{\mu\nu} = diag(-1,1,1,1)$ and $g^{0\,CY}_{mn}$ represents the Calabi–Yau metric. Due to its Ricci flatness this configuration automatically solves the Einstein equations when all other fields vanish. In order to identify the states in the effective four dimensional theory one have to consider variations of the ten dimensional metric about the background (2.26). In particular, the four dimensional graviton appears as fluctuations of the metric in the four space-time directions $\delta G_{\mu\nu}$ while the mixed components $\delta G_{\mu N}$ generally give rise to massive states as no harmonic one forms are present on a Calabi–Yau space. What is going to be interesting for us in the following are the internal fluctuations of the metric. In string theory (or in supergravity) there is no dynamical way to choose a particular manifold for performing the compactification. This means that for the ten dimensional theory all Calabi–Yau manifolds look the same and going from a Calabi–Yau manifold to another one which is infinitesimally close costs no energy. Thus all the fields which appear from the deformations of the internal Calabi–Yau manifold are flat directions of the potential and parameterize the vacuum degeneracy. Let us have a closer look and see how these deformations can be interpreted as fields in four dimensions.

The deformed metric should still satisfy the vacuum Einstein equation and thus we are lead to consider fluctuations δg_{mn} which satisfy

$$R_{mn}(g + \delta g) = 0 \ . \tag{2.27}$$

Expanding to the first order in δg one obtains the so called Lichnerowicz equation

$$\nabla_p \nabla^p \delta g_{mn} + 2R_{mpnq}\delta g^{pq} = 0 \ , \tag{2.28}$$

where $\delta g^{pq} = g^{mp} g^{nq} \delta g_{mn}$. As the Riemann tensor of Kähler manifolds has a very simple index structure (in particular only $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ and combinations obtained using its symmetry properties are non-vanishing) one can immediately see that the the equations for the $(2,0)$ metric variations $\delta g_{\alpha\beta}$ and for the $(1,1)$ ones $\delta g_{\alpha\bar{\beta}}$ decouple. It can be checked that the solutions to the above equations can be written as

$$\begin{aligned} \delta g_{\alpha\beta} &= z^a \, \Omega_{\alpha\gamma\delta} (\chi_a)^{\gamma\delta}_{\beta} \ , \\ \delta g_{\alpha\bar{\beta}} &= v^i \, (\omega_i)_{\alpha\bar{\beta}} \ , \end{aligned} \tag{2.29}$$

where Ω is the holomorphic $(3,0)$ form, χ are $(1,2)$ harmonic forms and ω_i are $(1,1)$ harmonic forms introduced before. From the Calabi–Yau point of view, z^a and v^i are constants parameterizing the above expansions, but from a four dimensional point of view they appear as scalar fields and they are precisely the moduli we were mentioning before. Their dynamics is described in four dimensions by nonlinear sigma models which have as target space the corresponding moduli space. One can check this by computing explicitly the ten dimensional Ricci scalar. However we do not perform this computation here, but just rely on the results in the literature [30, 35, 36].

The metric deformations in (2.29) have also a geometrical interpretation. Changing the metric by a $(2,0)$ piece renders the metric non-hermitian. This can obviously not be undone by a holomorphic coordinate transformation so the only solution to obtain again a hermitian metric is to change the complex structure. Thus the $(2,0)$ deformations of the metric are responsible for the so called complex structure deformations. On the other hand, changing the metric by a $(1,1)$ piece does not require any change in the complex structure as the new metric is still hermitian. However what changes in this case is the Kähler form and thus $\delta g_{\alpha\bar{\beta}}$ are termed Kähler class deformations.

In string theory together with the metric, in the NS-NS sector one finds an antisymmetric tensor field B_{MN} whose expansion in the harmonic $(1,1)$ forms produces a set of $h^{(1,1)}$ scalar fields which combine together with the Kähler class deformations to form the complex scalars $t^i = b^i + iv^i$. It is known [36] that the scalar manifold spanned by t^i is a special Kähler manifold and the same holds true for the manifold spanned by z^a. Knowing that one of the two sets of scalars enter in the vector multiplets in one of the type II theories compactified to four dimensions[8] this is consistent with the result obtained for $N = 2$ supergravity theories where the scalars in the vector multiplets span indeed a special Kähler manifold [37].

Before moving on let us make the one more remark. Due to the fact that the equations for the Kähler class and complex structure deformations decouple, the moduli space of Calabi–Yau manifolds can be written as

$$\mathcal{M} = \mathcal{M}_{1,1} \otimes \mathcal{M}_{2,1} \ . \tag{2.30}$$

We can in this way see that the geometrical interpretation of mirror symmetry is that it exchanges the (complexified) Kähler deformations with the complex structure ones.

2.3.3 Type IIA on CY$_3$

Let us present the first example of a Calabi–Yau compactification by choosing as starting point the ten dimensional type IIA supergravity briefly discussed in section 2.1.2. The computation we present here was first discussed in [30]. As we showed before Calabi–Yau manifolds break three quarters of the supersymmetry which is present in ten dimensions and so we expect to obtain after compactifying a type II theory an $N = 2$ supergravity in four dimensions. Thus we will write the final form of the action in the standard $N = 2$ way as presented in appendix B.

The first step in performing the KK reduction is to specify the background field configuration which has to be a solution to the ten dimensional equations of motion. For the metric we take the Ansatz (2.26) while for the matter fields we make the simplifying

[8] t^i are associated with the scalars in the vector multiplets in the case of type IIA theory, while z^a enter the vector multiplets in the case of type IIB theory

assumption that they are constant in the ten-dimensional background.[9] This implies that all the field strengths vanish and thus due to the fact that the Calabi–Yau metric has zero Ricci tensor, the choice (2.26) automatically satisfies the ten dimensional Einstein equations. Before turning to the KK reduction let us emphasize once more that the configuration we have just discussed is a supersymmetric one. This means that the four dimensional theory admits a supersymmetric Minkowski vacuum. This will however not be the case in the situations we will treat in the next chapters.

The reduction proceeds as explained in some detail in appendix D.1 by taking fluctuations around the background and deriving the dynamics for these fluctuations. As we are only interested in the low energy approximation we truncate out the fields which acquire a mass in the KK reduction due to the dependence on the internal manifold. This is usually done by keeping in the expansion of the ten dimensional fields only the massless modes. They turn out to correspond to harmonic forms on the internal manifold [22]. Thus the massless fields in four dimensions are completely determined by the topology of the compactification manifold.

It is instructive to have a closer look at the four dimensional massless spectrum obtained by compactifying type IIA theory on a Calabi–Yau three-fold and see how it arranges itself into $N = 2$ representations. We will use the following analysis in the more complicated cases which are discussed in the next chapters. Expanding the ten dimensional fields $\hat{A}_1$, $\hat{B}_2$ and $\hat{C}_3$ in the Calabi–Yau harmonic forms defined above we obtain

$$\begin{aligned} \hat{A}_1 &= A^0 \ , \\ \hat{B}_2 &= B_2 + b^i\omega_i \ , \\ \hat{C}_3 &= C_3 + A^i \wedge \omega_i + \xi^A\alpha_A - \tilde{\xi}_A\beta^A \ , \end{aligned} \tag{2.31}$$

where C_3 is a three-form, B_2 a two-form, (A^0, A^i) are one-forms and $b^i, \xi^A, \tilde{\xi}_A$ are scalar fields in $D = 4$. Furthermore we have to take into account the other massless modes which appear due to the fluctuations of the metric on the internal manifold, namely the scalars v^i and z^a (2.29). The Kähler class moduli v^i (2.29) combine with the scalars b^i defined in (2.31) into complex scalar fields $t^i = b^i + iv^i$ which together with the one-forms A^i from (2.31) form the bosonic components of $h^{(1,1)}$ vector multiplets (A^i, t^i). The complex structure deformations z^a and the scalars $\xi^a, \tilde{\xi}_a$ of (2.31) are members of $h^{(2,1)}$ hyper-multiplets while $\xi^0, \tilde{\xi}_0$ together with the dilaton ϕ and B_2 form the tensor multiplet. A^0 in (2.31) is the graviphoton which together with the four-dimensional metric $g_{\mu\nu}$ describes the bosonic components of the gravitational multiplet.

The next step in performing the compactification is the derivation of the low energy

[9] Strictly speaking this is not required to be the case for the internal directions and the background can in principle depend on the internal coordinates. The possibility of having non-trivial fields in the background was first considered in [9, 38]. However for a metric of the form (2.26) the ten dimensional equations of motion impose that these fields are in fact constants. It was shown in [29, 39, 40] that in order to incorporate such non-trivial fields one has to introduce a warp factor in the Ansatz (2.26). In the next chapter we will discuss some generalization of this setup in that we allow for some field strengths to have non-vanishing values in the background.

dynamics of the fields obtained in the expansion (2.31). This can be done by replacing the expansion (2.31) in the ten dimensional equations of motion for type IIA theory obtained from the action (2.2) and derive in this way the equations of motion for the four dimensional fields. This is however cumbersome as the equations of motion have a quite complicated form. A more transparent way would be to replace the expansion (2.31) directly in the action and perform the integration over the internal manifold. The drawback is that in principle this method is not guaranteed to work every time and one should really check that the action obtained in this way correctly reproduce the four dimension equations of motion. However for the cases we study here this second method always work and we will never show explicitly that the result gives the right equations of motion.

First we compute the field strengths (2.3) assuming the Ansatz (2.31). Using the fact that the harmonic forms are closed it is straightforward to obtain the following expansions

$$\begin{aligned}\hat{F}_2 &= \mathrm{d}A^0 \ , \\ \hat{H}_3 &= \mathrm{d}B_2 + \mathrm{d}b^i\omega_i \ , \\ \hat{F}_4 &= \mathrm{d}C_3 - B_2\wedge \mathrm{d}A^0 + (\mathrm{d}A^i - b^i\mathrm{d}A^0)\wedge\omega_i + \mathrm{d}\xi^A\alpha_A - \mathrm{d}\tilde{\xi}_A\beta^A \ .\end{aligned} \tag{2.32}$$

Now one plugs (2.31) and (2.32) into the action (2.2) and performs the integrals over the Calabi–Yau space. The different terms in the action (2.2) become

$$\begin{aligned}-\frac{1}{4}\int_{Y_3}\hat{H}_3\wedge *\hat{H}_3 &= -\frac{\mathcal{K}}{4}\mathrm{d}B_2\wedge *\mathrm{d}B_2 - \mathcal{K}g_{ij}\mathrm{d}b^i\wedge *\mathrm{d}b^j \ , \\ -\frac{1}{2}\int_{Y_3}\hat{F}_2\wedge *\hat{F}_2 &= -\frac{\mathcal{K}}{2}\mathrm{d}A^0\wedge *\mathrm{d}A^0 \ , \\ -\frac{1}{2}\int_{Y_3}\hat{F}_4\wedge *\hat{F}_4 &= -\frac{\mathcal{K}}{2}(\mathrm{d}C_3 - \mathrm{d}A^0\wedge B_2)\wedge *(\mathrm{d}C_3 - \mathrm{d}A^0\wedge B_2) \\ &\quad -2\mathcal{K}g_{ij}(\mathrm{d}A^i - \mathrm{d}A^0 b^i)\wedge *(\mathrm{d}A^j - \mathrm{d}A^0 b^j) \\ &\quad +\frac{1}{2}\left(\mathrm{Im}\,\mathcal{M}^{-1}\right)^{AB}\left[\mathrm{d}\tilde{\xi}_A - \mathcal{M}_{AC}\mathrm{d}\xi^C\right]\wedge *\left[\mathrm{d}\tilde{\xi}_B - \bar{\mathcal{M}}_{BD}\mathrm{d}\xi^D\right] \ , \\ \int_{Y_3}\mathcal{L}_{top} &= -\frac{1}{2}\Big[B_2\wedge \mathrm{d}(\xi^A\mathrm{d}\tilde{\xi}_A - \tilde{\xi}_A\mathrm{d}\xi^A) + b^i\mathrm{d}A^j\wedge \mathrm{d}A^k\mathcal{K}_{ijk} \\ &\qquad -b^ib^j\mathrm{d}A^k\wedge \mathrm{d}A^0\mathcal{K}_{ijk} + \frac{1}{3}b^ib^jb^k\mathrm{d}A^0\wedge \mathrm{d}A^0\mathcal{K}_{ijk}\Big] \ ,\end{aligned} \tag{2.33}$$

where $\mathcal{K}$ denotes the Calabi–Yau volume, g_{ij} is the metric on the space of Kähler deformations and $\mathcal{K}_{ijk}$ denotes the triple intersection number on a Calabi–Yau manifold Y_3

$$g_{ij} = \frac{1}{4\mathcal{K}}\int_{Y_3}\omega_i\wedge *\omega_j \ , \qquad \mathcal{K}_{ijk} = \int_{Y_3}\omega_i\wedge\omega_j\wedge\omega_k \ . \tag{2.34}$$

Moreover the matrix $\mathcal{M}$ is defined by the following relations

$$\begin{aligned}
\int_{Y_3} \alpha_A \wedge *\alpha_B &= -\left[(\mathrm{Im}\,\mathcal{M}) + (\mathrm{Re}\,\mathcal{M})(\mathrm{Im}\,\mathcal{M})^{-1}(\mathrm{Re}\,\mathcal{M})\right]_{AB} \ , \\
\int_{Y_3} \beta^A \wedge *\beta^B &= -\left[(\mathrm{Im}\,\mathcal{M})^{-1}\right]^{AB} \ , \\
\int_{Y_3} \alpha_A \wedge *\beta^B &= -\left[(\mathrm{Re}\,\mathcal{M})(\mathrm{Im}\,\mathcal{M})^{-1}\right]_A{}^B \ .
\end{aligned} \tag{2.35}$$

Furthermore, as explained in section 2.3.2 the gravitational sector produces the kinetic terms for the Kähler and complex structure moduli which have the form[10]

$$\int_{Y_3} -R^{(10)} * 1 \ \sim \ -R^{(4)} * 1 - g_{ij}\mathrm{d}v^i \wedge *\mathrm{d}v^j - g_{ab}\mathrm{d}z^a \wedge *\mathrm{d}\bar{z}^b \ , \tag{2.36}$$

where g_{ij} is given in (2.34) while $g_{a\bar{b}}$ is the metric on the complex structure moduli space and is given in (B.38).

Comparing with the standard $N = 2$ supergravity spectrum in four dimensions from table 2.3 we see that at this stage we do not have exactly the same fields as a three and a two-form field are present. The three-form is not dynamic as it carries no degrees of freedom in four dimensions. However, in general, its dualization produces a contribution to the cosmological constant. As shown in [14] this constant can be viewed as a specific RR-flux and we will take it into account properly in section 3.3. Here we choose this constant to be zero and hence we discard the contribution of C_3.[11] A two form on the other hand has a dual description in four dimensions in terms of a scalar and thus the only thing we need to do in order to recover the standard spectrum of $N = 2$ supergravity in 4 dimensions is to dualize it to an axion a. This dualization can be performed in the standard way as described in appendix D.2.1 and the result can be written as

$$\mathcal{L}_a = -\frac{e^{2\phi}}{4}\left[\mathrm{d}a - (\tilde{\xi}_A\mathrm{d}\xi^A - \xi^A\mathrm{d}\tilde{\xi}_A)\right] \wedge *\left[\mathrm{d}a - (\tilde{\xi}_A\mathrm{d}\xi^A - \xi^A\mathrm{d}\tilde{\xi}_A)\right] \ . \tag{2.37}$$

In general supergravity actions are written in the Einstein frame and for this one needs to rescale the metric with a factor $e^{2\phi}$ in order to absorb the extra $e^{-2\phi}$ which appears in front of the Einstein–Hilbert term. Note that ϕ denotes the four dimensional dilaton which is defined from the ten dimensional dilaton $\hat{\phi}$ as $e^{-2\phi} = \mathcal{K}e^{-2\hat{\phi}}$.

Finally we have to arrange the gauge field sector. For this note that in $N = 2$ supergravity in addition to the gauge fields coming from the vector-multiplets there is the graviphoton. In a component action it makes sense to treat all the spin one fields

[10] It is well known that this issue is more subtle [30] due to the Weyl rescaling which one should perform in four dimensions in order to incorporate the overall volume factor. However we will ignore this problem and we just use the fact that in the Einstein frame only the kinetic terms (2.36) appear.

[11] In fact what we are going to set to zero is the full gauge invariant combination $C_3 - \mathrm{d}A^0 \wedge B_2$. To see this one should perform the dualization of C_3 in a proper way as explained in the appendix D.2.2 and at the end set the constant which is dual to C_3 to zero.

on the same footing. Thus we introduce the collective notation $A^I = (A^0, A^i)$ where $I = (0, i) = 0, \dots, h^{(1,1)}$ where the index 0 denotes the graviphoton. With this additional piece of notation the final form of the low energy effective action becomes

$$\begin{aligned} S_{IIA} &= \int \Big[-\frac{1}{2} R * \mathbf{1} - g_{ij} \mathrm{d}t^i \wedge * \mathrm{d}\bar{t}^j - h_{uv} \mathrm{d}q^u \wedge * \mathrm{d}q^v \\ &\qquad + \frac{1}{2} \operatorname{Im} \mathcal{N}_{IJ} F^I \wedge * F^J + \frac{1}{2} \operatorname{Re} \mathcal{N}_{IJ} F^I \wedge F^J \Big], \end{aligned} \tag{2.38}$$

where the gauge coupling matrix $\mathcal{N}$ can be directly read off from (2.33)

$$\begin{aligned} \operatorname{Re} \mathcal{N}_{00} &= -\frac{1}{3} \mathcal{K}_{ijk} b^i b^j b^k , \qquad \operatorname{Im} \mathcal{N}_{00} = -\mathcal{K} + \left(\mathcal{K}_{ij} - \frac{1}{4} \frac{\mathcal{K}_i \mathcal{K}_j}{\mathcal{K}} \right) b^i b^j , \\ \operatorname{Re} \mathcal{N}_{i0} &= \frac{1}{2} \mathcal{K}_{ijk} b^j b^k , \qquad \operatorname{Im} \mathcal{N}_{i0} = -\left(\mathcal{K}_{ij} - \frac{1}{4} \frac{\mathcal{K}_i \mathcal{K}_j}{\mathcal{K}} \right) b^j , \\ \operatorname{Re} \mathcal{N}_{ij} &= -\mathcal{K}_{ijk} b^k , \qquad \operatorname{Im} \mathcal{N}_{ij} = \left(\mathcal{K}_{ij} - \frac{1}{4} \frac{\mathcal{K}_i \mathcal{K}_j}{\mathcal{K}} \right) , \end{aligned} \tag{2.39}$$

and h_{uv} is the σ-model metric for the scalars in the hyper-multiplets which can be written in the standard quaternionic form [41]

$$\begin{aligned} h_{uv} \mathrm{d}q^u \wedge * \mathrm{d}q^v &= \mathrm{d}\phi \wedge * \mathrm{d}\phi + g_{ab} \mathrm{d}z^a \wedge * \mathrm{d}\bar{z}^b \\ &\quad + \frac{e^{4\phi}}{4} \left[\mathrm{d}a - (\tilde{\xi}_A \mathrm{d}\xi^A - \xi^A \mathrm{d}\tilde{\xi}_A) \right] \wedge * \left[\mathrm{d}a - (\tilde{\xi}_A \mathrm{d}\xi^A - \xi^A \mathrm{d}\tilde{\xi}_A) \right] \\ &\quad - \frac{e^{2\phi}}{2} \left(\operatorname{Im} \mathcal{M}^{-1} \right)^{AB} \left[\mathrm{d}\tilde{\xi}_A - \mathcal{M}_{AC} \mathrm{d}\xi^C \right] \wedge * \left[\mathrm{d}\tilde{\xi}_B - \bar{\mathcal{M}}_{BD} \mathrm{d}\xi^D \right] . \end{aligned} \tag{2.40}$$

Comparing with the results presented in appendix B one observes that the effective action derived in (2.38) is precisely of the form of an $N = 2$ supergravity and the couplings (2.39) coincide with the ones derived on the supergravity side (B.27).

2.3.4 Type IIB on CY$_3$

Let us now discuss the second example of a KK reduction, namely the compactification of type IIB supergravity on a Calabi–Yau threefold which was previously discussed in [31, 32]. The calculations in this section are in many respects similar to the ones in the previous one so we will keep the discussion short highlighting only the major differences from the type IIA case. What we focus on is the relation to type IIA compactifications via mirror symmetry and so we consider that the compactification manifold in type IIB case is the mirror manifold of the one we chose in the previous section. However, in order not to overload the notation we use the same symbols to denote the Calabi–Yau quantities on both sides.

We start from the ten-dimensional type IIB theory presented in section 2.1.3[12] and we choose a background field configuration like in the previous section. More precisely except for the metric for which we assume the direct product structure (2.26) all the other fields are chosen to vanish in the background. As before we are interested in the massless spectrum which one obtains by expanding the ten dimensional matter fields $\hat{B}_2$, $\hat{C}_2$ and $\hat{A}_4$ in the Calabi–Yau harmonic forms

$$\begin{aligned} \hat{B}_2 &= B_2 + b^i\,\omega_i\ , \qquad \hat{C}_2 = C_2 + c^i\,\omega_i\ , \\ \hat{A}_4 &= D_2^i \wedge \omega_i + V^A \alpha_A - U_B \beta^B + \rho_i \tilde{\omega}^i\ , \end{aligned} \tag{2.41}$$

where ω_i, $\tilde{\omega}^i$ and (α_A, β^A) were introduced in (2.24) and (2.25). Let us see how these fields combine into $N = 2$ supermultiplets. Due to the self-duality of $\hat{F}_5$ one has to be more careful. In particular, this condition tells us that only half of the degrees of freedom encoded in $\hat{A}_4$ are physical. From the point of view of the four-dimensional theory the self-duality condition implies, as we will see in a while, that the four-dimensional fields obtained from the expansion of the four-form $\hat{A}_4$ (2.41) come in Poincaré dual pairs. More specific the two-forms D_2^i are the duals of the scalars ρ_i while the vector fields (V^A, U_A) are related by electric-magnetic duality. Thus it is completely equivalent to keep either of the dual fields, but for definiteness and in order to be closer to the $N = 2$ spectrum from table 2.3 we will choose as physical fields in four dimensions the scalars $\rho_i,\ i = 1, \ldots, h^{(1,1)}$ and the vector fields $V^A, A = 0, \ldots, h^{(2,1)}$. Taking also into account the Calabi–Yau moduli (2.29) one sees that the $N = 2$ spectrum consists of a gravitational multiplet with bosonic components $(g_{\mu\nu}, V^0)$, a double-tensor multiplet (B_2, C_2, ϕ, l), $h^{(1,1)}$ hyper-multiplets (ρ^i, v^i, b^i, c^i) and $h^{(2,1)}$ vector multiplets (V^a, z^a).[13]

Comparing with the type IIA spectrum in four dimensions we see that the number of hyper- and vector multiplets are exchanged and now the complex structure deformations z^a are the bosonic partners of the vector fields. As we will see at the end of this chapter this is precisely how mirror symmetry connects the two theories.

Let us now proceed with the KK reduction of type IIB action. The difference from type IIA theory comes because of the self-duality of $\hat{F}_5$. As we mentioned above this condition can not be obtained from the action (2.9), but rather has to be imposed as a separate constraint in order to obtain the correct equations of motion. Thus we have to make sure that the field strength $\hat{F}_5$ obtained from (2.10) by substituting the field expansion (2.41) satisfies

$$\hat{F}_5 = *\hat{F}_5\ . \tag{2.42}$$

Upon using the expressions for the Hodge duals of the harmonic forms on a Calabi–Yau manifold (B.29), (B.30), (B.40) and (B.42) the above condition splits in the following

[12] In fact, in order to have shorter formulae we redefine the field $\hat{A}_4$ such that its field strength becomes $\hat{F}_5 = \mathrm{d}\hat{A}_4 - \mathrm{d}\hat{B}_2 \wedge \hat{C}_2$. Note that with this the form of the action including the topological term are not changed.

[13] Note that keeping the two-forms D^i instead of the scalars ρ_i would mean that in the final spectrum we would have $h^{(1,1)}$ tensor multiplets replacing the $h^{(1,1)}$ hyper-multiplets.

two constraints for the four-dimensional fields

$$\begin{aligned} G_A &= \mathrm{Im}\,\mathcal{M}_{AB} * F^B + \mathrm{Re}\,\mathcal{M}_{AB} F^B \ , \\ \mathrm{d}\rho_i - \mathcal{K}_{ijk}\mathrm{d}b^j c^k &= 4\mathcal{K} g_{ij} * (\mathrm{d}D^j - \mathrm{d}b^j C_2 - \mathrm{d}c^j B_2) \ , \end{aligned} \tag{2.43}$$

where we have introduced the notation

$$F^A = \mathrm{d}V^A \ , \qquad\qquad G_A = \mathrm{d}U_A \ . \tag{2.44}$$

We see now what we explained in words before that the fields V^A and U_A are not independent and the same is true with the fields D^i and ρ_i. However, for deriving the four dimensional effective action it will be easier to treat them as independent fields and then impose the conditions (2.43) at a later stage in the calculation by adding appropriate Lagrange multiplier terms. Thus inserting (2.41) into (2.9), performing the integrals over the Calabi–Yau space using (2.24), (2.25), (2.34), (2.35) and keeping only the terms which contain the fields F^A, G_A, D^i or ρ_i we obtain

$$\begin{aligned} S(F^A, G_A, D^i, \rho_i) &= \frac{1}{4}(\mathrm{Im}\,\mathcal{M})^{-1AB}(G_A - F^C\mathcal{M}_{AC}) \wedge (*G_B - *F^D\bar{\mathcal{M}}_{BD}) \\ &\quad -\mathcal{K} g_{ij}(\mathrm{d}D^i - \mathrm{d}b^i \wedge C_2 - c^i\mathrm{d}B_2) \wedge *(\mathrm{d}D^j - \mathrm{d}b^j \wedge C_2 - c^j\mathrm{d}B_2) \\ &\quad -\frac{g^{ij}}{16\mathcal{K}}(\mathrm{d}\rho_i - \mathcal{K}_{ikl}\mathrm{d}b^k c^l) \wedge *(\mathrm{d}\rho_j - \mathcal{K}_{jmn}\mathrm{d}b^m c^n) \\ &\quad -\frac{1}{2}\mathcal{K}_{ijk}\mathrm{d}D^i \wedge \mathrm{d}b^j c^k - \frac{1}{2}\mathrm{d}\rho_i \wedge (c^i\mathrm{d}B_2 + \mathrm{d}b^i \wedge C_2) \ . \end{aligned} \tag{2.45}$$

As explained above, the correct action for these fields can be only obtained after imposing the self-duality conditions (2.43). The aim is to obtain these constraints from a variational principle and then integrate out the redundant fields from their equations of motion. To do this we add the following total derivatives to the action [42]

$$\delta S(F^A, G_A, D^i, \rho_i) = \frac{1}{2}F^A \wedge G_A + \frac{1}{2}\mathrm{d}D^i \wedge \mathrm{d}\rho_i \ . \tag{2.46}$$

One can easily see that the self-duality conditions (2.43) are now realized as equations of motion for the fields G_A and $\mathrm{d}D^i$. Replacing G_A and $\mathrm{d}D^i$ from (2.43) we obtain the following action for $F^A = \mathrm{d}V^A$ and $\mathrm{d}\rho_i$

$$\begin{aligned} S(V^A, \rho_i) &= \frac{1}{2}\mathrm{Im}\,\mathcal{M}_{AB}F^A \wedge *F^B + \frac{1}{2}\mathrm{Re}\,\mathcal{M}_{AB}F^A \wedge F^B \\ &\quad -\frac{g^{ij}}{2\mathcal{K}}(\mathrm{d}\rho_i - \frac{1}{2}\mathcal{K}_{ikl}\mathrm{d}b^k c^l) \wedge *(\mathrm{d}\rho_j - \frac{1}{2}\mathcal{K}_{jmn}\mathrm{d}b^m c^n) \\ &\quad -2\mathrm{d}\rho_i \wedge (\mathrm{d}b^i \wedge C_2 + c^i\mathrm{d}B_2) - \frac{1}{2}\mathrm{d}B_2 \wedge (\mathcal{K}_{ijk}c^i c^j \mathrm{d}b^k) \ , \end{aligned} \tag{2.47}$$

where we have further performed the rescaling $\rho_i \to 2\rho_i$.

