Calabi-Yau Fourfold Compactifications in String Theory

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Chapter 1

Introduction

1.1 Why string theory?

At the end of the 20th century theoretical physics is in a peculiar situation. Its two corner stones, the standard model of particle physics and the general theory of relativity, are capable to explain almost every experimental piece of data from particle and astrophysics with an impressive accuracy. Nevertheless the physical picture drawn by the standard model and general relativity is not complete. Thus their great success is turned into the main obstacle for progress in theoretical physics today. Extensions of the standard physical picture have to proceed without a clear experimental guidance. However one can make a virtue of necessity because the excellent agreement of the theory with current experiments puts severe constraints on any attempt to go beyond the standard model.

There are several good reasons to believe that the standard model of particle physics (SM) and the general theory of relativity (GR) have to be extended. Maybe the most severe one is the fact, that both theories neglect the lessons from the respective other one. GR is a classical theory neglecting all quantum aspects of matter as described by the SM. A look at Einstein's equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$ immediately shows that a quantum mechanical treatment of the matter part in $T_{\mu\nu}$ actually requires to treat also space and time quantum mechanically. Such a quantum mechanical treatment of gravity might also solve the problem of singularities encountered quite generically in solutions of GR, see e.g. [1]. Several different suggestions have been made for a quantization of gravity which are reviewed in [2]. None of them is yet completely satisfactory. The covariant approach for example tries to describe quantized gravity as a quantum field theory of metric fluctuations around a given background metric. This leads however to a non-renormalizable quantum field theory.

The other side of the coin is that the SM is formulated in flat Minkowski space ignoring all gravitational effects. Although this is perfectly justified in earth based experiments there have to be expected new effects in 'extreme' astrophysical situations as encountered e.g. in the early universe or near a black hole. A first impression of these new effects can be gained through the semi-classical methods of quantum field theory in curved space [3]. In this approach the quantum fields are defined on a curved background space-time which is however taken to be still a solution of the classical equations of GR. One effect predicted via this method is the Hawking radiation of a black hole.

A further indication for the incompleteness of the current physical picture is its high degree of arbitrariness. It has more than 20 free parameters whose values have to be determined by experiment and can not be extracted from the theory itself. As if this was not unsatisfactory enough it turns out that the parameters have to be balanced very accurately in order to lead to a universe containing galaxies, stars and biological life. The arguments supporting the specialty of the actual values of the parameters are nicely reviewed in [4]. If one is not content to blame this to a fortuitous choice of initial conditions one has to find a more fundamental theory which allows to tackle the question why the parameters have their actual values. It should also shed light on other characteristics of the SM which sofar remain unexplained, e.g. the particle spectrum, the gauge group and the fact that space-time appears to be four dimensional.

Several ideas have been proposed to get a handle on the arbitrariness of the SM. The most prominent ones are *grand unification* and *Kaluza-Klein theories*. The idea of grand unification is the embedding of the SM gauge group into a simple gauge group like $SU(5)$, $SO(10)$ or E_6. According to this scenario there is only one gauge group at some high energy scale, which can roughly be estimated to be of the order $m_{GUT} \sim \mathcal{O}(10^{16}\text{GeV})$. Through a generalized Higgs mechanism this gauge group is reduced to the $SU(3) \times SU(2) \times U(1)$ of the SM at this scale. Besides the reduction of the number of gauge couplings grand unified theories achieve a prediction of the electroweak mixing angle and many models also establish a relation between the masses of the b quark and the τ lepton, see [5] for an introduction. Another desirable effect of grand unification is that in many models the gauge coupling of the unified gauge group is asymptotically free, thus removing the possible inconsistencies related to the non-asymptotically free electroweak gauge couplings [6]. It turns out that the idea of grand unification is not consistent with the particle spectrum of the SM. However there is an extension of the SM in which a unification of all three couplings at high energies is in fact possible. This is the supersymmetric SM which is reviewed in [7]. It has $N = 1$ supersymmetry because extended supersymmetries can not accommodate the chiral structure of the SM. As supersymmetry is not observed in nature it must be broken if it is to play any role at all. This leads to a mass split between the bosonic and fermionic partners of the order of the supersymmetry breaking scale. If this scale is roughly the same as the electroweak scale, i.e. $m_{SUSY} \sim 100$ GeV, the supersymmetric partners of the SM particles could be heavy enough to escape detection in accelerator experiments so far. In addition such a low supersymmetry breaking scale could save the grand unification idea. Due to the additional light particles above m_{SUSY}, the renormalization group equations indeed indicate a unification of all three couplings at a scale $\sim 10^{16}$GeV within the minimal supersymmetric extension of the SM, see e.g. [5]. One should bear in mind however that the derivation of this result relies on the assumption that the supersymmetric SM remains a valid description of particle physics up to energies of order $\mathcal{O}(10^{16}\text{GeV})$.

Supersymmetry might also solve another problem of the SM, the *hierarchy problem*. If the SM was valid up to an energy scale Λ one would expect the Higgs mass to get quantum corrections proportional to Λ^2. Via the Higgs mechanism all masses of the SM particles are proportional to the Higgs mass. In order to explain why they are of the order of the electroweak scale there are basically two possibilities; either Λ is much larger than the electroweak scale and the bare value of the Higgs mass is fine tuned in an unnatural way to cancel the quantum corrections except for some 100 GeV, or Λ is of the same order as the electroweak scale. In this case the SM has to be replaced by another theory above the energy scale Λ. However in order not to run into the same fine tuning problem as before the Higgs mass should not get radiative corrections quadratic in the cutoff Λ' of the new theory.[1] This is possible if the successor theory is supersymmetric. Then the quantum corrections of the Higgs mass depend only logarithmically on Λ' which could be as large as the Planck mass $m_{Planck} \sim 10^{19}$GeV without leading to a fine tuning problem.

Another attempt to get a deeper understanding of the structure of the SM is made by Kaluza-Klein theories [8, 9]. They assume that space-time has more than four dimensions. In order to make contact with the real world only four of them should be large enough to be observable at the low energies accessible at today's accelerators. The attraction of Kaluza-Klein theories originates in the following fact. If the internal space has an isometry group G the low energy effective theory for the physics in $D = 4$ describes gravity coupled to $\tilde{G}$-gauge fields, for a subgroup $\tilde{G} \subset G$, even if the only force present in the higher dimensional theory is gravity.[2] This allows for an interpretation of Maxwell/Yang-Mills theories as arising from higher dimensional gravity theories. Additionally the value of the coupling constant is related to the size of the internal space and thus gets a 'deeper' geometrical origin. Unfortunately it has not been possible to extract the SM coupled to gravity from a higher dimensional gravity theory coupled to matter fields. Nevertheless the Kaluza-Klein idea has been revived in the context of string theory although with a slightly different motivation. For more details on Kaluza-Klein reduction see appendix C.

Most of the different approaches to extend the standard physical picture described so far merge naturally in string theory. Its basic step forward is the abandonment of the framework of local quantum field theory. Point particles are replaced by one dimensional objects - the strings. At low energies they appear point-like and the extended structure only becomes apparent at the string scale which is usually supposed to be close to the Planck scale. All different particles of the SM are interpreted as different vibrational modes of a string. Interestingly enough there is one mode which has the characteristics of a graviton and therefore string theory automatically 'contains' gravity. Upon quantization it gives a consistent perturbative quantum theory of gravity and thus overcomes the non-renormalizability encountered in the quantum field theory approach to quantum gravity. The extended nature of strings 'smears' out the location of interactions in a way that removes the ultraviolet divergences of field theory. Although this is a great improvement string theory has not yet fully solved the problem of quantizing gravity. It considers strings moving in a given background space-time. The gravitons in the string spectrum describe small fluctuations of this fixed background and string theory thus only provides a consistent perturbation theory of fluctuations around a given space-time. Nevertheless it might well be that this drawback is just due to the present formulation of the theory and its ability to describe quantum gravity at least perturbatively is a major motivation to take string theory seriously.

A relativistic quantum theory of one dimensional objects is strongly constrained by consistency requirements. It turns out that the strings have to move in a ten dimensional space-time. This immediately brings the Kaluza-Klein idea back into the game. Six of the ten dimensions have to be compactified in order to make contact to

[1] In this argument we assume of course that the Higgs boson of the SM is elementary and not a condensate of strongly interacting fermions as proposed in the *Technicolor* theories. It is however difficult to fit such theories to the experimental data taken at LEP.

[2] The question which subgroup $\tilde{G} \subset G$ can be obtained is in general a difficult question and can only be answered in a case by case analysis. For more details see appendix C.

our world. Furthermore there are only five consistent string theories possible at all which should be compared to the infinitely many consistent quantum field theories. We will see later that even these five string theories are not independent of each other but they seem to be just special limits of a single theory called *M-theory*. This high degree of uniqueness of the theory (also concerning the ten dimensional spectrum and gauge group) is spoiled however by the requirement for choosing a background around which to expand in the Kaluza-Klein reduction. Even restriction to space-times with four large dimensions leaves many possibilities, each one leading to different physics in the $D = 4$ effective theory. We have already mentioned that the gauge group of the lower dimensional theory depends on the internal space and as explained in appendix C the same holds for the spectrum. Moreover the physics described by the five consistent string theories is supersymmetric. However supersymmetry can be spontaneously broken by the background as we will explain below. Thus by a suitable choice of the compactification manifold the physics in $D = 4$ will be non-supersymmetric.

It has not been possible yet to find a background whose $D = 4$ effective theory is exactly the SM or its supersymmetric extension. Nevertheless many attractive features can be obtained, such as chiral gauge couplings, grand unified gauge groups like E_6 and three particle families. However even if one finds a background exactly reproducing the SM the question remains for a dynamical mechanism explaining why nature chose exactly this background. An answer to this question is most probably outside the scope of the present formulation of string theory.

We have now reviewed the reasons for considering string theory and briefly sketched some of its features and open problems but in the end any physical theory has to be judged according to its agreement with experiments. Some of its characteristics might be in the reach of future particle physics experiments. This is especially true for supersymmetry. There are theoretical and experimental indications that the lightest supersymmetric particle might well be discovered at the Large Hadron Collider (LHC) [5] whose start is planned for 2005. The discovery of extra dimensions at the LHC or other planned accelerators is less probable and depends on the size of the additional dimensions. Possible effects at collider experiments are reviewed in [10]. Extra dimensions should also manifest themselves in deviations from the $1/r^2$ Newton law [11]. However all these experiments only test aspects of the theory which are not intrinsically tied to string theory. The discovery of supersymmetry and extra dimensions would not be a proof for the existence of strings. If the string scale is indeed close to the Planck scale there seems to be only little hope to verify string theory itself in earth based experiments.[3] This implies that astrophysical observations might become more and more important to probe physics at very high energies. It has been an amazing insight of the last years that probably only 5 to 10 per cent of the energy in our universe consist of particles described by the SM (mainly baryons), see [13] for a review. The rest comprises the cosmological constant and some dark matter whose composition is yet unknown. One possible candidate is the lightest supersymmetric particle [14]. Another striking observation has been the discovery of ultra-high-energy cosmic rays with energies above 10^{20} GeV. They might originate in the decay of superheavy relic particles for which string theory offers some candidates, see [14] for an overview. A further astrophysical hint for physics beyond the SM comes from recent observations of the spectra of distant quasars which seem to indicate a time dependence of the fine

[3] It has however been realized recently that the string scale in one of the five consistent string theories might be much lower than the Planck scale, see [12] for a review.

structure constant [15]. Although this is again not a compulsory proof of string theory it is interesting in view of the fact that in string theory the coupling 'constants' are replaced by the expectation values of scalar fields and are thus allowed to vary in time. Finally astronomy is just at the verge of exploring completely new observational methods like neutrino and gravitational wave experiments which have the potential to give insight into those early stages of the universe which can not be probed by observations of the cosmic microwave background. Interestingly it could well be that supersymmetry gets unexpected support from the Laser Interferometric Space Antenna (LISA) planned for 2010 which is designed to observe the gravitational waves produced during the electroweak phase transition whose energy density is expected to be much higher in supersymmetric versions of the SM [16].

1.2 What is string theory?

In this section we want to give a more explicit introduction into string theory. Of course it has to be very sketchy and the interested reader is referred to the literature for further details [17–19].

We start by considering a string moving in a D-dimensional Minkowski space-time M_D with coordinates X^M. It can be described by the embedding of the string world-sheet into space-time, i.e. by a map from a two-dimensional surface Σ into M_D, $X^M(\sigma^1, \sigma^2) : \Sigma \to M_D$, where σ^α are the coordinates on Σ. In principle Σ is allowed to have boundaries apart from the incoming and outgoing strings. This leads to the description of open strings. Furthermore Σ can be oriented or unoriented. For simplicity we concentrate on oriented closed strings here and consider only surfaces Σ without boundaries, see figure 1.1 for the case of one incoming and one outgoing string. In

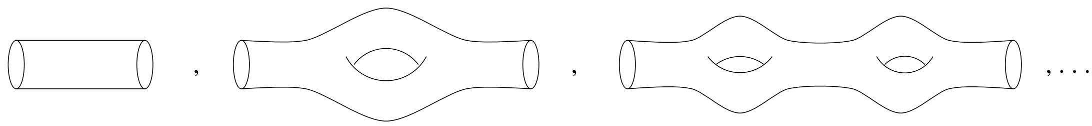

Figure 1.1: String world-sheets

analogy to the point particle the action determining the classical equations of motion for the string is taken to be proportional to the area of the world-sheet, i.e.

$$S_{NG} = \frac{1}{2\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{\det\left(\partial_\alpha X^M \partial_\beta X_M\right)} \ . \tag{1.1}$$

This is known as the *Nambu-Goto action*. The quantity $T = \frac{1}{2\pi\alpha'}$ is the string tension which determines the size of the string. The limit $\alpha' \to 0$ is the point particle limit in which the string contracts to a single point. The string scale alluded to in the last section is defined as $m_s = \alpha'^{-1/2}$.

The action (1.1) is classically equivalent to the *Polyakov action*

$$S_P = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{\gamma}\gamma^{\alpha\beta} \partial_\alpha X^M \partial_\beta X_M \ , \tag{1.2}$$

where $\gamma_{\alpha\beta}$ is the world-sheet metric. The Polyakov action has the advantage of being polynomial in the derivatives $\partial_\alpha X^M$. Therefore it is usually taken as the starting point in defining the quantum theory. A rigorous proof that both (1.1) and (1.2) lead to the same quantum theory is missing but it is generally believed to be the case.