For the rest of the action (2.9) which does not involve the field $\hat{A}_4$ the derivation of the four dimensional action proceeds as in the type IIA case by just replacing the expansion of the ten dimensional fields (2.41) into the ten dimensional action (2.9). One thus obtains

$$\begin{aligned} S &= \int e^{-2\phi}\Big(-\frac{1}{2}R * \mathbf{1} + 2\mathrm{d}\phi \wedge *\mathrm{d}\phi - \frac{1}{4}H_3 \wedge *H_3 - g_{ab}\mathrm{d}z^a \wedge *\mathrm{d}\bar{z}^b - g_{ij}\mathrm{d}t^i \wedge *\mathrm{d}\bar{t}^j\Big) \\ &\quad -\frac{1}{2}\mathcal{K}\,\mathrm{d}l \wedge *\mathrm{d}l - 2\mathcal{K}g_{ij}(\mathrm{d}c^i - l\mathrm{d}b^i) \wedge *(\mathrm{d}c^j - l\mathrm{d}b^j) \\ &\quad -\frac{1}{2}(\mathrm{d}C_2 - l\mathrm{d}B_2) \wedge *(\mathrm{d}C_2 - l\mathrm{d}B_2) + S(V^A, \rho_i)\ . \end{aligned} \tag{2.48}$$

Again in order to recover the usual $N = 2$ spectrum which we have described in table 2.3 we have to dualize the two two-forms C_2 and B_2 to scalars which we denote by h_1 and h_2 respectively. Following appendix D.2.1 one obtains for the action (2.48)

$$\begin{aligned} S &= \int e^{-2\phi}\Big(-\frac{1}{2}R^*\mathbf{1} + 2\mathrm{d}\phi \wedge *\mathrm{d}\phi - g_{ab}\mathrm{d}z^a \wedge *\mathrm{d}\bar{z}^b - g_{ij}\mathrm{d}t^i \wedge *\mathrm{d}\bar{t}^j\Big) \\ &\quad -\frac{1}{2}\mathcal{K}\,\mathrm{d}l \wedge *\mathrm{d}l - 2\mathcal{K}g_{ij}(\mathrm{d}c^i - l\mathrm{d}b^i) \wedge *(\mathrm{d}c^j - l\mathrm{d}b^j) \\ &\quad -\frac{1}{2}(\mathrm{d}h_1 - 2b^i\mathrm{d}\rho_i) \wedge *(\mathrm{d}h_1 - 2b^i\mathrm{d}\rho_i) \\ &\quad -\Big[\mathrm{d}h_2 + 2l(\mathrm{d}h_1 - 2b^i\mathrm{d}\rho_i) + 4c^i(\mathrm{d}\rho_i - \frac{1}{4}\mathcal{K}_{ijk}c^j\mathrm{d}b^k)\Big]^2 \\ &\quad +\frac{1}{2}\mathrm{Im}\,\mathcal{M}_{AB}F^A \wedge *F^B + \frac{1}{2}\mathrm{Re}\,\mathcal{M}_{AB}F^A \wedge F^B\ , \end{aligned} \tag{2.49}$$

The action above is the effective action in four dimensions which appears by compactifying type IIB supergravity on Calabi–Yau manifolds. Comparing it with the $N = 2$ standard supergravity action (B.20) one observes that it is not obvious that they coincide. To bring it in this form one has to further redefine the fields in the hyper-multiplets in order to put them in the quaternionic form given in [41]

$$\begin{aligned} \xi^0 &= l\ , \qquad \xi^i = lb^i - c^i\ , \\ \tilde{\xi}_i &= -2\rho_i - \frac{l}{2}\mathcal{K}_{ijk}b^jb^k + \mathcal{K}_{ijk}b^jc^k\ , \\ \tilde{\xi}_0 &= h_1 + \frac{l}{6}\mathcal{K}_{ijk}b^ib^jb^k - \frac{1}{2}\mathcal{K}_{ijk}b^ib^jc^k\ , \\ a &= h_2 + lh_1 + 2\rho_i(c^i - lb^i)\ . \end{aligned} \tag{2.50}$$

and rescale the metric by a factor $e^{2\phi}$ in order to go to the Einstein frame. With this the final form of the type IIB effective action in four dimensions reads

$$\begin{aligned} S_{IIB} &= \int \Big[-\frac{1}{2} R^* \mathbf{1} - g_{ab} dz^a \wedge * d\bar{z}^b - h_{uv} dq^u \wedge * dq^v \\ &\quad + \frac{1}{2} \operatorname{Im} \mathcal{M}_{IJ} F^I \wedge * F^J + \frac{1}{2} \operatorname{Re} \mathcal{M}_{IJ} F^I \wedge F^J \Big], \end{aligned} \tag{2.51}$$

where the metric h_{uv} is given by

$$\begin{aligned} h_{uv} dq^u \wedge * dq^v &= d\phi \wedge * d\phi + g_{ab} dz^a \wedge * d\bar{z}^b \\ &+ \frac{e^{4\phi}}{4} \Big[da - (\tilde{\xi}_A d\xi^A - \xi^A d\tilde{\xi}_A) \Big] \wedge * \Big[da - (\tilde{\xi}_A d\xi^A - \xi^A d\tilde{\xi}_A) \Big] \\ &- \frac{e^{2\phi}}{2} \left(\operatorname{Im} \mathcal{N}^{-1} \right)^{AB} \Big[d\tilde{\xi}_A - \mathcal{N}_{AC} d\xi^C \Big] \wedge * \Big[d\tilde{\xi}_B - \bar{\mathcal{N}}_{BD} d\xi^D \Big] . \end{aligned} \tag{2.52}$$

with the matrix $\mathcal{N}$ defined in (2.39).

One can see now that the redefinition (2.50) had two effects. First the effective action of type IIB supergravity compactified on a Calabi–Yau three-fold (2.51) has the standard $N = 2$ form (B.20). Secondly from the form (2.51) the relation to type IIA compactification (2.38) is straightforward via the following identifications

$$\left. \begin{array}{ccc} z^a & \leftrightarrow & t^i \\ \mathcal{N} & \leftrightarrow & \mathcal{M} \\ g_{ab} & \leftrightarrow & g_{ij} \end{array} \right\} \quad \longrightarrow \quad h^A_{uv} \leftrightarrow h^B_{uv} \ . \tag{2.53}$$

Thus we can identify equations (2.50) and (2.53) with the mirror map[14] which exchanges the type IIA compactification with the IIB one.

2.4 Discussion

Let us make a few comments on the results obtained in the last sections. The choice of a Calabi–Yau space as a compactification manifold was dictated on phenomenological grounds by requiring that the resulting four dimensional theory does not preserve all the ten dimensional supersymmetries, but only some of them. However the theories derived in (2.38) and (2.51) fail to reproduce some minimal features which are necessary for making contact with the standard model. For example both theories (2.38) and (2.51) have $N = 2$ supersymmetry which does not allow four dimensional chiral fermions. Moreover, one notices that the gauge fields are all Abelian and thus again is inconsistent with the observations that the standard model is based on a non-Abelian gauge group. Finally

[14]Note that one has to be careful with the basis in which the mirror map is considered. In order to identify the complex structure deformations with the Kähler ones like in (2.53) one has to make sure that these are the flat coordinates on the corresponding moduli spaces. However we do not enter such details and assume that the the scalar fields z^A and t^i obtained from the reduction of the Ricci scalar are the correct coordinates.

the actions (2.38) and (2.51) feature $h_{(1,1)} + h_{(2,1)}$ scalar fields which can be as large as hundreds. These scalar fields are all flat directions of the potential and thus their vacuum expectation values parameterize a huge vacuum degeneracy. Furthermore, the fact that there is no potential associated to these fields means that they are completely massless and thus again inconsistent with the present observation as no scalar particles were observed until the present energies.

Consequently in order to obtain more realistic models one would at least need to obtain a four dimensional potential for the scalar fields which could fix their vacuum expectation values, give them masses and break spontaneously supersymmetry.[15]

In the next section we are going to present a generalization of the setup described in this section in that some fields will be given a background value which will lead to the appearance of a scalar potential for some of the moduli.

[15]We have already argued that non-Abelian gauge groups arise when one introduces D-branes in the theory. Moreover dualities between different string theories have taught us that singularities of the internal manifold can also generate non-Abelian gauge groups [43]. However in the following we will not be concerned with these cases and we restrict ourselves to Abelian vector fields.

Chapter 3

Type IIA theory with fluxes

In this chapter we study a particular way of obtaining potentials for the moduli which appear in KK compactifications by turning on fluxes for various field strengths. This idea has first appeared in [44] for the case of type IIA compactifications on Calabi–Yau three-folds, in [45] for M-theory on Calabi–Yau four-folds and in [46] for type IIB again on Calabi–Yau three-folds. Several aspects of such compactifications including moduli stabilization, effective superpotentials or dualities have been discussed recently [14–16, 42, 47–76]. In this first chapter which is devoted to such generalized compactifications we discuss type IIA theory and the various possibilities for turning on fluxes. We show explicitly how to derive the low energy effective action in this case and that indeed the fluxes generate a potential for some of the moduli. Another aspect which we will treat in this chapter is the relation of such effective actions to known supergravities in four dimensions. We show in several cases that the theory we obtain is a gauged $N = 2$ supergravity and thus fluxes do not break explicitly supersymmetry. First we explain in section 3.1 how the idea of fluxes has first appeared and we try to give an intuitive picture of the modifications which are produced by turning on fluxes by considering a very simple example. In section 3.2 we come to the first example of interest namely turning on NS-NS fluxes in the type IIA compactification and then we continue by studying the effect of the RR fluxes in section 3.3.

3.1 Generalities

In a KK reduction one starts with with a higher dimensional theory and assumes that the the space in which it lives is a product of the four-dimensional space-time and a compact internal manifold (1.1). In order to obtain a four dimensional theory one expands the fields in a complete set of functions on the internal manifold and performs the integration over the internal coordinates. The fields which appear in this way generically have masses which are inverse proportional to the size of the compactification manifold. In a low energy approximation one truncates out the massive fields and all what is considered are the massless fields which correspond to the expansion in harmonic forms on the internal space. These fields come in general without a potential and thus their vacuum expectation values are free parameters of the theory. The natural question to ask is whether one can find a way to generate a potential for such fields which can lift the vacuum degeneracy.

The idea which we use in this work is based on the observation made in [77] that a certain dependence of the ten dimensional fields on the coordinates of the internal manifold can generate masses for some of the fields and even break supersymmetry. One of the first questions to ask is whether this is a consistent procedure, as it is well known that in general the KK truncation is valid only if one keeps all fields which do not depend on the internal directions and truncate out all the others. It was argued in [77] that as long as the dependence on the internal coordinates is introduced according to some global symmetry of the action no new (massive) modes from the KK towers are involved and thus the procedure is consistent. More intuitively, this symmetry is used to assure that the Lagrangian is independent of the internal coordinates even if the fields are allowed to depend on them and thus in the end one can still integrate out all the dependence on the internal manifold.

Let us see how this works in a simple example [78, 79]. Consider a real scalar field $\hat{\lambda}$ coupled to gravity in 5 dimensions[1]

$$S_\lambda = \int \left(-\hat{R} * \mathbf{1} - \frac{1}{2}\, \hat{\mathrm{d}}\hat{\lambda} \wedge *\hat{\mathrm{d}}\hat{\lambda} \right) \ . \tag{3.1}$$

This action is clearly invariant under

$$\lambda \to \lambda + a \ , \tag{3.2}$$

where a is a constant. Suppose we compactify this action on a circle and denote the coordinate on the circle by y. For the metric we use the same Ansatz as in D.3 while for the scalar we now assume

$$\lambda(x, y) = \lambda(x) + my \ . \tag{3.3}$$

It is instructive to decompose the 'field strength' of λ as

$$(\hat{\mathrm{d}}\hat{\lambda})_4 = \mathrm{d}\lambda \ , \qquad\qquad (\hat{\mathrm{d}}\hat{\lambda})_y = m\ \mathrm{d}y \ . \tag{3.4}$$

One notices that the four-dimensional part of $\hat{\mathrm{d}}\hat{\lambda}$ is the same as in a normal KK compactification, but now there is an additional piece pointing in the fifth direction which is constant. So one can see that even if the field itself is allowed to depend on the internal coordinate y, its field strength $\hat{\mathrm{d}}\hat{\lambda}$ is independent of y. However, integrating $\hat{\mathrm{d}}\hat{\lambda}$ over the circle one obtains a non-trivial flux through this circle which is proportional to m. This is why considering a generalized Ansatz like in (3.3) such that some field strength acquires a purely internal value will be called from now on 'turning on fluxes'.

In order to compute the lower-dimensional effective action one notices that only $\mathrm{d}\hat{\lambda}$ appears in the action (3.1) so that the additional y dependence introduced via (3.3)

[1] We use all the time the convention that hats indicate higher dimensional objects. Here in particular we also use a hat on the symbol for the higher dimensional exterior derivative i.e. $\hat{\mathrm{d}}$.

disappears from the Lagrangian and one can perform again the integral over the internal space as in the usual KK reductions. The result now can be written as

$$S = \int \left(-R * 1 - \frac{1}{6\phi^2} \mathrm{d}\phi \wedge * \mathrm{d}\phi - \frac{\phi}{4} F \wedge *F - \frac{1}{2} D\lambda \wedge * D\lambda - \frac{m^2}{2\phi} * 1 \right) , \tag{3.5}$$

where the only difference compared to ordinary KK reductions is that a covariant derivative appears $D\lambda = \mathrm{d}\lambda - mA$ and a potential term is generated. One can further check that under the residual diffeomorphism transformation the KK gauge field has the usual gauge transformation and the only way the action (3.5) can be invariant is if λ also transforms

$$\delta A_\mu = \mathrm{d}\theta, \qquad \delta\lambda = \theta , \tag{3.6}$$

Thus what we have realized by considering a deformed KK Ansatz like (3.3) is that the original shift symmetry of $\hat{\lambda}$ (3.2) is gauged using the KK vector field A.

Before we move on to study some more complicated situations let us note that the Ansatz (3.4) is not consistent with a Minkowski four dimensional space. The reason is that $(\hat{\mathrm{d}}\hat{\lambda})_y$ contributes to the five dimensional energy-momentum tensor and thus the equations of motion are no longer $\hat{R}_{MN} = 0$. This can also be seen from the action (3.5) as the potential term does not generally has to vanish in the background.

In the following we will generalize the above situation in a straightforward way. We consider compactifications of type II theories and allow for a very specific dependence on the internal coordinates similar to (3.3). The symmetry we use in this case will be the Abelian gauge symmetry associated to various p-form potentials present in the two theories.[2] As in the above example the field strength of the corresponding p-form potential acquires a purely internal value.

Since we are going to focus on Calabi–Yau compactifications we will not be able to naively choose a linear dependence as in (3.3). The flux (the purely internal value of the field strength of the field for which a generalized compactification is assumed) will rather turn out to be a harmonic form on the internal manifold, in our case the Calabi–Yau space.[3] To see this consider that we want to turn on a flux for the p-form field strength $F_p = \mathrm{d}C_{p-1}$. As the background must be a solutions of the ten dimensional equations of motion we find

$$\mathrm{d} * F_p = 0 , \tag{3.7}$$

if all other fields vanish. Together with the Bianchi identity $\mathrm{d}F_p = 0$ this leads to a solution for F_p which is a harmonic form on the internal manifold. Thus we can choose

$$F_p = m_i \omega_p^i , \tag{3.8}$$

where ω_p^i are harmonic p-forms on the internal manifold while m_i are constants parameterizing the flux F_p. In principle this is not the end of the story because the value of this

[2]If one thinks of λ in (3.1) as a 0-form potential the transformation (3.2) is nothing but the ordinary Abelian gauge transformation associated with p-form fields.

[3]Note that this is also true in the above example. Here the field strength is the one form $\mathrm{d}\lambda$ which on the internal manifold is $m\,\mathrm{d}y$ and thus proportional to the unique harmonic one-form on the circle.

field strength on the internal manifold gives rise to a non-vanishing energy-momentum tensor and thus backreacts on the compactification geometry inducing a non-trivial warp factor [29, 39, 40]. However, if we heuristically write the Einstein equations

$$R_{mn} \sim (F_p^2)_{mn} \ , \tag{3.9}$$

in the limit that the fluxes are small we can neglect the right hand side and we are left with the ordinary Calabi–Yau condition. On the other hand this is not always a reliable limit. It will be natural to interpret the fluxes which we turn on as electric and magnetic charges of the fields in the low energy effective theory. This suggest that in a consistent quantum theory the fluxes should be quantized due to Dirac quantization condition as was first noticed in [44]. This means that in string theory m_i in (3.8) are integer multiples of the string scale α' and thus there is no continuous limit $m_i \to 0$. In supergravity however, which is the $\alpha' \to 0$ limit of string theory, this can be done, and in the rest of this work we neglect this subtlety that the fluxes are quantized and we are going to work in the limit that they are small and do not influence the background metric. Moreover we assume that the light modes remain the same as in the no-flux case and so we only deal with the same fields as were appearing in the normal compactifications of type II theories and which we have seen in chapter 2.

Having discussed how to turn on fluxes in general we can start to compute low energy effective actions in the presence of fluxes. Before that let us come back to the issue of consistency which can play some role when fluxes for more fields are present. Turning on a flux for some p-form field strength like in (3.8) can be problematic as the fundamental field is the $p-1$ form potential C_{p-1} and not its field strength F_p. Clearly, assuming (3.8), C_{p-1} can not be globally defined as a harmonic form can not be exact. The best one can do is to say that since a harmonic form is closed one can locally write it as a total derivative. In our case this would mean to write $\omega^i = \mathrm{d}\xi^i$ locally. Note that the same happens in the above example as the term my in (3.3) is not globally defined on a circle. If the field C_{p-1} and not its field strength F_p appears in the action then we have to make sure that the non-local contributions coming from the above considerations cancel in the action. On the other hand if the field C_{p-1} appears in the action only via its Abelian field strength $F_p = \mathrm{d}C_{p-1}$ then a term (3.8) is harmless. In order to avoid problems coming from non-local terms in the action we are going to consider turning on fluxes for those fields which can appear simultaneously only via their Abelian field strength.

3.2 IIA with NS fluxes

Let us start by looking at Calabi–Yau compactifications of type IIA theory with non-trivial fluxes for fields in the NS-NS sector (in short NS fluxes). This is a clear example which is computationally not much more involved than the usual Calabi–Yau compactification presented in section 2.3.3. Thus we use it to explain how to obtain the low energy effective action when fluxes are turned on and how the resulting theory can be related to gauged supergravity.

3.2.1 The low energy effective action

In section 2.1.1 we have already introduced the spectrum for the type IIA supergravity. The bosonic fields in the NS-NS sector are the metric $\hat{g}_{MN}$ the dilaton $\hat{\phi}$ and the antisymmetric tensor $\hat{B}_2$. In all what follows we keep the dilaton to be constant on the internal manifold and thus the only form-field we have at our disposal to turn on fluxes is the B-field. So we consider a background configuration where the field strength of this field $\hat{H}_3 = \mathrm{d}\hat{B}_2$ has a non-trivial flux which is parameterized by the elements of the third cohomology group on the Calabi–Yau manifold Y, $H^3(Y)$

$$\hat{H}_3^{int} = p^A \alpha_A - q_A \beta^A \ , \tag{3.10}$$

where (p^A, q_A) are the $2h^{(2,1)} + 2$ flux parameters needed in order to completely specify the internal value of $\hat{H}_3$. (α_A, β^A), $A = 0, \ldots, h^{(2,1)}$ is the real basis for $H^3(Y)$ which is normalized as in (2.25). Except for this modification the background is the same as discussed in section 2.3.3. Thus we assume the metric (2.26) and take all other background field strengths except for $\hat{H}_3$ to vanish.

In order to see more clearly how the Ansatz (3.10) modifies the low energy action it is useful to make a field redefinition in the type IIA theory in ten dimensions such that the two-form field B_2 appears in the action only in the form of $H_3 = \mathrm{d}B_2$. This is achieved by performing the following transformation $\hat{C}_3 \to \hat{C}_3 + \hat{A}_1 \wedge \hat{B}_2$. The form of the action (2.2) remains the same, but now with the following definitions

$$\hat{F}_4 = \mathrm{d}\hat{C}_3 - \hat{A}_1 \wedge \hat{H}_3 \ , \qquad \hat{H}_3 = \mathrm{d}\hat{B}_2 \ , \qquad \mathcal{L}_{top} = \frac{1}{2}\hat{H}_3 \wedge \hat{C}_3 \wedge d\hat{C}_3 \, . \tag{3.11}$$

Note that the above field redefinitions also change the gauge transformations (2.5) into

$$\begin{aligned} \delta\hat{B}_2 &= \mathrm{d}\hat{\Lambda}_1 \, , & \delta\hat{C}_3 &= \mathrm{d}\hat{\Sigma}_2 \ , \\ \delta\hat{A}_1 &= \mathrm{d}\hat{\Theta} \, , & \delta\hat{C}_3 &= \hat{\Theta} \wedge \mathrm{d}\hat{B}_2 \, . \end{aligned} \tag{3.12}$$

Let us now see how the compactification with fluxes proceeds in this case. The key point to notice is that considering the non-trivial configuration (3.10) amounts to shift the expansion in harmonic forms of the field strength $\hat{H}_3$ (2.32) according to

$$\hat{H}_3 \to \hat{H}_3 + p^A \alpha_A - q_A \beta^A \ . \tag{3.13}$$

This is the reason why we needed the action to be expressed only in terms of the field strength $\hat{H}_3$ as the fluxes enter the compactified theory only via the field strengths for which non-trivial background values are considered. Beside this everything else remains as in section 2.3.3 and in particular we consider the same light spectrum and thus we again assume the same field expansions as in (2.31). What (3.13) does modify are the field strengths which now have different expansion from the ones in (2.32)

$$\begin{aligned} \hat{F}_2 &= \mathrm{d}A^0 \ , \\ \hat{H}_3 &= \mathrm{d}B_2 + \mathrm{d}b^i \omega_i + p^A \alpha_A - q_A \beta^A \ , \\ \hat{F}_4 &= \mathrm{d}C_3 - A^0 \wedge H_3 + (\mathrm{d}A^i - A^0 \mathrm{d}b^i) \wedge \omega_i + D\xi^A \wedge \alpha_A - D\tilde{\xi}_A \wedge \beta^A \ , \end{aligned} \tag{3.14}$$

where we denoted

$$D\xi^A = \mathrm{d}\xi^A - p^A A^0 \ , \qquad D\tilde{\xi}_A = \mathrm{d}\tilde{\xi}_A - q_A A^0 \, . \tag{3.15}$$

At this stage one notices that the effect of turning on fluxes (3.13), like in the example from section 3.1, is to gauge the previous isometries $\xi^A \to \xi^A + const$ and $\tilde{\xi}_A \to \tilde{\xi}_A + const$. Indeed one can directly check by 'compactifying' the gauge transformations (3.12) and taking into account (3.13) that the variations of the fields ξ^A and $\tilde{\xi}_A$ become

$$\delta A^0 = \mathrm{d}\Theta \ , \qquad \delta\xi^A = p^A\Theta \ , \qquad \delta\tilde{\xi}_A = q_A\Theta \ . \tag{3.16}$$

This motivates the notation (3.15) as these are gauge invariant quantities and moreover it is natural to interpret them as proper covariant derivatives. At this point this is the only difference compared to the massless case discussed in section 2.3.3. Furthermore the gauge invariance assures that in the final form of the action the normal derivatives will be replaced by the covariant ones (3.15). However these are not the only modifications produced by (3.13) and in order to see the full effect of the fluxes one should go deeper into the details of the compactification.

There is one more difference to the massless case presented in section 2.3.3 which one can immediately notice. Due to the terms in the expansion of $\hat{H}_3$ which lie completely in the Calabi–Yau space the kinetic term of $\hat{B}_2$ generates after performing the Calabi–Yau integrals a potential term in the four dimensional action

$$V_H = -\frac{1}{4}\frac{e^{-2\phi}}{\mathcal{K}}\,(q_A - \mathcal{M}_{AC}\,p^C)(\mathrm{Im}\,\mathcal{M})^{-1AB}(q_B - \bar{\mathcal{M}}_{BD}\,p^D) \ , \tag{3.17}$$

where $\mathcal{M}$ was defined in (2.35). Finally the topological term (3.11) produces a new interaction

$$\delta\mathcal{L}_{\rm top} = -(p^A\tilde{\xi}_A - q_A\xi^A)\,\mathrm{d}C_3 \ , \tag{3.18}$$

which upon the dualization of C_3 using the formulae in appendix D.2.2 leads to

$$\mathcal{L}_{C_3} \to -\frac{1}{2\mathcal{K}}(p^A\tilde{\xi}_A - q_A\xi^A)^2 \ *\mathbf{1} - (p^A\tilde{\xi}_A - q_A\xi^A)\,A^0\wedge H_3 \ . \tag{3.19}$$

Again as in section 2.3.3 we have put to zero the constant to which C_3 is dual. We did this anticipating that it plays a special role when RR fluxes are turned on and thus we will consider this constant properly in the next section. Note that unlike section 2.3.3 the dualization of C_3 has produced a non-trivial result even if the constant to which C_3 is dual was taken to be zero. This is so only due to the additional interaction for C_3 in (3.18). The first term in (3.19) gives another contribution to the scalar potential which now reads

$$V = -\frac{e^{-2\phi}}{4\,\mathcal{K}}\,(q - p\mathcal{M})(\mathrm{Im}\,\mathcal{M})^{-1}(q - p\bar{\mathcal{M}}) + \frac{1}{2\,\mathcal{K}}(p^A\tilde{\xi}_A - q_A\xi^A)^2 \ , \tag{3.20}$$

while the second term produces the right gaugings for the scalar which is dual to B_2. This is not difficult to see using the formulae in appendix D.2.1. The result of this dualization is

$$\mathcal{L}_{B_2} \to \mathcal{L}_a = -\frac{e^{2\phi}}{4}\,\Big[Da - (\tilde{\xi}_A D\xi^A - \xi^A D\tilde{\xi}_A)\Big]\wedge *\Big[Da - (\tilde{\xi}_A D\xi^A - \xi^A D\tilde{\xi}_A)\Big] \ , \tag{3.21}$$

where

$$Da = \mathrm{d}a + (p^A \tilde{\xi}_A - q_A \xi^A) A^0 \,. \tag{3.22}$$

So in addition to the scalars ξ^A and $\tilde{\xi}_A$ the axion is also charged as one can see from the above covariant derivative.

After redefining the gauge fields according to

$$A^i \to A^i - b^i A^0 \;, \tag{3.23}$$

and rescaling the metric with the dilaton factor $e^{2\phi}$ the action takes the form

$$\begin{aligned} S_{IIA} \;\; &= \int \Big[-\frac{1}{2} R * \mathbf{1} - g_{ij} \mathrm{d}t^i \wedge * \mathrm{d}\bar{t}^{\bar{j}} - h_{uv} Dq^u \wedge * Dq^v - V_E * \mathbf{1} \\ &\quad + \frac{1}{2} \,\mathrm{Im}\, \mathcal{N}_{IJ} F^I \wedge * F^J + \frac{1}{2} \,\mathrm{Re}\, \mathcal{N}_{IJ} F^I \wedge F^J \Big], \end{aligned} \tag{3.24}$$

where the gauge couplings are given by (2.39) and h_{uv} can be read from (2.40) while the Einstein frame potential reads

$$V_{\mathrm{E}} = -\frac{e^{2\phi}}{4\,\mathcal{K}} \,(q - p\mathcal{M})(\mathrm{Im}\,\mathcal{M})^{-1}(q - p\bar{\mathcal{M}}) + \frac{e^{4\phi}}{2\,\mathcal{K}} (p^A \tilde{\xi}_A - q_A \xi^A)^2 \;. \tag{3.25}$$

As one can easily see the fluxes did not change the form of the action (2.38) too much. The only differences are the covariant derivatives (3.15) and (3.22) and the appearance of a potential (3.25). One can also notice that the action (3.24) has a very similar form to the general $N = 2$ gauged supergravities (B.20). In the next section we will indeed show that the action (3.24) describes a gauged supergravity. Before we see this let us make one more comment regarding the potential (3.25). The matrix $\mathcal{M}$ defined in (2.35) depends on the complex structure moduli.[4] Thus the NS fluxes in type IIA compactifications lift the flat directions corresponding to the complex structure moduli. Furthermore the potential (3.25) also depends on the scalars ξ^A and $\tilde{\xi}_A$. As these scalars combine together with the complex structure deformations to form the hyper-multiplets we conclude that the potential lift all the flat directions corresponding to the scalars in the hyper-multiplets. One can also notice a dependence of the potential on the Calabi–Yau volume $\mathcal{K}$ and on the four dimensional dilaton ϕ. However it is easy to see that these directions are not stabilized and the potential has a run-away form with the minimum reached at infinity. Finally we note that due to the fact that the matrix $\mathrm{Im}\,\mathcal{M}$ is negative definite the potential is explicitly positive.

Relative to the moduli stabilization problem it is also worthwhile mentioning that it was argued [48, 72, 73] that the overall effect of turning on fluxes is the appearance of a superpotential. In analogy with these works the superpotential which is generated in the case of type IIA theory with NS fluxes is

$$W \sim \int_{Y_3} \big(C_3 + ie^{-\phi}\Omega\big) \wedge H_3 \;, \tag{3.26}$$

[4] An easy way to see this is to note that $\mathcal{M}$ plays the role of the gauge couplings in type IIB compactifications and thus should depend on the scalars in the vector multiplets which in this case are the complex structure deformations.

where it is understood that only the purely internal values of the above fields contribute to the integral. In general the superpotential was argued to be $W = \int \text{calibration} \wedge \text{flux}^5$ [73]. In the case above one can easily see that the term containing C_3 should be present as it will precisely give rise to the second term in the potential (3.25). As a second check one notices that the superpotential (3.26) nicely matches the one found in [81] for M-theory compactified on manifolds with G_2 holonomy with four form flux G_4. Indeed 'reducing' the G_2 structure to an $SU(3)$ one [82, 83] and considering a flux only for B_2 one immediately obtains the formula (3.26) for the superpotential.

3.2.2 Relation to gauged supergravity

Let us now discuss the relation between the theory obtained in (3.24) and the known $N = 2$ supergravities. As presented in appendix B there exist $N = 2$ supergravity theories with charged matter which are termed gauged supergravities. The difference from a normal (ungauged) supergravity is that some of the isometries of the scalar manifold are gauged in that the ordinary derivatives are replaced by covariant ones

$$D\phi^\alpha = \partial\phi^\alpha - k_I^\alpha(\phi) A^I \ , \tag{3.27}$$

where ϕ^α denotes a generic scalar field either from the hypermultiplets or from the vector-multiplets (the index α runs over all the scalars in the theory), $k_I^\alpha(\phi)$ is the killing vector corresponding to the isometry which is gauged and A^I denotes the vector fields. Supersymmetry further requires a scalar potential to be present which has the form

$$V_E = e^K X^I \bar{X}^J (g_{\bar{\imath}j}\, k_I^{\bar{\imath}} k_J^j + 4h_{uv}\, k_I^u k_J^v) - \left[\frac{1}{2}(\text{Im}\,\mathcal{N})^{-1IJ} + 4e^K X^I \bar{X}^J\right] P_I^x P_J^x \ , \tag{3.28}$$

where K denotes the Kähler potential, X^I are the (complex) scalars in the vector multiplets, the index u runs over the hyper-scalars, P_I^x are the Killing prepotentials introduced in (B.17), $\mathcal{N}$ denotes the gauge coupling matrix defined in (B.7) while $g_{\bar{\imath}j}$ and h_{uv} are the metrics on the two scalar manifolds.

To make a connection between the theory obtained in the previous section and $N = 2$ gauged supergravities the only thing to check is that the potential (3.25) can be derived using the general formula (3.28) and the Killing vectors defined in (3.27) which can be read off from (3.15) and (3.22). For this we would have to solve the equations for the Killing prepotentials (B.17) and then replace them in the general formula (3.28).

For the case at hand however, (3.28) considerably simplifies in the sense that the term which contains the Killing prepotentials vanishes. To see this we first note that since only one vector field A^0 participates in the gaugings (3.15) and (3.22), the only non-trivial components of the Killing prepotentials P_I^x in (B.17) are the ones for which $I = 0$. Using the expressions for the Kähler potential of the of the scalars in the vector multiplets

[5] Intuitively a calibration is a closed p-form which minimizes the volume of p-submanifolds. For a more precise definition see [80].

(B.25) and for the inverse gauge coupling matrix (B.33) and the fact that $X^0 = 1$ one immediately sees that

$$\frac{1}{2}(\mathrm{Im}\,\mathcal{N})^{-1\,00} + 4e^K X^0 \bar{X}^0 = 0 \; . \tag{3.29}$$

Inserting (3.29) into (B.19) the formula for the gauged supergravity potential in this case becomes

$$V_E = (4e^K h_{uv} k_I^u k_J^v + g_{\bar{\imath}j} k_I^{\bar{\imath}} k_J^j) X^I \bar{X}^J \; . \tag{3.30}$$

As the scalars in the vector multiplets are not charged the corresponding Killing vectors $k_I^i = 0$ vanish. Furthermore using again the fact that only one vector field, namely A^0 participates in the gaugings only the Killing vectors k_0^u survive and thus the above formula further simplifies to

$$V_E = 4e^K h_{uv} k_0^u k_0^v \; . \tag{3.31}$$

The remaining Killing vectors k_0^u can be immediately read off from the covariant derivatives (3.15) and (3.22) to be

$$k_0^{\xi^B} = p^B \, , \quad k_0^{\tilde{\xi}_B} = q_B \, , \quad k_0^a = (p^A \tilde{\xi}_A - q_A \xi^A) \; . \tag{3.32}$$

Using the metric components of the charged scalars from (2.40), the evaluation of (3.31) precisely results in the potential (3.25) and thus establishes the consistency with gauged supergravity.

3.3 IIA with RR fluxes

3.3.1 The low energy effective action

Let us continue our study about fluxes in type IIA compactifications by focusing on the fluxes for the fields in the RR sector. As we will see later it is natural to start not from the usual type IIA theory as we did in the previous section, but from its massive version briefly presented in section 2.1.2. Thus we consider the ten dimensional action (2.2) with the field strengths defined in (2.6) and the topological term in (2.7). With the action written in this form the RR fields A_1 and C_3 appear only via their Abelian field strengths $\mathrm{d}A_1$ and $\mathrm{d}C_3$ respectively for which we assume non-trivial internal values. For the background field configuration which we should specify before performing the KK reduction this means that we consider the metric (2.26), but now unlike in section 2.3.3 we take $\mathrm{d}A_1$ and $\mathrm{d}C_3$ to be non-vanishing.

As we explained before we assume the same zero modes as in the massless case and thus we are going to use the same field expansion as in (2.31). The difference comes from the fact that the field strengths $\mathrm{d}A_1$ and $\mathrm{d}C_3$ are shifted according to[6]

$$\mathrm{d}\hat{C}_3 \to \mathrm{d}\hat{C}_3 + e_i \tilde{\omega}^i \; , \qquad \mathrm{d}\hat{A}_1 \to \mathrm{d}\hat{A}_1 - m^i \omega_i \; . \tag{3.33}$$

[6]The minus sign in the last relation was chosen to make the symplectic invariance explicit later.

The flux parameters (e_i, m^i) are constants and ω_i and $\tilde{\omega}^i$ were introduced in (2.24) and form basis for the harmonic $(1,1)$ and $(2,2)$-forms respectively. As in the previous section the fluxes (3.33) enter in the compactification via the field strengths (2.6) for which the expansions in the Calabi–Yau harmonic forms (2.32) is replaced with

$$\begin{aligned}
\hat{H}_3 &= \mathrm{d}B_2 + \mathrm{d}b^i \omega_i \ , \\
\hat{F}_2 &= \mathrm{d}A^0 + mB_2 - (m^i - mb^i)\omega_i \ , \\
\hat{F}_4 &= \mathrm{d}C_3 - B_2 \wedge \mathrm{d}A^0 - \frac{m}{2}(B_2)^2 + (\mathrm{d}A^i - \mathrm{d}A^0 b^i + m^i B_2 - mB_2 b^i) \wedge \omega_i \\
&\quad + (\mathrm{d}\xi^A \alpha_A - \mathrm{d}\tilde{\xi}_A \beta^A) + (b^i m^j - \frac{1}{2} m b^i b^j)\mathcal{K}_{ijk}\tilde{\omega}^k + e_i \tilde{\omega}^i \ ,
\end{aligned} \tag{3.34}$$

where $\mathcal{K}_{ijk}$ is defined in (2.34) and furthermore we used (B.30).

The derivation of the four-dimensional effective action proceeds as in section 2.3.3 by inserting (3.34) into the ten dimensional action. Except for a couple of new contributions which we are going to discuss in the following the structure of the theory is the same and we are not going to repeat the calculations presented in section 2.3.3.

Let us study one by one the modifications produced by the fluxes. First in the field strength $\hat{F}_2$ of $\hat{A}_1$ one notices the presence of a term which points only along the internal manifold. This will give rise in the kinetic term of A_1 to a potential piece

$$V_1 = 2\mathcal{K}(m^i - mb^i)(m^j - mb^j)g_{ij} \ , \tag{3.35}$$

where $g_{ij}(v) = \frac{1}{4\mathcal{K}}\int_{Y_3} \omega_i \wedge * \omega_j$ is the metric on the space of Kähler deformations (2.34) and $\mathcal{K}$ denotes the volume of Y_3. In addition the following interaction and mass terms for B_2 arise

$$\delta\mathcal{L}_{\text{int}} = -m\,\mathcal{K} B_2 \wedge * \mathrm{d}A^0 - \frac{m^2 \mathcal{K}}{2}\, B_2 \wedge * B_2 \, . \tag{3.36}$$

Similarly, due to the additional term in F_4 which lies completely in the internal manifold, the kinetic term of $\hat{C}_3$ also contributes to the potential

$$V_3 = \frac{1}{8\,\mathcal{K}}(e_i + b^k m^l \mathcal{K}_{ikl} - \frac{1}{2} m b^k b^l \mathcal{K}_{ikl})(e_j + b^m m^n \mathcal{K}_{jmn} - \frac{1}{2} m b^m b^n \mathcal{K}_{jmn}) g^{ij} \ , \tag{3.37}$$

and to the interaction terms for B_2

$$\begin{aligned}
\delta\mathcal{L}_{\text{int}} &= -4\mathcal{K}(m^i - mb^i) B_2 \wedge * \left(\mathrm{d}A^j - \mathrm{d}A^0 b^j\right) g_{ij} \\
&\quad -2\mathcal{K}(m^i - mb^i)(m^j - mb^j) g_{ij} B_2 \wedge * B_2 \\
&\quad -\frac{\mathcal{K}}{2}\big(\mathrm{d}C_3 - B_2 \wedge \mathrm{d}A^0 - \frac{m}{2}(B_2)^2\big) \wedge * \big(\mathrm{d}C_3 - B_2 \wedge \mathrm{d}A^0 - \frac{m}{2}(B_2)^2\big) \, .
\end{aligned} \tag{3.38}$$

Here g^{ij} is defined as $g^{ij} = 4\mathcal{K}\int_{Y_3} \tilde{\omega}^i \wedge * \tilde{\omega}^j$ and denotes the inverse metric on the complexified Kähler cone. Finally in addition to (2.33) the topological terms (2.7) produce the following interactions due to (3.33)

$$\begin{aligned} \delta\mathcal{L}_{top} &= -B_2 \wedge \big(\mathrm{d}A^i e_i + b^i \mathrm{d}A^j m^k \mathcal{K}_{ijk} - b^i e_i \mathrm{d}A^0 - b^i b^j m^k \mathcal{K}_{ijk} \mathrm{d}A^0\big) \\ &\quad -\frac{1}{2}(2b^i e_i + b^i b^j m^k \mathcal{K}_{ijk} - \frac{m}{3} b^i b^j b^k \mathcal{K}_{ijk})\, \mathrm{d}C_3 + \frac{m}{2} B_2 \wedge (\mathrm{d}A^i - \mathrm{d}A^0 b^i) b^j b^k \mathcal{K}_{ijk} \\ &\quad -\frac{1}{2}(m^i e_i - m b^i e_i + b^i m^j m^k \mathcal{K}_{ijk} - \frac{3m}{2} b^i b^j m^k \mathcal{K}_{ijk} + \frac{m^2}{2} b^i b^j b^k \mathcal{K}_{ijk})(B_2)^2 \,. \end{aligned} \tag{3.39}$$

The above expressions arrange themselves into the form of an $N = 2$ massive supergravity with the parameters e and m playing the role of charges and masses. To see this we should first recover the normal $N = 2$ spectrum. In this process we do not set the three-form C_3 to zero as we did in sections 2.3.3 and 3.2, but rather dualize it to a constant e_0 which will turn out to come on the same footing as the other RR fluxes e_i which we turned on[7]. Using the formulae in the appendix D.2.2 one can write the action dual to C_3 as

$$\begin{aligned} \mathcal{L}_{C_3} \to \mathcal{L}_{e_0} &= -\frac{1}{2\mathcal{K}}\big(e_0 + e_i b^i + \frac{1}{2} b^i b^j m^k \mathcal{K}_{ijk} - \frac{m}{6} b^i b^j b^k \mathcal{K}_{ijk}\big)^2 \, {}^*\mathbf{1} \\ &\quad -\big(e_0 + e_i b^i + \frac{1}{2} b^i b^j m^k \mathcal{K}_{ijk} - \frac{m}{6} b^i b^j b^k \mathcal{K}_{ijk})(B_2 \wedge \mathrm{d}A^0 + \frac{m}{2}(B_2)^2\big) \,. \end{aligned} \tag{3.41}$$

Let us stress that the appearance of the parameter e_0 obtained by dualizing C_3 does not depend on the fact that we have turned on other fluxes. $\mathcal{L}_{e_0}$ does not vanish in the limit $m = m^i = e_j = 0$ and thus is also present in the compactification of massless type IIA supergravity without any fluxes turned on. From a supergravity point of view this parameter is completely arbitrary and one can choose to set it to zero and recover in this way the results of section 2.3.3.

Putting together the above results one can write the low-energy effective action of type IIA theory compactified on a Calabi–Yau three-fold in the presence of background

[7] This assertion should not be very surprising for the following reason. Consider the dual formulation of type IIA where instead of a three-form potential C_3 one deals with a five form C_5. The flux e_0 appears in this picture as the unique flux for $\mathrm{d}C_5$ which is proportional to the volume form on the Calabi–Yau manifold. Moreover, the fluxes e_i now come from the dualization of the $h^{(1,1)}$ three-forms C_3^i which one obtains in the expansion of C_5 in the harmonic $(1,1)$ forms on the Calabi–Yau space

$$C_5 \sim C_3^i \wedge \omega_i \,. \tag{3.40}$$

From these arguments it should be clear that there is indeed a relation between the constant e_0 and the fluxes e_i.

RR fluxes in the following form

$$\begin{aligned} S &= \int e^{-2\phi}\Big(-\frac{1}{2}R^*\mathbf{1} + 2\mathrm{d}\phi\wedge *\mathrm{d}\phi - \frac{1}{4}H_3\wedge *H_3 - g_{ab}\mathrm{d}z^a\wedge *\mathrm{d}\bar{z}^b - g_{ij}\mathrm{d}t^i\wedge *\mathrm{d}\bar{t}^{\bar{j}}\Big) \\ &\quad +\frac{1}{2}\left(\mathrm{Im}\,\mathcal{M}^{-1}\right)^{AB}\left[\mathrm{d}\tilde{\xi}_A + \mathcal{M}_{AC}\mathrm{d}\xi^C\right]\wedge *\left[\mathrm{d}\tilde{\xi}_B + \bar{\mathcal{M}}_{BD}\mathrm{d}\xi^D\right] \\ &\quad +\frac{1}{2}H_3\wedge(\xi^A\mathrm{d}\tilde{\xi}_A - \tilde{\xi}_A\mathrm{d}\xi^A) + \frac{1}{2}\mathrm{Im}\mathcal{N}_{IJ}F^I\wedge *F^J + \frac{1}{2}\mathrm{Re}\mathcal{N}_{IJ}F^I\wedge F^J \\ &\quad -B_2\wedge J_2 - \frac{1}{2}M^2\, B_2\wedge *B_2 - \frac{1}{2}M_T^2\, B_2\wedge B_2 - V\,, \end{aligned} \tag{3.42}$$

where $I = 0,\dots,h^{(1,1)}$ and $\mathcal{N}_{IJ},\mathcal{M}_{AB}$ are standard supergravity couplings defined in (2.39) and (2.35)). One notices that except for the two form which has not been yet dualized the first two lines are exactly like in the standard case presented in section 2.3.3. The effects of turning on fluxes can be seen only in the last line of (3.42) and consist of new couplings of the NS two-form B_2 and the quantities J_2, M^2, M_T^2 are found to be[8]

$$\begin{aligned} J_2 &= (e_I F^I - m^I G_I)\ , \\ M^2 &= -m^I\mathrm{Im}\mathcal{N}_{IJ}m^J\ , \\ M_T^2 &= -m^I\mathrm{Re}\mathcal{N}_{IJ}m^J + m^I e_I\ , \end{aligned} \tag{3.43}$$

where we denoted m by m^0 and introduced the vectors $m^I = (m^0, m^i)$, $e_I = (e_0, e_i)$. Furthermore, we introduced the magnetic duals of $F^I \equiv dA^I$ by

$$G_I \equiv \mathrm{Im}\mathcal{N}_{IJ} * F^J + \mathrm{Re}\mathcal{N}_{IJ}F^J\ . \tag{3.44}$$

Finally the string frame potential in (3.42) is found to be

$$V = -\frac{1}{2}(e_I - \mathcal{N}_{IK}m^K)\,(\mathrm{Im}\,\mathcal{N})^{-1IJ}\,(e_J - \bar{\mathcal{N}}_{JL}m^L)\ , \tag{3.45}$$

where $(\mathrm{Im}\,\mathcal{N})^{-1}$ is given in (B.33). In contrast to the case of NS fluxes the potential found above depends on the complexified Kähler deformations via the matrix $\mathcal{N}$. In the Einstein frame potential the dilaton appears as an overall factor and thus again the potential has the run-away behavior in this direction.

Let us again write down the superpotential which corresponds to this case [48,72,73]

$$W \sim e_0 + \int_{Y_3} K\wedge F_4 + \int_{Y_3} K\wedge K\wedge F_2 + m^0\int_{Y_3} K\wedge K\wedge K\ , \tag{3.46}$$

where K is the complexified Kähler form (B.21). In this case a relation to M-theory is harder to see as in particular m^0 does not have an interpretation in M-theory while the two form field strength F_2 can not be interpreted as a flux as it has a geometric origin. However the second term in the superpotential (3.46) can be obtained again from [81] by reducing the G_2 structure to an $SU(3)$ one [82,83].

[8]Note that M^2 is positive since in our conventions $\mathrm{Im}\mathcal{N}_{IJ}$ is negative definite.

3.3.2 Relation to gauged supergravity

Let us now turn to study the connection between the theory obtained in the previous section and gauged supergravities. For the case at hand this is a more delicate issue due to the non-standard couplings which appeared in (3.42).

An important point in understanding the structure of the theory are the symmetries of the action (3.42). Beside the normal gauge invariance associated to the vector fields $\delta A^I = \mathrm{d}\Theta^I$ which is manifest in (3.42) there is also a Stuckelberg transformations associated to the two-form B_2 which also leaves the action inert

$$\delta B_2 = \mathrm{d}\Lambda \ , \quad \delta C_3 = \Lambda \wedge \mathrm{d}A^0 \ , \quad \delta A^I = -m^I \Lambda \ . \tag{3.47}$$

The way to see how these transformations appear is by simply reducing the ten-dimensional gauge transformations (2.8) to four dimensions and taking into account the fact that we have turned on fluxes (3.33). Note that these transformations are very similar to the original ten-dimensional ones as one can go to a gauge where the B-field eats one of the vector fields and effectively becomes massive. The only difference is that now we have $h^{(1,1)} + 1$ mass parameters and we can gauge away any of the vector fields.

It is a remarkable fact that both in the action (3.42) and in the gauge transformations (3.47) the constants m and e_0 naturally combine with the $2h^{(1,1)}$ flux parameters e_i and m^i. As we will see in a while this fact turns out to be crucial for preserving the symplectic invariance of the theory. To see this let us first introduce the notation

$$\begin{aligned} \check{F}^I &\equiv \mathrm{d}A^I + m^I B_2 \ , B \\ \check{G}_I &\equiv \mathrm{Re}\mathcal{N}_{IJ}\check{F}^J + \mathrm{Im}\mathcal{N}_{IJ} * \check{F}^J \ . \end{aligned} \tag{3.48}$$

With these definitions one can rewrite the Bianchi identities and the equations of motion for the vector fields as

$$\begin{aligned} \mathrm{d}\,\mathrm{d}A^I &= \mathrm{d}\check{F}^I - m^I dB_2 = 0 \ , \\ \frac{\partial \mathcal{L}}{\partial A^I} &= \mathrm{d}\check{G}_I - e_I dB_2 = 0 \ . \end{aligned} \tag{3.49}$$

Furthermore, the equation of motion for B_2 reads

$$\frac{\partial \mathcal{L}}{\partial B_2} = \frac{1}{2}\mathrm{d}(e^{-2\phi} * dB_2) + m^I \check{G}_I - e_I \check{F}^I = 0 \ . \tag{3.50}$$

Now one can observe that the equations of motion derived above are invariant under symplectic transformations under which $(\check{F}^i, \check{G}_I)$ and (m^I, e_I) transform as symplectic vectors i.e.

$$\begin{pmatrix} m^I \\ e_I \end{pmatrix} \rightarrow \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} m^I \\ e_I \end{pmatrix} , \qquad \begin{pmatrix} \check{F}^I \\ \check{G}_I \end{pmatrix} \rightarrow \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} \check{F}^I \\ \check{G}_I \end{pmatrix} , \tag{3.51}$$

where U, V, W, Z are $(h^{(1,1)} + 1) \times (h^{(1,1)} + 1)$ matrices satisfying (B.12) such that the $2(h^{(1,1)}+1) \times 2(h^{(1,1)}+1)$ matrix from the above equation is a symplectic matrix. This is a highly non-trivial and somehow unexpected result. As explained in appendix B $N = 2$ supergravities enjoy the property that they are invariant under transformations which rotate the electric and magnetic field strengths into one another. However, when gauging some of the isometries of the scalar manifold in order to obtain a gauged supergravity one needs to make a choice for an electric basis and so gaugings break the symplectic invariance explicitly. In other words it is impossible to have fields which carry both electric and magnetic charges coupled to $N = 2$ supergravity[9]. What the above results tell us is that one can avoid these problems if instead of considering scalar fields which are charged under both electric and magnetic fields one couples a two-form field, which in four dimensions carries the same number of degrees of freedom as a real scalar, to both electric and magnetic field strengths as in (3.42). In this way there is no need to fix the symplectic basis and one can consistently couple dyonic objects (two-forms) to $N = 2$ supergravity in a symplectic invariant fashion.

Let us now investigate the relation between the action derived in (3.42) and the standard gauged $N = 2$ supergravity as summarized in appendix B. A straightforward approach as in the case of NS fluxes will not be possible as now the action (3.42) features new ingredients like a mass for the two form field B_2 and couplings to both electric and magnetic field strengths. As such a supergravity was not constructed until now we will not be able to give a rigorous proof of the relation between the theory obtained in (3.42) and $N = 2$ supergravity. We will rather try to show some evidence that such a relation should indeed exist.

Let us first observe that the masses for B_2 and the couplings to the magnetic field strengths vanish for $m^I = 0$.[10] Since we have established the symplectic invariance of the theory we can always perform a symplectic transformation on the vector (m^I, e_I) and go to a basis where all m^I vanish[11]

$$\begin{pmatrix} m^I \\ e_I \end{pmatrix} \to \begin{pmatrix} 0 \\ e'_I \end{pmatrix} . \tag{3.52}$$

In this basis the 'new' couplings considerably simplify and from (3.43) and (3.45) one immediately obtains

$$M = M_T = 0 , \qquad J_2 = e'_I F'^I , \qquad V = -\frac{1}{2} e'_I \left(\mathrm{Im}\,\mathcal{N}'\right)^{-1IJ} e'_J , \tag{3.53}$$

[9] An attempt to cure this problem was done in [46] and we will come back to this work later.

[10] At a first sight it seems that if only the magnetic charges are present the B_2 field is still massive. However this is just an artifact of the basis we have chosen to describe the gauge fields. To be more precise in the case all the electric fluxes vanish we are basically describing (magnetic) charged fields using the (electric) dual vector fields. If we go to the magnetic basis for the vector fields all the masses for the B_2 field disappear and we are left with normal couplings like in (3.53). We will elaborate more on this topic in the last chapter where the problem will become more stringent.

[11] We should stress again that from a pure supergravity point of view the fluxes m^I, e_I are just continuous parameters and so there always exist an $Sp(2h^{(1,1)} + 2, \mathbf{R})$ transformation such that the rotated magnetic fluxes vanish. In a quantum theory however, the fluxes become quantized and the $Sp(2h^{(1,1)} + 2, \mathbf{R})$ invariance is generically broken to $Sp(2h^{(1,1)} + 2, \mathbf{Z})$. In this case, it is impossible to set the magnetic charges to zero by an $Sp(2h^{(1,1)} + 2, \mathbf{Z})$ rotation.

where the prime indicates the rotated basis. The drawback of this basis is that also the gauge couplings $\mathcal{N}$ of the action (3.42) change according to (B.14) and the relation to the prepotential as given in (B.7) is more complicated. So we have the choice to work either with the standard gauge couplings and a set of complicated interactions of B_2 or to transform to a new basis where the gauge couplings are more complicated but B_2 remains massless. In this latter basis the consistency with gauged supergravity is easily established so let us first discuss this case.

For $m^I = 0$, B_2 is massless and thus can be dualized to a scalar a as in appendix D.2.1. After a Weyl rescaling $g_{\mu\nu} \to e^{2\phi} g_{\mu\nu}$ the dual action reads

$$\begin{aligned} S &= \int \Big[-\frac{1}{2} R^* \mathbf{1} + \frac{1}{2} \mathrm{Im}\, \mathcal{N}'_{IJ} F'^I \wedge *F'^J + \frac{1}{2} \mathrm{Re}\, \mathcal{N}'_{IJ} F'^I \wedge F'^J \\ &\qquad - g_{ij} \mathrm{d}t^i \wedge * \mathrm{d}\bar{t}^{\bar{j}} - h_{uv} Dq^u \wedge * Dq^v - V_E \Big] \, , \end{aligned} \tag{3.54}$$

where the h_{uv} denotes the standard quaternionic metric [41] which is given in (2.40). The only scalar which is charged in the above action is the dual of the NS two-form B_2 and its covariant derivative is given by

$$Da = \mathrm{d}a + 2e'_I A'^I \, . \tag{3.55}$$

V_E represents the potential in the Einstein frame and is given by

$$V_E = -\frac{e^{4\phi}}{2} e'_I \left(\mathrm{Im}\, \mathcal{N}' \right)^{-1 IJ} e'_J \, . \tag{3.56}$$

In order to establish the consistency with gauged $N = 2$ supergravity we need to show that the potential (3.56) is consistent with the general form (3.28) known from gauged supergravity. Let us first note that only one scalar a in the hyper-multiplets carries gauge charge while in the vector multiplet sector all scalars remain neutral. In terms of the Killing vectors defined in (3.27) equation (3.55) implies

$$k_I^u = -2e'_I \delta^{ua} \, , \qquad k_I^i = 0 \, . \tag{3.57}$$

Inserted into (3.28) and using (2.40) one arrives at

$$V_E = -\frac{1}{2} \left[(\mathrm{Im}\, \mathcal{N}')^{-1} \right]^{IJ} P_I^x P_J^x + 4 e^K X^I \bar{X}^J (e^{4\phi} e'_I e'_J - P_I^x P_J^x) \, . \tag{3.58}$$

We are left with the computation of the Killing prepotentials P_I^x defined in (B.17). Following [46] one first observes that for the constant (field independent) Killing vectors as in (3.57) equations (B.17) are solved by

$$P_I^x = k_I^u \omega_u^x \, , \quad x = 1, 2, 3 \, , \tag{3.59}$$

where ω_u^x is the $SU(2)$ connection on the quaternionic manifold. For the case at hand ω_u^x has been computed in [41] and here we only need their result $\omega_a^x = -\frac{1}{2} e^{2\phi} \delta^{3x}$. Inserted into (3.59) using (3.57) we obtain

$$P_I^1 = P_I^2 = 0 \, , \quad P_I^3 = -e^{2\phi} e'_I \, . \tag{3.60}$$

One can now see that the last term in (3.58) vanishes while the first one reproduces the potential (3.56). This establishes the consistency with $N = 2$ gauged supergravity.

Let us return to the discussion of the action in the unrotated basis where both e_I and m^I are non-zero. In this case B_2 is massive and the relation with the standard gauged supergravity is not obvious and, as far as we know, has not been discussed in the literature. However, one can use the fact that a massive two-form in $d = 4$ is Poincaré dual to a massive vector [84–86]. This generic duality is briefly summarized in appendix D.2.3. In the following we perform the duality transformation and display the dual action in terms of only vector fields.

Starting from the action (3.42) it is straightforward to apply the results in appendix D.2.3. Denoting by A^H the dual of the massive B_2 the resulting action reads

$$\begin{aligned} S \;=\; & \int e^{-2\phi}\Big(-\frac{1}{2}R * \mathbf{1} + 2\mathrm{d}\phi \wedge *\mathrm{d}\phi - g_{ab}\mathrm{d}z^a \wedge *\mathrm{d}\bar{z}^b - g_{ij}\mathrm{d}t^i \wedge *\mathrm{d}\bar{t}^j \Big) \\ & + \frac{1}{2}\left(\mathrm{Im}\,\mathcal{M}^{-1}\right)^{AB}\Big[\mathrm{d}\tilde{\xi}_A - \mathcal{M}_{AC}\mathrm{d}\xi^C\Big] \wedge *\Big[\mathrm{d}\tilde{\xi}_B - \bar{\mathcal{M}}_{BD}\mathrm{d}\xi^D\Big] - V \\ & + \frac{1}{2}\mathrm{Im}\,\mathcal{N}_{IJ}F^I \wedge *F^J + \frac{1}{2}\mathrm{Re}\mathcal{N}_{IJ}F^I \wedge F^J - e^{2\phi}A^H \wedge *A^H \\ & - \frac{1}{2}\frac{M^2}{M^4 + M_T^4}\left(F^H - J_2'\right) \wedge *\left(F^H - J_2'\right) \\ & + \frac{1}{2}\frac{M_T^2}{M^4 + M_T^4}\left(F^H - J_2'\right) \wedge \left(F^H - J_2'\right) , \end{aligned} \tag{3.61}$$

where

$$F^H = \mathrm{d}A^H \ , \qquad J_2' = J_2 - \frac{1}{2}\mathrm{d}(\tilde{\xi}_A\mathrm{d}\xi^A - \xi^A\mathrm{d}\tilde{\xi}_A) \,, \tag{3.62}$$

and the quantities M, M_T and J_2 are defined in (3.43). The above action contains an explicit mass term for the vector field A_H which can equivalently be written as the covariant derivative of a Goldstone boson

$$e^{2\phi}A^H \wedge *A^H = \frac{1}{4}e^{2\phi}Da \wedge *Da \ , \tag{3.63}$$

where

$$Da = \mathrm{d}a + 2A'^H \,. \tag{3.64}$$

(A'^H denotes the gauge transformed vector potential.) Inserting (3.63) into (3.61) and absorbing $\frac{1}{2}(\tilde{\xi}_A\mathrm{d}\xi^A - \xi^A\mathrm{d}\tilde{\xi}_A)$ into a further redefinition of A^H results in

$$\begin{aligned} S \;=\; & \int -\frac{1}{2}R^*\mathbf{1} - g_{ij}\mathrm{d}t^i \wedge *\mathrm{d}\bar{t}^j - h_{uv}Dq^u \wedge *Dq^v - V_E \\ & + \frac{1}{2}\,\mathrm{Im}\,\hat{\mathcal{N}}_{\hat{I}\hat{J}}F^{\hat{I}} \wedge *F^{\hat{J}} + \frac{1}{2}\,\mathrm{Re}\,\hat{\mathcal{N}}_{\hat{I}\hat{J}}F^{\hat{I}} \wedge F^{\hat{J}} \ , \end{aligned} \tag{3.65}$$

where also a Weyl rescaling $g_{\mu\nu} \to e^{2\phi}g_{\mu\nu}$ has been performed and we introduced the index $\hat{I} = (I, H)$. V_E is the Weyl rescaled potential related to V of (3.45) by $V_E = e^{4\phi}V$.

$h_{uv}Dq^u \wedge *Dq^v$ is again the standard quaternionic metric given in (2.40) with the only difference that (3.55) is replaced by (3.64). Moreover, the 'new' $(h^{(1,1)}+2) \times (h^{(1,1)}+2)$ dimensional gauge coupling matrix $\hat{\mathcal{N}}_{\hat{I}\hat{J}}$ is given by

$$\begin{aligned} \hat{\mathcal{N}}_{IJ} &= \mathcal{N}_{IJ} - i\,\mu\left(e_I - \mathcal{N}_{IK}m^K\right)\left(e_J - \mathcal{N}_{JL}m^L\right)\,, \qquad \hat{\mathcal{N}}_{HH} = -i\,\mu\,, \\ \hat{\mathcal{N}}_{IH} &= i\,\mu\left(e_I - \mathcal{N}_{IK}m^k\right)\,, \qquad\qquad \mu \equiv \frac{M^2 + iM_T^2}{M^4 + M_T^4}\,. \end{aligned} \tag{3.66}$$

One easily shows that $\hat{\mathcal{N}}_{IJ}m^J = e_I$ and hence $\mathrm{Im}\,\hat{\mathcal{N}}_{\hat{I}\hat{J}}$ has a null vector while $\mathrm{Re}\,\hat{\mathcal{N}}_{\hat{I}\hat{J}}$ has one constant eigenvalue. This implies that one (linear combination) of the vector fields only has a topological coupling.

The dualization of B_2 resulted in an additional massive vector A^H and we chose to write the mass term as the coupling of a Goldstone boson a. The number of physical degrees of freedom is of course unchanged since the action (3.61)/(3.65) is still invariant under the gauge transformations (3.47) which after dualization become

$$\delta A^I = -m^I \Lambda_1\,, \qquad \delta A^H = 0\,. \tag{3.67}$$

A^H being the Poincaré dual of H_3 is invariant under (3.47) but one of the other $h^{(1,1)}+1$ vector fields in (3.65) can be gauged away by (3.67). In this 'unitary gauge' the symplectic invariance is lost. Thus the theory can be formulated in terms of only vector fields but symplectic invariance demands the presence of an additional auxiliary vector field with only topological couplings. In any physical gauge the symplectic invariance is broken.