The Polyakov action has the following symmetries: (a) D-dimensional Poincaré invariance, (b) invariance under diffeomorphisms of the world-sheet and (c) two-dimensional Weyl invariance, i.e. it is invariant under $\gamma'_{\alpha\beta}(\sigma) = \exp(\omega(\sigma))\gamma_{\alpha\beta}(\sigma)$ for an arbitrary function $\omega(\sigma)$. Weyl invariance plays a crucial role in string theory, because it is generically anomalous under quantization. Demanding Weyl invariance also in the quantum theory in order to avoid violation of unitarity imposes severe constraints on the theory.

The transition to the quantum theory proceeds through first quantization, i.e. the coordinates X^M are promoted to operators. As they depend on the world-sheet coordinates σ^α they can be considered as quantum fields in the two-dimensional 'space-time' Σ.[4] As in usual field theory there are several ways of quantization. We do not want to go into the details here and just focus on the results. The spectrum of the quantum theory consists of the vibrational excitations of the string. It turns out that a string theory based on the action (1.2) alone has some unwanted features. Its spectrum contains a state with negative mass (the tachyon) but no states that transform as spinors under the D-dimensional Lorentz group, which could be interpreted as space-time fermions. To get rid of these nuisances one has to introduce fermionic degrees of freedom on the world-sheet (i.e. fields that transform as spinors under the two-dimensional Lorentz group, but actually like X^M as vectors under the D-dimensional Lorentz group) and extend (1.2) in a supersymmetric way. The conformal invariance of (1.2) is thus enhanced to an $N = 1$ superconformal invariance. Demanding that this invariance is maintained after quantization, i.e. that there is no Weyl anomaly in the quantum theory, constrains the dimension of the space-time uniquely to be $D = 10$ which we will assume from now on. The spectrum of the superconformal field theory can no longer be completely visualized as vibrational modes of the string moving in ten-dimensional Minkowski space because of the additional fermionic degrees of freedom. Nevertheless all states of the theory carry quantum numbers of the ten-dimensional Lorentz group and can be interpreted as 'particles' in space-time. Furthermore to eliminate the tachyon one has to perform a suitable truncation of the spectrum known as the *GSO projection*. The remaining states consist of a set of massless particles and an infinite tower of massive excitations whose masses are quantized in units of the string scale $\alpha'^{-1/2}$. As one usually assumes this to be of the order of the Planck mass these states are extremely heavy.

A very intriguing feature of string theory is that scattering amplitudes of the ten-dimensional particles can be calculated by correlation functions in the two-dimensional superconformal field theory. In order to evaluate the full amplitude one has to sum over all world-sheet topologies with the same incoming and outgoing string states. This is analogous to the loop expansion in quantum field theory. The first two terms of the expansion in topologies for a four-point amplitude are shown in figure 1.2. The incoming and outgoing strings should actually be extended to infinity, so that the pictures in figure 1.2 are conformally equivalent to a sphere and a torus with point like insertions of the external string states. In two-dimensional conformal field theories there is a one-to-one correspondence between states and operators. To every state there is a so called *vertex operator* which creates the state by acting on the vacuum.

[4]In writing (1.2) we have used a Euclidian signature for $\gamma_{\alpha\beta}$. It can be verified that this leads to the same quantum theory as a Minkowskian signature.

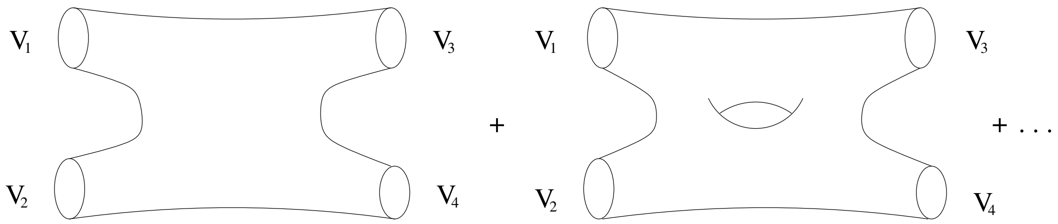

Figure 1.2: Four-point amplitude

An n-point amplitude is then schematically given by

$$\mathcal{A}(V_1,\ldots,V_n) = \sum_{\text{topologies}} \int \frac{\mathcal{D}X\mathcal{D}\gamma\mathcal{D}\psi}{\text{Vol(gauge)}} e^{-S} \prod_{i=1}^{n} \int_{\Sigma} d^2\sigma_i \sqrt{\gamma(\sigma_i)}\; V_i(\sigma_i)\ , \tag{1.3}$$

where the $V_i(\sigma_i)$ are the vertex operators corresponding to the string states whose scattering amplitude one wants to calculate. They are inserted into the world-sheet Σ at σ_i. Furthermore the sum runs over all topologically different Riemann surfaces Σ, S is the action of the superconformal field theory, ψ denotes collectively all its fermionic degrees of freedom and the measure has to be divided by the volume of the world-sheet gauge group.

There are several possibilities to introduce world-sheet fermions in (1.2) and to perform the GSO projection. It turns out however that basically two further consistency restrictions eliminate most of them. First one has to demand *modular invariance*. The one loop partition function of the string theory, i.e. the amplitude (1.3) with zero external string states and Σ taken to be the torus, has to be invariant under the large diffeomorphisms of the torus which can not continuously be deformed to the identity, see section B.8. The second requirement has to do with the fact that the spectrum of a string theory can contain non-Abelian gauge potentials and one has to demand that there are no anomalies in the corresponding gauge symmetries. Taken together the restrictions are strong enough that there are only five consistent string theories in $D = 10$ Minkowski space-time. Their massless bosonic spectra and gauge groups are collected in table 1.1. One sees that every string theory contains a graviton g_{MN} and a

Type	Massless bosonic spectrum	Gauge group G	N
Heterotic $E_8 \times E_8$	$g_{MN}, B_{MN}, \Phi, A^a_M$	$E_8 \times E_8$	1
Heterotic $SO(32)$	$g_{MN}, B_{MN}, \Phi, A^a_M$	$SO(32)$	1
I	$g_{MN}, \Phi, A^a_M, A_{MN}$	$SO(32)$	1
IIA	$g_{MN}, B_{MN}, \Phi, A_M, A_{MNP}$	$U(1)$	2
IIB	$g_{MN}, B_{MN}, \Phi, A, A_{MN}, A_{MNPQ}$	-	2

Table 1.1: The five consistent string theories in $D = 10$

scalar Φ called the *dilaton*. Furthermore all string theories except the type I are based on closed strings only and their spectrum includes an antisymmetric tensor gauge field B_{MN} which is called the *NS B-field*. Besides this 'universal' part of the spectrum each string theory has its individual massless bosonic excitations, consisting of non-Abelian gauge fields A^a_M, $a = 1,\ldots,\dim G$, or antisymmetric p-form gauge fields $A_{M_1\ldots M_p}$, the so called *RR p-forms*. Strings do not carry any charge of the RR p-form gauge fields.[5] However it has been one of the major recent discoveries that string theory contains actually further objects than just strings and these carry charge of the RR gauge fields. They are called *D-branes* or *Dp-branes*, where p denotes the number of their spatial dimensions. Which values for p occur depends on the string theory under consideration. We will have more to say about D-branes in section 1.4.

In addition to the bosonic part displayed in table 1.1 the spectrum of each of the five string theories contains fermions. Among the massless fermions there are one or two with spin 3/2 which can be interpreted as gravitinos. Their presence indicates that not only the auxiliary two-dimensional world-sheet theory is supersymmetric but also the resulting space-time physics.[6] The number of gravitinos in each string theory is shown in the last column of table 1.1.

We now want to leave the unphysical situation of a ten-dimensional Minkowski space and consider strings moving in phenomenologically more interesting space-times. Apart from the metric one can also give non-trivial background values to the dilaton and (in the heterotic and type II string) the NS B-field.[7] This leads to the following generalization of the Polyakov action

$$S_s = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{\gamma} \Big[\Big(\gamma^{\alpha\beta} g_{MN}(X) + i\epsilon^{\alpha\beta} B_{MN}(X) \Big) \partial_\alpha X^M \partial_\beta X^N + \alpha' R \Phi(X) \Big] \ , \quad (1.4)$$

which has the form of a non-linear sigma model. R is the two-dimensional curvature scalar and, as above, (1.4) has to be accompanied by appropriate fermionic terms to yield a viable string theory. The functions $g_{MN}(X), B_{MN}(X)$ and $\Phi(X)$ can be interpreted as the couplings of the two-dimensional field theory. The vanishing of their β-functions $\beta^g_{MN} = \beta^B_{MN} = \beta^\Phi = 0$ renders the sigma model Weyl invariant and one is tempted to interpret these conditions as equations of motion for the space-time fields g_{MN}, B_{MN} and Φ. The β-functions have a loop expansion in the two-dimensional field theory which is equivalent to an expansion in α'. In a target space with characteristic radius R_c and also $B_{MN}(X)$ and $\Phi(X)$ varying on the same scale R_c the effective dimensionless coupling of the world-sheet theory is $\sqrt{\alpha'}R_c^{-1}$. Terms with more than two derivatives in the β-functions are of higher order in the $\sqrt{\alpha'}R_c^{-1}$ expansion. Thus if $\sqrt{\alpha'}R_c^{-1} \ll 1$ perturbation theory in the two-dimensional theory is valid and it is possible to truncate the equations of motion at the two derivative level. This is known as the regime of *low energy effective field theory*. Furthermore in this limit it is allowed to neglect the heavy string modes and consider only the massless spectrum.

One possible choice of background is given by $B_{MN} = 0$ and $\Phi = \Phi_0 = \text{const}$. In this case the leading terms of the β-functions are $\beta^g_{MN} = \alpha' R_{MN} + \mathcal{O}(\alpha'^2)$, $\beta^B_{MN} = \mathcal{O}(\alpha'^2)$ and $\beta^\Phi_{MN} \sim (D-10) + \mathcal{O}(\alpha'^2)$. Thus the β-function for the metric indeed reproduces

[5] This distinguishes the antisymmetric 2-form of type I from the NS B-field.

[6] Supersymmetry of the world-sheet theory is a necessary but not a sufficient requirement to obtain supersymmetry in space-time. The so called *type 0A* and *type 0B* theories have supersymmetry on the world-sheet and fulfill all mentioned consistency requirements except the demand for a tachyon free spectrum. Nevertheless their spectrum contains no fermions at all and is therefore certainly not invariant under space-time supersymmetry.

[7] It is also straightforward to consider backgrounds for the vector fields in the Cartan subalgebra of the gauge group in the heterotic theories, which we will not do here though. Backgrounds for the RR p-form fields are however hard to describe in the framework we discussed so far. There is an alternative (and equivalent) formulation of string theory, the so called *Green-Schwarz formulation*, in which such backgrounds can be included as well [20].

to lowest order the vacuum Einstein equations whereas the dilaton β-function gives the constraint on the space-time dimensionality as in Minkowski space. However string theory also predicts corrections to Einstein gravity coming from higher derivative terms involving higher powers of the Riemann tensor. They are due to the heavy string excitations and will play a crucial role in section 4.

There is an alternative way to derive equations of motion for the massless space-time fields. One can calculate their n-point functions using (1.3) now with S the supersymmetric version of (1.4). One then determines the effective space-time action by demanding that its classical scattering amplitudes should reproduce these n-point functions and uses this effective action to derive the equations of motion. It turns out that the leading terms in an α'-expansion, the low energy effective theories, describe ten-dimensional supergravities, either type I supergravity in case of the heterotic and type I string theory or type IIA/B supergravity in case of type IIA/B string theory. Both approaches have been used to calculate a correction to the Einstein-Hilbert action $\sim \alpha'^3 R^4$, i.e. proportional to a contraction of four space-time Riemann-tensors. In [21] this has been achieved by deriving the metric β-function to four-loop order and in [22] by calculating the scattering of four gravitons at tree-level (i.e. $\Sigma = S^2$ in (1.4)). It has been shown in [23] that both results indeed agree.

We now come to a subtle point. The β-functions of the sigma model (1.4) are completely determined by the short distance physics on the world-sheet. Thus they should not depend on the topology of the world-sheet. If $\beta = 0$ was the whole story one would not expect any contributions to the equations of motion from higher genus Riemann surfaces in the scattering amplitude approach. This turns out to be incorrect as has been verified in an explicit calculation of the one-loop scattering amplitude of four gravitons in [24] leading to a correction of the coefficient of the term $\sim R^4$ mentioned in the last paragraph. In fact the integration over the world-sheet metric γ in (1.3) is another source for a Weyl anomaly of the string theory. In order to maintain Weyl invariance it is therefore not enough to demand that the two-dimensional sigma model is Weyl invariant for a fixed γ and the conditions of vanishing β-functions get loop corrections.[8] This is known as the *Fischler-Susskind mechanism* and is reviewed in chapter 9 of [19].

At this point one comment is in order. If the dilaton background is constant, $\Phi = \Phi_0$, the last term in (1.4) is just $\Phi_0 \chi$, where χ is the Euler number of the Riemann surface Σ, see section B.1. This is related to the genus g of Σ via $\chi = 2 - 2g$. A constant background value for the dilaton leads to a relative weighting of the different topologies in (1.3) with $e^{-\Phi_0 \chi}$. Thus $e^{2\Phi_0}$ is the loop counting parameter and as every additional loop entails two additional string splitting- respectively joining-vertices, e^{Φ_0} is proportional to the closed string coupling constant. We thus see that the loop corrections to the effective equations of motion are negligible if e^{Φ_0} is small. However a priori there is no reason why this should be the case. The question of what determines the value of the string coupling constant has not been settled so far and might require new insights into the nature of non-perturbative string theory.

[8]The leading terms of the effective action with up to two derivatives are however not corrected by loop effects.

1.3 String theory on Calabi-Yau manifolds

From a phenomenological standpoint the most interesting choice for the metric in (1.4) has a product structure[9]

$$g_{MN} = \begin{pmatrix} \eta^{(4)}_{\mu\nu} & 0 \\ 0 & g^{(6)}_{ab} \end{pmatrix} , \tag{1.5}$$

where $\eta^{(4)}_{\mu\nu}$ is the four-dimensional Minkowski metric and $g^{(6)}_{ab}$ is the metric of a six-dimensional internal manifold which should be small enough not to come into conflict with our world looking four-dimensional at low energies. Furthermore in order to maintain four-dimensional Lorentz invariance one should also choose $B_{\mu\nu} = B_{\mu a} = 0$ and $g^{(6)}_{ab}$, B_{ab} and Φ should only depend on the internal coordinates. Accompanied by a similar split of the fermions the non-linear sigma model thus divides into an internal and an external superconformal field theory. This allows for a generalization. One can replace the internal part of the non-linear sigma model by an arbitrary unitary superconformal field theory which fulfills the requirement of rendering the full theory Weyl invariant. It is a very interesting question how to give a geometrical interpretation in this case. We do not go into the details here but refer to [25] for an overview.