To conclude this section let us discuss another aspect of the dualization of the massive B-field. Equations (3.49) can be solved for $\check{F}^I$ and $\check{G}_I$ in terms of electric and magnetic potentials A^I and $\tilde{A}_I$

$$\check{F}^I = m^I B_2 + \mathrm{d}A^I\,, \quad \check{G}_I = e_I B_2 + \mathrm{d}\tilde{A}_I\,. \tag{3.68}$$

Now the equation of motion for B_2 becomes

$$\frac{1}{2}d(e^{-2\phi} * \mathrm{d}B_2) + m^I \mathrm{d}\tilde{A}_I - e_I \mathrm{d}A^I = 0\,. \tag{3.69}$$

This suggests that we can introduce a scalar field a (the dual of B_2) which obeys

$$e^{-2\phi} * \mathrm{d}B_2 = Da \equiv \mathrm{d}a - 2m^I\tilde{A}_I + 2e_I A^I\,. \tag{3.70}$$

This definition has the feature that it maintains explicitly the symplectic invariance closely related to the proposal of [46, 49]. However, in (3.68) B_2 and not $\mathrm{d}B_2$ appears and thus it is not possible to give an action in terms of the dual scalar a with electric and magnetic couplings. Nevertheless, one can compute the electric and magnetic Killing prepotentials corresponding to the gauging (3.70) as suggested in [46, 49]. They are very similar to the ones found only for the electrically charged particles (3.60)

$$P_I^3 = e^{2\phi}e_I, \qquad \tilde{P}^{I3} = e^{2\phi}m^I, \qquad P_I^1 = P_I^2 = \tilde{P}^{I1} = \tilde{P}^{I2} = 0\,. \tag{3.71}$$

Using the formula for the potential suggested in [46]

$$\begin{aligned} V_E &= 4e^K X^I \bar{X}^J h_{uv}(k_I^u - \tilde{k}^{uK}\mathcal{N}_{KI})(k_I^v - \tilde{k}^{vK}\bar{\mathcal{N}}_{KI}) \\ &\quad - \left[\frac{1}{2}(\mathrm{Im}\,\mathcal{N})^{-1IJ} + 4e^K X^I \bar{X}^J\right](P_I^x - \tilde{P}^{Kx}\mathcal{N}_{KI})(P_J^x - \tilde{P}^{Kx}\bar{\mathcal{N}}_{KJ})\ , \end{aligned} \tag{3.72}$$

which is the symplectic invariant extension of (3.28), one immediately recovers the potential obtained in (3.45).

This concludes our analysis about fluxes in type IIA theory. We will next move on to type IIB case and we will mainly be interested in how mirror symmetry relates the two theories when fluxes are turned on.

Chapter 4

Type IIB theory with fluxes

Having discussed the effect of fluxes in type IIA compactifications in the last chapter we now turn our attention to type IIB theory. The main reason for this is mirror symmetry as we want to study how the fluxes modify it. We present in the first two sections type IIB compactifications with RR and NS-NS fluxes turned on and in the last section we start the discussion about mirror symmetry at the level of the four-dimensional effective actions. In order to obtain the full picture it will turn out that we need some generalization of the notion of flux we have introduced in the previous chapter and this will motivate the work in the last chapter.

4.1 IIB with RR fluxes

Let us first see what is the effect of RR fluxes in type IIB compactifications. Similar to the previous chapter we rely on the usual Calabi–Yau compactifications of type IIB theory presented in section 2.3.4 and do not repeat once more the common details. Rather we emphasize the differences which appear when turning on fluxes.

In order to turn on RR fluxes we perform the field redefinition $A_4 \to A_4 + \frac{1}{2} B_2 \wedge C_2$ which has the effect that the field C_2 appears in the action only as $\mathrm{d}C_2$. The form of the action (2.9) is not modified, but now the field strengths have a different form

$$\hat{H}_3 = \mathrm{d}\hat{B}_2\,, \qquad \hat{F}_3 = \mathrm{d}\hat{C}_2 - l\mathrm{d}\hat{B}_2\,, \qquad \hat{F}_5 = \mathrm{d}\hat{A}_4 + \hat{B}_2 \wedge \mathrm{d}\hat{C}_2\,. \tag{4.1}$$

As we have discussed at length in the case of type IIA theory, turning on fluxes does not change the light spectrum and thus one assumes again the field expansions (2.41). The only modification is that the field strength of C_2 is shifted according to[1]

$$\mathrm{d}\hat{C}_2 \to \mathrm{d}\hat{C}_2 + m^A \alpha_A - e_A \beta^A\,, \tag{4.2}$$

[1]Even if the RR sector of type IIB theory also comprises zero and four-form potentials l and A_4 with one and five-form field strengths one can not use these fields to turn on fluxes in a Calabi–Yau compactification as there are no harmonic one or five-forms on such a space. Thus the only possibility when we want to turn on RR fluxes is to consider the field strength of the RR two-form C_2.

where e_A, m^A are the constant background fluxes. With this, the expansion of the field strengths $\hat{F}_3$ in the Calabi–Yau harmonic forms turns into

$$\hat{F}_3 = \mathrm{d}C_2 - lH_3 + (\mathrm{d}c^i - l\mathrm{d}b^i)\,\omega_i + m^A\alpha_A - e_A\beta^A\,. \tag{4.3}$$

As in the other cases presented in the last chapter the compactification proceeds by inserting the field expansions (2.41) and (4.2) into the action and performing the integrals over the Calabi–Yau space. Let us shortly see what are the modifications compared to section 2.3.4 when fluxes are turned on. First of all due to the term in the expansion of $\hat{F}_3$ (4.3) which points only in the internal directions the kinetic term of C_2 produces a potential which reads

$$V = -\frac{1}{2}\big(e_A - m^C\mathcal{M}_{CA}\big)(\mathrm{Im}\,\mathcal{M})^{-1AB}\big(e_B - \bar{\mathcal{M}}_{BD}m^D\big)\,, \tag{4.4}$$

where the matrix $\mathcal{M}$ was defined in (2.35). Furthermore, the fluxes (4.2) enter the expansion of $\hat{F}_5$ which now becomes

$$\hat{F}_5 = (\mathrm{d}D_2^i + b^i\mathrm{d}C_2 + B_2\wedge\mathrm{d}c^i)\wedge\omega_i + \check{F}^A\alpha_A - \check{G}_A\beta^A + (\mathrm{d}\rho_i + b^j\mathrm{d}c^k\mathcal{K}_{ijk})\wedge\tilde{\omega}^i\,, \tag{4.5}$$

where we used the definitions

$$\check{F}^A \equiv F^A + m^AB_2\,, \qquad \check{G}_A \equiv G_A + e_AB_2\,. \tag{4.6}$$

Finally, the topological term produces a Green-Schwarz type interaction

$$\delta\mathcal{L}_{\mathrm{top}} = -\frac{1}{2}(F^Ae_A - G_Am^A)\wedge B_2\,. \tag{4.7}$$

Let us pause for a while and analyze the effect of the fluxes at this stage. Comparing with section 3.2 the above formulae have a similar structure. First of all (4.6) is very similar to (3.15) with the only difference that now we deal with higher degree form-fields. In the same way (4.7) is the analog of (3.18) while the potential (4.4) corresponds to (3.17). The analogy we have just seen has no deep meaning and only denotes a similar structure which arises during the calculations. The only reason for which we mentioned it is to stress that as in section 3.2 from this point on the calculation is very similar to the usual Calabi–Yau compactification. The difference is that in the end the field strengths F^A are replaced by the gauge invariant[2] quantity $\check{F}^A$ from (4.6).

As in section 2.3.4 just substituting the field expansions (2.41) and (4.3) into the action does not lead to the correct dynamics in four dimensions. For this one has to further impose the self-duality condition on $\hat{F}_5$ at the level of the four-dimensional fields. Note that the only modification in the expansion of $\hat{F}_5$ is that the electric and magnetic field strengths F^A and G_A are replaced by the quantities $\check{F}^A$ and $\check{G}_A$ defined in (4.6). Thus the self-duality condition involving D^i and ρ_i which is given in the second line of

[2]The interesting gauge invariance of (4.1) is now $\delta\hat{B}_2 = \mathrm{d}\Lambda_1$, $\delta\hat{A}_4 = -\Lambda_1\wedge\mathrm{d}\hat{C}_2$. Considering (4.2) this leads to $\delta B_2 = \mathrm{d}\Lambda_1$, $\delta F^A = -m^A\mathrm{d}\Lambda_1$, $\delta G_A = -e_A\mathrm{d}\Lambda_1$ and thus the quantities $\check{F}^A$ and $\check{G}_A$ are indeed gauge invariant.

(2.43) is not modified and so the reduction of this sector is not affected by the fluxes. Consequently we are going to restrict our attention to the gauge field sector and assume the other results from section 2.3.4. For the fields F^A and G_A the self-duality condition becomes

$$\check{G}_A = \mathrm{Im}\,\mathcal{M}_{AB} * \check{F}^B + \mathrm{Re}\,\mathcal{M}_{AB}\check{F}^B\,, \tag{4.8}$$

and as in section 2.3.4 this can be obtained as the equation of motion of G_A if one adds to the action the total derivative term $\frac{1}{2}F^A \wedge G_A$. Eliminating $\check{G}_A$ from (4.8) and after taking into account (4.7) one is left with the following Lagrangian for the gauge fields

$$\begin{aligned}\mathcal{L}(F) &= \frac{1}{2}F^A \wedge G_A - \frac{1}{2}(F^A e_A - G_A m^A) \wedge B_2 \qquad (4.9)\\ &= \frac{1}{2}\mathrm{Im}\,\mathcal{M}_{AB}\check{F}^A \wedge *\check{F}^B + \frac{1}{2}\mathrm{Re}\,\mathcal{M}_{AB}\check{F}^A \wedge \check{F}^B - \frac{1}{2}(F^A e_A + \check{F}^A e_A) \wedge B_2\,.\end{aligned}$$

After performing the field redefinitions (2.50), except for the the last one as B_2 has not been yet dualized to the axion a, one obtains the action

$$\begin{aligned}S \;=\; \int e^{-2\phi}\Big(&-\frac{1}{2}R*\mathbf{1} + 2\mathrm{d}\phi\wedge *\mathrm{d}\phi - \frac{1}{4}H_3\wedge *H_3 - g_{ab}\mathrm{d}z^a\wedge *\mathrm{d}\bar{z}^b - g_{ij}\mathrm{d}t^i\wedge *\mathrm{d}\bar{t}^j\Big)\\ &+\frac{1}{2}\left(\mathrm{Im}\,\mathcal{N}^{-1}\right)^{IJ}\left[\mathrm{d}\tilde{\xi}_I + \mathcal{N}_{IK}\mathrm{d}\xi^K\right]\wedge *\left[\mathrm{d}\tilde{\xi}_J + \bar{\mathcal{N}}_{JL}\mathrm{d}\xi^L\right] \qquad (4.10)\\ &+\frac{1}{2}H_3\wedge(\tilde{\xi}_A\mathrm{d}\xi^A - \xi_A\mathrm{d}\tilde{\xi}_A) + \frac{1}{2}\mathrm{Im}\,\mathcal{M}_{AB}\check{F}^A\wedge *\check{F}^B + \frac{1}{2}\mathrm{Re}\,\mathcal{M}_{AB}\check{F}^A\wedge\check{F}^B\\ &-\frac{1}{2}B_2\wedge(\check{F}^A + \mathrm{d}V^A)e_A - V\;,\end{aligned}$$

where V is given in (4.4).

Let us make a few comments on the result obtained above. First of all one notices that in the action (4.10) we do not have only the standard $N = 2$ fields described in table 2.3. The reason for this is that the antisymmetric tensor field B_2 is massive and can not be dualized to a scalar. This is very similar to what we have already encountered in the previous chapter in the case of type IIA theory with RR fluxes. In fact we will see in section 4.3 that these two theories are related by mirror symmetry and thus all the arguments presented in section 3.3 to relate the theory with a massive two-form field to some gauge supergravity also apply here. It is worth noting that the potential (4.4) depends on the complex structure moduli (which are now members of the vector multiplets) and this is in agreement with mirror symmetry as in the case of type IIA with RR fluxes the potential (3.45) was a function of the Kähler moduli. This dependence on the complex structure moduli can be also seen from the corresponding superpotential [48]

$$W \sim \int_{Y_3}\Omega\wedge\mathrm{d}C_2\;. \tag{4.11}$$

4.2 IIB with NS fluxes

Let us now come to the subject of turning on NS fluxes in type IIB theory which will motivate part of the work in the next chapter. Due to the $SL(2,\mathbf{R})$ symmetry which rotates the two two-forms into one another (2.14), considering NS fluxes instead of RR ones in type IIB theory does not bring in any new features. Apart from different factors of the dilaton, the results from the previous section will be valid if one exchanges the two fields B_2 and C_2.

As a consequence now the RR two-form field C_2 becomes massive. For completeness we record the final result without going through all the details of the compactification. As in the previous section the modifications caused by the fluxes appear in the compactification via

$$\mathrm{d}\hat{B}_2 = \mathrm{d}B_2 + \mathrm{d}b^i \wedge \omega_i + \tilde{m}^A \alpha_A - \tilde{e}_A \beta^A \ . \tag{4.12}$$

Performing the KK reduction as before one ends up with the following effective action in four dimensions

$$\begin{aligned}
S_{IIB}^{(4)} \ = \ & \int -\frac{1}{2} R *\mathbf{1} - g_{ab}\mathrm{d}z^a \wedge *\mathrm{d}\bar{z}^b - g_{ij}\mathrm{d}t^i \wedge *\mathrm{d}\bar{t}^j - \mathrm{d}\phi \wedge *\mathrm{d}\phi \\
& -\frac{1}{4} e^{-4\phi}\mathrm{d}B_2 \wedge *\mathrm{d}B_2 - \frac{1}{2} e^{-2\phi}\mathcal{K}\left(\mathrm{d}C_2 - l\mathrm{d}B_2\right) \wedge *\left(\mathrm{d}C_2 - l\mathrm{d}B_2\right) \\
& -\frac{1}{2}\mathcal{K}e^{2\phi}\mathrm{d}l \wedge *\mathrm{d}l - 2\mathcal{K}e^{2\phi} g_{ij}\left(\mathrm{d}c^i - l\mathrm{d}b^i\right) \wedge *\left(\mathrm{d}c^j - l\mathrm{d}b^j\right) \\
& -\frac{e^{2\phi}}{2\mathcal{K}} g^{-1\,ij}\left(\mathrm{d}\rho_i - \frac{1}{2}\mathcal{K}_{ikl}c^k\mathrm{d}b^l\right) \wedge *\left(\mathrm{d}\rho_j - \frac{1}{2}\mathcal{K}_{jmn}c^m\mathrm{d}b^n\right) \\
& +2\left(\mathrm{d}b^i \wedge C_2 + c^i\mathrm{d}B_2\right) \wedge \left(\mathrm{d}\rho_i - \frac{1}{2}\mathcal{K}_{ijk}c^j\mathrm{d}b^k\right) + \frac{1}{2}\mathcal{K}_{ijk}c^i c^j dB_2 \wedge \mathrm{d}b^k \\
& +\frac{1}{2}\mathrm{Re}\,\mathcal{M}_{AB}\check{F}^A \wedge \check{F}^B + \frac{1}{2}\mathrm{Im}\,\mathcal{M}_{AB}\check{F}^A \wedge *\check{F}^B + \frac{1}{2}\tilde{e}_A\left(F^A + \check{F}^A\right) \wedge C_2 \\
& +\frac{1}{2}e^{4\phi}\left(l^2 + \frac{e^{-2\phi}}{2\mathcal{K}}\right)\left(\tilde{e} - \mathcal{M}\tilde{m}\right)_A \mathrm{Im}\,\mathcal{M}^{-1\,AB}\left(\tilde{e} - \bar{\mathcal{M}}\tilde{m}\right)_B *\mathbf{1}\ ,
\end{aligned} \tag{4.13}$$

where $\check{F}^A = F^A - \tilde{m}^A C_2$. The only reason this action looks more complicated than (4.10) is because now C_2 is massive and its dualization is not straightforward as it was before and thus performing the field redefinitions (2.50) is more cumbersome in this case. As later on we will need this action for the particular case when the magnetic fluxes vanish let us see how the action simplifies under such assumptions. First of all we note that when $m^A = 0$ we have $\check{F}^A = F^A = \mathrm{d}A^A$ and thus all mass terms for C_2 drop out. One can now dualize C_2 and B_2 to scalars as we did before and perform the field redefinitions

(2.50). In this way one obtains for the four-dimensional action

$$\begin{aligned} S^{(4)}_{IIB} &= \int -\frac{1}{2}R*\mathbf{1} - g_{ab}\mathrm{d}z^a \wedge *\mathrm{d}\bar{z}^b - h_{uv}Dq^u \wedge *Dq^v - V_E *\mathbf{1} \\ &\quad +\frac{1}{2}\mathrm{Re}\,\mathcal{M}_{AB}F^A \wedge F^B + \frac{1}{2}\mathrm{Im}\,\mathcal{M}_{AB}F^A \wedge *F^B \ , \end{aligned} \tag{4.14}$$

where the quaternionic metric h_{uv} is given by (2.52) while the potential (in the Einstein frame) reads

$$V_E = -\frac{1}{2}\,e^{4\phi}\Big(l^2 + \frac{e^{-2\phi}}{2\mathcal{K}}\Big)\,\tilde{e}_A\left[(\mathrm{Im}\,\mathcal{M})^{-1}\right]^{AB}\tilde{e}_B. \tag{4.15}$$

The presence of the electric fluxes has gauged some of the isometries of the hyper-scalars as can be seen from the covariant derivatives

$$Da = \mathrm{d}a - \xi^0\tilde{e}_A V^A \ , \qquad D\tilde{\xi}_0 = \mathrm{d}\tilde{\xi}_0 + \tilde{e}_A V^A \ , \qquad D\tilde{\xi}_i = \mathrm{d}\tilde{\xi}_i \ , \qquad D\xi^I = \mathrm{d}\xi^I \ . \tag{4.16}$$

Again as in the previous section the scalar potential depends on the complex structure moduli. In addition the potential also depends on the RR scalar l and on the dilaton. This dependence can be summarized in a superpotential which has the form [48]

$$W \sim \tau \int_{Y_3} \Omega \wedge H_3 \ , \tag{4.17}$$

where τ denotes the complex dilaton $\tau = l + ie^{-\phi}$ introduced in section 2.1.3. One sees that unlike the type IIA case in type IIB the superpotential for the RR and NS-NS fluxes are very similar. If one turns on both RR and NS-NS fluxes the total superpotential can be written as

$$W \sim \int_{Y_3} \Omega \wedge (\mathrm{d}C_2 + \tau \mathrm{d}B_2) = \int_{Y_3} \Omega \wedge G_3 \ . \tag{4.18}$$

Thus the NS-NS fluxes complexify the RR ones using the complex type IIB coupling τ.

4.3 Mirror symmetry with fluxes

Let us now come to the main point of this chapter namely mirror symmetry. We have already seen in chapter 2 that for normal Calabi–Yau compactification there is a precise relation (2.50) and (2.53) which maps the low energy effective action of type IIA in the one of type IIB theory. Thus the natural question which arises is whether this map is still valid when fluxes are turned on.

We have already anticipated that in the case of RR fluxes this is true and indeed one can easily see that modulo the identifications (2.53) the low energy actions (3.42) and (4.10) are the same. To have a complete list of transformations we also have to add to the relations (2.53) the fact that the fluxes on the two sides are be mapped into one another[3]

$$e_I \leftrightarrow e_A\ , \qquad\qquad m^I \leftrightarrow m^A\ . \tag{4.19}$$

Note that in this last relation crucially depends on the extra two parameters m^0 and e_0 which were coming from the ten dimensional mass m and the constant dual to C_3 in four dimensions respectively.

It is also instructive to check mirror symmetry using the superpotentials which we wrote for these cases (3.46) and (4.11). First of all one notices that in the type IIA case the superpotential depends on the Kähler moduli while in type IIB case it depends on the complex structure ones. Performing the integrals over the Calabi–Yau manifold using (2.24), (2.25), (B.23), (B.26) and (B.35) it is not hard to see that

$$\begin{aligned} W_A &\sim e_I t^I - m^I \mathcal{F}_I(t)\ , \\ W_B &\sim e_A z^A - m^A \mathcal{F}_A(z)\ , \end{aligned} \tag{4.20}$$

where t^I denote the complexified Kähler moduli in the case of type IIA compactification, z^A are complex structure moduli in the type IIB case and by $\mathcal{F}(t/z)$ we denoted the prepotential in type IIA and type IIB respectively. Clearly now as mirror symmetry exchanges the Kähler and the complex structure moduli and also the prepotentials the two superpotentials are mapped into one another provided we further assume the map (4.19).

Let us now turn our attention to the case when NS fluxes are present. In this case the low energy effective actions describing the two theories do not look similar anymore. Even without a detailed study one can claim that the two theories are not the same by applying a simple counting argument. It is easy to see that in type IIA case one has $2(h^{(2,1)}+1)$ parameters, and the same holds for type IIB. However as mirror symmetry exchanges the odd and even cohomologies these numbers can not be the same on the two sides. For mirror symmetry to work one would also need $2(h^{(1,1)}+1)$ on both sides or in other words one needs even forms field strengths in the NS sector. Moreover both potentials (3.25) and (4.15) depend on the complex structure moduli of the corresponding compactification and again these moduli are not related by mirror symmetry.

To solve this problem we have to generalize the above procedure of turning on fluxes in that we should allow for a different class of manifolds in the compactification and we will see how this works in the next chapter.

[3]Recall that e_I and m^I, $I = 0, \ldots, h^{(1,1)}$ denote the RR fluxes in type IIA theory, while e_A and m^A $A = 0, \ldots, h^{(2,1)}$ denote the RR fluxes in type IIB theory. As $h^{(1,1)}(Y) = h^{(2,1)}(\tilde{Y})$ the identification (4.19) is indeed consistent.

Chapter 5

Mirror symmetry with NS fluxes

In the last chapter we have started the discussion of mirror symmetry when fluxes are turned on. For the case of RR fluxes we found no real obstruction to mirror symmetry, but for the NS fluxes the situation remained unclear and we postponed its discussion for this chapter. We will see that the 'missing' fluxes come from considering different type of manifolds which are termed *half-flat* manifolds with $SU(3)$ structure. After introducing the main mathematical ideas in section 5.1 we perform the KK reduction on half-flat manifolds and show that in this way we obtain the effective actions which were derived in the previous chapters for type IIA and type IIB theories with NS fluxes. However the argument we present holds only for half of the fluxes while in order to reproduce all the NS-NS fluxes it seems that one has to further generalize the internal manifold. We discuss this issue in section 5.3.

5.1 Manifolds with $SU(3)$ structure

Let us start by reviewing once more why mirror symmetry did not work when NS fluxes were turned on in the way we did in the last chapters. First of all we have shown that the fluxes have to be harmonic forms on the internal manifold (3.8). The NS sectors of the two type II theories are identical and in the matter part beside the dilaton one finds only the NS two-form B_2. Thus it appears that there is only one way to turn on fluxes in the NS sector, namely for the three-form field strength $H_3 = \mathrm{d}B_2$

$$H_3 = m^A \alpha_A - e_A \beta^A \ . \tag{5.1}$$

This can be done in both type IIA and type IIB theories, but the two situations can not be related by mirror symmetry as mirror symmetry exchanges odd and even cohomologies. To restore it one would rather need even-form fluxes or in other words even-form field strengths. As we do not expect mirror symmetry to mix NS-NS and RR sectors[1] it is clear that we need a generalization of the notion of flux introduced in the last chapter

[1] In fact we have already seen that the RR fluxes in type II theories are mirror symmetric to one another.

in order to find the mirror partners of the NS three-form fluxes. Thus our task for this chapter will be to find configurations which are mirror to Calabi–Yau manifolds with NS fluxes turned on. Note that by 'mirror configuration' we understand the physical picture where type IIA and type IIB theories compactified on such mirror configurations lead to the same physics in four dimensions.

We have just argued that the two-form field B_2 can not reproduce the fluxes we need for mirror symmetry. The other choices in the NS-NS sector are the dilaton and the metric. As it is hard to imagine that all the flux parameters can appear from the dilaton we are naturally lead to consider that the metric itself has to be deformed in such a way that it reproduces the missing flux parameters. In a recent paper [50] it was proposed that the manifold which could produce the mirror NS fluxes should not be complex, but only almost complex and the even-form fluxes should come from the lack of integrability of the almost complex structure. The solution to this problem was found in [16] and the manifold considered in this case was indeed non-complex. In this section we present in detail these ideas following [16, 17].

Let us first see what choices we have for the internal manifold which can reproduce the configuration mirror to (5.1). Throughout this argument we are going to use mirror symmetry as a guiding principle. The crucial observation which will allow us to find a restricted class of internal manifolds is about the amount of supersymmetry preserved by the four-dimensional action. As we argued in the previous chapters, when fluxes are turned on, the low energy effective action still has $N = 2$ supersymmetry even if in general a supersymmetric Minkowski ground state does not exist. Thus, we would first of all need that the manifold we are looking for preserves $N = 2$ supersymmetry. As explained in section 2.3 this requirement is quite restrictive as it implies that the manifold has $SU(3)$ structure or equivalently that is has a globally defined nowhere-vanishing spinor. We will see in a while that such manifolds are classified according to their so called *intrinsic torsion* and this will turn out to be the ingredient which has the potential to reproduce the mirror NS fluxes and which was missing in the simple case of Calabi–Yau manifolds. Before we actually see how these fluxes appear if we take into account a non-vanishing intrinsic torsion we make a short detour into the mathematical description of these manifolds. A more general approach can be found in appendix C or in the existing literature [68, 87–91].

We start from the fact that manifolds with $SU(3)$ structure posses a globally defined spinor which we denote by η. This spinor is in general not covariantly constant with respect to the Levi–Civita connection[2] as it was the case for Calabi–Yau manifolds and thus $\nabla\eta$ will give a measure of the deviation from a Calabi–Yau manifold. One can nevertheless find [87, 88] a new connection $\nabla^{(T)}$ which satisfies

$$\nabla^{(T)}_m \eta = \nabla_m \eta - \frac{1}{4} \kappa^0{}_{mnp} \Gamma^{np} \eta = 0 \ , \tag{5.2}$$

where Γ^{np} is the antisymmetrized product of gamma matrices and $\kappa^0{}_{mnp}$ denotes the intrinsic contorsion tensor. (for a more detailed discussion see appendix C)

[2]We have implicitly assumed that we have chosen a metric g on the manifold which is invariant under the the action of the structure group.

The existence of a globally defined spinor is not a very intuitive picture of the manifolds with $SU(3)$ structure. Fortunately there exist an alternative description in terms of $SU(3)$ invariant tensors which are constructed out of the the spinor η. For definiteness we choose η to be a Majorana spinor and we use the gamma matrix conventions presented in appendix A. We define a (real) two-form J with the components

$$J_{mn} = -i\eta^\dagger \Gamma_7 \Gamma_{mn}\eta \ , \tag{5.3}$$

and a (complex) three-form Ω

$$\Omega = \Omega^+ + i\Omega^- \ , \tag{5.4}$$

where

$$\Omega^+_{mnp} = -i\eta^\dagger \Gamma_{mnp}\eta \ , \qquad \Omega^-_{mnp} = -iJ_m{}^q J_n{}^r J_p{}^s\, \Omega^+_{qrs} \ . \tag{5.5}$$

$\Gamma_{m_1 \dots m_p}$ denote antisymmetrized products of p gamma matrices which are defined in (A.12) and the indices are raised and lowered with the metric g. Due to the gamma matrix algebra the quantities defined above enjoy a couple of nice properties similar to the corresponding ones (complex structure and holomorphic three form) defined on a Calabi–Yau manifold. In particular one can show that [29, 39]

$$J_m{}^p J_p{}^n = -\delta_m{}^n \ , \qquad J_m{}^p J_n{}^r g_{pr} = g_{mn} \ , \tag{5.6}$$

which tells us that J is an almost complex structure and the $SU(3)$ invariant metric g is hermitian with respect to it. Moreover, as explained in the appendix this allows us to introduce (p, q) forms and one can further show that

$$J_m{}^q J_n{}^r\, \Omega_{qrp} = -\Omega_{mnp} \ , \tag{5.7}$$

which means that the first two indices of Ω are of the same complex type with respect to the almost complex structure J. Due to the antisymmetry in all indices of Ω one can in fact show that all the three indices of Ω are of the same type and it is conventional to consider that Ω is a $(3, 0)$ form.

Up to this point everything looks like on an ordinary Calabi–Yau manifold. The difference comes from the fact that now the Levi–Civita connection does not have $SU(3)$ holonomy anymore or in other words the spinor η satisfies (5.2) for some non-vanishing tensor κ^0. This immediately implies that the quantities J and Ω are covariantly constant with respect to the same connection with torsion namely

$$\begin{aligned} \nabla^{(T)}_m J_{np} &= \nabla_m J_{np} - \kappa^0{}_{mn}{}^r J_{rp} - \kappa^0{}_{mp}{}^r J_{nr} = 0 \ , \\ \nabla^{(T)}_m \Omega_{nmp} &= \nabla_m \Omega_{npq} - \kappa^0{}_{mn}{}^r \Omega_{rpq} - \kappa^0{}_{mp}{}^r \Omega_{nrq} - \kappa^0{}_{mq}{}^r \Omega_{npr} = 0 \ . \end{aligned} \tag{5.8}$$

Upon antisymmetrizing the free indices in the above expressions one obtains

$$\begin{aligned} \mathrm{d}J_{mnp} &= 6T^0_{[mn}{}^r J_{|r|p]} \ , \\ \mathrm{d}\Omega_{mnpq} &= 12T^0_{[mn}{}^r \Omega_{|r|pq]} \ , \end{aligned} \tag{5.9}$$

where $T^0_{mnp} = \frac{1}{2}(\kappa^0{}_{mnp} - \kappa^0{}_{nmp})$ denotes the intrinsic torsion. So unlike on a Calabi–Yau manifold J and Ω are no longer closed and we see again that the intrinsic (con)torsion is indeed the obstruction for a general manifold with $SU(3)$ structure to be Calabi–Yau. As J and Ω completely describe the manifold with $SU(3)$ structure these relations can be inverted to obtain the intrinsic torsion in terms of the exterior derivatives of J and Ω. This was done in [87] where the classification of manifolds with $SU(3)$ structure was given according to the different $SU(3)$ irreducible components in which the intrinsic torsion, T^0 splits. Decomposing the intrinsic torsion in $SU(3)$ representations one finds

$$T^0 \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 \ , \tag{5.10}$$

with the corresponding parts of T^0 labeled by T_i with $i = 1, \ldots, 5$ and where the representations corresponding to the different $\mathcal{W}_i$ are given in table 5.1.

Component	Interpretation	$SU(3)$-representation
$\mathcal{W}_1$	$J \wedge \mathrm{d}\Omega$ or $\Omega \wedge \mathrm{d}J$	$\mathbf{1} \oplus \mathbf{1}$
$\mathcal{W}_2$	$(\mathrm{d}\Omega)_0^{2,2}$	$\mathbf{8} \oplus \mathbf{8}$
$\mathcal{W}_3$	$(\mathrm{d}J)_0^{2,1} + (\mathrm{d}J)_0^{1,2}$	$\mathbf{6} \oplus \bar{\mathbf{6}}$
$\mathcal{W}_4$	$J \wedge \mathrm{d}J$	$\mathbf{3} \oplus \bar{\mathbf{3}}$
$\mathcal{W}_5$	$\mathrm{d}\Omega^{3,1}$	$\mathbf{3} \oplus \bar{\mathbf{3}}$

Table 5.1: The five classes of the intrinsic torsion of a space with $SU(3)$ structure.

The second column of table 5.1, gives an interpretation of each component of T^0 in terms of exterior derivatives of J and Ω. The superscripts refer to projecting onto a particular (p, q)-type, while the 0 subscript refers to the irreducible $SU(3)$ representation with any trace part proportional to J^n removed (see appendix C). A further discussion on the above result is presented in appendix and here we just pause to make one more comment which will be significant later.

Such manifolds are in general not complex manifolds. The obstruction to finding a true complex structure is given by the Nijenhuis tensor $N_{mn}{}^p$ which is defined in (C.4). It was found in [87] that the Nijenhuis tensor of a manifold with $SU(3)$ structure is determined in terms of the first two torsion classes $\mathcal{W}_1$ and $\mathcal{W}_2$. One can intuitively understand this as follows. According to the table 5.1 the classes $\mathcal{W}_1$ and $\mathcal{W}_2$ are given by the $(3, 0)$ part of $\mathrm{d}J$ and the $(2, 2)$ of $\mathrm{d}\Omega$ respectively. These components vanish identically if the manifold is complex as in such a case the exterior derivative can only increase by 1 either of the two degrees p or q of a (p, q) form. In other words one really needs a non-complex space in order to have the differential of a $(1, 1)$ form to be of $(3, 0)$ type or the differential of a $(3, 0)$ form to be of $(2, 2)$ type.

Let us now turn to see how to actually choose the internal manifold which has a chance to reproduce the mirror NS fluxes. Asking for supersymmetry of the low-energy effective action we were led to consider manifolds with $SU(3)$ structure which in turn are classified according to table 5.1. What this means is that now we have to find a way to choose the right manifold (the right torsion classes) which can reproduce the mirror partners of the NS fluxes (5.1). This is not straightforward and we are going to spend some time to understand what exactly fixes our choice. Moreover, at the moment we lack a rigorous procedure which allows us to select in a unique way the appropriate manifold and so we are going to present a set of motivations which point to a special class of manifolds with $SU(3)$ structure. They are called *half-flat* manifolds with $SU(3)$ structure and in the next sections we will show that performing the KK reduction on such manifolds one precisely recovers the mirror of the NS fluxes (5.1).

As we have stressed before, it is the intrinsic torsion which distinguishes a general manifold with $SU(3)$ structure from a Calabi–Yau space. Thus the missing fluxes should be provided by the intrinsic torsion of the particular manifold which is chosen, or equivalently by the derivatives $\mathrm{d}J$ and $\mathrm{d}\Omega$. As the usual fluxes we have seen in the last chapter are elements of the Calabi–Yau cohomology groups $H^{p,q}(Y)$ in order to restore mirror symmetry we need to match the $SU(3)$ representations of the cohomology groups $H^{p,q}(Y)$ with the $SU(3)$ representations of the intrinsic torsion. In other words this means that that we have to keep those torsion classes which correspond to the Calabi–Yau cohomology groups. In particular this suggests to set

$$T_4 = T_5 = 0 \ . \tag{5.11}$$

since the corresponding $H^{3,2}(Y)$ and $H^{3,1}(Y)$ groups vanish on Y. On the other hand $T_{1,2,3}$ can be non-zero as the corresponding cohomologies do exist on Y. There is one further intuitive argument which allows us to restrict the torsion classes. Note that in general mirror symmetry exchanges the Kähler and the complex structure moduli. The former are given by the expansion of the complexified Kähler form $K = B_2 + iJ$ on the internal manifold while the latter by the holomorphic $(3,0)$ form $\Omega = \Omega^+ + i\Omega^-$. Hence, we can say that under mirror symmetry Ω and K are exchanged. Turning on a flux for the NS three-form field strength $H_3 = \mathrm{d}B_2$ means that B_2 (i.e. the real part of K) on the internal manifold is no longer closed. This in turn suggests that at least half of the components of Ω, say Ω^+, are no longer closed. Since Ω^- remains closed we expect that half of the torsion components in $\mathcal{W}_1 \oplus \mathcal{W}_2$ vanish. Together with (5.11) the constraints on the torsion can be written as

$$\begin{aligned} \mathrm{d}\Omega^- &= 0, \\ \mathrm{d}\,(J \wedge J) &= 0. \end{aligned} \tag{5.12}$$

Manifolds satisfying these conditions are known in the mathematical literature as half-flat manifolds with $SU(3)$ structure and in the following we are going to consider them as the candidates to obtain the mirror NS fluxes. Note that these manifolds have a non-vanishing component of the intrinsic torsion in $\mathcal{W}_1 \oplus \mathcal{W}_2$. From the above discussion it means that that half-flat manifolds are non-complex (only almost complex) in agreement with the suggestion made in [50]. Moreover one notices that a four-form, $\mathrm{d}\Omega$, has already appeared in the NS sector and this was precisely what we were missing before in the case of a Calabi–Yau manifold.

As we will see in the next section this can not be the end of the story as such manifolds will only reproduce half of the missing fluxes. At the end of this chapter we will try to find what is the appropriate generalization which reproduces all the missing fluxes.

5.2 Type II theories on half-flat manifolds

In the last section we have presented a series of arguments which led us to the conclusion that the half-flat manifolds represent the configuration mirror to the NS fluxes (5.1). To see how this exactly comes about we perform in this section the KK reduction of the two type II theories on such manifolds. However, as pointed out before the half-flat manifolds can only reproduce half of the fluxes from (5.1) which we call electric fluxes,[3] while the origin of the other half is much harder to test.

Based on mirror symmetry we make a couple of assumptions about the topology and moduli space of half-flat manifolds and then we apply these results in order to derive in sections 5.2.2 and 5.2.3 the low energy effective actions of type II theories on such spaces. For concreteness we have in mind the following picture. Consider a pair of mirror manifolds Y and $\tilde{Y}$. We define $\hat{Y}$ to be the (half-flat) manifold such that type IIA compactified on $\hat{Y}$ is the same as type IIB on $\tilde{Y}$ when NS fluxes are turned on. Equivalently, what we will discuss in section 5.2.3, type IIB on $\hat{Y}$ is the same as type IIA on $\tilde{Y}$ with NS fluxes turned on. Thus in this picture in the limit that the fluxes go to zero the manifolds Y and $\hat{Y}$ coincide. As we will shortly see this limit is a bit more delicate, but what is important to keep in mind is that we treat $\hat{Y}$ as a (small) deformation of the manifold Y which is the true geometric mirror of $\tilde{Y}$.

5.2.1 Mirror symmetry and half-flat manifolds

We have already noted that on a general manifold with $SU(3)$ structure there exist a four-form, $\mathrm{d}\Omega$, which can play the role of the NS four-form field strength which we need to restore mirror symmetry. In order to find the $h^{(1,1)}$ fluxes required by mirror symmetry we have to expand this four-form in a similar way we expanded the four-form fluxes. Consequently we set

$$\mathrm{d}\Omega = e_i \tilde{\omega}^i \ , \qquad i = 1, \ldots, h^{(1,1)}(Y) \ , \tag{5.13}$$

where $\tilde{\omega}^i$ is a basis for $(2,2)$ forms on $\hat{Y}$ which in the limit of small torsion should locally coincide with the basis for harmonic $(2,2)$ forms on Y, while e_i are constants parameterizing the flux. One can also introduce the dual basis ω_i, $i = 1, \ldots, h^{(1,1)}(Y)$ for the $(1,1)$ forms and a basis for the three-forms (α_A, β^A) $A = 0, \ldots, h^{(2,1)}(Y)$. The key point here, imposed by mirror symmetry, is that these forms should obey the same relations as on the undeformed Calabi–Yau manifold (2.24) (2.25) and (B.23). Moreover, mirror symmetry further restricts the manifold $\hat{Y}$, in that its moduli space of metrics have to coincide with the moduli space of the original Calabi–Yau manifold Y. In particular we should be able to expand J and Ω in the moduli in the same way as on a Calabi–Yau

[3] Formally by electric fluxes we mean the flux parameters e_A in (5.1). We stress once more that the name comes only from the four dimensional interpretation as these parameters naturally appear as electric charges. However this is a basis dependent notion as going to the dual magnetic basis for the gauge fields the electric and magnetic charges are also interchanged.

space (B.21) and (B.35)

$$\begin{aligned} \Omega &= z^A \alpha_A - \mathcal{F}_A \beta^A , \qquad A = 0, 1, \ldots, h^{(2,1)}(Y) , \\ J &= v^i \omega_i , \qquad i = 1, \ldots, h^{(1,1)}(Y) , \end{aligned} \tag{5.14}$$

where $z^A = (1, z^a)$ with $a = 1, \ldots, h^{(2,1)}(Y)$ and the z^a are the scalar fields corresponding to the deformations of the complex structure ($\mathcal{F}_A$ is the corresponding prepotential), while the v^i are the (real) scalar fields corresponding to the Kähler deformations. In order to write (5.14) we have adopted the symplectic basis which is appropriate for the moduli problem where the norm of Ω is not fixed, but one can instead choose one of the z's (in the case above z^0) to be constant. Clearly if we impose (5.13) and keep the expansion (5.14) the forms (α_A, β^A) can not be all closed (harmonic). Inserting (5.14) into (5.13), we have

$$\mathrm{d}\Omega = z^A \mathrm{d}\alpha_A - \mathcal{F}_A \mathrm{d}\beta^A = e_i \tilde{\omega}^i . \tag{5.15}$$

As the fluxes should not depend on the specific point in the moduli space we are looking at this is only possible if we have

$$\mathrm{d}\alpha_0 = e_i \, \tilde{\omega}^i , \qquad \mathrm{d}\alpha_a = \mathrm{d}\beta^A = 0 , \tag{5.16}$$

where α_0 is singled out since it is the only direction in Ω which is independent of z^a.[4] Furthermore, inserting (5.16) into (2.24) and integrating by parts gives

$$e_i = \int \omega_i \wedge d\alpha_0 = - \int \mathrm{d}\omega_i \wedge \alpha_0 . \tag{5.17}$$

Thus, consistency requires

$$\mathrm{d}\omega_i = e_i \beta^0 , \qquad \mathrm{d}\tilde{\omega}^i = 0 , \tag{5.18}$$

where the second equation follows from (5.16).[5]

As we can easily see from the above equations the forms ω_i are not harmonic as they are not closed anymore. The same is true for the basis (α_A, β^A) as they are not coclosed anymore. To see this note that the Hodge star (B.40) on any of these forms necessarily involves α_0 which obeys (5.16).

One can nevertheless construct harmonic forms out of the ω_i as their derivative is proportional to the same three-form β^0. Suppose for instance that e_1 is non-zero then the linear combinations

$$\omega_i' = \omega_i - \frac{e_i}{e_1} \omega_1 , \qquad i \neq 1 , \tag{5.19}$$

are harmonic in that they satisfy

$$\mathrm{d}\omega_i' = \mathrm{d}^\dagger \omega_i' = 0 , \tag{5.20}$$

[4]Of course this corresponds to a specific choice of the symplectic basis of H^3. It is the same choice which is conventionally used in establishing the mirror map without fluxes.

[5]Strictly speaking also $d\omega_i = e_i \beta^0 + a^A \alpha_A + b_a \beta^a$ for some arbitrary coefficients a^A, b_a solves (5.17). However by a similar argument as presented for the exterior derivative of ω_i one can see that any non-vanishing such coefficient produces a nonzero derivative of α_a or/and β^A contradicting (5.16). From this one concludes that the only solution of (5.17) together with (5.16) is (5.18).

where we used $\mathrm{d}^\dagger \omega'_i = *\mathrm{d}*\omega'_i \sim *\mathrm{d}\tilde{\omega}'^i$. Thus, there are still at least $h^{(1,1)}(Y)-1$ harmonic forms ω'_i on $\hat{Y}$. The same argument can be repeated for H^3 where one finds $2h^{(2,1)}$ harmonic forms or in other words the dimension of H^3 has changed by two and we have together

$$h^{(2)}(\hat{Y}) = h^{(1,1)}(Y) - 1 \ , \qquad h^{(3)}(\hat{Y}) = h^{(3)}(Y) - 2 \ . \tag{5.21}$$

Physically this can be understood from the fact that some of the scalar fields gain a mass proportional to the flux parameters and no longer appear as zero modes of the compactification. Similarly, from mirror symmetry we do not expect the occurrence of new zero modes on $\hat{Y}$ as these would correspond to additional new massless fields in the effective action. Thus the new manifold $\hat{Y}$ is topologically different from Y and so there is no continuous limit to pass from Y to $\hat{Y}$ which is the analog of the flux quantization condition we have discussed in chapter 3. Recall that in order to compute the low energy effective action we needed that the fluxes are small and we overcame this difficulty by considering the large volume limit, case in which the fluxes can be regarded as continuous parameters and can be chosen arbitrary small. In the case of the half-flat manifold this is not obviously possible as the manifolds with and without flux are topologically different. Mirror symmetry tells us that the right limit should be the large complex structure and we will assume that in this case the torsion can be made locally small and the two manifolds are locally the same.

Let us summarize the results obtained so far. Requiring that the half-flat manifold $\hat{Y}$ is in fact mirror to a Calabi–Yau $\tilde{Y}$ when fluxes for the NS three-form field strength are turned on we have conjectured the existence of a set of forms on $\hat{Y}$ satisfying the conditions (5.16) and (5.18) which essentially encode information about its topology. Moreover assuming the same moduli expansions we immediately obtain

$$\mathrm{d}J = v^i e_i \beta^0 \ . \tag{5.22}$$

Using the standard $SU(3)$ relation $J \wedge \Omega = 0$ which implies that $\omega_i \wedge \alpha^A = \omega_i \wedge \beta^A = 0$ for all A and i one finds $J \wedge \mathrm{d}J = 0$ proving in this way the consistency with the half-flatness assumption. [6] Furthermore, since $\mathrm{d}J$ and $\mathrm{d}\Omega$ completely determine the intrinsic torsion T^0, we see that all the components of T^0 are given in terms of the constants e_i without the need for any additional information.

5.2.2 Type IIA on a half-flat manifold

Having discussed the topology of the half-flat manifolds we can turn to study compactifications on such spaces and we focus at the beginning on the type IIA case. Thus we

[6] It would be interesting to calculate the moduli space of half-flat metrics on $\hat{Y}$ directly and see that it agreed with, or at least had a subspace, of the form given by (5.14) together with (5.16) and (5.18).

assume a background metric of diagonal form[7]

$$\hat{G} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g^{\hat{Y}}_{mn} \end{pmatrix} , \tag{5.24}$$

where $g^{\hat{Y}}$ is the $SU(3)$ invariant metric on the half-flat manifold. As far as the matter fields are concerned we consider turning on a very specific flux for the NS three-form field H_3

$$(H_3)^{int} = e_0 \beta^0 . \tag{5.25}$$

The reason for this is that it appears that the half-flat manifolds can only reproduce $h^{(1,1)}$ flux parameters e_i (5.13), but on the other hand in order to recover the mirror of all electric fluxes (5.1) we need one more parameter. Thus the claim is that e_0 in (5.25) precisely plays the role of this last flux parameter. The reason for picking such a specific flux is that in the final result this additional parameter e_0 combines in the right way with the other fluxes e_i coming from (5.13). Beside (5.24) and (5.25) all other fields are taken to be trivial in the background.

There is one more thing we need to discuss before starting the computation of the effective action: the light modes in four dimensions. For this we use again mirror symmetry as guiding principle. This requires that the light spectrum obtained in usual Calabi–Yau compactifications is not modified. We also know from section 4.2 that some of the fields acquire masses proportional to the fluxes and thus we should allow for fields which have masses of order (flux)^2. Moreover one can immediately notice that the Laplace operator on the forms discussed in the last section is precisely of order (flux)^2 and so it is legitimate to expand the matter fields in these forms. Thus we are going to perform a non-standard KK compactification where the forms in which we expand are not harmonic anymore, but satisfy

$$\mathrm{d}\alpha_0 = e_i \tilde{\omega}^i , \qquad \mathrm{d}\alpha_a = \mathrm{d}\beta^A = 0 , \qquad \mathrm{d}\omega_i = e_i \beta^0 , \qquad \mathrm{d}\tilde{\omega}^i = 0 . \tag{5.26}$$

However we continue to demand that these forms have identical intersection numbers as on the Calabi–Yau or in other words obey unmodified (2.24) and (2.25).

Formally, all what this means is that we assume the same field expansions as in (2.31) but now we take into account the additional relations (5.25) and also (5.26). Because of the former equation we again start from the formulation of type IIA which is appropriate for turning on NS fluxes (3.11). Using the above relations it is easy to see that the expansions of the field strengths become

$$\begin{aligned} \hat{H}_3 &= \mathrm{d}B_2 + \mathrm{d}b^i \omega_i + (e_i b^i + e_0)\beta^0 , \\ \hat{F}_4 &= (\mathrm{d}C_3 - A^0 \wedge \mathrm{d}B_2) + (\mathrm{d}A^i - A^0 \mathrm{d}b^i) \wedge \omega_i + D\xi^A \alpha_A - D\tilde{\xi}_A \beta^A + \xi^0 e_i \tilde{\omega}^i , \end{aligned} \tag{5.27}$$

[7] Strictly speaking this background is not a solution of the ten-dimensional equations of motion because the half-flat manifolds are not Ricci flat. However in the small torsion limit one can write the Riemann tensor of the manifold $\hat{Y}$

$$\hat{R} = R_{CY} + O((T^0)^2) . \tag{5.23}$$

In this way we can think of the terms of order $(T^0)^2$ as being a small correction to the Einstein equations as in (3.9).

where the covariant derivatives are given by

$$D\tilde{\xi}_0 = \mathrm{d}\tilde{\xi}_0 + e_i(A^i + b^i A^0) + e_0 A^0, \qquad D\xi^A = \mathrm{d}\xi^A, \qquad D\tilde{\xi}_a = \mathrm{d}\tilde{\xi}_a. \tag{5.28}$$

This formula is one of the major consequences of compactifying on $\hat{Y}$ (in particular of expanding the ten-dimensional fields in forms which are not harmonic) as one of the scalars, $\tilde{\xi}_0$, becomes charged.

From here on the compactification proceeds as in the other cases when we have turned on fluxes by inserting (5.27) together with (2.31) back into the action (2.2) and performing the integrals over $\hat{Y}$ using (2.24), (2.25), (2.34) and (2.35). Again as in the other cases we only outline the modifications which appear in comparison to the usual compactification presented in section (2.3.3).

Let us first concentrate on the structure of the action leaving the computation of the scalar potential for the end of this section. Then the only new features we encounter come from the topological term

$$\begin{aligned} \frac{1}{2}\int_{\hat{Y}} \hat{H}_3 \wedge \hat{C}_3 \wedge \mathrm{d}\hat{C}_3 \;=\; & \frac{\xi^0}{2}\mathrm{d}B_2 \wedge A^i e_i + \frac{1}{2}\mathrm{d}B_2 \wedge \left(\xi^0(\mathrm{d}\tilde{\xi}_0 + e_i A^i) + \xi^a \mathrm{d}\tilde{\xi}_a - \tilde{\xi}_A \mathrm{d}\xi^A\right) \\ & + \frac{\xi^0}{2} e_i \mathrm{d}b^i \wedge C_3 + \frac{1}{2}\mathrm{d}b^i \wedge A^j \wedge \mathrm{d}A^k \mathcal{K}_{ijk} \\ & - \frac{\xi^0}{2}(e_i b^i + e_0)\mathrm{d}C_3 - \frac{1}{2}(e_i b^i + e_0) \wedge C_3 \wedge \mathrm{d}\xi^0 \;, \end{aligned} \tag{5.29}$$

where $\mathcal{K}_{ijk}$ is defined in (2.34). Before we attempt to write the form of the four-dimensional effective action we should again perform the dualization of the three-form. Using the results in appendix D.2.2 we obtain for dual action[8]

$$S_{dual} = -\frac{(\xi^0)^2}{2\mathcal{K}}(e_i b^i + e_0)^2 - \xi^0\,(e_i b^i + e_0) A^0 \wedge \mathrm{d}B_2\,. \tag{5.30}$$

One further dualizes B_2 to a scalar field denoted by a and taking into account the Green-Schwarz type interactions (5.29) and (5.30) one obtains that this scalar becomes charged. Redefining the gauge fields as $A^i \to A^i - b^i A^0$ and after going to the Einstein frame one obtains the effective action to be

$$\begin{aligned} S_{IIA} \;=\; & \int \Big[-\frac{1}{2}R *\mathbf{1} - g_{ij}\mathrm{d}t^i \wedge *\mathrm{d}\bar{t}^{\bar{j}} - h_{uv} Dq^u \wedge *Dq^v \\ & + \frac{1}{2}\,\mathrm{Im}\,\mathcal{N}_{IJ} F^I \wedge *F^J + \frac{1}{2}\,\mathrm{Re}\,\mathcal{N}_{IJ} F^I \wedge F^J - V_{IIA} *\mathbf{1}\Big], \end{aligned} \tag{5.31}$$

where the gauge coupling matrix $\mathcal{N}_{IJ}$ and the metrics g_{ij}, h_{uv} are the same as in the case discussed in section (2.3.3). Among the covariant derivatives of the hyper-multiplet scalars Dq^u the only non-trivial ones are

$$Da = \mathrm{d}a + \xi^0 e_I A^I\,; \qquad D\tilde{\xi}_0 = \mathrm{d}\tilde{\xi}_0 + e_I A^I\,, \tag{5.32}$$

[8]Note that after performing the dualization we have again set to zero the constant to which C_3 is dual.

where $e_I,\ I=0,\dots h^{(1,1)}$ is a collective notation for e_0 and $e_i,\ i=1,\dots h^{(1,1)}$.

Before discussing the potential let us note that the action (5.31) already has the form expected from mirror symmetry with the action derived in (4.14). In particular the forms α_0 and β^0 in (5.26) single out the two scalars $\xi^0, \tilde{\xi}_0$ from the expansion of $\hat{C}_3$. ξ^0 maps under mirror symmetry (2.50) to the RR scalar l which is already present in the $D=10$ type IIB theory while $\tilde{\xi}_0$ maps to the charged RR scalar in type IIB. Moreover, using these identifications one observes that the gauging (5.32) is precisely what one obtains in the type IIB case with NS electric fluxes turned on (4.16). Finally we see that the extra flux parameter e_0 does indeed combine with the e_i defined in (5.13) justifying our choice for this flux.

What we are now left to check in order to prove that the action (5.31) is the mirror of (4.14) is that that the two scalar potentials also agree. In the case of type IIA compactified on $\hat{Y}$ one can identify four distinct contributions to the potential: from the kinetic terms of $\hat{B}_2$ and $\hat{C}_3$, from the dualization of C_3 in 4 dimensions and from the Ricci scalar of $\hat{Y}$. Let us study these contributions one by one. We go directly to the four-dimensional Einstein frame which amounts to multiplying every term in the potential by a factor $e^{4\phi}$ coming from the rescaling of $\sqrt{-g}$, ϕ being the four-dimensional dilaton which is related to the ten-dimensional dilaton $\hat{\phi}$ by $e^{-2\phi}=e^{-2\hat{\phi}}\mathcal{K}$.

Using (5.27) we see that the kinetic term of $\hat{B}_2$ in (2.2) gives the following contribution to the potential

$$V_1=\frac{e^{2\phi}}{4\mathcal{K}}(e_ib^i+e_0)^2\int_{\hat{Y}}\beta^0\wedge*\beta^0=-\frac{e^{-2\phi}}{4\mathcal{K}}(e_ib^i+e_0)^2\left[(\mathrm{Im}\,\mathcal{M})^{-1}\right]^{00}\ , \tag{5.33}$$

where the integral over $\hat{Y}$ was performed using (2.35). Similarly, the kinetic term of $\hat{C}_3$ produces the following piece in the potential

$$V_2=e^{4\phi}\,\frac{(\xi^0)^2}{8\mathcal{K}}\,e_ie_jg^{ij}\ , \tag{5.34}$$

where g^{ij} arises after integrating over $\hat{Y}$ using (B.28). Furthermore, (5.30) also produces a term in the potential

$$V_3=e^{4\phi}\,\frac{(\xi^0)^2}{2\mathcal{K}}(e_ib^i+e_0)^2\ . \tag{5.35}$$

The last contribution to the potential is due to the fact that the half-flat manifolds are no longer Ricci flat. The computation of the Ricci scalar is quite involved and so we displayed it in the appendix C.2. Here we just recall the final result for the Ricci scalar

$$R_{hf}=-\frac{1}{8}\,e_ie_jg^{ij}\left[(\mathrm{Im}\,\mathcal{M})^{-1}\right]^{00}\ . \tag{5.36}$$

Taking into account the factor $\frac{e^{-2\hat{\phi}}}{2}$ which multiplies the Ricci scalar in the ten-dimensional action (2.2) and the factor $e^{4\phi}$ coming from the four-dimensional Weyl rescaling one obtains the contribution to the potential coming from the gravity sector to be

$$V_g=-\frac{e^{2\phi}}{16\mathcal{K}}\,e_ie_jg^{ij}\left[(\mathrm{Im}\,\mathcal{M})^{-1}\right]^{00}\ . \tag{5.37}$$

Combining (5.33), (5.34), (5.35) and (5.37) after using (B.33) we can finally write the entire potential which appears in the compactification of type IIA supergravity on $\hat{Y}$

$$V_{IIA} = -\frac{e^{4\phi}}{2}\left((\xi^0)^2 - \frac{e^{-2\phi}}{2}\left[(\mathrm{Im}\,\mathcal{M})^{-1}\right]^{00}\right) e_I e_J \left[(\mathrm{Im}\mathcal{N})^{-1}\right]^{IJ} . \tag{5.38}$$

In order to compare this potential to the one obtained in type IIB case (4.15) we should first see how the formula (5.38) changes under the mirror map. First, from (2.50) we learn that ξ^0 corresponds to the ten dimensional scalar of type IIB theory, l. Then from mirror symmetry know that the gauge coupling matrices $\mathcal{M}$ and $\mathcal{N}$ are mapped into one another (2.53). In particular this means that[9]

$$\left[(\mathrm{Im}\,\mathcal{M}_A)^{-1}\right]^{00} \leftrightarrow \left[(\mathrm{Im}\,\mathcal{N}_B)^{-1}\right]^{00} = -\frac{1}{\mathcal{K}_B} . \tag{5.39}$$

where we used the expression for $(\mathrm{Im}\,\mathcal{N})^{-1}$ from (B.33). With this observation it can be easily seen that the type IIA potential (5.38) is precisely mapped into the type IIB one (4.15) provided one identifies the electric flux parameters $e_I \leftrightarrow \tilde{e}_A$ and the four-dimensional dilatons on the two sides. In this way we established that the low energy effective action obtained by compactifying type IIA supergravity on a half-flat manifold coincides with the one obtained by turning on electric NS fluxes in the Calabi–Yau compactification of type IIB supergravity.

It is also interesting to write a superpotential for this case as we did in all other situations. The natural guess is

$$W \sim \int_{\hat{Y}} K \wedge \left(\mathrm{d}C_3 + i e^{-\phi}\mathrm{d}\Omega^+\right) \ , \tag{5.40}$$

where K is the complexified Kähler form $K = B_2 + iJ$. As in the other case when we had NS fluxes in type IIA theory (3.26) one notices again the presence of the RR three-form C_3 in the superpotential. This term is needed now in order to reproduce the ξ^0 term in the potential (5.38). For consistency, one can again explicitly check that this superpotential is precisely the mirror of (4.17) when the magnetic fluxes are set to zero.

5.2.3 Type IIB on a half-flat manifold

At the beginning of section 5.2 we have defined $\hat{Y}$ to be the manifold which could reproduce the mirror NS fluxes. In the previous section we have already seen that type IIA compactified on half-flat manifolds with $SU(3)$ structure is mirror to type IIB with NS electric fluxes turned on and thus we can say that $\hat{Y}$ is such a half-flat manifold. There is one immediate check which one can imagine, namely that the above definition of $\hat{Y}$ should not depend on whether we consider type IIA or type IIB theory. More specific using the same $\hat{Y}$ one should be able to also reproduce the NS (electric) fluxes

[9] In order to avoid confusions we have added the label A/B to specify the fact that the corresponding quantity appears in type IIA/IIB theory.

in type IIA theory. This is what we want to do in this section namely to show that type IIB compactified on a half-flat manifold leads to the same effective action as type IIA compactified on a Calabi–Yau manifold with NS fluxes turned on.

We take the compactification setup to be the same as in the last section (5.24) and again perform the field expansion in the forms (5.26). We again turn on the NS three-form flux along β^0 which once more will turn out to provide the last flux parameter needed in order to restore mirror symmetry when all electric NS fluxes are turned on. Thus considering the usual field expansions (2.41) one derives the following expressions for the field strengths $\hat{H}_3$, $\hat{F}_3$ and $\hat{F}_5$

$$\begin{aligned} \hat{H}_3 &= \mathrm{d}B_2 + \mathrm{d}b^i \omega_i + (e_i b^i + e_0)\beta^0 \ , \\ \hat{F}_3 &= (\mathrm{d}C_2 - l\mathrm{d}B_2) + (\mathrm{d}c^i - l\mathrm{d}b^i) \wedge \omega_i + e_i(c^i - lb^i)\beta^0 - le_0\beta^0 \ , \\ \hat{F}_5 &= (\mathrm{d}D^i - \mathrm{d}b^i \wedge C_2 - c^i \mathrm{d}B_2) \wedge \omega_i + (D\rho_i - \mathcal{K}_{ijk} c^j \mathrm{d}b^k) \wedge \tilde{\omega}^i + F^A \wedge \alpha_A - \check{G}_A \wedge \beta^A \ , \end{aligned} \tag{5.41}$$

where we have defined

$$\begin{aligned} D\rho_i &= \mathrm{d}\rho_i - e_i V^0 \ , \\ F^A &= \mathrm{d}V^A \ , \quad G_A = \mathrm{d}U_A \ , \\ \check{G}_0 &= G_0 - e_i(D^i - b^i C_2) + e_0 C_2 \ ; \quad \check{G}_a = G_a \ . \end{aligned} \tag{5.42}$$

The topological term is also going to be modified because of the non-standard algebra of the forms we consider (5.16) and (5.18). Using these relations and the field expansions (2.41) one derives

$$\begin{aligned} -\frac{1}{2}\int \hat{A}_4 \wedge \mathrm{d}\hat{B}_2 \wedge \mathrm{d}\hat{C}_2 &= -\frac{1}{2}\mathcal{K}_{ijk} D^i \wedge \mathrm{d}b^j \wedge \mathrm{d}c^k - \frac{1}{2}\rho_i - \big(\mathrm{d}B_2 \wedge \mathrm{d}c^i + \mathrm{d}b^i \wedge \mathrm{d}C_2\big) \\ &\quad + \frac{1}{2} e_i V^0 \wedge \big(c^i \mathrm{d}B_2 - b^i \mathrm{d}C_2\big) - \frac{1}{2} e_0 V^0 \wedge \mathrm{d}C_2 \ . \end{aligned} \tag{5.43}$$

From this point on the derivation of the low energy effective action proceeds as in section 2.3.4 or 4.1 with the only difference that the fields ρ_i appear with a non-trivial covariant derivative (5.42). In particular one has to impose the self-duality condition on $\hat{F}_5$. Due to the fact that the forms ω_i, $\tilde{\omega}^i$, α_A, β^A are taken to obey the same relations as on a normal Calabi–Yau manifold (B.29), (B.40), (B.42) for the four-dimensional fields this condition reduces to a form which is very similar to (2.43)

$$\begin{aligned} \check{G}_A &= \mathrm{Im}\,\mathcal{M}_{AB} * F^B + \mathrm{Re}\,\mathcal{M}_{AB} F^B \ , \\ D\rho_i - \mathcal{K}_{ijk} c^j \mathrm{d}b^k &= 4\mathcal{K} g_{ij} * (\mathrm{d}D^j - \mathrm{d}b^j \wedge C_2 - c^j \mathrm{d}B_2) \ . \end{aligned} \tag{5.44}$$

Again these constraints can be obtained as equations of motion for the fields $\mathrm{d}D^i$ and G_A after adding the Lagrange multipliers (2.46). Eliminating now these redundant fields, after going to the Einstein frame and performing the mirror map (2.50) one obtains the following effective action

$$S = \int \Big[-\frac{1}{2}R^*\mathbf{1} - g_{ab}\mathrm{d}z^a \wedge *\mathrm{d}\bar{z}^b - \tilde{h}_{uv} Dq^u \wedge *Dq^v - V_{IIB} * \mathbf{1} + \frac{1}{2}\operatorname{Im}\mathcal{M}_{AB}F^A \wedge *F^B + \frac{1}{2}\operatorname{Re}\mathcal{M}_{AB}F^A \wedge F^B \Big] , \quad (5.45)$$

where the gauge couplings and the sigma model metrics are exactly as in the massless compactification presented in section 2.3.4 The non-trivial covariant derivatives are

$$D\tilde{\xi}_I = \mathrm{d}\tilde{\xi}_I - e_I V^0 \; ; \qquad Da = \mathrm{d}a - e_I V^0 \xi^I \; , \quad (5.46)$$

while all the other fields remain neutral.