In case the internal manifold is small enough all string excitations can be interpreted as four-dimensional particles. If for example the only excited modes of the string come from the internal conformal field theory the corresponding particle will look like a scalar from the four-dimensional point of view but also vector, tensor and spinor fields arise. Again there is a set of massless modes and an infinite tower of massive ones. These comprise now also massive Kaluza-Klein modes. In principle one could derive the four-dimensional effective action governing the massless particles by the scattering amplitude approach, using in (1.4) the ansatz (1.5) for the metric and B_{ab} and Φ as specified in the last paragraph. This is however realizable only in simple cases. An easier way is to use the ten-dimensional effective action and perform a Kaluza-Klein reduction around this background as described in more detail in appendix C.

Regarding our discussion of the possible phenomenological importance of supersymmetry in the low energy four-dimensional physics it is natural to ask what properties the internal metric has to have in order to lead to a supersymmetric effective action in $D = 4$. We first answer this question in the case $H_{abc} = 0$, where H is the field strength of the antisymmetric tensor. Furthermore we assume that we are in the large radius limit, i.e. $\sqrt{\alpha'}R_c^{-1} \ll 1$. Thus the low energy physics is well approximated by either type I or type IIA/B supergravity. In field theory the requirement for an unbroken supersymmetry in a given background is (at the classical level) that the variations of the elementary fermion fields under supersymmetry transformations vanish. It turns out that two necessary conditions are a constant dilaton and the presence of a covariantly constant spinor ζ on the internal manifold. This last requirement has an important consequence. It implies

$$[\nabla_m, \nabla_n]\zeta = \frac{1}{4} R_{mnpq}\Gamma^{pq}\zeta = 0 \; . \tag{1.6}$$

This is only possible if the holonomy group of the internal space is a subgroup of $SU(3)$. As reviewed in section B.5 this means that the internal metric must be Kähler

[9] However, in view of the recent developments brane world scenarios provide another promising starting point for phenomenology.

and Ricci-flat, i.e. the internal manifold has to be a *Calabi-Yau* manifold.[10] We restrict our discussion to the case that the holonomy group is exactly $SU(3)$ and not a subgroup thereof. Then the four-dimensional low energy theory has minimal $N = 1$ supersymmetry in the heterotic and type I theories and extended $N = 2$ supersymmetry in the type II theories.[11]

Actually for the heterotic and type I string there are further conditions coming from the variations of the gauginos. These conditions constrain possible backgrounds for the vector fields. They are most easily expressed after introducing complex coordinates on the Calabi-Yau manifold and read $F_{ij} = F_{\bar{\imath}\bar{\jmath}} = 0$ and $g^{i\bar{\jmath}} F_{i\bar{\jmath}} = 0$. The first two conditions mean that the gauge field background has to be a holomorphic connection of a holomorphic gauge bundle. The allowed gauge bundles are further restricted by the second condition to the subclass of the so called *stable holomorphic vector bundles*.[12] One might wonder if it is possible to just set $F = 0$ and fulfill these conditions trivially. This is however not the case. The reason is that the 'field strength' H, whose background we set to zero, is actually not given merely by the exterior derivative of the antisymmetric tensor in the case of heterotic and type I string theory. The cancellation of gauge and gravitational anomalies requires a modification by Chern-Simons terms which implies

$$dH = \frac{\alpha'}{4}\Big[\mathrm{tr}(R \wedge R) - \mathrm{tr}(F \wedge F)\Big] \ , \tag{1.7}$$

where $\mathrm{tr}(R \wedge R)$ and $\mathrm{tr}(F \wedge F)$ are the second Chern classes of the tangent respectively gauge bundle. For Calabi-Yau manifolds with exact $SU(3)$ holonomy the second Chern class is non-trivial. Thus also the gauge field has to be non-trivial in order to satisfy (1.7). In the case $H = 0$ there is only one possibility known; the spin and the gauge connection basically have to be identified which is called *embedding the spin connection into the gauge connection*. Thus there is a non-trivial $SU(3)$ gauge field in the background leaving only $E_6 \times E_8$ respectively $SO(26)$ as the gauge groups in $D = 4$.[13]

Let us now have a closer look at the massless spectrum in four dimensions. We concentrate on the bosonic part only because the fermionic part is then fixed by supersymmetry. As explained in appendix C it can be derived by expanding the ten-dimensional bosonic fields in terms of the harmonic forms of the internal manifold and is therefore determined by its topology. A graviton, a dilaton and an antisymmetric tensor are always present in the four-dimensional spectrum arising from the corresponding ten-dimensional fields, see table 1.1. However the reduction of the metric and the antisymmetric tensor yield more massless scalars. The supersymmetry conditions do not impose any restrictions on the complex structure or the Kähler class of the Calabi-Yau metric. This implies that any choice is equally good in (1.5) and one could instead let these geometrical moduli of the internal space vary over four-dimensional space-time.

[10] For a mathematical introduction to Calabi-Yau manifolds see appendix B and the references given there.

[11] In view of the above mentioned generalization let us just note, that a general internal superconformal field theory has to have an extended superconformal symmetry in order to lead to space-time supersymmetry in $D = 4$, i.e. it must be a so called $(0, 2)$-theory in the heterotic case and a $(2, 2)$-theory in the type II case.

[12] See [26] for the mathematical background.

[13] From a phenomenological point of view the case of an unbroken $SO(26)$ is not very appealing as it turns out that all charged matter in the $D = 4$ effective theory transforms in real representations of $SO(26)$. On the other hand E_6 is one of the grand unified gauge groups and the $E_6 \times E_8$ heterotic theory has been intensively studied in the context of 'string phenomenology'.

They are thus promoted to moduli fields in the massless spectrum of the effective field theory. These purely geometrical moduli arising from the parameters of the Ricci-flat internal metric are accompanied in string theory with further moduli fields emerging from the Kaluza-Klein reduction of the other ten-dimensional fields. The antisymmetric tensor for example leads to as many massless scalars in the four-dimensional theory as there are Kähler moduli. Supersymmetry requires to combine these two sets of real scalars into one set of complex scalars, the so called *complexified Kähler moduli*. The part of the massless spectrum we have described up to now is universal and appears in the Calabi-Yau reduction of any of the five string theories: A graviton $g_{\mu\nu}$, a dilaton ϕ, an antisymmetric tensor $B_{\mu\nu}$, $h^{2,1}$ complex scalars describing the complex structure deformations of the internal space and $h^{1,1}$ complex Kähler moduli, where $h^{2,1}$ and $h^{1,1}$ are the two non-trivial Hodge numbers of the Calabi-Yau manifold, see appendix B. Furthermore the antisymmetric tensor can be traded for a scalar a.[14]

The remainder of the massless four-dimensional spectrum depends on the string theory with which one starts. To simplify the discussion we shall concentrate here on the three cases which are most relevant for the rest of the thesis: the type II theories and the heterotic $E_8 \times E_8$ theory.

In the four-dimensional type II theories additional massless fields arise from the reduction of the RR p-forms. For the details we refer to the literature and only give the result here, see e.g. [19]. Supersymmetry requires the whole spectrum to fit into $N = 2$ supersymmetry multiplets. The relevant massless multiplets and their bosonic content are the

1. supergravity multiplet: 1 graviton and 1 vector,
2. vector multiplet: 1 vector and 1 complex scalar,
3. hypermultiplet: 4 real scalars.

It turns out that in the type IIA theory the reduction of the RR fields leads to $h^{1,1} + 1$ vectors, which combine with the graviton and the complex Kähler moduli into one supergravity and $h^{1,1}$ vector multiplets, and to $h^{2,1} + 1$ complex scalars, which combine with the dilaton, the axion a and the complex structure moduli into $h^{2,1} + 1$ hypermultiplets. In the type IIB theory the situation is somehow reverse in that the Kähler moduli sit in $h^{1,1}$ hypermultiplets and the complex structure moduli are the complex scalars of $h^{2,1}$ vector multiplets. This will play an important role momentarily when we discuss mirror symmetry. Again there is one further hypermultiplet comprising the dilaton and axion and the supergravity multiplet.

All the massless scalars are indeed moduli of the string compactification because they do not have a potential in the effective theory. Thus one could give them arbitrary constant background values. The set of all possible background values is known as the moduli space of the theory. In supergravity theories the moduli space can not be arbitrary. In particular $N = 2$ supergravity in $D = 4$ restricts the moduli space to be the product of a special Kähler manifold spanned by the vector multiplet scalars and a quaternionic Kähler manifold for the scalars of the hypermultiplets, see [27] for a review. Both definitions can be found in appendix B.

The presence of the massless scalars in the spectrum is in conflict with experiment because so far no massless scalars have been observed in nature. In order to make

[14] This duality transformation in $D = 4$ is always possible when the effective action only depends on the field strength of $B_{\mu\nu}$. Moreover the dual scalar is an axion as it has a *Peccei-Quinn shift symmetry*, i.e. the dual action is invariant under $a \to a + \text{const}$.

any sense the Calabi-Yau scenario has to be extended by a mechanism which generates a potential for the scalars to render them massive. A possible way to achieve this is discussed in chapter 4.

We now briefly turn to the massless spectrum of the heterotic $E_8 \times E_8$ in $D = 4$. Here the additional fields originate in the dimensional reduction of the ten-dimensional gauge bosons. Again we refer to [19] for the details. As in the case of type II compactifications the spectrum has to fit into multiplets, this time of $N = 1$ supersymmetry. The relevant massless multiplets are the supergravity multiplet, whose bosonic content is just the graviton, the vector multiplet comprising a vector and the chiral multiplet with a complex scalar as bosonic content. We have already seen that the remaining gauge group in $D = 4$ is $E_6 \times E_8$. E_6 has a 27-dimensional complex representation which we denote by $\mathbf{27}$. It turns out that besides gauge bosons in the adjoint representation of $E_6 \times E_8$ the spectrum contains $h^{2,1}$ chiral multiplets in the $\mathbf{27}$ and $h^{1,1}$ chiral multiplets in the $\overline{\mathbf{27}}$.[15] Moreover there are some further scalars neutral under the gauge group whose number is not given by a Hodge number. It is however determined by the dimension of an appropriate cohomology group, see [17] for details. In many cases these scalars are moduli of the string compactification. Giving them a non-vanishing vacuum expectation value amounts to deforming the gauge bundle of the background gauge fields. The same can be achieved by vacuum expectation values for certain of the charged fields. In this way even the rank of the gauge group can be reduced. According to the supersymmetry conditions and the Bianchi identity (1.7) the background gauge fields can in general be chosen as holomorphic connections in any stable holomorphic vector bundle whose second Chern class equals the second Chern class of the Calabi-Yau manifold.[16] If one departs from embedding the spin connection into the gauge connection the effective four-dimensional gauge group (and the charged matter spectrum) is generically different. Furthermore the choice of vanishing H is in general no longer possible. Aspects of non-vanishing H are discussed in [17, 30].

To summarize there are the following moduli in the heterotic theory sitting in chiral multiplets: $h^{2,1}$ complex structure moduli, $h^{1,1}$ complexified Kähler moduli, the complex dilaton consisting of the dilaton and the axion and a number of gauge bundle moduli. Altogether they span a moduli space which is constrained by supersymmetry to be a Kähler manifold.

Before we turn in the next section to some more recent developments which have elucidated the relationship between the single string theories, let us mention one subtle point of Calabi-Yau compactifications. In view of the importance of higher derivative terms for our discussion in chapter 4 we briefly want to see how they modify the discussion of this section. We have seen in the last section that there is, among others, a correction to the Einstein-Hilbert action $\sim \alpha'^3 R^4$ arising at string tree- and loop-level. It modifies the Einstein equation such that Ricci-flat metrics are no longer solutions. It has been argued in [31] that it is always possible to modify the Ricci-flat metric on a Calabi-Yau manifold in a way that the new metric fulfils the exact equations of motion. It is still a Kähler metric whose Kähler form is in the same Kähler class as the Kähler form of the Ricci-flat metric. As the metric is no longer Ricci-flat its holonomy

[15] These states are neutral under the E_8 gauge bosons and contain the quarks and leptons in a grand unified scenario based on E_6.

[16] The moduli spaces of such vector bundles are generally very hard to analyze. Only partial results are known in the case that the manifold is elliptically fibered, i.e. it has a fibration structure with a torus as the fiber, see e.g. [28, 29] for details.

group is not contained in $SU(3)$ and there is no covariantly constant spinor anymore. However as reviewed in section B.6 there always exists a gauge covariantly constant spinor such that $(\nabla - \frac{i}{2}A)\zeta = 0$. As the obstruction to a Ricci-flat metric arises at order $\mathcal{O}(\alpha'^3)$ also A is of the same order. This modified constraint on ζ should arise from corrected supersymmetry transformations in order to be consistent with unbroken supersymmetry in $D = 4$ [32, 33]. Indeed the existence of higher derivative terms in the effective action requires correction terms to the supersymmetry transformations starting at order $\mathcal{O}(\alpha'^3)$. Thus it seems that starting with the Ricci-flat metric one can modify the solution order by order in α' maintaining a supersymmetric four-dimensional effective theory.

1.4 Dualities and M-theory

We have seen in section 1.2 that there are five consistent string theories in $D = 10$. In the last section we considered them in non-trivial backgrounds which were chosen by hand in order to make contact with our four-dimensional world. Up to now there is no satisfactory explanation why exactly four space-time dimensions are expanding and six remained small after the Big Bang. Other backgrounds with less or more extended dimensions are mathematically on equal footing. The only requirement is that these backgrounds have to obey the equations of motion of string theory. A given background leads to a distinctive effective theory governing the space-time physics in an expansion around this background as it determines the spectrum, the gauge group, the amount of supersymmetry and the form of the interactions. For a given string theory we can think of all possible backgrounds as constituting a huge moduli space. These backgrounds encompass not only the geometry in which the strings move but also nontrivial vacuum expectation values for the various fields of the spectrum including the dilaton. As we have discussed at the end of section 1.2 the latter is related to the coupling constant of the string theory.