The potential appearing in (5.45) is straightforward to compute. Using the fact that it is generated by the terms in the field strengths (5.41) which reside completely in the internal manifold one writes

$$V_{IIB} = -\frac{e^{-2\phi}}{4\mathcal{K}}(e_i b^i + e_0)^2 \left[(\operatorname{Im}\mathcal{M})^{-1}\right]^{00} - \frac{1}{2}\Big[e_i(c^i - lb^i) - le_0\Big]^2 \left[(\operatorname{Im}\mathcal{M})^{-1}\right]^{00} + V_g \; , \quad (5.47)$$

where V_g denotes again the potential term which arises from the Einstein-Hilbert term due to the fact that the compactification manifold is not Ricci flat (5.36). Using (5.37) and (B.33) one can write the potential in the following form

$$\left[\frac{\mathrm{e}^{2\phi}}{4} e_I e_J \left(\operatorname{Im}\mathcal{N}^{-1}\right)^{IJ} - \frac{\mathrm{e}^{4\phi}}{2}(e_I \xi^I)^2\right] \left[(\operatorname{Im}\mathcal{M})^{-1}\right]^{00} \; . \quad (5.48)$$

Once again we see that the theory obtained in (5.45), (5.46) and (5.47) is the same as the one obtained by compactifying the type IIA theory with non-trivial electric NS fluxes turned on from section 3.2. In particular the gaugings (5.46) have the same form with the ones found in (3.15) and (3.22) if one sets the magnetic fluxes $p^A = 0$. Moreover it can be seen that in this limit also the potentials (5.47) and (3.20) coincide when one takes into account that the gauge coupling matrices are exchanged $\mathcal{N} \leftrightarrow \mathcal{M}$ and uses the mirror version of (5.39)

The final thing to do is to write again a superpotential which corresponds to this case

$$W \sim \int_{\hat{Y}} \left(F_3 + ie^{-\phi}\mathrm{d}K\right) \wedge \Omega \; , \quad (5.49)$$

where $K = B_2 + iJ$ and again the term with F_3 is needed in order to reproduce the ξ dependence in the potential. Needless to say it can be checked that (5.49) is indeed the mirror of (3.26).

To summarize the results obtained in this section, we have seen that the low-energy effective action of type IIA theory compactified on $\hat{Y}$ is precisely the mirror of the effective action obtained in section 4.2 for type IIB theory compactified on $\tilde{Y}$ in the presence of

NS electric fluxes. Moreover we have shown that also the reverse situation holds namely the low energy effective action of type IIB compactified on $\hat{Y}$ coincides with the one obtained for type IIA compactified on Y with NS electric fluxes turned on. This is our final argument that the half-flat manifold $\hat{Y}$ is the right compactification manifold for obtaining the mirror partners of the NS electric fluxes (5.1). In particular the interplay between the gravity and the matter sector which resulted in the potentials (5.38) and (5.47) provided a highly nontrivial check of this assumption.

5.3 Magnetic fluxes

The success of the last section where we obtained the mirror configuration of the Calabi–Yau compactifications with NS fluxes is partly faded by the fact that we were able to recover only half of the wanted fluxes. We have denoted these fluxes as electric ones because in the low energy theory they appear as electric charges for some of the fields. The story of the second half of the fluxes in (5.1) is far more involved and we will try in this section to present a couple of ideas which lead to some generalization of the notion of half-flat manifolds.

First let us discuss the obstacles one encounters when trying to reproduce the magnetic fluxes. Naively one could say that looking at spaces which satisfy $\mathrm{d}\Omega^- = 0$ reduces in an arbitrary way the number of flux parameters by a factor of 2. This statement is not obviously wrong and we will even try to justify later that indeed this is the correct generalization. However from the point of view of the last section choosing a manifold for which $\mathrm{d}\Omega$ is not real complicates things tremendously. The reason for this is that in such cases we no longer have a simple interpretation in terms of the basis (α_A, β^A) as in (5.16). Trying to naively apply the argument of the last section that the fluxes should not depend on the specific point in the moduli space would mean to incorporate the full $\mathrm{d}\Omega$ in the derivative of α_0. However, this is not possible since α_0 is a real form while the above identification would mean that its derivative is complex. Thus we find ourselves right from the beginning in the situation that we can not verify our conjecture because there is no easy way to compute the low energy effective action. One can do a little better than that in the sense that there are other quantities like the superpotentials which can be computed without specifying anything about the three-form basis (α_A, β^A). We will be more specific at the end of this section where we will try to guess the properties of the manifold which can reproduce both electric and magnetic fluxes.

In order to find the proper generalization one can also try to use a 'bottom-up' approach. We know how the low energy-effective action when both electric and magnetic NS fluxes are turned on looks like and so one can ask what is the right compactification manifold which can reproduce this effective action. In particular, one can derive in this way the specific algebra which the basis forms should satisfy. As we will see in a while one again runs immediately into problems. Let us consider for definiteness that we want to reproduce the type IIB theory with all NS fluxes turned on[10]. We have already given

[10]Similar arguments though a bit more technically involved also apply for the case of type IIA with

in section 4.2 the low energy effective action one obtains when all NS fluxes are turned on and let us see what problems we encounter when trying to reproduce this theory. From the action (4.13) one immediately notices that the RR two-form C_2 becomes massive. We would thus need that the same thing happens in type IIA on some appropriately chosen manifold with $SU(3)$ structure. However type IIA theory compactified on a Calabi–Yau manifold features no RR two-form and if we insist to keep the same light spectrum as we had in usual Calabi–Yau compactifications there is no way one can obtain a massive two form in the RR sector. We can nevertheless try to consider field expansions in some yet unknown one-form, but it is hard to see how one can restore mirror symmetry in this case so we will drop the idea right from the beginning. The most probable thing which can happen is that we do not work in the right basis. We have already seen in section 3.3.2 that a massive two form in four dimensions can have different interpretations. One can think that the right picture which we have to reproduce is not the one where the massive two form is present explicitly, but one of the dual pictures. The one where the massive two form is traded for a massive vector in four dimensions is still not easy to find on the type IIA side. The reason is that in this case there is an extra (redundant) gauge field and moreover the gauge couplings are modified according to (3.66). First of all there is no way to obtain extra fields unless one considers different forms to expand in. This would again mean that mirror symmetry is going to be difficult to restore. Second of all there is no reason that the previous gauge couplings are modified once we introduce some extra form to expand in. However, from (3.66) the gauge coupling matrix seem to be modified.

The last description we gave for a massive two-form in section 3.3.2 was one where the massive two-form is dualized to a scalar which is both electrically and magnetically charged. We do not want to go into the details of this last possibility as we do not understand completely how to obtain this situation on the mirror type IIA side. We only remark that this is not improbable as there exist a formulation of type IIA theory where both the usual fields and their Poincare duals are present [61, 92, 93]. As in type IIB, in this other formulation of type IIA theory one has to impose a duality relation between the fields in order to obtain the correct equations of motion.

There is one more aspect which is worth presenting regarding the magnetic fluxes. In the case of the type IIB theory compactified with NS fluxes turned on it is pure convention to call some of the fluxes electric and some other magnetic. What we really mean is that the two situations when we have only electric and only magnetic fluxes are perfectly similar. From the effective action (4.13) this fact is not so clear as it looks like the field C_2 is still massive when the electric fluxes are set to zero while in the case that the magnetic fluxes vanish C_2 is clearly massless. However this is only an artifact of the vector field basis we have chosen. In other words we are trying to describe magnetic charges using electric vector fields. Going to the dual picture where the gauge fields are replaced with their magnetic duals one can easily check that C_2 is not anymore massive and the whole action can be written as in (4.14). However now the gauge couplings have

NS fluxes.

to be modified according to

$$\mathcal{I} \longrightarrow \left(\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R}\right)^{-1} , \qquad \mathcal{R} \longrightarrow \left(\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R}\right)^{-1} \mathcal{R}\mathcal{I}^{-1} , \tag{5.50}$$

where by $\mathcal{R}$ and $\mathcal{I}$ we have denoted the real and imaginary part of the matrix $\mathcal{M}$ respectively. In order to further map this action to a mirror IIA version and in particular in order to have the kinetic terms for the hyper-scalars in the standard quaternionic form (2.52) one needs to exchange the scalar fields ξ and $\tilde{\xi}$ in the definition of the mirror map (2.50). This means that on the type IIA side the scalar which plays the role of the ten dimensional type IIB scalar will be $\tilde{\xi}_0$ in the case when we want to reproduce the magnetic fluxes rather than ξ^0 which we found in the previous sections. In order to obtain such an effective action we would now need that in the relations (5.16) and (5.18) one exchanges α_0 with β^0. From this kind of picture we gave here one notices an immediate problem when one tries to turn on both electric and magnetic fluxes: there are two scalars ξ^0 and $\tilde{\xi}_0$ which have to play the role of the ten dimensional type IIB scalar at the same time. It becomes again clear that one has to find a less naive way to turn on both electric and magnetic fluxes. It is somehow curious that the internal geometry knows already about these problems as we can not make both $d\alpha_0$ and $d\beta^0$ non-vanishing and assuming consistency with Calabi–Yau intersection numbers without violating $dd\omega_i = 0$.

Let us now turn to analyze this situation from the superpotentials perspective. Again we try to reproduce type IIB with NS fluxes. For this case the superpotential was given in (4.17). We argued that in the mirror picture one needs even form fluxes and indeed one can see from (5.40) that the four-form $d\Omega^+$ enters the superpotential. The claim is now that what one further needs is a two form flux F_2^{mag} which should have an expansion like

$$F_2^{mag} = m^i \omega_i , \tag{5.51}$$

and the corresponding superpotential would be

$$W \sim \int_{\hat{Y}} F_2^{mag} \wedge K \wedge K . \tag{5.52}$$

The reason to write this is just analogy with what happens in type IIB theory where the NS fluxes complexify the RR ones $G_3 = dC_2 + \tau H_3$ (4.18). So we would now need a two-form which is constructed out of defining forms of a manifold with $SU(3)$ structure J and Ω. A natural candidate is then $F_2^{mag} = d^\dagger\Omega^+ \sim *d\Omega^-$. It is not surprising that with this choice one can partly reproduce the superpotential (4.17). The difficult part now is to obtain the correct τ dependence in this superpotential. Once more we should stress that with the assumption that the magnetic fluxes come from $d^\dagger\Omega^+$ it is difficult to translate this condition in some algebra of the basis forms and thus it is almost impossible to attempt to compute the low energy effective action.

Chapter 6

Conclusions

In this work we have studied fluxes in type II string compactifications on Calabi–Yau threefolds. In chapter 3 we focused on type IIA theory and we described how to obtain the low energy effective action in such compactifications with fluxes. The first important result was to show that the bosonic action when NS fluxes are turned on is a particular case is an $N = 2$ gauged supergravity in four dimensions. Thus the fluxes do not break explicitly supersymmetry, but instead turn a normal supergravity into a gauged/massive one where the flux parameters play the role of masses and charges. The second important result in this chapter was obtained for case of RR fluxes when the compactification led to a massive two-form in four dimensions (3.42). The interesting feature is that this two-form couples to both electric and magnetic field strengths in such a way that the symplectic invariance of $N = 2$ supergravities is still maintained (3.49). Relying on the results for the NS fluxes we conclude that the action we obtained when RR fluxes are turned on is a new type of $N = 2$ gauged supergravity which in contrast to the known ones is still symplectic invariant. The first essential fact for this symmetry to be preserved is that the isometry which is gauged in this case corresponds to a two-form which can naturally couple to the field strengths of the vector fields in comparison to the scalars which can only couple to the gauge potentials. Thus, the dualization of the two-form in four dimensions leads as we have seen in section 3.3.2 to an explicit breaking of the symplectic invariance. The second issue which is crucial for preserving the symplectic invariance is the fact that in addition to the $2\,h^{(1,1)}$ RR fluxes in the four-dimensional theory one finds two more parameters: m, the mass parameter of the starting massive type IIA theory and e_0, the constant which is the dual of the three-form C_3 in four dimensions. Recall that in type IIA compactified on a Calabi–Yau threefold one finds $h^{(1,1)}$ vector multiplets and the symplectic group in this case is $Sp(2(h^{(1,1)} + 1)$, where the extra "+1" comes from the graviphoton. A theory with charged particles which is invariant under this symmetry must have $h^{(1,1)} + 1$ electric and $h^{(1,1)} + 1$ magnetic charges which transform in the fundamental representation of $Sp(2(h^{(1,1)} + 1)$ and the two parameters discussed above together with the ordinary $2\,h^{(1,1)}$ RR fluxes precisely behave in this way.

In chapter 4 we turned our attention to type IIB theory with fluxes having in mind the relation to type IIA via mirror symmetry. For the case of RR fluxes this was straightforward to check at the level of low energy effective actions. However we stress again the important role played by the additional parameters m and e_0 discussed above as together with the usual $2\,h^{(1,1)}$ RR fluxes they are the mirror of the $2\,(h^{(2,1)+1})$ RR fluxes from type IIB theory. For the case of NS-NS fluxes the situation is more complicated as one has on both sides three-form fluxes which can not be mapped into one another by mirror symmetry. We have proposed in chapter 5 a solution to this problem by choosing as compactification manifold a different space than a Calabi–Yau threefold. In order to have the right amount of supersymmetry in four dimensions, namely $N = 2$, these manifolds must have first of all an $SU(3)$ structure. $SU(3)$ structures have been classified according to their intrinsic torsion (see table 5.1) and as Calabi–Yau manifolds are torsion free this seems to be the right generalization of Calabi–Yau manifolds which can reproduce the mirror of the NS fluxes. Guided by mirror symmetry we have argued that the intrinsic torsion has to obey some further constraints and for the case at hand we have found that in addition to the $SU(3)$ structure the manifold has to satisfy the half-flat conditions $\mathrm{d}(J \wedge J) = \mathrm{d}\Omega^- = 0$. Using again mirror symmetry we find that these manifolds have smaller cohomology groups (5.21), but the moduli space of metrics on such spaces should be the same as the moduli space of ordinary Calabi–Yau manifolds. With these assumptions we have performed the KK reduction of type IIA/B theories and showed that the low energy effective action obtained in this way precisely coincides with the one of type IIB/A when electric NS fluxes are turned on. For the magnetic fluxes a generalization of the half-flat manifolds is required in that $\mathrm{d}^\dagger\Omega^+ \sim \mathrm{d}\Omega^- \neq 0$. However an explicit test of this proposal is not known at the moment.

Another important aspect of turning on fluxes is that a potential is generated which partially lifts the vacuum degeneracy. We have briefly pointed out that for type IIA with NS-NS fluxes the complex structure moduli are lifted while in type IIA with RR fluxes the potential depends on the Kähler moduli. In type IIB both RR and NS-NS fluxes lift only the complex structure moduli. These results can be easily seen by working with the superpotentials (3.26), (3.46), (4.17) and (4.11).

A couple of open questions still remain. First it is interesting to continue the analysis in order to obtain the mirror of the magnetic NS fluxes. Furthermore many of the results of chapter 5 are not satisfactory as they were imposed using mirror symmetry as an argument. Thus it would be interesting to have an independent mathematical derivation of the topology and moduli space of half-flat manifolds. Another important aspect is to use the analysis in this work in some phenomenological interesting models which have at most $N = 1$ supersymmetry and feature non-Abelian gauge groups. In such cases a more detailed study of the moduli stabilization problem has to be done and in particular to find a way to fix both the complex structure and the Kähler moduli at the same time.

Appendix A

Conventions and notations

Throughout this thesis we use the following conventions.

- The space-time metric has signature $(-,+,+,\ldots)$.
- The components of a differential p-form are defined as follows

$$A_p = \frac{1}{p!} A_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} \, . \tag{A.1}$$

- A hat on a p-form, e.g. $\hat{A}_p$ denotes differential forms in $d = 10$. p-forms without the hat are four-dimensional quantities.
- The Hodge operation $*$ is defined in such a way that

$$\mathrm{d}A_p \wedge *\mathrm{d}A_p = \frac{\sqrt{-g}}{(p+1)!} (\mathrm{d}A)_{\mu_1 \ldots \mu_{p+1}} (\mathrm{d}A)^{\mu_1 \ldots \mu_{p+1}} \mathrm{d}^d x \ , \tag{A.2}$$

  reproduces the correct kinetic term for a p-form in d space-time dimensions. In particular we denote $*\mathbf{1} = \sqrt{-g}\, \mathrm{d}^d x$.

- After compactification the Hodge operation splits into a Hodge-star on the four-dimensional space and another one acting on the internal Calabi–Yau space. For example, in the expansion of a p form one encounters terms like $\hat{A}_p = \cdots + A_{p-k}\omega_k + \cdots$, where ω_k is some harmonic k form on the internal space. The Hodge dual is given by

$$*\hat{A}_p = \cdots + (-1)^{k(p-k)} \, * A_{p-k} * \omega_k + \cdots \ , \tag{A.3}$$

  where the first $*$ on the RHS acts only in space-time while the second acts only in the internal space. The $(-1)^{k(p-k)}$ assures that the kinetic term of $\hat{A}_p$ produces

$$\int_{Y_3} \hat{A}_p \wedge *\hat{A}_p = \cdots + A_{p-k} \wedge *A_{p-k} \int_{Y_3} \omega_k \wedge *\omega_k + \cdots \, . \tag{A.4}$$

- The indices $i, j, k, \dots$ label harmonic $(1,1)$ and $(2,2)$ forms on the Calabi–Yau threefold and run from 1 to $h^{(1,1)}$; the indices $I, J, \dots$ label the vector fields in type IIA compactifications and include the zero $I = 0, 1, \dots, h^{(1,1)}$. The indices $a,\ b\dots$ run from 1 to $h^{(2,1)}$ and label $(2,1)$-forms on Y_3. The indices $A,\ B\dots$ include the zero and label all three-forms including the $(3,0)$-form, i.e. $A = 0, 1, \dots, h^{(2,1)}$. $A, B, \dots$ also label vector fields in type IIB compactifications.

- Indices $m,\ n,\ p, \dots = 1, \dots, 6$ label real internal coordinates. When we use complex coordinates we label them with $\alpha, \beta = 1, 2, 3,\ \bar\alpha, \bar\beta = 1, 2, 3$.

- The Riemann curvature tensor is defined as
$$R_{mnp}{}^{q} = \partial_m \phi_{np}{}^{q} - \partial_n \phi_{mp}{}^{q} - \phi_{mp}{}^{r}\phi_{nr}{}^{q} + \phi_{np}{}^{r}\phi_{mr}{}^{q}, \tag{A.5}$$
where ϕ denotes a general connection that contains two contributions $\phi_{mn}{}^{p} = \Gamma_{mn}{}^{p} + \kappa_{mn}{}^{p}$ where $\Gamma_{mn}{}^{p} = \Gamma_{nm}{}^{p}$ denote the Christoffel symbols and $\kappa_{mn}{}^{p}$ is the contorsion which we define more precisely in appendix C.1. For the Ricci tensor we use $R_{np} = R_{nmp}{}^{m}$ and thus in these conventions the Ricci scalar of a sphere is negative.

- We define the ϵ-symbol to be $\epsilon^{123456} = +1$. The indices are lowered with the metric. It follows that in terms of 'complex indices' one has, as a result of the $SU(3)$ structure,
$$\epsilon^{\alpha\beta\gamma\bar\alpha\bar\beta\bar\gamma} = -i\epsilon^{\alpha\beta\gamma}\,\epsilon^{\bar\alpha\bar\beta\bar\gamma}\ . \tag{A.6}$$
where similarly $\epsilon^{123} = \epsilon^{\bar1\bar2\bar3} = +1$.

- For the gamma matrices we use the conventions from [18]. In particular the gamma matrices on the internal space are chosen to be hermitian matrices satisfying
$$\{\Gamma_m, \Gamma_n\} = 2g_{mn}\ . \tag{A.7}$$
The chirality operator Γ_7 is defined as
$$\Gamma_7 = i\Gamma_1 \dots \Gamma_6 = \frac{i}{6!}\epsilon_{m_1\dots m_6}\Gamma^{m_1}\dots\Gamma^{m_6}\ . \tag{A.8}$$
Majorana spinors on the six-dimensional internal space can be defined if we adopt the following conventions for the charge conjugation matrix $\mathcal{C}$
$$\mathcal{C}^T = \mathcal{C}\ , \qquad \Gamma_m^T = -\mathcal{C}\Gamma_m\mathcal{C}^{-1}\ , \tag{A.9}$$
while the Majorana condition on a spinor η reads
$$\eta^\dagger = \eta^T\mathcal{C}\ . \tag{A.10}$$
Symmetry properties of the gamma matrices and $\mathcal{C}$ with the above conventions imply that for a commuting Majorana spinor η the following quantities vanish [18]
$$\eta^\dagger\Gamma_{(1)}\eta = \eta^\dagger\Gamma_{(2)}\eta = \eta^\dagger\Gamma_{(5)}\eta = \eta^\dagger\Gamma_{(6)}\eta = 0\ , \tag{A.11}$$
where by $\Gamma_{(n)}$ we have denoted the antisymmetric product of n gamma matrices
$$\Gamma_{(n)} = \Gamma_{m_1\dots m_n} = \Gamma_{[m_1}\dots\Gamma_{m_n]}\ . \tag{A.12}$$

Appendix B

$N = 2$ (gauged) supergravity in four dimensions

B.1 Generalities

As $N = 2$ (gauged) supergravities are the central theme of this work it is useful to put together the most important formulae which we encounter throughout the thesis. For a more detailed discussion of the subject we refer the reader to [94–100].

We have discussed the spectrum of $N = 2$ theories in table 2.3 and we have argued that in generic cases one deals with supergravity (gravity multiplet) coupled to n_V vector and n_H hyper-multiplets. We do not discuss the more exotic cases of tensor or double tensor multiplets as these can in general be dualized to hyper-multiplets.[1] The vector multiplets contain n_V complex scalars $t^i, i = 1, \ldots, n_V$ while the hyper-multiplets contain $4n_H$ real scalars $q^u, u = 1, \ldots, 4n_H$. $N = 2$ supersymmetry requires that the scalar manifold factorizes

$$\mathcal{M} = \mathcal{M}_V \otimes \mathcal{M}_H \ , \tag{B.1}$$

where the component $\mathcal{M}_V$ is a special Kähler manifold spanned by the scalars t^i while $\mathcal{M}_H$ is a quaternionic manifold spanned by the scalars q^u.

A special Kähler manifold is a Kähler manifold whose geometry obeys an additional constraint [94]. This constraint states that the Kähler potential K is not an arbitrary real function but determined in terms of a holomorphic prepotential $\mathcal{F}$ according to

$$K = -\ln\Big(i\bar{X}^I(\bar{t})\mathcal{F}_I(X) - iX^I(t)\bar{\mathcal{F}}_I(\bar{X})\Big) \ , \tag{B.2}$$

where $X^I, I = 0, \ldots, n_V$ are $(n_V + 1)$ holomorphic functions of the t^i. $\mathcal{F}_I$ abbreviates the derivative, i.e. $\mathcal{F}_I \equiv \frac{\partial \mathcal{F}(X)}{\partial X^I}$ and $\mathcal{F}(X)$ is a homogeneous function of X^I of degree 2, i.e. $X^I\mathcal{F}_I = 2\mathcal{F}$.

[1]The cases when these multiplets are massive and the dualization to hyper-multiplets is not possible have not been discussed in the literature.

The $4n_H$ scalars $q^u, u = 1, \ldots, 4n_H$ in the hyper-multiplets are coordinates on a quaternionic manifold [95]. This implies the existence of three almost complex structures $(J^x)_v^w, x = 1, 2, 3$ which satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy} + i\epsilon^{xyz} J^z \ . \tag{B.3}$$

Associated with the complex structures there is a triplet of fundamental (Kähler) forms

$$K^x_{uv} = h_{uw} (J^x)_v^w \ , \tag{B.4}$$

where h_{uw} is the quaternionic metric. The holonomy group of a quaternionic manifold is $Sp(2) \times Sp(2n_h)$ and K^x is identified with the field strength of the $Sp(2) \sim SU(2)$ connection ω^x, i.e.

$$K^x \ = \ \mathrm{d}\omega^x + \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z \ . \tag{B.5}$$

A quaternionic metric h_{uv} together with a holomorphic prepotential $\mathcal{F}$ specifies uniquely the (ungauged) supergravity action. In particular its bosonic part has the form

$$S = \int \Big[-\frac{1}{2} R * \mathbf{1} - g_{i\bar{j}} \mathrm{d}t^i \wedge * \mathrm{d}\bar{t}^{\bar{j}} - h_{uv} \mathrm{d}q^u \wedge * \mathrm{d}q^v + \frac{1}{2} \mathrm{Im}\, \mathcal{N}_{IJ} F^I \wedge * F^J + \frac{1}{2} \mathrm{Re}\, \mathcal{N}_{IJ} F^I \wedge F^J \Big] \ , \tag{B.6}$$

where $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K, \ F^I = dA^I$ (F^0 denotes the field strength of the graviphoton) and the gauge coupling functions are given by

$$\mathcal{N}_{IJ} = \bar{\mathcal{F}}_{IJ} + 2i \ \frac{\mathrm{Im}\mathcal{F}_{IK} \mathrm{Im}\mathcal{F}_{JL} X^K X^L}{\mathrm{Im}\mathcal{F}_{LK} X^K X^L} \ . \tag{B.7}$$

It is interesting to note that $N = 2$ supergravities are invariant under generalized electric-magnetic duality transformations. This symmetry however is not a symmetry of the of the action but only of the equations of motion and Bianchi identities. To see this we introduce the magnetic dual field strengths

$$G_I = \frac{\partial \mathcal{L}}{\partial F^I} = \mathrm{Re}\, \mathcal{N}_{IJ} F^J + \mathrm{Im}\, \mathcal{N}_{IJ} * F^J \ , \tag{B.8}$$

and thus the equations of motion become

$$0 = \frac{\partial \mathcal{L}}{\partial A^I} = \mathrm{d}G_I \ , \tag{B.9}$$

while the Bianchi identities read

$$\mathrm{d}F^I = 0 \ . \tag{B.10}$$

These equations are invariant under the generalized duality rotations[2]

$$\begin{pmatrix} F^I \\ G_I \end{pmatrix} \to \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} F^I \\ G_I \end{pmatrix} \ , \tag{B.11}$$

[2]This is often stated in terms of the self-dual and anti-self-dual part of the field strength $F^{\pm J}$ and the dual quantities $G_I^+ \equiv \mathcal{N}_{IJ} F^{+J}, \ G_I^- \equiv \bar{\mathcal{N}}_{IJ} F^{-J}$.

where U, V, W and Z are constant, real, $(n_V+1)\times(n_V+1)$ matrices which obey

$$\begin{aligned} U^{\mathrm{T}}V - W^{\mathrm{T}}Z \quad &= V^{\mathrm{T}}U - Z^{\mathrm{T}}W = \mathbf{1}\,, \\ U^{\mathrm{T}}W = W^{\mathrm{T}}U\,, \qquad & Z^{\mathrm{T}}V = V^{\mathrm{T}}Z\,, \end{aligned} \tag{B.12}$$

which means that the full matrix which appears in (B.11) is symplectic

$$\mathcal{O} \;= \begin{pmatrix} U & Z \\ W & V \end{pmatrix}, \qquad \mathcal{O} \in Sp(2n_V+2)\ . \tag{B.13}$$

(F^I, G_I) form a $(2n_V+2)$ symplectic vector and the same is true for $(X^I, \mathcal{F}_I)$. Clearly, the Kähler potential (B.2) is invariant under this symplectic transformation. Finally, the matrix $\mathcal{N}$ transforms according to

$$\mathcal{N} \to (V\mathcal{N}+W)\,(U+Z\mathcal{N})^{-1}\,. \tag{B.14}$$

Let us now turn to gauged $N=2$ supergravity [96]. One can gauge the isometries on the scalar manifold $\mathcal{M}$. Such isometries are generated by the Killing vectors $k_I^u(q)$, $k_I^i(t)$

$$\delta q^u \;=\; \Lambda^I k_I^u(q)\ , \qquad \delta t^i \;=\; \Lambda^I k_I^i(t)\ . \tag{B.15}$$

$k_I^u(q)$, $k_I^i(t)$ satisfy the Killing equations which in $N=2$ supergravity can be solved in terms of four Killing prepotentials (P_I, P_I^x). The Killing vectors on $\mathcal{M}_V$ are holomorphic and obey

$$k_I^i(t) = g^{i\bar{j}}\partial_{\bar{j}} P_I\ , \tag{B.16}$$

while the Killing vectors on $\mathcal{M}_H$ are determined by a triplet of Killing prepotentials $P_I^x(q)$ via

$$k_I^u\,K_{uv}^x \;=\; -D_v P_I^x \equiv -(\partial_v P_I^x + \epsilon^{xyz}\omega_v^y P_I^z)\ . \tag{B.17}$$

Gauging the isometries (B.15) requires the replacement of ordinary derivatives by covariant derivatives in the action

$$\partial_\mu q^u \to D_\mu q^u = \partial_\mu q^u - k_I^u A_\mu^I\ , \qquad \partial_\mu t^i \to D_\mu t^i = \partial_\mu t^i - k_I^i A_\mu^I\ . \tag{B.18}$$

Furthermore the potential

$$\begin{aligned} V_E \;&=\; e^K\Big[X^I\bar{X}^J(g_{\bar{\imath}j}\,k_I^{\bar{\imath}}k_J^j + 4h_{uv}\,k_I^u k_J^v) + g^{i\bar{j}}D_iX^I D_{\bar{j}}\bar{X}^J P_I^x P_J^x - 3X^I\bar{X}^J P_I^x P_J^x\Big] \\ &=\; e^K X^I\bar{X}^J(g_{\bar{\imath}j}\,k_I^{\bar{\imath}}k_J^j + 4h_{uv}\,k_I^u k_J^v) - \Big[\frac{1}{2}(\mathrm{Im}\,\mathcal{N})^{-1IJ} + 4e^K X^I\bar{X}^J\Big] P_I^x P_J^x\ , \end{aligned} \tag{B.19}$$

has to be added to the action in order to preserve supersymmetry. The bosonic part of the action of gauged $N=2$ supergravity is then given by

$$S = \int -\frac{1}{2}R^*\mathbf{1} - g_{i\bar{j}}Dt^i\wedge{*}D\bar{t}^{\bar{j}} - h_{uv}Dq^u\wedge{*}Dq^v + \frac{1}{2}\,\mathrm{Im}\,\mathcal{N}_{IJ}F^I\wedge{*}F^J + \frac{1}{2}\,\mathrm{Re}\,\mathcal{N}_{IJ}F^I\wedge F^J - V_E\ . \tag{B.20}$$

The symplectic invariance of the ungauged theory is generically broken since the action now explicitly depends on the gauge potentials A^I through the covariant derivatives Dt^i, Dq^u.

B.2 N=2 supergravities from Calabi–Yau threefolds

We have seen in section 2.3 that Calabi–Yau compactifications of type II theories lead to $N = 2$ supergravities in four dimensions. A crucial fact for this which has deep connections with the structure of $N = 2$ theories is that the moduli space of Calabi–Yau manifolds is a direct product of two special Kähler manifolds (2.30). In particular any of these factors can appear as the scalar manifold for the vector multiplets in one of the type II compactifications. In order to fill the gap left in the main text we present now a couple of facts about the geometry of these moduli spaces.

B.2.1 The complexified Kähler cone

According to section 2.3.2, the Kähler class deformations together with the zero modes of B_2 are harmonic $(1,1)$-forms on Y_3. Hence both the Kähler form J and B_2 components on the internal manifold can be expanded in the basis of $(1,1)$ forms introduced in (2.24)

$$B_2 + iJ \;=\; (b^j + iv^j)\omega_j \;\equiv\; t^j\omega_j\ , \qquad j = 1,\dots,h^{(1,1)}\ . \tag{B.21}$$

Let us denote by $\mathcal{K}_{ijk}$ the triple intersection number on Y_3

$$\mathcal{K}_{ijk} = \int_{Y_3} \omega_i \wedge \omega_j \wedge \omega_k\ . \tag{B.22}$$

Then it is useful to define the following quantities:

$$\begin{aligned} \mathcal{K}_{ij} &= \int_{Y_3} \omega^i \wedge \omega^j \wedge J = \mathcal{K}_{ijk}v^k \\ \mathcal{K}_{i} &= \int_{Y_3} \omega_i \wedge J \wedge J = \mathcal{K}_{ijk}v^j v^k\ , \\ \mathcal{K} &= \frac{1}{6}\int_{Y_3} J\wedge J\wedge J = \frac{1}{6}\mathcal{K}_{ijk}v^i v^j v^k\ , \end{aligned} \tag{B.23}$$

Note that we have introduced the factor $\frac{1}{6}$ in the last definition so that $\mathcal{K}$ is precisely the volume of Y_3. The metric on the complexified Kähler cone $\mathcal{M}_{1,1}$ is Kähler, i.e. $g_{ij} = \partial_i\bar{\partial}_j K$ and given by [30, 35]

$$\begin{aligned} g_{ij} \;&=\; \frac{1}{4\mathcal{K}}\int_{Y_3} \omega_i \wedge *\omega_j = -\frac{1}{4}\left(\frac{\mathcal{K}_{ij}}{\mathcal{K}} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{\mathcal{K}^2}\right) \\ &=\; -\partial_i\partial_j\big(-\ln 8\,\mathcal{K}\big) = \partial_i\bar{\partial}_j\big(-\ln 8\,\mathcal{K}\big)\ . \end{aligned} \tag{B.24}$$

Furthermore, the Kähler potential K is determined in terms of a holomorphic prepotential $\mathcal{F}$ via

$$e^{-K} = 8\,\mathcal{K} = i\left(\bar{X}^I\mathcal{F}_I - X^I\bar{\mathcal{F}}_I\right)\ , \qquad \mathcal{F}_I \equiv \partial_I\mathcal{F}\ , \qquad I = 0,\dots,h^{(1,1)}\ , \tag{B.25}$$

where

$$\mathcal{F} = -\frac{1}{3!}\frac{\mathcal{K}_{ijk}X^iX^jX^k}{X^0}\ . \tag{B.26}$$

The complexified Kähler class deformations t^i are the so called special coordinates related to the X^I via $X^I = (1, t^i)$. (X^I, F_I) transforms as a symplectic vector under (B.13) and K is a symplectic invariant.

From a physical point of view the scalars t^i are the bosonic partners of gauge fields A^I for the case of type IIA compactification. Thus, using the standard $N = 2$ formula (B.7) it is possible to compute the couplings of the vector fields in the low energy action from the geometry of the t^is described above. Inserting the prepotential (B.26) in (B.7) one obtains

$$\begin{aligned}
\mathrm{Re}\,\mathcal{N}_{00} &= -\frac{1}{3}\mathcal{K}_{ijk}b^ib^jb^k\ , &\quad \mathrm{Im}\,\mathcal{N}_{00} &= -\mathcal{K} + \left(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{\mathcal{K}}\right)b^ib^j\ ,\\
\mathrm{Re}\,\mathcal{N}_{i0} &= \frac{1}{2}\mathcal{K}_{ijk}b^jb^k\ , &\quad \mathrm{Im}\,\mathcal{N}_{i0} &= -\left(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{\mathcal{K}}\right)b^j\ ,\\
\mathrm{Re}\,\mathcal{N}_{ij} &= -\mathcal{K}_{ijk}b^k\ , &\quad \mathrm{Im}\,\mathcal{N}_{ij} &= \left(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{\mathcal{K}}\right)\ .
\end{aligned} \tag{B.27}$$

For the different calculations in this thesis it is useful to introduce the inverse matrices g^{ij} and $(\mathrm{Im}\,\mathcal{N})^{-1IJ}$. Using (2.24) it is not difficult to see that

$$g^{ij} = 4\mathcal{K}\int_{Y_3}\tilde{\omega}^i \wedge *\tilde{\omega}^j\ , \tag{B.28}$$

or equivalently

$$*\omega_i = 4\mathcal{K}g_{ij}\tilde{\omega}^j\ , \quad *\tilde{\omega}^i = \frac{1}{4\mathcal{K}}g^{ij}\omega_j\ . \tag{B.29}$$

Furthermore we have

$$\omega_i \wedge \omega_j \sim \mathcal{K}_{ijk}\tilde{\omega}^k\ , \tag{B.30}$$

where the symbol $\sim$ denotes the fact that the quantities are in the same cohomology class. Introducing $\mathcal{K}^{ij}$ via

$$\mathcal{K}^{ij}\mathcal{K}_{jk} = \delta^i_k\ , \tag{B.31}$$

one obtains

$$g^{ij} = -4\mathcal{K}\left(\mathcal{K}^{ij} - \frac{v^iv^j}{2\,\mathcal{K}}\right)\ . \tag{B.32}$$

Finally the inverse gauge coupling matrix has the form

$$(\mathrm{Im}\,\mathcal{N})^{-1} = \begin{pmatrix} -\frac{1}{\mathcal{K}} & -\frac{b^i}{\mathcal{K}} \\ -\frac{b^i}{\mathcal{K}} & \mathcal{K}^{ij} - \frac{b^ib^j}{\mathcal{K}} - \frac{v^iv^j}{2\,\mathcal{K}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\mathcal{K}} & -\frac{b^i}{\mathcal{K}} \\ -\frac{b^i}{\mathcal{K}} & -\frac{g^{ij}}{4\mathcal{K}} - \frac{b^ib^j}{\mathcal{K}} \end{pmatrix}\ . \tag{B.33}$$

B.2.2 The special geometry of H^3

Let us now turn our attention to the complex structure moduli space. We have seen that the complex structure moduli are given by the variations of the metric which are of $(2,0)+(0,2)$ type. Using a well known theorem by Torelli [101] one can describe this space in a simpler way by just looking at the holomorphic $(3,0)$ form Ω. Consider the basis for $H^3(Y)$ introduced in (2.25). Then the above cited theorem states that the complex structure moduli space is given by the space of all possible periods

$$Z^A = \int_Y \Omega \wedge \beta^A \ , \qquad \mathcal{G}_A = \int_Y \Omega \wedge \alpha_A \ . \tag{B.34}$$

It is not hard to have an intuitive picture of this result. Consider the expansion of Ω in the basis (2.25)

$$\Omega = Z^A \alpha_A - \mathcal{G}_A \beta^A \ . \tag{B.35}$$

By deforming the complex structure one should also deform Ω as by definition is holomorphic and of type $(3,0)$ which are complex structure dependent. On the other hand, the basis (α_A, β^A) is real and thus the periods Z^A and $\mathcal{G}_A$ should change when the complex structure is deformed.

It turns out [35] that $\mathcal{G}_A$ are functions of Z^A and determined in terms of a homogeneous function of degree two $\mathcal{G}(Z)$ as

$$\mathcal{G}_A = \frac{\partial \mathcal{G}}{\partial Z^A} \equiv \partial_A \mathcal{G} \ . \tag{B.36}$$

Furthermore, Ω is homogeneous of degree one in Z, i.e. $\Omega = Z^A \partial_A \Omega$ with

$$\partial_A \Omega = \alpha_A - \mathcal{G}_{AB} \beta^B \ . \tag{B.37}$$

The deformations of the complex structure z^a, $a = 1, \dots, h^{(2,1)}$ which reside in $H^{2,1}(Y_3)$ defined in (2.29) are related to the coordinates Z^A via $z^a = Z^a/Z^0$ or in other words one can choose $Z^A = (1, z^a)$. The metric $g_{a\bar{b}}$ on the space of complex structure deformations $\mathcal{M}_{2,1}$ is Kähler

$$g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K \ , \tag{B.38}$$

with the Kähler potential K given by

$$K = -\ln i \int_{Y_3} \Omega \wedge \bar{\Omega} = -\ln i \left(\bar{Z}^A \mathcal{G}_A - Z^A \bar{\mathcal{G}}_A \right) \ . \tag{B.39}$$

As we see K is determined in terms of the holomorphic prepotential $\mathcal{G}(Z)$ and hence $\mathcal{M}_{2,1}$ is a special Kähler manifold.

As in the previous section let us comment on the symplectic structure of this geometry. First of all it is easy to see that the basis (α_A, β^A) (2.25) for $H^3(Y)$ is defined only up to a symplectic rotation. In order to keep Ω from (B.35) invariant $(Z^A, \mathcal{G}_A)$ must transform as a symplectic vector. Note that in this way the Kähler potential (B.39) is explicitly invariant under these transformations.

Finally, let us discuss the action of the Hodge $*$ on the basis (2.25). $*\alpha_A$ and $*\beta^B$ are both three-forms again so that they can be expanded in terms of α and β according to

$$*\alpha_A = A_A{}^B\,\alpha_B + B_{AB}\,\beta^B\ , \qquad *\beta^A = C^{AB}\,\alpha_B + D^A{}_B\,\beta^B\ . \tag{B.40}$$

Using (2.25) one derives

$$\begin{aligned} B_{AB} = \int_{Y_3} \alpha_A \wedge *\alpha_B \ &= \ \int_{Y_3} \alpha_B \wedge *\alpha_A = B_{BA}\ , \\ C^{AB} = -\int_{Y_3} \beta^A \wedge *\beta^B \ &= \ -\int_{Y_3} \beta^B \wedge *\beta^A = C^{BA}\ , \\ A_A{}^B = -\int_{Y_3} \beta^B \wedge *\alpha_A \ &= \ -\int_{Y_3} \alpha_A \wedge *\beta^B = -D^B{}_A\ . \end{aligned} \tag{B.41}$$

Furthermore, the matrices $A,\ B,\ C$ can be determined in terms of the matrix $\mathcal{M}$ [97,102] defined in (B.7) using the prepotential $\mathcal{G}$ from (B.36) and which will give the gauge couplings in the case of the type IIB theory compactified on a Calabi–Yau manifold

$$\begin{aligned} A &= (\mathrm{Re}\,\mathcal{M})\,(\mathrm{Im}\,\mathcal{M})^{-1}\ , \\ B &= -\,(\mathrm{Im}\,\mathcal{M}) - (\mathrm{Re}\,\mathcal{M})\,(\mathrm{Im}\,\mathcal{M})^{-1}\,(\mathrm{Re}\,\mathcal{M})\ , \\ C &= (\mathrm{Im}\,\mathcal{M})^{-1}\ . \end{aligned} \tag{B.42}$$

Appendix C

G-structures

C.1 Complex and almost complex manifolds

In this section we assemble a few facts about G-structures as taken from the mathematical literature where one also finds the proofs omitted here. (See, for example, [87, 88, 90, 91, 103, 104].) We concentrate on the example of manifolds with $SU(3)$-structure which are the most significant ones for this thesis.

C.1.1 Almost Hermitian manifolds

Before discussing G-structures in general, let us recall the definition of an almost Hermitian manifold. This allows us to introduce some useful concepts, and, as we subsequently will see, provides us with a classic example of a G-structure.

A manifold of real dimension $2n$ is called *almost complex* if it admits a globally defined tensor field $J_m{}^n$ which obeys

$$J_m{}^p J_p{}^n = -\delta_m{}^n \ . \tag{C.1}$$

A metric g_{mn} on such a manifold is called Hermitian if it satisfies

$$J_m{}^p J_n{}^r g_{pr} = g_{mn} \ . \tag{C.2}$$

An almost complex manifold endowed with a Hermitian metric is called an *almost Hermitian manifold*. The relation (C.2) implies that $J_{mn} = J_m{}^p g_{pn}$ is a non-degenerate 2-form which is called *the fundamental form*.

On any even-dimensional manifold one can locally introduce complex coordinates. However, complex manifolds have to satisfy in addition that, first, the introduction of complex coordinates on different patches is consistent, and second that the transition functions between different patches are holomorphic functions of the complex coordinates. The first condition corresponds to the existence of an almost complex structure. The second condition is an integrability condition, implying that there are coordinations such that the almost complex structure takes the form

$$J = \begin{pmatrix} i\mathbf{1_{n\times n}} & 0 \\ 0 & -i\mathbf{1_{n\times n}} \end{pmatrix} . \tag{C.3}$$

The integrability condition is satisfied if and only if the Nijenhuis tensor ${N_{mn}}^p$ vanishes. It is defined as

$$\begin{aligned} {N_{mn}}^p &= {J_m}^q \left(\partial_q {J_n}^p - \partial_n {J_q}^p\right) - {J_n}^q \left(\partial_q {J_m}^p - \partial_m {J_q}^p\right) \\ &= {J_m}^q \left(\nabla_q {J_n}^p - \nabla_n {J_q}^p\right) - {J_n}^q \left(\nabla_q {J_m}^p - \nabla_m {J_q}^p\right) \ , \end{aligned} \tag{C.4}$$

where ∇ denotes the covariant derivative with respect to the Levi–Civita connection.

One can also consider an even stronger condition where $\nabla_m J_{np} = 0$. This implies ${N_{mn}}^p = 0$ but in addition that $\mathrm{d}J = 0$ and means we have a *Kähler manifold*. In particular, it implies that the holonomy of the Levi–Civita connection ∇ is $U(n)$.

Even if there is no coordinate system where it can be put in the form (C.3), any almost complex structure obeying (C.1) has eigenvalues $\pm i$. Thus even for non-integrable almost complex structures one can define the projection operators

$$(P^{\pm})_m{}^n = \frac{1}{2}(\delta^n_m \mp i {J_m}^n) \ , \tag{C.5}$$

which project onto the two eigenspaces, and satisfy

$$P^{\pm}P^{\pm} = P^{\pm} \ , \qquad P^+P^- = 0 \ . \tag{C.6}$$

On an almost complex manifold one can define (p,q) projected components $\omega^{p,q}$ of a real $(p+q)$-form ω^{p+q} by using (C.5)

$$\omega^{p,q}_{m_1\dots m_{p+q}} = (P^+)_{m_1}{}^{n_1}\dots(P^+)_{m_p}{}^{n_p}(P^-)_{m_{p+1}}{}^{n_{p+1}}\dots(P^-)_{m_{p+q}}{}^{n_{p+q}}\omega^{p+q}_{n_1\dots n_{p+q}} \ . \tag{C.7}$$

Furthermore, a real $(p+q)$-form is of the type (p,q) if it satisfies

$$\omega_{m_1\dots m_p n_1\dots n_q} = (P^+)_{m_1}{}^{r_1}\dots(P^+)_{m_q}{}^{r_p}(P^-)_{n_1}{}^{s_1}\dots(P^-)_{n_q}{}^{s_q}\omega_{r_1\dots r_p s_1\dots s_q} \ . \tag{C.8}$$

In analogy with complex manifolds we denote the projections on the subspace of eigenvalue $+i$ with an unbarred index α and the projection on the subspace of eigenvalue $-i$ with a barred index $\bar\alpha$. For example the hermitian metric of an almost Hermitian manifold is of type $(1,1)$ and has one barred and one unbarred index. Thus, raising and lowering indices using this hermitian metric converts holomorphic indices into anti-holomorphic ones and vice versa. Moreover the contraction of a holomorphic and an anti-holomorphic index vanishes, i.e. given V_m which is of type $(1,0)$ and W^n which is of type $(0,1)$, the product $V_m W^m$ is zero. Similarly, on an almost hermitian manifold of real dimension $2n$ forms of type $(p,0)$ vanish for $p > n$. Finally, derivatives of (p,q)-forms pick up extra pieces compared to complex manifolds precisely because J is not constant. One finds [104]

$$\mathrm{d}\omega^{(p,q)} = (\mathrm{d}\omega)^{(p-1,q+2)} + (\mathrm{d}\omega)^{(p,q+1)} + (\mathrm{d}\omega)^{(p+1,q)} + (\mathrm{d}\omega)^{(p+2,q-1)} \ . \tag{C.9}$$

C.1.2 G-structures and G-invariant tensors

An orthonormal frame on a d-dimensional Riemannian manifold M is given by a basis of vectors e_i, with $i = 1, \ldots, d$, satisfying $e_i^m e_j^n g_{mn} = \delta_{ij}$. The set of all orthonormal frames is known as the frame bundle. In general, the structure group of the frame bundle is the group of rotations $O(d)$ (or $SO(d)$ is M is orientable). The manifold has a G-structure if the structure group of the frame bundle is not completely general but can be reduced to $G \subset O(d)$. For example, in the case of an almost Hermitian manifold of dimension $d = 2n$, in turns out one can always introduce a complex frame and as a result the structure group reduces to $U(n)$.

An alternative and sometimes more convenient way to define G-structures is via G-invariant tensors, or, if M is spin, G-invariant spinors. A non-vanishing, globally defined tensor or spinor ξ is G-invariant if it is invariant under $G \subset O(d)$ rotations of the orthonormal frame. In the case of almost Hermitian structure, the two-form J is an $U(n)$-invariant tensor. Since the invariant tensor ξ is globally defined, by considering the set of frames for which ξ takes the same fixed form, one can see that the structure group of the frame bundle must then reduce to G (or a subgroup of G). Thus the existence of ξ implies we have a G-structure. Typically, the converse is also true. Recall that, relative to an orthonormal frame, tensors of a given type form the vector space for a given representation of $O(d)$ (or $Spin(d)$ for spinors). If the structure group of the frame bundle is reduced to $G \subset O(d)$, this representation can be decomposed into irreducible representations of G. In the case of almost complex manifolds, this corresponds to the decomposition under the $P^{\pm}$ projections (C.5). Typically there will be some tensor or spinor that will have a component in this decomposition which is invariant under G. The corresponding vector bundle of this component must be trivial, and thus will admit a globally defined non-vanishing section ξ. In other words, we have a globally defined non-vanishing G-invariant tensor or spinor.

To see this in more detail in the almost complex structure example, recall that we had a globally defined fundamental two-form J. Let us specialize for definiteness to a six-manifold, though the argument is quite general. Two-forms are in the adjoint representation **15** of $SO(6)$ which decomposes under $U(3)$ as

$$\mathbf{15} = \mathbf{1} + \mathbf{8} + (\mathbf{3} + \bar{\mathbf{3}}) \ . \tag{C.10}$$

There is indeed a singlet in the decomposition and so given a $U(3)$-structure we necessarily have a globally defined invariant two-form, which is precisely the fundamental two-form J. Conversely, given a metric and a non-degenerate two-form J, we have an almost Hermitian manifold and consequently an $U(3)$-structure.

In this paper we are interested in $SU(3)$-structure. In this case we find two invariant tensors. First we have the fundamental form J as above. In addition, we find an invariant complex three-form Ω. Three-forms are in the **20** representation of $SO(6)$, giving two singlets in the decomposition under $SU(3)$,

$$\begin{aligned} \mathbf{15} &= \mathbf{1} + \mathbf{8} + \mathbf{3} + \bar{\mathbf{3}} \quad \Rightarrow \quad J \ , \\ \mathbf{20} &= \mathbf{1} + \mathbf{1} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{6} + \bar{\mathbf{6}} \quad \Rightarrow \quad \Omega = \Omega^+ + \mathrm{i}\Omega^- \ . \end{aligned} \tag{C.11}$$

In addition, since there is no singlet in the decomposition of a five-form, one finds that

$$J \wedge \Omega = 0 \ . \tag{C.12}$$

Similarly, a six-form is a singlet of $SU(3)$, so we also must have that $J \wedge J \wedge J$ is proportional to $\Omega \wedge \bar{\Omega}$. The usual convention is to set

$$J \wedge J \wedge J = \frac{3\mathrm{i}}{4} \Omega \wedge \bar{\Omega} \ , \tag{C.13}$$

Conversely, a non-degenerate J and Ω satisfying (C.12) and (C.13) implies that M has $SU(3)$-structure.

We can similarly ask what happens to spinors for a structure group $SU(3)$. In this case we have the isomorphism $Spin(6) \cong SU(4)$ and the four-dimensional spinor representation decomposes as

$$\mathbf{4} = \mathbf{1} + \mathbf{3} \quad \Rightarrow \quad \eta \ . \tag{C.14}$$

We find one singlet in the decomposition, implying the existence of a globally defined invariant spinor η. Again, the converse is also true. A metric and a globally defined spinor η implies that M has $SU(3)$-structure.

C.1.3 Intrinsic torsion

One would like to have some classification of G-structures. In particular, one would like a generalization of the notion of a Kähler manifold where the holonomy of the Levi–Civita connection reduces to $U(n)$. Such a classification exists in terms of the *intrinsic torsion*. Let us start by recalling the definition of torsion and contorsion on a Riemannian manifold (M, g).

Given any metric compatible connection ∇' on (M, g), i.e. one satisfying $\nabla'_m g_{np} = 0$, one can define the Riemann curvature tensor and the torsion tensor as follows

$$[\nabla'_m, \nabla'_n] V_p = -R_{mnp}{}^q V_q - 2T_{mn}{}^r \nabla'_r V_p \ , \tag{C.15}$$

where V is an arbitrary vector field. The Levi-Civita connection is the unique torsionless connection compatible with the metric and is given by the usual expression in terms of Christoffel symbols $\Gamma_{mn}{}^p = \Gamma_{nm}{}^p$. Let us denote by ∇ the covariant derivative with respect to the Levi-Civita connection while a connection with torsion is denoted by $\nabla^{(T)}$. Any metric compatible connection can be written in terms of the Levi-Civita connection

$$\nabla^{(T)} = \nabla + \kappa \ , \tag{C.16}$$

where $\kappa_{mn}{}^p$ is the contorsion tensor. Metric compatibility implies

$$\kappa_{mnp} = -\kappa_{mpn} \ , \quad \text{where} \quad \kappa_{mnp} = \kappa_{mn}{}^r g_{rp} \ . \tag{C.17}$$

Inserting (C.17) into (C.15) one finds a one-to-one correspondence between the torsion and the contorsion

$$\begin{aligned} T_{mn}{}^p &= \frac{1}{2}(\kappa_{mn}{}^p - \kappa_{nm}{}^p) \equiv \kappa_{[mn]}{}^p \ , \\ \kappa_{mnp} &= T_{mnp} + T_{pmn} + T_{pnm} \ . \end{aligned} \tag{C.18}$$

These relations tell us that given a torsion tensor T there exist a unique connection $\nabla^{(T)}$ whose torsion is precisely T.

Now suppose M has a G-structure. In general the Levi-Civita connection does not preserve the G-invariant tensors (or spinor) ξ. In other words, $\nabla\xi \neq 0$. However, one can show [88], that there always exist some other connection $\nabla^{(T)}$ which is compatible with the G structure so that

$$\nabla^{(T)}\xi = 0 \ . \tag{C.19}$$

Thus for instance, on an almost Hermitian manifold one can always find $\nabla^{(T)}$ such that $\nabla^{(T)}J = 0$. On a manifold with $SU(3)$-structure, it means we can always find $\nabla^{(T)}$ such that both $\nabla^{(T)}J = 0$ and $\nabla^{(T)}\Omega = 0$. Since the existence of $SU(3)$-structure is also equivalent to the existence of an invariant spinor η, this is equivalent to the condition $\nabla^{(T)}\eta = 0$.

Let κ be the contorsion tensor corresponding to $\nabla^{(T)}$. From the symmetries (C.17), we see that κ is an element of $\Lambda^1 \otimes \Lambda^2$ where Λ^n is the space of n-forms. Alternatively, since $\Lambda^2 \cong so(d)$, it is more natural to think of $\kappa_{mn}{}^p$ as one-form with values in the Lie-algebra $so(d)$ that is $\Lambda^1 \otimes so(d)$. Given the existence of a G-structure, we can decompose $so(d)$ into a part in the Lie algebra g of $G \subset SO(d)$ and an orthogonal piece $g^\perp = so(d)/g$. The contorsion κ splits according into

$$\kappa = \kappa^0 + \kappa^g \ , \tag{C.20}$$

where κ^0 is the part in $\Lambda^1 \otimes g^\perp$. Since an invariant tensor (or spinor) ξ is fixed under G rotations, the action of g on ξ vanishes and we have, by definition,

$$\nabla^{(T)}\xi = \left(\nabla + \kappa^0 + \kappa^g\right)\xi = \left(\nabla + \kappa^0\right)\xi = 0 \ . \tag{C.21}$$

Thus, any two G-compatible connections must differ by a piece proportional to κ^g and they have a common term κ^0 in $\Lambda^1 \otimes g^\perp$ called the "intrinsic contorsion". Recall that there is an isomorphism (C.18) between κ and T. It is more conventional in the mathematics literature to define the corresponding torsion

$$T^0_{mn}{}^p = \kappa^0_{[mn]}{}^p \in \Lambda^1 \otimes g^\perp \ , \tag{C.22}$$

known as the *intrinsic torsion*.

From the relation (C.21) it is clear that the intrinsic contorsion, or equivalently torsion, is independent of the choice of G-compatible connection. Basically it is a measure of the degree to which $\nabla\xi$ fails to vanish and as such is a measure solely of the G-structure itself. Furthermore, one can decompose κ^0 into irreducible G representations. This provides a classification of G-structures in terms of which representations appear in the decomposition. In particular, in the special case where κ^0 vanishes so that $\nabla\xi = 0$, one says that the structure is "torsion-free". For an almost Hermitian structure this is equivalent to requiring that the manifold is complex and Kähler. In particular, it implies that the holonomy of the Levi–Civita connection is contained in G.

Let us consider the decomposition of T^0 in the case of $SU(3)$-structure. The relevant representations are

$$\Lambda^1 \sim \mathbf{3} \oplus \bar{\mathbf{3}} \ , \qquad g \sim \mathbf{8} \ , \qquad g^\perp \sim \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \ . \tag{C.23}$$

Thus the intrinsic torsion, which is an element of $\Lambda^1 \otimes su(3)^\perp$, can be decomposed into the following $SU(3)$ representations

$$\begin{aligned}\Lambda^1 \otimes su(3)^\perp &= (\mathbf{3} \oplus \bar{\mathbf{3}}) \otimes (\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}) \\ &= (\mathbf{1} \oplus \mathbf{1}) \oplus (\mathbf{8} \oplus \mathbf{8}) \oplus (\mathbf{6} \oplus \bar{\mathbf{6}}) \oplus (\mathbf{3} \oplus \bar{\mathbf{3}}) \oplus (\mathbf{3} \oplus \bar{\mathbf{3}})' \ .\end{aligned} \tag{C.24}$$

The terms in parentheses on the second line correspond precisely to the five classes $\mathcal{W}_1, \dots, \mathcal{W}_5$ presented in table 5.1. We label the component of T^0 in each class by $T_1, \dots, T_5$.