By continuously changing the values of the moduli one can deform the background and move around in the moduli space. In one corner of the moduli space are the Calabi-Yau compactifications discussed in the last section. Here moving around in the moduli space could for example mean a change in the complex structure and the Kähler class of the Calabi-Yau manifold. A priori one might expect that this large moduli space has many connection components. For instance, it could be that topologically distinct Calabi-Yau spaces can not continuously be deformed into each other. This depends however on the underlying concept of continuity. As reviewed in [25] it turns out that the resulting effective space-time physics behaves totally smooth in certain topology changing processes. String theory smoothes out deformations which look singular from the point of view of classical geometry. It might therefore well be that the whole moduli space is connected in this sense.

The picture of string theory we have drawn so far looks as follows: There are five different string theories each endowed with its own moduli space of viable backgrounds. Although the dynamical mechanisms which choose the background still have to be unraveled the reduction to only five consistent theories is a great achievement in the attempt to reduce the arbitrariness of the SM. However this is not the whole story yet. Rather it turns out that the moduli spaces of all the ‘different’ string theories are actually identical. There seems to be only one huge moduli space of one underlying theory and the backgrounds of a specific string theory alluded to in the foregoing discussion are just different 'coordinates' on this single moduli space. This certainly requires some explanation. Again we refer to [19] for the details and restrict ourselves to the results.

It is known since some time now that the type IIA theory compactified on a circle of radius R describes exactly the same physics as the type IIB theory compactified on a circle of radius α'/R. Thus by varying the radius from infinity to zero one can continuously deform the type IIA to the type IIB theory. The same holds true for the two heterotic theories. This is a special case of the so called *T-duality* and establishes that indeed the heterotic respectively type II theories have the same moduli spaces.

This phenomenon that different backgrounds can nevertheless lead to the same physics is a typical property of string theory and has no analog in theories based on point particles. Another example of this astonishing feature is *mirror symmetry* which has also found far reaching applications in mathematics. It has been noticed that in many cases the compactification of type IIA theory on a Calabi-Yau manifold leads to exactly the same effective theory as the compactification of type IIB theory on a topologically different manifold, the mirror Calabi-Yau. In this equivalent picture the complex structure moduli fields describe the same massless excitations as the complexified Kähler moduli in the type IIA compactification and vice versa. Thus the complex structure and Kähler moduli change their role - an instance which has led to great progress in calculating certain quantities in the low energy effective action; for a review see [34].[17]

Now consider the type I theory at small string coupling and slowly increase the value of the coupling constant. We have already mentioned in section 1.2 the existence of Dp-branes in string theory. The type I theory contains a D1-brane, i.e. a second object with one spatial dimension besides the fundamental string. It is also called the *D-string*. One characteristic of D-branes is that their tension is inversely proportional to the string coupling constant. Thus in the limit of weak coupling they are very heavy and can not be produced at low energies. That is why the (fundamental) strings are the most relevant objects in string theory in this limit. However with increasing coupling constant the situation changes. Surprisingly the fundamental type I string starts to decay and plays no role anymore at very strong coupling. On the other hand the D-string becomes lighter and lighter and is the relevant object at strong coupling. Due to supersymmetry it is possible to extrapolate its excitation spectrum to the regime of strong coupling. It turns out to be identical to the spectrum of a weakly coupled heterotic $SO(32)$ string. It has therefore been conjectured that both theories describe the same physics. This proposal has not been proven rigorously but further arguments can be given in favor of it. Interestingly it relates the strong coupling limit of one theory to the weak coupling limit of another one. This is termed *S-duality*.

The discussion of the strongly coupled type I string has made two things obvious. First it has shown that some strong coupling effects of a given string theory might be within the reach of the current techniques. This is possible if one finds a dual description of the same physics in terms of a weakly coupled theory. For the type I and the heterotic $SO(32)$ theories we have found that they are S-dual to each other. The type IIB theory

[17] Let us mention that we have used the expression *Calabi-Yau manifold* to denote six dimensional manifolds so far. As explained in appendix B the definition of a Calabi-Yau manifold is more general. $2n$ dimensional Calabi-Yau spaces are called Calabi-Yau $n$folds and mirror symmetry also appears in compactifications on them. However, only if n is odd does it relate the type IIA to the type IIB theory. For n even it relates compactifications of the same type II theory on topologically different manifolds.

is weak/strong self-dual and for the type IIA and heterotic $E_8 \times E_8$ theory we will identify the strong coupling limit in a moment. Beforehand let us remark a second lesson from the strongly coupled type I theory. In our description of string theory in the previous sections we have implicitly assumed a small coupling constant. Here we see that this restriction is not only necessary because of the enormous difficulties arising in higher loop calculations but also because the whole approach using strings as the fundamental degrees of freedom is only valid in the weak coupling regime of the corresponding string theory. With increasing coupling other objects like the D-branes become light. It is not clear at the time of writing what really are the fundamental degrees of freedom of the theory at a generic point in moduli space. Only if our universe resides at a point in the moduli space at which one of the five string theories is weakly coupled it can be described by perturbative string theory in a way outlined in the foregoing sections.

We still have to motivate that also the moduli spaces for the type II and heterotic string theories are connected. This will also shed new light on their strong coupling behavior. The type IIA theory contains D0-branes, i.e. particle like objects. Again their mass is inversely proportional to the coupling constant g. A bound state of n D0-branes has a mass $\sim n/g$. This matches exactly the spectrum of a Kaluza-Klein compactification on a circle of radius $\sim g$, see appendix C. Thus one is tempted to interpret the strong coupling limit as a decompactification limit in which an eleventh dimension opens up. This is very surprising at first sight as we have argued in section 1.2 that string theory restricts space-time to be ten-dimensional. Again the solution to this puzzle lies in our wrong assumption that strings are the fundamental degrees of freedom of the theory. In fact if the scenario describing the type IIA string theory as the Kaluza-Klein reduction of an 11-dimensional theory is to make any sense this theory should have a low energy limit which reduces to the one of type IIA string theory, i.e. to IIA supergravity. There is only one candidate - 11-dimensional supergravity. Its bosonic spectrum consists of a metric and a three-form gauge potential, see section 2.2. The corresponding object carrying the charge of this gauge field has two spatial dimensions and is called a *membrane*. In order to reproduce the IIA strings in the Kaluza-Klein scenario one has to consider membranes which are wrapped around the circular dimension. Thus the IIA strings only appear one-dimensional at weak coupling. Increasing the value of the coupling constant opens up the eleventh dimension and the strings expand into two-dimensional tubes thus circumventing the restriction to ten dimensions arising from string theory.

A similar picture arises in the strong coupling limit of the heterotic $E_8 \times E_8$ theory. One can again argue that an eleventh dimension appears which this time has the form of an interval instead of a circle. Now the length of the interval is $\sim g$ and the string expands into a cylindrical membrane one of whose spatial dimensions is extended along the interval.

We see that both the weakly coupled type IIA and heterotic $E_8 \times E_8$ string theory are embedded in the moduli space of an 11-dimensional theory. This is the missing link to establish the statement made earlier in this section that there is only one underlying theory whose moduli space embraces those of all string theories, which is schematically shown in figure 1.3. This theory is called *M-theory*. At certain corners of the moduli space it looks effectively ten-dimensional and can be described by a weakly coupled string theory. Its low energy limit is given by 11-dimensional supergravity but its precise non-perturbative formulation still has to be found.

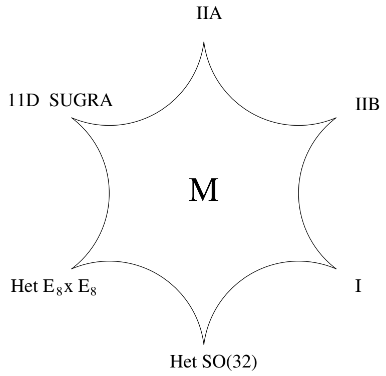

Figure 1.3: Moduli space of M-theory

1.5 Topic and organization of the thesis

After this tour de force through string theory we now come to the main concern of the thesis. We have already mentioned that the heterotic $E_8 \times E_8$ theory has attained a lot of interest in the search for a string theoretic version of the (supersymmetric) SM as its Calabi-Yau compactifications give rise to interesting gauge groups and minimal $N = 1$ supersymmetry in $D = 4$. However, despite considerable efforts a number of serious problems remain within the framework of the perturbative heterotic string. Among them are the missing of a satisfactory mechanism for supersymmetry breaking at low energy and for stabilizing the moduli such as the dilaton and the Kähler- and complex structure moduli in a Calabi-Yau compactification. Non-perturbative effects of the heterotic string are expected to play a crucial role in finding the solutions. Some of these non-perturbative features are captured by a construction known as *F-theory* compactified on elliptically fibered fourfolds [35].

There are several ways to define F-theory. The definition we will use is the following. An elliptically fibered fourfold is a torus fibration over a (complex) three dimensional base space B_3, which we denote by $Y_4 \xrightarrow{T^2} B_3$. F-theory compactified on Y_4 is then defined as the compactification of type IIB string theory on B_3, where the complex dilaton $A + ie^{-\Phi}$ varies over B_3 in the same way as the complex structure of the torus fiber varies in Y_4. However, in order to fulfill the equations of motion 7-branes have to be present in the type IIB background. It has been argued in [35] that in case Y_4 is not only elliptically fibered but in addition K3-fibered over a two-dimensional base B_2, i.e. $Y_4 \xrightarrow{\mathrm{K3}} B_2$, this F-theory compactification is a dual description of a compactification of the heterotic string on an elliptically fibered Calabi-Yau threefold $Y_3 \xrightarrow{T^2} B_2$ with the same base B_2.[18] These F-theory compactifications can include certain non-perturbative effects of the heterotic string. The four-dimensional spectrum can for example contain more antisymmetric tensors and a larger gauge group than possible in the perturbative framework. These and further aspects of F-theory on Calabi-Yau fourfolds are discussed in [28, 29, 35–72]. However, because of the presence of the 7-branes F-theory vacua are difficult to handle.

Closely related to the heterotic $N = 1$ theories in $D = 4$ are their circle and torus compactifications to $D = 3$ and $D = 2$ which are the main focus of the thesis. In these

[18] In addition one has to specify a gauge bundle on the heterotic side.

cases some of the non-perturbative features are described by M-theory or type IIA string theory compactified on a Calabi-Yau fourfold Y_4 [35]. This leads to the following chain of dualities for the heterotic string with four supercharges

$$\begin{array}{lccc} \text{D=4:} & \text{F}/Y_4 & \cong & \text{Het}/Y_3 \\ & \downarrow & & \downarrow \\ \text{D=3:} & \text{M}/Y_4 & \cong & \text{Het}/(Y_3 \times S^1) \\ & \downarrow & & \downarrow \\ \text{D=2:} & \text{IIA}/Y_4 & \cong & \text{Het}/(Y_3 \times T^2) \end{array} \quad . \tag{1.8}$$

The further compactification to $D = 3$ and $D = 2$ has two main advantages. First the low energy regimes of M- and IIA string theory are much easier to describe as in F-theory because they are given by supergravity theories. Therefore we can base our analysis on a Kaluza-Klein reduction of the corresponding effective actions. Second in $D = 3$ and $D = 2$ the vector multiplet contains a real respectively complex scalar in the adjoint representation of the gauge group and thus - contrary to the situation in $D = 4$ - a Coulomb branch exists. Working on this Coulomb branch simplifies the analysis considerably as we will see.

Aspects of M/type IIA theory on Calabi-Yau fourfolds have been discussed by other authors in [36, 41, 46, 67, 71, 73–81]. Furthermore due to the close relationship between the compactifications of F/M/IIA theory on Calabi-Yau fourfolds many results from the F-theory literature cited in the last but one paragraph are also relevant for the three- and two-dimensional cases. Thus a lot is known about the spectrum in $D = 3$ and $D = 2$, the moduli space, the duality map to the moduli space of the heterotic string and non-perturbative superpotentials stemming from wrapped 5-branes. In contrast to this the emphasis of our thesis lies on a study of the low energy effective actions arising in a Kaluza-Klein reduction of type IIA and 11-dimensional supergravity on Calabi-Yau fourfolds. In $D = 3$ the moduli space of theories with four supercharges is restricted to be a Kähler manifold [82], whereas in $D = 2$ it is in general not Kähler anymore [83]. Nevertheless it has a related structure and is determined by two real functions. Only very limited results had been derived for the Kähler potential in $D = 3$ and the functions determining the moduli space in $D = 2$. For the type IIA compactification they have been given in [81] neglecting the moduli coming from the three-form potential. In our thesis we fill this gap by deriving the three-dimensional Kähler potential and the relevant two-dimensional functions for the moduli including those coming from expanding the three-form potential.

One of the aims of the thesis is to facilitate the identification of those Calabi-Yau fourfolds which lead to a perturbative heterotic dual when used in (1.8). This is a necessary first step if one wants to extract non-perturbative effects. Our analysis is based on a comparison of the low energy effective Lagrangians in $D = 3$ and $D = 2$, but we ignore all charged matter multiplets and only focus on the moduli. In spirit it is very close to a similar analysis carried out for the duality of type IIA on a Calabi-Yau threefold Y_3 and the heterotic string on K3$\times T^2$ in [84, 85].

In our comparison of the low energy effective theories we first restrict ourselves to the case of Calabi-Yau fourfolds with vanishing Euler number $\chi = 0$. Consistency in compactifications of M/type IIA theory on Calabi-Yau fourfolds with $\chi \neq 0$ requires backgrounds with non-vanishing fluxes or space-time filling membranes/strings [73, 74, 76]. We then extend the analysis to the case $\chi \neq 0$ by including non-trivial fluxes. They have the additional merit to generate a non-vanishing potential for the moduli fields in the low energy effective theory and thus provide a mechanism to stabilize at least some of them. The corresponding superpotentials have been proposed in [67, 80]. Here we derive the potential through a Kaluza-Klein reduction. This derivation requires to consider higher derivative terms in the eleven/ten-dimensional Lagrangian [73, 86]. In practice we mainly focus on the three-dimensional case, where we furthermore show that fluxes can be used to break supersymmetry spontaneously. An investigation of the potential in $D = 2$ and a translation of the results to the heterotic string are still in progress.