In the case of $SU(3)$-structure, each component T_i can be related to a particular component in the $SU(3)$ decomposition of $\mathrm{d}J$ and $\mathrm{d}\Omega$. From (C.21), we have

$$\begin{aligned}\mathrm{d}J_{mnp} &= 6 T^0_{[mn}{}^r J_{r|p]} \ , \\ \mathrm{d}\Omega_{mnpq} &= 12 T^0_{[mn}{}^r \Omega_{r|pq]} \ .\end{aligned} \tag{C.25}$$

Since J and Ω are $SU(3)$ singlets, $\mathrm{d}J$ and $\mathrm{d}\Omega$ are both elements of $\Lambda^1 \otimes su(3)^\perp$. Put another way, the contractions with J and Ω in (C.25) simply project onto different $SU(3)$ representations of T^0. We can see which representations appear simply by decomposing the real three-form $\mathrm{d}J$ and complex four-form $\mathrm{d}\Omega$ under $SU(3)$. We have,

$$\begin{aligned}\mathrm{d}J &= \left[(\mathrm{d}J)^{3,0} + (\mathrm{d}J)^{0,3}\right] + \left[(\mathrm{d}J)^{2,1}_0 + (\mathrm{d}J)^{1,2}_0\right] + \left[(\mathrm{d}J)^{1,0} + (\mathrm{d}J)^{0,1}\right] \ , \\ \mathbf{20} &= (\mathbf{1} \oplus \mathbf{1}) \oplus (\mathbf{6} \oplus \bar{\mathbf{6}}) \oplus (\mathbf{3} \oplus \bar{\mathbf{3}}) \ ,\end{aligned} \tag{C.26}$$

and

$$\begin{aligned}\mathrm{d}\Omega &= (\mathrm{d}\Omega)^{3,1} + (\mathrm{d}\Omega)^{2,2}_0 + (\mathrm{d}\Omega)^{0,0} \ , \\ \mathbf{24} &= (\mathbf{3} \oplus \bar{\mathbf{3}})' \oplus (\mathbf{8} \oplus \mathbf{8}) \oplus (\mathbf{1} \oplus \mathbf{1}) \ .\end{aligned} \tag{C.27}$$

The superscripts in the decomposition of $\mathrm{d}J$ and $\mathrm{d}\Omega$ refer to the (p,q)-type of the form. The 0 subscript refers to the irreducible $SU(3)$ representation where the trace part, proportional to J^n has been removed. Thus in particular, the traceless parts $(\mathrm{d}J)^{2,1}_0$ and $(\mathrm{d}\Omega)^{2,2}_0$ satisfy $J \wedge (\mathrm{d}J)^{2,1}_0 = 0$ and $J \wedge (\mathrm{d}\Omega)^{2,2}_0 = 0$ respectively. The trace parts on the other hand, have the form $(\mathrm{d}J)^{1,0} = \alpha \wedge J$ and $(\mathrm{d}\Omega)^{0,0} = \beta J \wedge J$, with $\alpha \sim *(J \wedge \mathrm{d}J)$ and $\beta \sim *(J \wedge \mathrm{d}\Omega)$ respectively. Note that a generic complex four-form has 30 components. However, since Ω is a $(3,0)$-form, from (C.9) we see that $\mathrm{d}\Omega$ has no $(1,3)$ part, and so only has 24 components. Comparing (C.26) and (C.27) with (C.24) we see that

$$\mathrm{d}J \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \ , \quad \mathrm{d}\Omega \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_5 \ , \tag{C.28}$$

and as advertised, $\mathrm{d}J$ and $\mathrm{d}\Omega$ together include all the components T_i. Note that the singlet component T_1 can be expressed either in terms of $(\mathrm{d}J)^{0,3}$, corresponding to $\Omega \wedge \mathrm{d}J$ or in terms of $(\mathrm{d}\Omega)^{0,0}$ corresponding to $J \wedge \mathrm{d}\Omega$. This is simply a result of the relation (C.12) which implies that $\Omega \wedge \mathrm{d}J = J \wedge \mathrm{d}\Omega$.

C.2 The Ricci scalar of half-flat manifolds

The simplest way to derive the Ricci scalar for the manifold considered in chapter 5 is by using the integrability condition one can derive from the Killing spinor equation (5.2)

$$R^{(T)}_{mnpq}\Gamma^{pq}\eta = 0, \tag{C.29}$$

where the Riemann tensor of the connection with torsion is given by (A.5)

$$R^{(T)}_{mnpq} = R(\Gamma)_{mnpq} + \nabla_m \kappa^0{}_{npq} - \nabla_n \kappa^0{}_{mpq} - \kappa^0{}_{mp}{}^r \kappa^0{}_{nrq} + \kappa^0{}_{np}{}^r \kappa^0{}_{mrq} \ . \tag{C.30}$$

Here $R(\Gamma)_{mnpq}$ represents the usual Riemann tensor for the Levi-Civita connection and the covariant derivatives are again with respect to the Levi-Civita connection. For definiteness we choose the solution of the Killing spinor equation (5.2) to be a Majorana spinor.[1] Multiplying (C.29) by Γ^n and summing over n one obtains

$$R^{(T)}_{mnpq}\Gamma^{npq}\eta - 2R^{(T)}_{mn}\Gamma^n\eta = 0 \ . \tag{C.31}$$

Contracting from the left with $\eta^\dagger\Gamma^m$ and using the conventions for the Majorana spinors (A.11) one derives

$$2R^{(T)} = R^{(T)}_{mnpq}\eta^\dagger\Gamma^{mnpq}\eta \ . \tag{C.32}$$

where $R^{(T)}$ represents the Ricci scalar which can be defined from the Riemann tensor (C.30). Expressing $R^{(T)}_{mnpq}$ in terms of $R(\Gamma)_{mnpq}$ from (C.30), using the Bianchi identity $R(\Gamma)_{m[npq]} = 0$ and the fact that the contorsion is traceless $\kappa^0{}_{mn}{}^m = \kappa^{0m}{}_{mn} = 0$ which holds for half flat manifolds one can derive the formula for the Ricci scalar of the Levi-Civita connection

$$R = -\kappa^0{}_{mnp}\kappa^{0npm} - \frac{1}{2}\epsilon^{mnpqrs}(\nabla_m\kappa^0{}_{npq} - \kappa^0{}_{mp}{}^l\kappa^0{}_{nlq})J_{rs} \ . \tag{C.33}$$

Clearly in order to evaluate this expression we need the components of the intrinsic contorsion for the case we are dealing with. As we have presented in chapter 5 the intrinsic torsion (and thus also the intrinsic contorsion) can be uniquely determined in terms of the exterior derivatives of the fundamental form $\mathrm{d}J$ and of the $(3,0)$ form $\mathrm{d}\Omega$. By definition for half-flat manifolds the imaginary part of $T^-_{1\oplus 2}$, and the components T_4 and T_5 vanish and so we only have to determine $T^+_{1\oplus 2}$ and T_3.

From table 5.1 one sees that $T_{1\oplus 2}$ is in the same representation as a complex $(2,2)$ form F. Consequently we write

$$(T_{1\oplus 2})_{mn}{}^p = F_{mnrs}\Omega^{rsp} + \bar{F}_{mnrs}\bar{\Omega}^{rsp} \ . \tag{C.34}$$

The half flatness condition $T^-_{1\oplus 2} = 0$ just imposes that F is real ($F = \bar{F}$) so that

$$(T_{1\oplus 2})_{mn}{}^p \ = \ (T^+_{1\oplus 2})_{mn}{}^p \ = \ 2F^{(2,2)}_{mnrs}\,\Omega^{+rsp} \ , \tag{C.35}$$

[1] The results are independent of the choice of the spinor, but the derivations may be more involved.

where we have used (5.4). Explicitly, from the relations (C.25) one has that F is related to $\mathrm{d}\Omega$ by

$$F^{(2,2)}_{mnrs} \equiv \frac{1}{4||\Omega||^2}\,(\mathrm{d}\Omega)^{2,2}_{mnrs} = \frac{1}{4||\Omega||^2}\,(\mathrm{d}\Omega^+)^{2,2}_{mnrs}\ . \tag{C.36}$$

Similarly we see from table 5.1 that the component T_3 of the torsion is given by the traceless $(2,1)+(1,2)$ part of $\mathrm{d}J$. Expanding $\mathrm{d}J$ in forms of definite type we obtain

$$(\mathrm{d}J) = (\mathrm{d}J)^{(3,0)+(0,3)} + (\mathrm{d}J)^{(2,1)+(1,2)}\ . \tag{C.37}$$

The $(3,0)$ and $(0,3)$ parts are $SU(3)$ singlets thus proportional to Ω and $\bar{\Omega}$ respectively and from table 5.1 one sees that they can be completely determined in terms of the T_1 part of the torsion. Contracting (C.37) first with Ω and then with $\bar{\Omega}$ one obtains

$$(\mathrm{d}J)^{(3,0)+(0,3)} = 4F\Omega^-\ , \tag{C.38}$$

where by F we have denoted the trace of the four form

$$F \equiv F_{\alpha\beta}{}^{\alpha\beta}\ . \tag{C.39}$$

Using (C.38) equation (C.37) becomes

$$(\mathrm{d}J)_{mnp} = 4F(\Omega^-)_{mnp} + 6(T_3)_{[mn}{}^{r}J_{|r|p]}\ . \tag{C.40}$$

Equations (C.35), (C.36) and (C.40) give us the torsion components for a general half flat manifold. To obtain now the torsion components in terms of the fluxes we should replace $\mathrm{d}\Omega$ from (5.13). Using [105]

$$(\tilde{\omega}^i)_{\alpha\beta}{}^{\alpha\beta} = \frac{2v^i}{\mathcal{K}}\ , \tag{C.41}$$

one obtains for the trace of the four-form (C.35)

$$F \equiv F_{\alpha\beta}{}^{\alpha\beta} = \frac{e_i v^i}{2\mathcal{K}||\Omega||^2}\ . \tag{C.42}$$

Having derived the expressions for the non-vanishing torsion components we can now attempt to compute the Ricci scalar of half-flat manifolds using (C.33). In order to simplify the formulas we evaluate (C.33) term by term. The strategy will be to express first the contorsion κ^0 in terms of the torsion T^0 (C.18) and then go to complex indices splitting the torsion in its component parts[2] $T_{1\oplus 2}$ and T_3 which are of definite type with respect to the almost complex structure J.

The first term can be written as

$$A \equiv -\kappa^0{}_{mnp}\kappa^{0npm} = -(T^0_{mnp} + T^0_{pmn} + T^0_{pnm})(T^0)^{npm} = T^0_{mnp}(T^0)^{mnp} - 2T^0_{mnp}(T^0)^{npm}. \tag{C.43}$$

[2] Note that for the components of the intrinsic torsion T^0 we use only the notation T_i and drop the superscript 0.

Using the fact that the first two indices of the torsion T^0 are of the same type one obtains

$$A = (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\alpha\beta\gamma} - 2(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\beta\gamma\alpha} + (T_3)_{\alpha\beta\bar{\gamma}}(T_3)^{\alpha\beta\bar{\gamma}} + c.c. \ , \tag{C.44}$$

where *c.c.* denotes complex conjugation.

The second term can be computed if one takes into account that the four-dimensional effective action appears after one integrates the ten-dimensional action over the internal space, in this case $\hat{Y}$. Thus the second term in (C.33) can be integrated by parts to give[3]

$$B \equiv -\frac{1}{2}\epsilon^{mnpqrs}(\nabla_m \kappa^0{}_{npq})J_{rs} \sim \frac{1}{2}\epsilon^{mnpqrs}\kappa^0{}_{npq}\nabla_m J_{rs}. \tag{C.45}$$

Using (5.8) and (C.18) we obtain after going to complex indices

$$\begin{aligned} B &= -\epsilon^{mnpqrs}(T^0)_{mnp}(T^0)_{qr}{}^{t}J_{ts} \qquad \text{(C.46)} \\ &= -\epsilon^{\alpha\beta\gamma\bar{\alpha}\bar{\beta}\bar{\gamma}}(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})_{\bar{\alpha}\bar{\beta}}{}^{\delta}J_{\delta\bar{\gamma}} - \epsilon^{\alpha\beta\bar{\gamma}\bar{\alpha}\bar{\beta}\gamma}(T_3)_{\alpha\beta\bar{\gamma}}(T_3)_{\bar{\alpha}\bar{\beta}}{}^{\bar{\delta}}J_{\bar{\delta}\gamma} + c.c. \ . \end{aligned}$$

The six-dimensional ϵ symbol splits as

$$\epsilon^{\alpha\beta\gamma\bar{\alpha}\bar{\beta}\bar{\gamma}} = -i\epsilon^{\alpha\beta\gamma}\epsilon^{\bar{\alpha}\bar{\beta}\bar{\gamma}} \ , \tag{C.47}$$

and after some algebra involving the three-dimensional ϵ symbol one finds

$$B = -2(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\alpha\beta\gamma} - 4(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\beta\gamma\alpha} - 2(T_3)_{\alpha\beta\bar{\gamma}}(T_3)^{\alpha\beta\bar{\gamma}} + c.c. \ . \tag{C.48}$$

In the same way one obtains for the last term

$$C \equiv \frac{1}{2}\epsilon^{mnpqrs}\kappa^0{}_{mp}{}^{t}\kappa^0{}_{ntq}J_{rs} = 2(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\alpha\beta\gamma} + 2(T_3)_{\alpha\beta\bar{\gamma}}(T_3)^{\alpha\beta\bar{\gamma}} + c.c. \ . \tag{C.49}$$

Collecting the results from (C.44), (C.48) and (C.49) the formula for the Ricci scalar (C.33) becomes

$$R = (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\alpha\beta\gamma} - 6(T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\beta\gamma\alpha} + (T_3)_{\alpha\beta\bar{\gamma}}(T_3)^{\alpha\beta\bar{\gamma}} + c.c. \ . \tag{C.50}$$

The first two terms in the above expression can be straightforwardly computed using (C.35), (5.13) and (C.41). After a little algebra we find

$$\begin{aligned} (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\alpha\beta\gamma} &= \frac{e_i e_j}{8||\Omega||^2}(\tilde{\omega}^i)_{\alpha\beta\bar{\alpha}\bar{\beta}}(\tilde{\omega}^j)^{\alpha\beta\bar{\alpha}\bar{\beta}} \ , \qquad \text{(C.51)} \\ (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\beta\gamma\alpha} &= -\frac{e_i e_j}{8||\Omega||^2}(\tilde{\omega}^i)_{\alpha\beta\bar{\alpha}\bar{\beta}}(\tilde{\omega}^j)^{\alpha\beta\bar{\alpha}\bar{\beta}} + \frac{e_i e_j}{4||\Omega||^2}(*\tilde{\omega}^i)_{\alpha\bar{\beta}}(*\tilde{\omega}^j)^{\alpha\bar{\beta}} + \frac{(e_i v^i)^2}{4||\Omega||^2\mathcal{K}^2} \ . \end{aligned}$$

[3]Strictly speaking in 10 dimensions the Ricci scalar comes multiplied with a dilaton factor (2.2). However in all what we are doing we consider that the dilaton is constant over the internal space so it still make sense to speak about integration by parts without introducing additional factors with derivatives of the dilaton.

Integrating (C.51) over $\hat{Y}$ we obtain

$$\begin{aligned} \int_{\hat{Y}} (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\alpha\beta\gamma} &= \frac{e_i e_j g^{ij}}{8||\Omega||^2\mathcal{K}} \;, \\ \int_{\hat{Y}} (T_{1\oplus 2})_{\alpha\beta\gamma}(T_{1\oplus 2})^{\beta\gamma\alpha} &= -\frac{e_i e_j g^{ij}}{16||\Omega||^2\mathcal{K}} + \frac{(e_i v^i)^2}{4||\Omega||^2\mathcal{K}} \;. \end{aligned} \tag{C.52}$$

Finally, we have to compute the third term in (C.50). For this we take the square of (C.40) using the fact that the terms on the RHS do not mix as they carry indices of different types. Inserting (C.42) and dJ of (5.22) we obtain

$$(e_i v^i)^2 \int_{\hat{Y}} \beta^0 \wedge *\beta^0 = 2i \left(\frac{e_i v^i}{||\Omega||^2\mathcal{K}}\right)^2 \int_{\hat{Y}} \Omega \wedge \bar{\Omega} + 2\int_{\hat{Y}} (T_3)_{mnp}(T_3)^{mnp} \;. \tag{C.53}$$

The integral which appears on the LHS is given by

$$\int \beta^0 \wedge *\beta^0 = -\left[(\mathrm{Im}\,\mathcal{M})^{-1}\right]^{00} = \frac{8}{||\Omega||^2\mathcal{K}} \;, \tag{C.54}$$

where the first equation follows from (B.40) and (B.42) while the second equation is less obvious. The simplest way to see this is by using a mirror symmetry argument. We know that under mirror symmetry the gauge couplings $\mathcal{M}$ and $\mathcal{N}$ are mapped into one another. This also means that $(\mathrm{Im}\,\mathcal{M})^{-1}$ is mapped into $(\mathrm{Im}\,\mathcal{N})^{-1}$ and this matrix is given in (B.33) for a Calabi–Yau space. From here one sees that the element $[(\mathrm{Im}\,\mathcal{N})^{-1}]^{00}$ is just the inverse volume of the mirror Calabi–Yau space. Using again mirror symmetry and the fact that the Kähler potential of the Kähler moduli (B.24) is mapped into the Kähler potential of the complex structure moduli (B.39) we end up with the RHS of the above equation.

Now we can write (C.53) as

$$\int_{\hat{Y}} (T_3)_{mnp}(T_3)^{mnp} = 3\frac{(e_i v^i)^2}{||\Omega||^2\mathcal{K}} \;, \tag{C.55}$$

or in complex indices

$$\int_{\hat{Y}} (T_3)_{\alpha\beta\bar{\gamma}}(T_3)^{\alpha\beta\bar{\gamma}} = \frac{3}{2}\,\frac{(e_i v^i)^2}{||\Omega||^2\mathcal{K}} \;. \tag{C.56}$$

Inserting (C.52) and (C.56) into (C.50) and taking into account that all the terms in (C.52) and (C.56) are explicitly real such that the term '*c.c.*' in (C.50) just introduces one more factor of 2 we obtain the final form of the Ricci scalar

$$R = -\frac{1}{8}\, e_i e_j g^{ij} \left[(\mathrm{Im}\,\mathcal{M})^{-1}\right]^{00} \;, \tag{C.57}$$

where we have used again (C.54).

Appendix D

D.1 Kaluza–Klein reductions

As KK reductions play an important role in this thesis let us outline few of its most important features in a simple example, namely compactification on a circle. At the end we sketch the steps which are needed in order to generalize this to arbitrary internal manifolds. Thus this appendix is not intended to be an extensive review but rather to provide the main tools for computing low energy effective actions via the KK reduction. For a detailed discussion we refer the reader to the existing literature (see for example [106] and references therein).

We start from the Einstein-Hilbert action in five dimensions

$$S = -\int d^5x\sqrt{-G}\hat{R}\ , \tag{D.1}$$

and we want to see what is the interpretation from a four-dimensional point of view. This theory is clearly invariant under translations along any of the five coordinates and thus we can assume that one of the space directions, which we denote by y, is wrapped along a circle of radius ρ. The equations of motion in five dimensions read $\hat{R}_{MN} = 0$ and the simplest solution which is consistent with the above Ansatz is the Minkowski metric

$$G_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}\ . \tag{D.2}$$

This is a trivial example of the so called 'spontaneous compactification' where the higher-dimensional theory admits a solution which is a direct product of two spaces which do not talk to each other. Note that this is a necessary condition for a consistent compactification. We are now interested to study the dynamics of the theory in this vacuum. In other words we consider fluctuations about this vacuum and impose the five-dimensional equations of motion. For this we choose the following parameterization of the five-dimensional metric

$$G_{MN} = \phi^{-1/3}\begin{pmatrix} g_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix}\ , \tag{D.3}$$

where $g_{\mu\nu}$ is the four-dimensional metric, $A_\mu \equiv G_{\mu 5}$, is a vector field from the four dimensional point of view and $\phi \equiv g_{55}$ is a scalar field. These fields should be regarded as small variations around the background (D.2) or in other words their vacuum vacuum expectation values are given by

$$< g_{\mu\nu} > = \eta_{\mu\nu} \ , \qquad < A_\mu > = 0 \ , \qquad < \phi > = 1 \ . \tag{D.4}$$

For the simple Ansatz (D.2) it is not hard to see that the five dimensional equations of motion impose that the components of the metric (D.3) have to be eigenvectors of the Laplace operator in five dimensions. Furthermore in order to extract information about the four-dimensional physics we expand the fields in (D.3) as

$$g_{\mu\nu} = \sum_{n=-\infty}^{\infty} g_{\mu\nu}^{(n)} e^{\frac{2\pi niy}{\rho}} \ , \qquad A_\mu = \sum_{n=-\infty}^{\infty} A_\mu^{(n)} e^{\frac{2\pi niy}{\rho}} \ , \qquad \phi = \sum_{n=-\infty}^{\infty} \phi^{(n)} e^{n\frac{2\pi iy}{\rho}} \ . \tag{D.5}$$

The Laplace operator splits into a space-time and an internal part and so using the expansion (D.5) one generically obtaines

$$0 = \partial_M \partial^M \phi = \partial_\mu \partial^\mu \phi + \partial_y \partial_y \phi = \sum_{n=-\infty}^{\infty} \left[\partial_\mu \partial^\mu - \left(\frac{2\pi n}{\rho} \right)^2 \right] \phi^{(n)} e^{n\frac{2\pi iy}{\rho}} \ . \tag{D.6}$$

Thus from a four-dimensional point of view the fields in (D.5) have masses of order $m_n^2 \sim \frac{n^2}{\rho^2}$. If the size of the compactification circle is small the masses of the fields coming from the compactification are large and in a low energy approximation they can be neglected. Truncating out the massive modes effectively means to consider that the dependence of the five dimensional fields on the fifth coordinate is trivial. Inserting this back into the action and performing the integral over the fifth dimension one finds

$$S = \int d^4x \sqrt{-g} \ \left[-R - \frac{1}{4} \phi F_{\mu\nu} F^{\mu\nu} - \frac{1}{6\phi^2} \partial_\mu \phi \, \partial^\mu \phi \right] \ , \tag{D.7}$$

where in order to simplify the notation we have dropped the superscript (0) on the fields. We have also rescaled the four dimensional metric and the field ϕ to absorb the volume factor which appears after performing the integral over y and we have furthermore used the notation $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It is interesting to note that the general coordinate invariance in the fifth direction has transformed into a four dimensional Abelian gauge transformation $A_\mu \to A_\mu + \partial_\mu \lambda$.

If in five dimensions one also has matter fields the procedure is very similar to what we have just described. The generic equation of motion for the matter fields is

$$\partial_M \partial^M \Phi = 0 \ , \tag{D.8}$$

which under the assumption (D.2) again splits into a four dimensional part and an internal one. For a low energy approximation we are again interested in the massless modes and so we expand all the matter fields in harmonic functions on the circle and only keep the fields which correspond to zero eigenvalue.

This procedure can be easily generalized to arbitrary theories in arbitrary dimensions and for any number of compact dimensions. The main steps are exactly as in the above example

1. write the spontaneous compactification Ansatz

2. expand fields around this solution

3. identify the massless modes in lower dimensions

4. truncate the spectrum to only the massless modes and perform the integral over the internals space

Let us briefly sketch what are the differences one may encounter in such a generalization. First of all for a general theory it is not clear that it exhibits spontaneous compactification. If this is true and the internal manifold is flat (ie is a torus) then it is straightforward to generalize the above example. The only difference is that now there will be harmonic functions for each internal direction. If the manifold on the other hand is curved then one has to be more careful. For the matter fields the situation is again similar in that the massless modes in lower dimensions correspond to harmonic forms on the internal manifold.[1] For the fields which appear due to fluctuations of the metric in the internal directions the analysis is more complicated and a general prescription does not exist. For Ricci flat manifolds there is nevertheless a systematic way to extract information about the zero modes and in section 2.3.2 we have already presented an explicit example when the internal manifold is a Calabi–Yau space.

D.2 Poincaré dualities

For an arbitrary p-form in d dimensions one always has the choice to describe the action in terms of a Poincaré dual form. The nature of the dual form differs in the massless and massive case. A massless p-form in d dimensions describes $\binom{d-2}{p}$ physical degrees of freedom while a massive p-form in d dimensions contains $\binom{d-1}{p}$ degrees of freedom. The difference can be easily understood from a generalized Higgs mechanism where a p-form 'eats' a massless $p-1$-form and thus the number of degrees of freedom change by $\binom{d-2}{p-1}$. Therefore a massless p-form in d dimensions is dual to a $(d-p-2)$-form while a massive p-form is dual to a $(d-p-1)$-form. A massless $(d-1)$-form is special in that it is dual to a constant.

In $d=4$ this implies that a massless three-form is dual to a constant, a massless two-form is dual to a scalar while a massive 2-from is dual to a vector (a 1-form). As these cases in four dimensions appear repeatedly after compactifying the ten dimensional string/supergravity theories it is useful how to obtain the dynamics of the dual fields and so we will discuss these cases in turn.

[1] Note that by harmonic form we mean a form which satisfies $\Delta\omega = \mathrm{d}^\dagger \mathrm{d}\omega + \mathrm{d}\mathrm{d}^\dagger\omega = 0$ and not forms which are eigenvectors of the above Laplace operator with non-zero eigenvalues.

D.2.1 Dualization of a massless B_2

Let us first consider the dualization of a massless two-form B_2 with field strength $H_3 = dB_2$ to a scalar a. We start from the generic action

$$S_{H_3} = -\int \left[\frac{g}{4} H_3 \wedge *H_3 - \frac{1}{2} H_3 \wedge J_1\right], \tag{D.9}$$

where g is an arbitrary function of the scalars while J_1 is a generic 1-form depending on the scalars and possibly some gauge field A_1. The dualization can be carried out by introducing a scalar field a as a Lagrange multiplier and adding the term $H_3 \wedge da$ to S_{H_3}. Treating H_3 as an independent three-form (not being dB_2) the equation of motion for a implies $H_3 = dB_2$ while the equation of motion for H_3 reads ${}^*H_3 = \frac{1}{g}(da + J_1)$. Inserted back into the action (D.9) one obtains the dual action

$$S_a = -\int \frac{1}{4g}(da + J_1) \wedge *(da + J_1) \ . \tag{D.10}$$

There is an another way of treating the dualizations which turns out to be useful in understanding the dualization of a three-form in four dimensions. Consider the equation of motion for B_2

$$d(g\,{}^*H_3 - J_1) = 0, \tag{D.11}$$

which can be derived from (D.9). It is solved by

$$g^*H_3 - J_1 = da\,, \tag{D.12}$$

with a being some arbitrary scalar field. The equation of motion for this field is dictated by the Bianchi identity of H_3

$$0 = dH_3 = d\left[\frac{1}{g}\,{}^*(da + J_1)\right], \tag{D.13}$$

which in turn can be obtained from the action (D.10). This implies that the two ways described for the dualization of B_2 are equivalent.

D.2.2 Dualization of the three-form

Next we consider the dualization of a three-form in 4 dimensions. We start from a generic action for a three-form C_3 possibly coupled to two-forms, 1-forms and scalars

$$S_{C_3} = -\int \left[\frac{g}{4}(dC_3 - J_4) \wedge *(dC_3 - J_4) + \frac{h}{2} dC_3\right], \tag{D.14}$$

where g, h denote two arbitrary scalar functions and J_4 is an 4-form which can depend on the two-forms, 1-forms and scalars present in the spectrum.

For the field strength of a three-form in 4 dimensions there is no proper Bianchi identity since no 5-forms exist. That is why the second way of dualizing forms presented in the previous section, by exchanging the equation of motion with the Bianchi identity, can not work in this case. The only consistent way to proceed is to add a Lagrange multiplier to the action (D.14) [107]

$$S_{C_3} = -\int \left[\frac{g}{4}(dC_3 - J_4)\wedge *(dC_3 - J_4) + \frac{h}{2}dC_3 + \frac{e_0}{2}dC_3\right] , \qquad \text{(D.15)}$$

where e_0 is a constant. The equation of motion for dC_3 imply

$$\frac{g}{2}\, {}^*(dC_3 - J_4) = -\frac{h+e_0}{2} . \qquad \text{(D.16)}$$

Inserted back into the action (D.14) and using ${}^{**}dC_3 = -dC_3$ one obtains

$$S_{e_0} = -\int \left[\frac{1}{4g}(h+e_0)^2\, {}^*\mathbf{1} + \frac{1}{2}(h+e_0)J_4\right] . \qquad \text{(D.17)}$$

As we see a potential for the scalar fields is induced and e_0 play the role of a cosmological constant.

D.2.3 Dualization of a massive two-form

Finally, let us dicuss the dualization of the a massive two-form B_2 [84–86]. We start from a generic action

$$S_{B_2} = -\int \left[gH_3 \wedge *H_3 + M^2 B_2 \wedge *B_2 + M_T^2 B_2 \wedge B_2 + B_2 \wedge J_2\right] , \qquad \text{(D.18)}$$

where g, M, M_T can be field dependent couplings and J_2 is a two-form which can depend on the gauge potential A_1 and/or some scalar fields. (J_2 does not depend on B_2.) We can treat B_2 and H_3 as independent fields and ensure $H_3 = dB_2$ by the equations of motion. This is achieved in the action

$$S'_{B_2} = -\int \left[-gH_3 \wedge *H_3 + 2gH_3 \wedge *dB_2 + M^2 B_2 \wedge *B_2 + M_T^2 B_2 \wedge B_2 + B_2 \wedge J_2\right] , \qquad \text{(D.19)}$$

which indeed has $H_3 = dB_2$ as the equation of motion for H_3. So by inserting $H_3 = dB_2$ into (D.19) we obtain (D.18). On the other hand one can eliminate B_2 through its equation of motion and obtain an action expressed only in terms of H_3. The equation of motion for B_2 from (D.19) is

$$2M^2\, {}^*B_2 + 2M_T^2 B_2 + J_2 - 2d\, {}^*(gH_3) = 0 , \qquad \text{(D.20)}$$

which is solved by

$$\begin{aligned} {}^*B_2 &= \frac{1}{M^4 + M_T^4}\left[M^2 d\, {}^*(gH_3) + M_T^2\, {}^*d\, {}^*(gH_3) - \frac{M^2}{2}J_2 - \frac{M_T^2}{2}\, {}^*J_2\right] \quad \text{or} \\ B_2 &= \frac{1}{M^4 + M_T^4}\left[M_T^2 d\, {}^*(gH_3) - M^2\, {}^*d\, {}^*(gH_3) + \frac{M^2}{2}\, {}^*J_2 - \frac{M_T^2}{2}J_2\right] . \end{aligned} \qquad \text{(D.21)}$$

Inserted back into the action (D.19) results in

$$S''_{B_2} = \int \Big[gH_3 \wedge *H_3 - \frac{M^2}{M^4 + M_T^4} \left(d^*(gH_3) - \frac{1}{2} J_2 \right) \wedge * \left(d^*(gH_3) - \frac{1}{2} J_2 \right)$$
$$+ \frac{M_T^2}{M^4 + M_T^4} \left(d^*(gH_3) - \frac{1}{2} J_2 \right) \wedge \left(d^*(gH_3) - \frac{1}{2} J_2 \right) \Big]. \quad \text{(D.22)}$$

We can now replace H_3 by its Poincaré dual one-form $A^H = g\,{}^*H_3$ and the dual action for the massive field A^H is

$$S_{A^H} = -\int \Big[\frac{1}{g} A^H \wedge *A^H + \frac{M^2}{M^4 + M_T^4} \left(dA^H - \frac{1}{2} J_2 \right) \wedge * \left(dA^H - \frac{1}{2} J_2 \right)$$
$$- \frac{M_T^2}{M^4 + M_T^4} \left(dA^H - \frac{1}{2} J_2 \right) \wedge \left(dA^H - \frac{1}{2} J_2 \right) \Big]. \quad \text{(D.23)}$$

As promised this is the action for a massive one-form A^H.

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