More precisely the organization of the thesis is as follows. In chapter 2 we consider heterotic - M-theory duality in $D = 3$ at the level of low energy effective Lagrangians. On the heterotic side we derive the effective Lagrangian by a circle compactification of a general $N = 1$, $D = 4$ heterotic theory. On the M-theory side we compactify 11-dimensional supergravity on a Calabi-Yau fourfold with $\chi = 0$ and vanishing 4-form flux. We derive the Kähler potentials for both theories and compare them to each other. We show that for fourfolds which are K3-fibrations over a complex two-dimensional base B_2 the two Kähler potentials agree in the limit of large B_2 and weak heterotic coupling. Finally, the F-theory limit is discussed. In chapter 3 we apply the same procedure in one dimension less. The heterotic theory in $D = 2$ is derived via a torus compactification of a general $N = 1$, $D = 4$ heterotic theory and the type IIA compactification on a Calabi-Yau fourfold with $\chi = 0$ is performed. It is again possible to establish a duality map if we specify to the same kind of fourfolds as in $D = 3$. We then generalize our reduction of 11-dimensional supergravity to Calabi-Yau fourfolds with $\chi \neq 0$ and 4-form flux in chapter 4. There we also make some comments concerning the reduction of type IIA supergravity in the background of non-vanishing RR fluxes and about the duality to the heterotic theory in these cases. Chapter 5 contains our conclusion. Three appendices are following. The first assembles our notation and conventions, the second gives an introduction to complex manifolds in general and to Calabi-Yau spaces in particular and the third appendix reviews the Kaluza-Klein idea and contains some technical details of the compactifications.

Let us finally mention that the results of chapter 2 and 3 have been published in [87, 88] and those of chapter 4 are partly described in [89].

Chapter 2

$D = 3$ effective theories with four supercharges

In this chapter we investigate the low energy effective action of heterotic string theories compactified to $D = 3$ with four unbroken supercharges and study their duality to Calabi-Yau fourfold compactifications of 11-dimensional supergravity.

2.1 Heterotic theories in $D = 3$

2.1.1 $D = 4, N = 1$ heterotic theories

Our starting point is a generic effective supergravity Lagrangian in $D = 4$ with $N = 1$ supersymmetry. Such theories can be constructed as compactifications of the heterotic string on Calabi-Yau threefolds Y_3 or more generally from appropriate $(0, 2)$ superconformal field theories. For our purpose it is sufficient to focus only on the vector multiplets and the chiral moduli multiplets and ignore all charged matter multiplets. The effective Lagrangian in this case reads [90][1]

$$\begin{aligned}\mathcal{L}^{(4)} \quad = \quad & \sqrt{-g^{(4)}}\left(\frac{1}{2}R^{(4)} - G^{(4)}_{\bar{I}J}(\Phi,\bar{\Phi})\partial_m\bar{\Phi}^{\bar{I}}\partial^m\Phi^J - \frac{1}{4}\mathrm{Re}f_{ab}(\Phi)F^a_{mn}F^{bmn}\right. \\ & \left. +\frac{1}{4}\mathrm{Im}f_{ab}(\Phi)F^a_{mn}\tilde{F}^{bmn} + \dots\right) , \qquad (2.1)\end{aligned}$$

where $m, n = 0, \dots, 3$, Φ^I are the moduli fields and F^a_{mn} are the field strengths of the gauge bosons A^a_m. The index a labels the generators of the gauge group G and thus $a = 1, \dots, \dim(G)$. $N = 1$ supersymmetry requires that the $f_{ab}(\Phi)$ are holomorphic functions of the moduli which are further constrained by gauge invariance. The metric $G^{(4)}_{\bar{I}J}$ has to be a Kähler metric, that is

$$G^{(4)}_{\bar{I}J} = \bar{\partial}_{\bar{I}}\partial_J K^{(4)}_{\mathrm{het}}(\Phi,\bar{\Phi}) , \qquad (2.2)$$

where $K^{(4)}_{\mathrm{het}}$ is the Kähler potential.

Perturbative heterotic string theory imposes further constraints on the functions $K^{(4)}$ and f_{ab}. First of all the rank r of the gauge group G is bounded by the central

[1] We neglect the possibility of anomalous $U(1)$ gauge factors with appropriate four-dimensional Green-Schwarz terms.

charge c of the left moving (bosonic) conformal field theory on the world-sheet. For heterotic strings in $D = 4$ one has $c = 22$ and hence

$$\mathrm{r}(G) \leq 22 \ . \tag{2.3}$$

Secondly, the holomorphic f_{ab} are universal at the string tree level and determined by the heterotic dilaton $e^{\Phi^{(4)}_{\mathrm{het}}}$ and the axion a which is the dual of the antisymmetric tensor B_{mn}. The two scalars are combined into a complex $S = e^{-2\Phi^{(4)}_{\mathrm{het}}} + ia$ and one has

$$f_{ab} = k_a S \delta_{ab} + \dots \ , \tag{2.4}$$

where k is the level of the Kac-Moody algebra. The $\dots$ denote perturbative and non-perturbative quantum corrections which are suppressed in the large S (weak coupling) limit; they play no role in our discussion.

The metric of the moduli also simplifies at the string tree level. Using the notation $\Phi^I = (S, \phi^i)$ with $i = 1, \dots, n_4, I = 0, \dots, n_4$ one has

$$K^{(4)}_{\mathrm{het}} = -\ln(S + \bar{S}) + \tilde{K}^{(4)}_{\mathrm{het}}(\phi, \bar{\phi}) + \dots \ , \tag{2.5}$$

where $\tilde{K}^{(4)}(\phi, \bar{\phi})$ is the tree level Kähler potential for all moduli except the dilaton. It is a model dependent function and does not enjoy any generic properties. For Calabi-Yau compactifications with the standard embedding of the spin connection $\tilde{K}^{(4)}$ splits into a sum

$$\tilde{K}^{(4)}_{\mathrm{het}} = K^{(4)}_{1,1} + K^{(4)}_{2,1} \ , \tag{2.6}$$

where $K^{(4)}_{1,1}(K^{(4)}_{2,1})$ is the Kähler potential for the $(1,1)$-moduli ($(2,1)$-moduli) of Y_3. For future reference we also need to recall that in the large volume limit of Calabi-Yau compactifications $K^{(4)}_{1,1}$ is given by [91–93]

$$K^{(4)}_{1,1} = -\ln[d_{ABC}(t + \bar{t})^A (t + \bar{t})^B (t + \bar{t})^C] = -\ln[\mathrm{Vol}(Y_3)] \ , \tag{2.7}$$

where d_{ABC} are the classical intersection numbers, t^A the $(1,1)$-moduli and $\mathrm{Vol}(Y_3)$ is the classical volume of the compactification manifold Y_3. For $K^{(4)}_{2,1}$ one has instead [92–94]

$$K^{(4)}_{2,1} = -\ln\left[i \int_{Y_3} \Omega \wedge \bar{\Omega}\right] \ , \tag{2.8}$$

where Ω is the unique $(3,0)$-form.

Finally, let us note that the couplings of the dilaton in (2.4) and (2.5) are largely determined by the fact that $e^{\Phi^{(4)}_{\mathrm{het}}}$ organizes the string perturbation theory and that in perturbation theory there is a continuous Peccei-Quinn (PQ) symmetry $S \to S + i\gamma, \gamma \in \mathbb{R}$, shifting the axion a.

2.1.2 $D = 4, N = 1$ supergravity compactified on S^1

Let us reduce the Lagrangian (2.1) to $D = 3$ on a circle S^1. This does not break any supercharges so that the theory continues to have 4 real supercharges. In $D = 3$ this corresponds to $N = 2$ supersymmetry since the irreducible Majorana spinor has 2 real components.

For the S^1-reduction to $D = 3$ we use the Ansatz [95]:

$$g^{(4)}_{mn} = \begin{pmatrix} g^{(3)}_{\mu\nu} + r^2 B_\mu B_\nu & r^2 B_\mu \\ r^2 B_\nu & r^2 \end{pmatrix} , \qquad A^a_m = \left(A^a_\mu + B_\mu \zeta^a, \zeta^a\right) , \tag{2.9}$$

where $\mu, \nu = 0, 1, 2$ and r is the radius (measured in the $D = 4$ Einstein metric) of the S^1. The reduction procedure follows closely [95] where four-dimensional $N = 2$ vacua are considered. The details are shown in appendix C.4 and here we only give the results.

In $D = 3$ the vector multiplets contain real scalars ζ^a in the adjoint representation of G. Thus there is an additional component of the moduli space (a Coulomb branch) spanned by the scalar fields lying in the Cartan subalgebra of G and at a generic point in this moduli space the gauge group is $[U(1)]^{\mathrm{r}(G)}$. In order to make the notation not too heavy we continue to label the $U(1)$ gauge multiplets by the index a although in the previous section the same index ran over all gauge generators; thus $a = 1, \ldots, \mathrm{r}(G)$ henceforth.

In $D = 3$ an Abelian vector without Chern-Simons interactions is dual to a scalar and thus a vector multiplet is dual to a chiral multiplet. Technically this duality is achieved by adding Lagrange multipliers C^a, b to the three-dimensional Lagrangian and eliminating A^a_μ, B_μ by their equations of motion (see appendix C.4). In this dual picture all supermultiplets are chiral and thus their scalar fields have to parameterize a Kähler manifold [82]. It turns out that the Kähler structure only becomes manifest after introducing the coordinates

$$\begin{aligned} D^a &\equiv -f_{ab}(\Phi)\zeta^b + iC^a , \\ T &\equiv r^2 + ib + \frac{1}{2}(\mathrm{Re}f)^{-1}_{ab}(\Phi) D^a (D^b + \bar{D}^b) . \end{aligned} \tag{2.10}$$

Combining all $n_4 + \mathrm{r}(G) + 2$ scalar fields into the coordinate vector $Z^\Sigma = (S, \phi^i, D^a, T)$, $\Sigma = 0, \ldots, n_4 + \mathrm{r}(G) + 1$ the three-dimensional Lagrangian can be written as

$$\mathcal{L}^{(3)} = \sqrt{-g^{(3)}} \left(\frac{1}{2} R^{(3)} - G_{\bar{\Lambda}\Sigma} \partial_\mu \bar{Z}^{\bar{\Lambda}} \partial^\mu Z^\Sigma \right) , \tag{2.11}$$

where $G_{\bar{\Lambda}\Sigma}$ obeys

$$\begin{aligned} G_{\bar{\Lambda}\Sigma} &= \bar{\partial}_{\bar{\Lambda}} \partial_\Sigma K^{(3)}_{\mathrm{het}} , \\ K^{(3)}_{\mathrm{het}} &= K^{(4)}(\Phi, \bar{\Phi}) - \ln[T + \bar{T} - \frac{1}{2}(D + \bar{D})^a (\mathrm{Re}f)^{-1}_{ab} (D + \bar{D})^b] . \end{aligned} \tag{2.12}$$

Note that the argument of the logarithm is given by the square of the compactification radius, as can be seen from (2.10):

$$T + \bar{T} - \frac{1}{2}(D + \bar{D})^a (\mathrm{Re}f)^{-1}_{ab} (D + \bar{D})^b = 2r^2 . \tag{2.13}$$

2.1.3 The heterotic $D = 3$ low energy effective Lagrangian

So far the reduction did not use any input from string theory. Inserting the string tree level properties displayed in (2.4) and (2.5) into $K^{(3)}_{\mathrm{het}}$ of (2.12) yields

$$K^{(3)}_{\mathrm{het}} = \tilde{K}^{(4)}(\phi, \bar{\phi}) - \ln\left[(T + \bar{T})(S + \bar{S}) - (D^a + \bar{D}^a)^2\right] . \tag{2.14}$$

We will see in section 2.3 that the duality to the M-theory vacua is more naturally expressed in the coordinates $S' = \frac{1}{2}(S+T)$ and $T' = \frac{1}{2}(S-T)$ in which the Kähler potential looks like

$$K_{\text{het}}^{(3)} = \tilde{K}^{(4)}(\phi, \bar{\phi}) - \ln[(S' + \bar{S}')^2 - (T' + \bar{T}')^2 - (D^a + \bar{D}^a)^2] \ . \tag{2.15}$$

The next step is to identify the three-dimensional dilaton. The relation to the four-dimensional dilaton is as usually

$$e^{-2\Phi_{\text{het}}^{(3)}} = r_s e^{-2\Phi_{\text{het}}^{(4)}} \ , \tag{2.16}$$

where r_s is the radius of S^1 measured in the four-dimensional string frame metric. In the reduction procedure we used the metric in the Einstein frame (2.9) which is related to the metric in the string frame by the Weyl rescaling $g^{(4)} = e^{-2\Phi_{\text{het}}^{(4)}} g_s^{(4)}$. This implies the following relation among the radii

$$r_s = r e^{\Phi_{\text{het}}^{(4)}} \ . \tag{2.17}$$

Combining (2.16) and (2.17) results in

$$e^{2\Phi_{\text{het}}^{(3)}} = \frac{e^{\Phi_{\text{het}}^{(4)}}}{r} \ . \tag{2.18}$$

Using (2.13) and $e^{-2\Phi_{\text{het}}^{(4)}} = \frac{1}{2}(S+\bar{S})$ we also derive

$$e^{-4\Phi_{\text{het}}^{(3)}} = (T+\bar{T})(S+\bar{S}) - (D^a + \bar{D}^a)^2 \ . \tag{2.19}$$

The three-dimensional dilaton governs the perturbation series in $D = 3$, as can be seen by reducing the four-dimensional Lagrangian in the string frame. The result can be found in appendix C.4. The Kähler potential of (2.14) is only valid to lowest order in r and $e^{\Phi_{\text{het}}^{(3)}}$ and gets perturbative and non-perturbative corrections.

Finally let us discuss the symmetries of the compactified theory. First of all there is the PQ symmetry associated with the four-dimensional axion a discussed in section 2.1.1. Furthermore, there are $\mathrm{r}(G)+1$ Abelian gauge symmetries associated with the $\mathrm{r}(G)+1$ gauge bosons A_μ^a, B_μ. In the dual Lagrangian these symmetries appear as continuous PQ symmetries acting on the dual scalars. Finally, the scalars ζ^a in the three-dimensional vector multiplets 'inherit' another PQ symmetry from the four-dimensional gauge invariance. So altogether (2.11), (2.12) are invariant under the following $2\mathrm{r}(G)+2$ PQ symmetries (with parameters $\gamma, \tilde{\gamma}, \gamma^a, \hat{\gamma}^a$)

$$\begin{array}{rcll} a & \to & a + \gamma \ , & \\ b & \to & b + \tilde{\gamma} \ , & \\ C^a & \to & C^a + \gamma^a \ , & b \to b + \gamma^a \zeta^a \ , \\ \zeta^a & \to & \zeta^a + \hat{\gamma}^a \ , & b \to b + \hat{\gamma}^a C^a \ . \end{array} \tag{2.20}$$

These PQ symmetries are exact in perturbation theory but broken to discrete subgroups non-perturbatively. In addition, there is the standard T-duality which acts on the radius of the S^1 and sends $r_s \to r_s^{-1}$ while keeping $e^{\Phi_{\text{het}}^{(3)}}$ fixed. Using (2.17) and (2.18) this corresponds to

$$e^{\Phi_{\text{het}}^{(4)}} \leftrightarrow r^{-1} \ , \tag{2.21}$$

and leaves (2.14) invariant.

2.2 M-theory compactified on Calabi-Yau fourfolds

As we have discussed in the introduction M-theory has been suggested as the strong coupling limit of type IIA string theory [96]. Even though a satisfactory formulation of M-theory has not been established its low energy limit is known to be 11-dimensional supergravity. In this section we perform the compactification of this low energy limit on Calabi-Yau fourfolds and obtain an effective Lagrangian in $D = 3$.

The starting point is the 11-dimensional supergravity action [97]:

$$\mathcal{S}^{(11)} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g^{(11)}} \left(R^{(11)} - \frac{1}{2}|F_4|^2 \right) - \frac{1}{12\kappa_{11}^2} \int A_3 \wedge F_4 \wedge F_4 \ , \tag{2.22}$$

where A_3 is a three-form, F_4 its field strength and $g^{(11)}$ the determinant of the 11-dimensional metric (more details of the notation used are given in appendix A). The Lagrangian (2.22) is the leading order contribution in a derivative expansion. One of the next to leading order terms in this expansion has been deduced from a one-loop computation in the type IIA theory [98] and then extrapolated to the 11-dimensional theory. It is associated to the sigma-model anomaly of the six-dimensional 5-brane world-volume [99]. It reads[2]

$$\begin{aligned} \delta\mathcal{S}_1^{(11)} &= -T_2 \int A_3 \wedge X_8 \\ \text{with} \qquad X_8 &= \frac{1}{(2\pi)^4} \left(-\frac{1}{768} (\text{tr}R^2)^2 + \frac{1}{192} \text{tr}R^4 \right) \ , \end{aligned} \tag{2.23}$$

where $T_2 \equiv (2\pi)^{2/3}(2\kappa_{11}^2)^{-1/3}$ is the membrane tension and R denotes the curvature two-form (B.6) so that the product R^n involves both a matrix and a wedge product. The term (2.23) leads to an important constraint for Calabi-Yau fourfold reductions of (2.22). It induces a potential tadpole term for the three-form A_3 rendering the resulting vacuum inconsistent [74]. The coefficient of the anomaly is set by the Euler number χ of Y_4

$$\int_{Y_4} X_8 = -\frac{\chi}{24} = -\frac{1}{4}(8 + h^{1,1} + h^{1,3} - h^{1,2}) \ . \tag{2.24}$$

Thus it can be avoided by choosing compactification manifolds with $\chi = 0$. However, the anomaly can also be cancelled by considering backgrounds with n space-time filling membranes or turning on non-trival F_4-flux [73,74,76].[3] In this case (and setting $T_2 \equiv 1$, i.e. $\kappa_{11}^2 \equiv 2\pi^2$)

$$\frac{\chi}{24} = n + \frac{1}{8\pi^2} \int_{Y_4} F_4 \wedge F_4 \tag{2.25}$$

has to hold for consistency. Backgrounds with space-time filling membranes are known to be dual to heterotic vacua in non-trivial 5-brane backgrounds [28, 55, 69]. Since we are for the moment primarily interested in identifying the perturbative heterotic string among the fourfold compactifications we choose to consider $n = 0$ in this chapter. Furthermore, we leave the case of nontrival F_4-flux to chapter 4 and consider fourfolds that satisfy $\chi = 0$ as compactification manifolds in the following. Moreover, we set $\kappa_{11} \equiv 1$ for the rest of the chapter.

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[2] We use an index '1' as there are further corrections becoming important in chapter 4.

[3] A further contribution to the anomaly related to M5-branes wrapped around three-cycles in the Calabi-Yau has been discussed in [41].

We now perform a lowest order Kaluza-Klein reduction of the 11-dimensional supergravity on Calabi-Yau fourfolds.[4] The relevant facts about Calabi-Yau fourfolds Y_4 are collected in appendix B. Following the procedure outlined in appendix C we take for the 11-dimensional metric the Ansatz

$$g^{(11)}_{MN}(x,y) = \begin{pmatrix} g^{(3)}_{\mu\nu}(x) & 0 \\ 0 & g^{(8)}_{ab}(x,y) \end{pmatrix}, \tag{2.26}$$

where $x^\mu(\mu = 0,1,2)$ denote the coordinates of three-dimensional Minkowski space and $y^a(a = 3,\ldots,10)$ denote the internal Calabi-Yau coordinates. As explained in section C.1 the metric

$$g^{(8)}_{ab} = \hat{g}^{(8)}_{ab}(\langle M\rangle) + \delta g^{(8)}_{ab}(\delta M(x)) \tag{2.27}$$

splits into a background solution, depending on the vacuum expectation values of the metric moduli, and small fluctuations which are induced by variations of the moduli. These fluctuations are given by the zero modes of the Lichnerowicz operator for $\hat{g}$. It has been shown in [103] that there are two different kinds of zero modes which are given in terms of non-trivial harmonic forms on Y_4. This can be best understood if we introduce complex coordinates ξ^j $(j = 1,\ldots,4)$ for Y_4 defining $\xi^j = \frac{1}{\sqrt{2}}(y^{2j+1}+iy^{2j+2})$.[5] For the deformation of the Kähler form one has[6]

$$i\delta g_{i\bar{j}} = \sum_{A=1}^{h^{1,1}} \delta M^A(x)\, e_{Ai\bar{j}}\ , \tag{2.28}$$

where e_A is an appropriate basis of $H^{1,1}(Y_4)$ and $M^A(x)$ are the corresponding (real) moduli. For the deformations of the complex structure one has

$$\delta g_{\bar{i}\bar{j}} = \sum_{\alpha=1}^{h^{3,1}} \delta Z^\alpha(x)\, b_{\alpha\bar{i}\bar{j}}\ , \tag{2.29}$$

where $Z^\alpha(x)$ are complex moduli and $b_{\alpha\bar{i}\bar{j}}$ is related to the basis Φ_α of $H^{3,1}(Y_4)$ by an appropriate contraction with the anti-holomorphic four-form $\bar{\Omega}$ on Y_4 [93]:

$$b_{\alpha\bar{i}\bar{j}} = -\frac{1}{3|\Omega|^2}\bar{\Omega}_{\bar{i}}{}^{klm}\Phi_{\alpha klm\bar{j}}\ , \qquad |\Omega|^2 \equiv \frac{1}{4!}\Omega_{ijkl}\bar{\Omega}^{ijkl}\ . \tag{2.30}$$

Finally, the three-form A_3 is expanded in terms of the (1,1)-forms e_A and the (2,1)-forms $\Psi_{Iij\bar{k}}$. More precisely

$$A_{\mu i\bar{j}} = \sum_{A=1}^{h^{1,1}} A^A_\mu(x)\, e_{Ai\bar{j}}\ , \qquad A_{ij\bar{k}} = \sum_{I=1}^{h^{2,1}} N^I(x)\,\Psi_{Iij\bar{k}}\ , \qquad A_{\bar{i}\bar{j}k} = \sum_{I=1}^{h^{2,1}} \bar{N}^{\bar{J}}(x)\,\bar{\Psi}_{\bar{J}\bar{i}\bar{j}k}\ . \tag{2.31}$$

So altogether the compactification leads to $h^{1,1}$ vector multiplets (A^A_μ, M^A) and $h^{2,1} + h^{3,1}$ chiral multiplets $(N^I), (Z^\alpha)$.

[4] A similar analysis for IIA compactifications on threefolds can be found in [100] while compactification of 11-dimensional supergravity on threefolds was considered in [101, 102].

[5] The corresponding integration measures are related according to $d^8\xi \equiv d^4\xi \wedge d^4\bar{\xi} = d^8y$.

[6] In the following we omit the superscript (8) at the internal metric.

On the space of $(1,1)$-forms one defines the metric [104][7]

$$G_{AB} \equiv \frac{1}{2\mathcal{V}} \int_{Y_4} e_A \wedge \star e_B \; = -\frac{1}{2\mathcal{V}} \int_{Y_4} d^8\xi \sqrt{g} e_{Ai\bar{\jmath}} e_{Bk\bar{m}} g^{i\bar{m}} g^{k\bar{\jmath}} \; , \tag{2.32}$$

where $\mathcal{V}$ is the volume of Y_4

$$\mathcal{V} = \int_{Y_4} d^8\xi \sqrt{g} = \frac{1}{4!} \int_{Y_4} J \wedge J \wedge J \wedge J = \frac{1}{4!} d_{ABCD} M^A M^B M^C M^D \; . \tag{2.33}$$

J is the Kähler form of the Calabi-Yau fourfold

$$J = i g_{i\bar{\jmath}} d\xi^i \wedge d\bar{\xi}^{\bar{\jmath}} = M^A e_A \tag{2.34}$$

and

$$d_{ABCD} = \int_{Y_4} e_A \wedge e_B \wedge e_C \wedge e_D \tag{2.35}$$

are the classical intersection numbers of Y_4, see section B.1.

On the space of $(3,1)$-forms one defines the metric [94]

$$G_{\alpha\bar{\beta}} \equiv -\frac{\int_{Y_4} \Phi_\alpha \wedge \bar{\Phi}_{\bar{\beta}}}{\int_{Y_4} \Omega \wedge \bar{\Omega}} = \frac{1}{4\mathcal{V}} \int_{Y_4} d^8\xi \sqrt{g} b_{\alpha\bar{\jmath}\bar{m}} \bar{b}_{\bar{\beta}ik} g^{i\bar{\jmath}} g^{k\bar{m}} = \partial_\alpha \bar{\partial}_{\bar{\beta}} K_{3,1} \; , \tag{2.36}$$

which is a Kähler metric with Kähler potential

$$K_{3,1} = -\ln \Big[\int_{Y_4} \Omega \wedge \bar{\Omega} \Big] . \tag{2.37}$$

Finally, on the space of $(2,1)$-forms we define a metric $G_{I\bar{J}}$ and intersection numbers $d_{AI\bar{J}}$

$$\begin{aligned} G_{I\bar{J}} \;&\equiv\; \frac{1}{2} \int_{Y_4} \Psi_I \wedge \bar{\star} \Psi_J = \frac{1}{4} \int_{Y_4} d^8\xi \sqrt{g} \Psi_{Ii\jmath\bar{k}} \bar{\Psi}_{\bar{J}l\bar{m}\bar{n}} g^{i\bar{m}} g^{j\bar{n}} g^{l\bar{k}}, \\ d_{AI\bar{J}} \;&\equiv\; \int_{Y_4} e_A \wedge \Psi_I \wedge \bar{\Psi}_{\bar{J}} = \frac{1}{4} \int_{Y_4} d^8\xi \sqrt{g} \epsilon^{ikls} \epsilon^{\bar{\jmath}\bar{m}\bar{n}\bar{r}} e_{Ai\bar{\jmath}} \Psi_{Ikl\bar{m}} \bar{\Psi}_{\bar{J}s\bar{n}\bar{r}} \; . \end{aligned} \tag{2.38}$$

The two quantities are related via[8]

$$G_{I\bar{J}} = -\frac{i}{2} d_{AI\bar{J}} M^A \; , \quad \text{or} \quad d_{AI\bar{J}} = 2i \frac{\partial G_{I\bar{J}}}{\partial M^A} \; . \tag{2.39}$$

In the following it is convenient to define a metric independent of M^A as

$$\hat{G}_{I\bar{J}} = -\frac{i}{2} c^A d_{AI\bar{J}} \; , \tag{2.40}$$

where c^A are constant real vectors with no vanishing entries but otherwise arbitrary.

It is important to notice that G_{AB} and d_{ABCD} are independent of the complex structure but $G_{I\bar{J}}$ and $d_{AI\bar{J}}$ do depend on Z^α and $\bar{Z}^{\bar{\alpha}}$. This dependence is not known

[7] For the definition of the Hodge $\star$- and the related Hodge $\bar{\star}$-operators used below see (B.20), (B.21).

[8] This can be checked noticing that $\bar{\star}\Psi_J = \frac{1}{2}\Big(\bar{\Psi}_{\bar{J}i\bar{\jmath}\bar{k}} g^{i\bar{\jmath}} g_{l\bar{m}} g_{n\bar{o}} + \bar{\Psi}_{\bar{J}l\bar{k}\bar{m}} g_{n\bar{o}}\Big) d\xi^l \wedge d\xi^n \wedge d\bar{\xi}^{\bar{k}} \wedge d\bar{\xi}^{\bar{m}} \wedge d\bar{\xi}^{\bar{o}}$. The first term corresponds to a term $\tilde{\Psi}_J \wedge J \wedge J$ with a globally defined (0,1)-form $\tilde{\Psi}_J = \frac{1}{2} \bar{\Psi}_{\bar{J}i\bar{\jmath}\bar{k}} g^{i\bar{\jmath}} d\bar{\xi}^{\bar{k}}$, see (B.37) and (B.39). As $h^{0,1} = 0$ for a Calabi-Yau fourfold the first term is actually trivial.

generically since it depends on the particular Y_4 under consideration. However, as we will show it is not necessary to know the complex structure dependence of $G_{I\bar{J}}$ and $d_{AI\bar{J}}$ explicitly in order to determine the Kähler potential.[9]

The basis Ψ^I of $(2,1)$-forms can locally be chosen to depend holomorphically on the complex structure (see section B.7) or in other words

$$\bar{\partial}_{\bar{Z}^{\bar{\alpha}}}\Psi_I = 0 \ , \qquad \partial_{Z^\alpha}\Psi_I \neq 0 \ . \tag{2.41}$$

The derivative $\partial_{Z^\alpha}\Psi^I$ can be expanded into $(1,2)$- and $(2,1)$-forms with complex-structure dependent coefficient functions σ and τ

$$\partial_{Z^\alpha}\Psi_I = \sigma_{\alpha I}{}^K(Z,\bar{Z})\,\Psi_K + \tau_{\alpha I}{}^{\bar{L}}(Z,\bar{Z})\,\bar{\Psi}_{\bar{L}} \ . \tag{2.42}$$

Note that τ is not the complex conjugate of σ but an independent function. Indices I and $\bar{J}$ are raised with δ^{IK} and $\delta^{\bar{J}\bar{K}}$. Differentiating (2.42) with respect to $\bar{Z}^{\bar{\alpha}}$ results in the following differential constraints for σ and τ

$$\bar{\partial}_{\bar{Z}^{\bar{\beta}}}\,\sigma_{\alpha I}{}^K = -\tau_{\alpha I}{}^{\bar{L}}\bar{\tau}_{\bar{\beta}\bar{L}}{}^K \ , \qquad \bar{\partial}_{\bar{Z}^{\bar{\beta}}}\,\tau_{\alpha I}{}^{\bar{K}} = -\tau_{\alpha I}{}^{\bar{L}}\bar{\sigma}_{\bar{\beta}\bar{L}}{}^{\bar{K}} \ . \tag{2.43}$$

From (2.38), (2.39), (2.40) and (2.42) it immediately follows that the complex structure dependence of $G_{I\bar{J}}, \hat{G}_{I\bar{J}}, d_{AI\bar{J}}$ is constrained by the differential equations[10]

$$\partial_{Z^\alpha}G_{I\bar{J}} = \sigma_{\alpha I}{}^K G_{K\bar{J}} \ , \qquad \partial_{Z^\alpha}\hat{G}_{I\bar{J}} = \sigma_{\alpha I}{}^K \hat{G}_{K\bar{J}} \ , \qquad \partial_{Z^\alpha}d_{AI\bar{J}} = \sigma_{\alpha I}{}^K\, d_{AK\bar{J}} \ . \tag{2.44}$$

The next step is to insert (2.26)-(2.31) into (2.22). The details of this reduction are presented in appendix C and here we only summarize the results. The vectors A^A_μ are again dualized to scalar fields denoted by P^A. So after dualization the vector multiplet becomes a chiral multiplet with the (real) scalars (M^A, P^A) and there are altogether $h^{1,1}+h^{1,2}+h^{1,3}$ chiral multiplets. Supersymmetry requires that the Lagrangian of these chiral multiplets must be expressible in the form

$$\mathcal{L}^{(3)} = \sqrt{-g^{(3)}}\left(\frac{1}{2}R^{(3)} - G_{\bar{\Lambda}\Sigma}\partial_\mu\bar{Z}^{\bar{\Lambda}}\partial^\mu Z^\Sigma\right) \ , \tag{2.45}$$

where $\Lambda,\Sigma = 1,\dots,h^{1,1}+h^{1,2}+h^{1,3}$ and $G_{\bar{\Lambda}\Sigma}$ is a Kähler metric $G_{\bar{\Lambda}\Sigma} = \bar{\partial}_{\bar{\Lambda}}\partial_\Sigma K^{(3)}_{\mathrm{M}}$. However, it turns out that the scalar fields which appear naturally in (2.28)-(2.31) in the expansion of the harmonic forms on Y_4 are not the appropriate Kähler coordinates Z^Σ. Rather a set of field redefinitions has to be performed in order to cast the three-dimensional Lagrangian into the form (2.45). The proper Kähler coordinates are $T^A, \hat{N}^I, Z^\alpha$ defined as

$$T^A = \frac{1}{\sqrt{8}}\Big(iP^A + \mathcal{V}G^A{}_B M^B - \frac{i}{4}d^A{}_{M\bar{L}}\hat{G}^{-1}{}_{\bar{J}}{}^M\hat{G}^{-1\bar{L}}{}_I\hat{N}^I\bar{\hat{N}}^{\bar{J}} + \omega^A{}_{IK}\hat{N}^I\hat{N}^K\Big), \tag{2.46}$$
$$\hat{N}^I = \hat{G}^I{}_{\bar{J}}(Z^\alpha,\bar{Z}^{\bar{\alpha}})\,\bar{N}^{\bar{J}} \ , \tag{2.47}$$

while the Z^α are unchanged. Like I and $\bar{J}$ also the indices A are raised with δ^{AB}. The ω_{AIK} are functions of Z^α and $\bar{Z}^{\bar{\alpha}}$ which have to obey

$$\bar{\partial}_{\bar{Z}^{\bar{\alpha}}}\omega_{AIK} = -\frac{i}{4}\hat{G}^{-1\bar{L}}{}_I\hat{G}^{-1\bar{J}}{}_K d_{AM\bar{L}}\bar{\tau}_{\bar{\alpha}\bar{J}}{}^M \ , \tag{2.48}$$

[9]Similarly, $K_{3,1}$ is generically known only in terms of an integral over the holomorphic $(4,0)$-form Ω but not necessarily explicitly (see (2.37)).

[10]It would be nice to have a better geometrical understanding of (2.43) and (2.44).

but are otherwise unconstrained. In terms of $T^A, \hat{N}^I, Z^\alpha$ the metric is Kähler with the Kähler potential

$$K_{\rm M}^{(3)} = K_{3,1} - \ln\left[\Xi^A \mathcal{V} G^{-1}_{AB} \Xi^B\right] , \tag{2.49}$$

where

$$\Xi^A \equiv \Big(T^A + \bar{T}^A + \frac{i}{2\sqrt{8}} d^A{}_{M\bar{L}} \hat{G}^{-1}{}_{\bar{J}}{}^M \hat{G}^{-1\bar{L}}{}_I \hat{N}^I \bar{\hat{N}}^{\bar{J}} - \frac{1}{\sqrt{8}}\omega^A{}_{IK}\hat{N}^I \hat{N}^K - \frac{1}{\sqrt{8}}\bar{\omega}^A{}_{\bar{J}\bar{L}} \bar{\hat{N}}^{\bar{J}} \bar{\hat{N}}^{\bar{L}}\Big). \tag{2.50}$$

(Note that $\mathcal{V}$ and G_{AB} in (2.49) have to be expressed in terms of the Kähler coordinates.)

In geometrical terms the argument of the logarithm is just the cube of the volume of the Calabi-Yau fourfold measured in the M-theory metric, which can be checked by inserting (2.46) into (2.50),

$$K_{\rm M}^{(3)} = K_{3,1} - 3\ln\mathcal{V}. \tag{2.51}$$

As we will see in the next section the duality to the heterotic vacua is more naturally expressed in terms of rescaled variables $\tilde{M}^A$

$$M^A \to \tilde{M}^A = \mathcal{V}^{1/2} M^A \ . \tag{2.52}$$

Using the rescaled Kähler form $\tilde{J} = \tilde{M}^A e^A$ in (2.33) and (C.37) one finds

$$\tilde{\mathcal{V}} = \mathcal{V}^3, \qquad \tilde{G}_{AB} = \mathcal{V}^{-1} G_{AB} \, , \tag{2.53}$$

where $\tilde{\mathcal{V}}$ and $\tilde{G}_{AB}$ have the same functional dependence on $\tilde{M}^A$ as $\mathcal{V}$ and G_{AB} have on M^A. In terms of the rescaled variables the Kähler potential becomes

$$K_M^{(3)} = K_{3,1} - \ln\left[\Xi^A \tilde{G}^{-1}_{AB} \Xi^B\right] = K_{3,1} - \ln\tilde{\mathcal{V}} \ . \tag{2.54}$$

Even though $K_M^{(3)}$ is the sum of two terms the moduli space does not factorize. When expressed in terms of the proper Kähler coordinates the second term in (2.54) does depend on $T^A, \hat{N}^I, Z^\alpha$ and therefore the metric is not block diagonal. However, from (2.50) we learn that for $\hat{N}^I = 0$ the moduli space does factorize locally and the Kähler potential becomes

$$\begin{aligned} K_M^{(3)} &= K_{3,1} + K_{1,1} \ , \\ K_{1,1} &= -\ln\left[(T^A + \bar{T}^A)\tilde{G}^{-1}_{AB}(T^B + \bar{T}^B)\right] \, , \\ K_{3,1} &= -\ln\left[\int_{Y_4} \Omega\wedge\bar{\Omega}\right] \, . \end{aligned} \tag{2.55}$$

Finally, let us discuss the continuous PQ-symmetries of the M-theory vacua in the large volume limit. First of all the scalars P^A which arise from dualizing the vectors A_μ^A inherit a PQ-symmetry from gauge invariance. The Kähler potential (2.49) is invariant under the shifts

$$P^A \to P^A + \tilde{\gamma}^A \ , \tag{2.56}$$

where $\tilde{\gamma}^A$ are arbitrary real constants. Secondly, the N^I arise from expanding the three-form in (2.31) and as a consequence they inherit part of the three-form gauge invariance. Specifically, the three-dimensional Lagrangian is invariant under the shift

$$N^I \Psi_I + \bar{N}^{\bar{I}} \bar{\Psi}_{\bar{I}} \to N^I \Psi_I + \bar{N}^{\bar{I}} \bar{\Psi}_{\bar{I}} + \text{const.} \, . \tag{2.57}$$

In terms of N^I this amounts to

$$N^I \to N^I + \gamma^I(Z,\bar{Z}) \ , \qquad \bar{N}^{\bar{I}} \to \bar{N}^{\bar{I}} + \bar{\gamma}^{\bar{I}}(Z,\bar{Z}) \ , \tag{2.58}$$

where the γ^I depend on the complex structure and as a consequence of (2.42) have to satisfy

$$\partial_{Z^\alpha}\gamma^J = -\gamma^I \sigma_{\alpha I}{}^J \ , \qquad \bar{\partial}_{\bar{Z}^{\bar{\alpha}}}\, \gamma^J = -\bar{\gamma}^{\bar{I}} \bar{\tau}_{\bar{\alpha}\bar{I}}{}^J \ . \tag{2.59}$$

The redefinition of the N^I in (2.47) is precisely such that it renders the PQ-symmetry holomorphic

$$\hat{N}^I \to \hat{N}^I + \hat{\gamma}^I(Z) \ , \tag{2.60}$$

where $\hat{\gamma}^I = \hat{G}^I{}_{\bar{J}}\bar{\gamma}^{\bar{J}}$ obeys

$$\bar{\partial}_{\bar{Z}^{\bar{\alpha}}}\hat{\gamma}^I = 0 \ , \qquad \partial_{Z^\alpha}\hat{\gamma}^I = \sigma_\alpha{}^I{}_K \hat{\gamma}^K - \bar{\hat{\gamma}}^{\bar{L}} \hat{G}^{-1}{}_{\bar{L}}{}^K \tau_{\alpha K}{}^{\bar{N}} \hat{G}^I{}_{\bar{N}} \ . \tag{2.61}$$

The corresponding PQ transformations of the fields T^A are

$$\begin{aligned} T^A \quad \to \quad & T^A - \frac{i}{2\sqrt{8}} d^A{}_{M\bar{L}} \hat{G}^{-1}{}_{\bar{J}}{}^M \hat{G}^{-1\bar{L}}{}_I \hat{N}^I \bar{\hat{\gamma}}^{\bar{J}} - \frac{i}{4\sqrt{8}} d^A{}_{M\bar{L}} \hat{G}^{-1}{}_{\bar{J}}{}^M \hat{G}^{-1\bar{L}}{}_I \hat{\gamma}^I \bar{\hat{\gamma}}^{\bar{J}} \\ & + \frac{1}{\sqrt{8}} \omega^A{}_{IK} \hat{\gamma}^I \hat{N}^K + \frac{1}{2\sqrt{8}} \omega^A{}_{IK} \hat{\gamma}^I \hat{\gamma}^K + \frac{1}{\sqrt{8}} \bar{\omega}^A{}_{\bar{I}\bar{K}} \bar{\hat{\gamma}}^{\bar{I}} \bar{\hat{N}}^{\bar{K}} + \frac{1}{2\sqrt{8}} \bar{\omega}^A{}_{\bar{I}\bar{K}} \bar{\hat{\gamma}}^{\bar{I}} \bar{\hat{\gamma}}^{\bar{K}}. \end{aligned} \tag{2.62}$$

Thus the P^A also transform[11] according to

$$P^A \to P^A - \frac{1}{4} d^A{}_{M\bar{L}} \hat{G}^{-1}{}_{\bar{J}}{}^M \hat{G}^{-1\bar{L}}{}_I \hat{N}^I \bar{\hat{\gamma}}^{\bar{J}} + \frac{1}{4} d^A{}_{M\bar{L}} \hat{G}^{-1}{}_{\bar{J}}{}^M \hat{G}^{-1\bar{L}}{}_I \hat{\gamma}^I \bar{\hat{N}}^{\bar{J}}. \tag{2.63}$$

So altogether we have $h^{1,1} + 2h^{2,1}$ continuous PQ-symmetries in the large volume limit of the M-theory compactification.

2.3 Duality

2.3.1 Heterotic – M-theory duality in $D = 3$

The next step is to investigate the dual relation between the heterotic and the M-theory Lagrangians. A first guidance can be obtained by considering the seven-dimensional duality (M/K3 $\simeq$ Het/T^3) [96] and fibering it over a common complex two-dimensional base B_2. For large B_2 one can apply the adiabatic argument [105] and conclude that M-theory compactified on a K3-fibered fourfold should be dual to the heterotic string compactified on an elliptically fibered threefold. The seven-dimensional string coupling constant $e^{\Phi^{(7)}_{\text{het}}}$ is related to the volume $\mathcal{V}_{K3}$ of the K3 measured in the (11-dimensional) M-theory metric and the respective space-time metrics $(g_7^{\text{het}}, g_7^M)$ differ by a power of $e^{\Phi^{(7)}_{\text{het}}}$ [96]

$$e^{4/3\Phi^{(7)}_{\text{het}}} = \mathcal{V}_{K3} \ , \qquad g_7^{\text{het}} = e^{4/3\Phi^{(7)}_{\text{het}}} g_7^M \ . \tag{2.64}$$

Using the adiabatic argument one derives

$$e^{-2\Phi^{(3)}_{\text{het}}} = \mathcal{V}^{\text{het}}_{B_2} e^{-2\Phi^{(7)}_{\text{het}}} = \mathcal{V}_{B_2}\, e^{2/3\Phi^{(7)}_{\text{het}}} = \mathcal{V}_{B_2}\, \mathcal{V}_{K3}^{1/2} \ , \tag{2.65}$$

[11] This transformation is necessary in order to render (C.44) invariant under (2.60).

where the volume of the base B_2 measured in the heterotic string frame metric $\mathcal{V}_{B_2}^{\rm het}$ is related to the volume $\mathcal{V}_{B_2}$ measured in the M-theory metric via $\mathcal{V}_{B_2}^{\rm het} = \mathcal{V}_{B_2} e^{8/3\Phi^{(7)}_{\rm het}}$ as can be seen from (2.64). Equation (2.65) implies that a large volume of the Calabi-Yau fourfold corresponds to heterotic weak coupling. Since the compactification of M-theory performed in the previous section is valid for large fourfolds it is legitimate to compare it to a weakly coupled heterotic string. Or in other words the two limits – large Y_4 in M-theory and weak coupling in the heterotic string – are mutually compatible. In particular duality requires that the respective Kähler potentials have to agree in this limit

$$K^{(3)}_{\rm M} = K^{(3)}_{\rm het} \ . \tag{2.66}$$

For perturbative heterotic vacua where $K^{(3)}_{\rm het}$ is given by (2.12) the equality between the Kähler potentials is not automatically satisfied but rather puts a constraint on the intersection numbers d_{ABCD} or more generally on the Calabi-Yau fourfold. It would be desirable to find those fourfolds which correspond to a perturbative heterotic vacuum exactly as was done in [85] for type IIA compactified on Calabi-Yau threefolds. Here we only make the first step in this direction in that we show that for the $(1,1)$-moduli of K3-fibered fourfolds (2.66) holds in the large base limit.

As the base B_2 we take in both theories a Hirzebruch surface $\mathbb{F}_n$ with n even and freeze the values of the scalars $\hat{N}^I, Z^\alpha$ on the M-theory side. In this case the M-theory Kähler potential simplifies $K^{(3)}_{\rm M} = K_{1,1}$, where $K_{1,1}$ is given in (2.55). The particular geometry of the Calabi-Yau fourfold we are considering affects the form of its intersection numbers and thus in view of (C.37) also the form of $\tilde{G}_{AB}$. But (C.37) determines $\tilde{G}_{AB}$ in terms of the coordinates $\tilde{M}^A$. In order to find a map between the heterotic and the M-theory variables one has to compare the Kähler potentials expressed in the Kähler coordinates. In general it is not possible to invert (2.46) and explicitly express $\tilde{M}^A$ (and thus $\tilde{G}_{AB}$) in terms of the T^A. However, as we will see shortly the explicit relation between $\tilde{M}^A$ and T^A can be obtained in the large base limit.

In this limit the intersection numbers of Y_4 enjoy specific properties. In order to understand this we need to know the group of its divisors[12], i.e. $H_6(Y_4, \mathbb{Z})$. A similar analysis for K3 fibered threefolds has been performed in [106]. Let $\pi : Y_4 \to B_2$ denote the projection map of the K3 fibered fourfold. One has three different contributions to $H_6(Y_4, \mathbb{Z})$:

1. Each divisor C of the base B_2 contributes a divisor D of Y_4 via its pullback $D = \pi^\star C$.

2. Each monodromy invariant algebraic two-cycle G_i in the generic fiber F_0 gives a divisor D_i of Y_4 if it is transported around the base.

3. When the K3 fiber degenerates over a locus of codimension one in the base in such a way that the degenerate fiber is reducible, the volumes of its components can be varied independently. Therefore such reducible "bad fibers" B_a contribute further elements to $H_6(Y_4, \mathbb{Z})$.

[12] For a definition see footnote 22 in section B.10.

The intersection of two divisors of the first kind can be traced back to the intersection of the corresponding divisors in the base:

$$(D_1 \cdot D_2)_{Y_4} = (\pi^\star C_1 \cdot \pi^\star C_2)_{Y_4} = \pi^\star (C_1 \cdot C_2)_{B_2}, \tag{2.67}$$

which is not a number, but a four-cycle in the Calabi-Yau fourfold. The subscript indicates the space within which we are considering the intersection theory. As explained in [107] $\mathbb{F}_n$ (with n even) is topologically the product of two S^2's, whose areas are two Kähler parameters U and V. The corresponding divisors C_U and C_V have the intersection numbers $(C_U \cdot C_U)_{B_2} = (C_V \cdot C_V)_{B_2} = 0$ and $(C_U \cdot C_V)_{B_2} = 1$. In view of (2.67) the intersections of the two related divisors of the fourfold are either zero or the generic K3-fiber F_0, i.e. $(D_U \cdot D_U)_{Y_4} = 0$, $(D_V \cdot D_V)_{Y_4} = 0$ and $(D_U \cdot D_V)_{Y_4} = F_0$. From this fact follows that the intersection of three and four D_U, D_V automatically vanishes and that $D_U \cdot D_V$ has no intersection with divisors coming from bad fibers B_a. Finally, we have

$$(D_U \cdot D_V \cdot D_i \cdot D_j)_{Y_4} = (G_i \cdot G_j)_{F_0}, \tag{2.68}$$

where G_i, G_j and D_i, D_j are defined in the second entry of the list above. In particular G_i and G_j are dual to elements of the (monodromy invariant) part of the Picard lattice of the generic fiber which has signature $(+, -, \ldots, -)$, see [106] or section B.9. Thus it is possible to choose the divisors of F_0 in a way that the right hand side of (2.68) is given by $\eta_{ij} = \mathrm{diag}(+, -, \ldots, -)$.

In the large base limit the (rescaled) volume $\tilde{\mathcal{V}} = \frac{1}{4!} d_{ABCD} \tilde{M}^A \tilde{M}^B \tilde{M}^C \tilde{M}^D$ is dominated by those terms which contain a maximal number of base moduli. From what we have just said it is clear that the leading contribution is

$$\tilde{\mathcal{V}} = \frac{1}{2} d_{UVij} \tilde{U}\tilde{V}\tilde{M}^i\tilde{M}^j = \frac{1}{2}\tilde{U}\tilde{V}\eta_{ij}\tilde{M}^i\tilde{M}^j, \tag{2.69}$$

where $\tilde{U}\tilde{V}$ is the (rescaled) volume of the base and $\frac{1}{2}\eta_{ij}\tilde{M}^i\tilde{M}^j$ that of the generic fiber. Obviously we are in an adiabatic regime, in which to leading order the volume is given by the product of the base and fiber volumes $\tilde{\mathcal{V}}_{Y_4} = \tilde{\mathcal{V}}_{B_2}\tilde{\mathcal{V}}_{K3}$. Using (2.69) and (C.37) we can compute $\tilde{G}_{AB}$ for the moduli $\tilde{U}$ and $\tilde{V}$ of the base and the moduli $\tilde{M}^i$ of the generic fiber in the large base limit[13]

$$\tilde{G}_{AB} = \begin{pmatrix} \frac{1}{2\tilde{U}^2} & 0 & 0 \\ 0 & \frac{1}{2\tilde{V}^2} & 0 \\ 0 & 0 & -\frac{\eta_{ij}}{\eta_{kl}\tilde{M}^k\tilde{M}^l} + \frac{2\eta_{ik}\tilde{M}^k\eta_{jl}\tilde{M}^l}{(\eta_{kl}\tilde{M}^k\tilde{M}^l)^2} \end{pmatrix}. \tag{2.70}$$

Inserted into (2.46) using (2.53) one obtains

$$\begin{aligned} T^U + \bar{T}^U &= \frac{1}{4}\left(\frac{\tilde{V}}{\tilde{U}}\right)^{1/2} (\eta_{kl}\tilde{M}^k\tilde{M}^l)^{1/2} \\ T^V + \bar{T}^V &= \frac{1}{4}\left(\frac{\tilde{U}}{\tilde{V}}\right)^{1/2} (\eta_{kl}\tilde{M}^k\tilde{M}^l)^{1/2} \\ T^i + \bar{T}^i &= \frac{1}{2}(\tilde{U}\tilde{V})^{1/2}\frac{\eta_{ij}\tilde{M}^j}{(\eta_{kl}\tilde{M}^k\tilde{M}^l)^{1/2}}. \end{aligned} \tag{2.71}$$

[13] At this point we neglect the possibility of moduli arising from bad fibers and discuss their contribution at the end of the section.

In connection with (2.69) this implies

$$\tilde{\mathcal{V}} = 32(T^U + \bar{T}^U)(T^V + \bar{T}^V)(T^i + \bar{T}^i)\eta_{ij}(T^j + \bar{T}^j) \ . \tag{2.72}$$

Thus in the large B_2 limit the volume also factorizes in the Kähler coordinates T^A. The M-theory Kähler potential reads in this limit

$$K^{(3)}_{\text{M}} = -\ln \tilde{\mathcal{V}} = -\ln \Big[(T^U + \bar{T}^U)(T^V + \bar{T}^V)(T^i + \bar{T}^i)\eta_{ij}(T^j + \bar{T}^j)\Big] \ , \tag{2.73}$$

and can now be compared to the $K^{(3)}_{\text{het}}$ of the heterotic vacua specified in (2.15).

On the heterotic side we take the large base limit together with the large S-limit (weak four-dimensional coupling). Thus only S', T', D^a and the $(1,1)$-moduli U, V of the base B_2 have to be taken into account while all other $(1,1)$-moduli, all $(2,1)$- and all gauge bundle moduli can be frozen at generic values. In elliptically fibered threefolds Y_3 with the Hirzebruch surface $\mathbb{F}_n$ as a base the Kähler potential of U and V simplifies to $\tilde{K}^{(4)} = -\ln[(U + \bar{U})(V + \bar{V})]$ [107, 108]. So altogether we have on the heterotic side

$$K^{(3)}_{\text{het}} = -\ln[(U + \bar{U})(V + \bar{V})] - \ln[(S' + \bar{S}')^2 - (T' + \bar{T}')^2 - (D^a + \bar{D}^a)^2] \ . \tag{2.74}$$

The two Kähler potentials agree if one identifies

$$(S', T', D^a) \leftrightarrow (T^i) \ , \qquad (U, V) \leftrightarrow (T^U, T^V) \ . \tag{2.75}$$

We see that the moduli which parameterize the base B_2 are identified on both sides and the $(1,1)$-moduli of the generic K3-fiber on the M-theory side are identified with the four-dimensional dilaton, the radius of S^1 and the scalars D^a related to the heterotic vector multiplets. Note that the correspondence (2.75) implies according to (2.19) and (2.71)

$$e^{-4\Phi^{(3)}_{\text{het}}} \sim (T^i + \bar{T}^i)\eta_{ij}(T^j + \bar{T}^j) \sim \tilde{\mathcal{V}}_{B_2} = \mathcal{V}_{B_2}\mathcal{V}, \tag{2.76}$$

which agrees with (2.65) obtained in the adiabatic regime. Furthermore the number of moduli T^i on the M-theory side is bounded by the maximal rank of the Picard group of the generic fiber, i.e. by $h^{(1,1)}(F_0) = 20$. This is consistent with the bound on the heterotic gauge group given in (2.3). In fact it is a little lower but that could be related to the fact that our analysis is based on purely classical geometry. (The same issue arises in type IIA vacua compactified on threefolds [85].)

So far we neglected moduli arising from reducible bad K3 fibres in the fourfold, i.e. we considered no divisors of the third kind. In the context of the duality between the heterotic string compactified on $K3 \times T^2$ and the type IIA string on K3-fibered Y_3 the reducible bad K3 fibres were shown to be related to non-perturbative physics on the heterotic side [85]. The same is true for M- and F-theory compactifications on K3-fibered threefolds Y_3 [107, 109]. For those K3-fibered fourfolds Y_4 which can be adiabatically obtained as K3-fibered threefolds over $\mathbb{P}^1$ it follows that the reducible bad K3 fibres also correspond to non-perturbative physics on the heterotic side [62]. Finally, the identification of the $(2,1)$- and $(3,1)$- moduli of Y_4 with the elliptic fiber, the $(2,1)$- and the bundle-moduli of Y_3 on the heterotic side is discussed in [28, 48, 54, 60, 62].

2.3.2 F-theory limit

Ultimately, one is interested in lifting the M-theory/heterotic duality discussed so far to four space-time dimensions. This amounts to a simple decompactification on the heterotic side while in the dual theory one has to take the F-theory limit [35]. The following discussion of the F-theory limit is strongly inspired by [106] where a similar decompactification limit is discussed in the context of type IIA/heterotic duality in four dimensions.

Taking the F-theory limit requires that the Calabi-Yau fourfold is elliptically fibered. Here we focus on those fourfolds which are in addition K3-fibered. Thus the F-theory limit for this restricted class of fourfolds requires K3-fibres which themselves are elliptically fibered over a base $\mathbb{P}^1$. The first step is to blow down[14] any rational curves in the generic K3-fiber which take the Picard number above its minimal value of 2. (The two moduli which are always present arise from the base $\mathbb{P}^1$ and the elliptic fiber.) The map (2.75) implies that on the heterotic side this corresponds to freezing the scalars corresponding to the real part of D^a.[15] After blowing down the Picard lattice is given by the even self-dual lattice $\Gamma_{1,1}$, which we suppose to be generated by the null vector v and its dual $v^\star$. The Kähler form of the generic K3-fiber is given by

$$J = \sqrt{\frac{1}{2\beta}}\, v^\star + \sqrt{\frac{\beta}{2}}\, v \ , \qquad \beta \in \mathbb{R}^+ \ , \tag{2.77}$$

where the coefficients have been chosen such that the volume of the generic fiber in M-theory units is of order one. As we are only interested in the qualitative features of the F-theory limit we do not keep track of any constants of order one in the following. The class of the base of the K3 is given by $v^\star - v$ (having self-intersection number -2) and the class of the elliptic fiber by v (with self-intersection number 0). The choice $d_{UVij} = \eta_{ij} = (+,-,\dots,-)$ in our previous discussion corresponds to choosing the divisors of the generic K3-fiber in such a way that the forms representing the cohomology classes of the Poincaré duals of these divisors are given by $A = \frac{1}{\sqrt{2}}(v + v^\star)$ and $B = \frac{1}{\sqrt{2}}(v - v^\star)$. One verifies that they obey $A.A = 1$, $B.B = -1$ and $A.B = 0$. ($A.A \equiv \int_{K3} A \wedge A$, etc.) Expanded in this basis the Kähler form of the generic K3-fiber (2.77) is given by

$$J = \frac{1}{2}\Big(\frac{1}{\sqrt{\beta}} + \sqrt{\beta}\Big) A + \frac{1}{2}\Big(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\Big) B \ . \tag{2.78}$$

This implies the identification $M^1 = \frac{1}{2}\Big(\sqrt{\beta} + \frac{1}{\sqrt{\beta}}\Big)$ and $M^2 = \frac{1}{2}\Big(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\Big)$. Inserted into (2.71) using (2.76) and the fact that the volume of the K3 is of order one we derive

$$\begin{aligned} T^1 + \bar{T}^1 &\sim e^{-2\Phi^{(3)}_{\text{het}}} \left(\sqrt{\beta} + \frac{1}{\sqrt{\beta}}\right) \ , \\ T^2 + \bar{T}^2 &\sim -e^{-2\Phi^{(3)}_{\text{het}}} \left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right) \ . \end{aligned} \tag{2.79}$$

From the map (2.75) we expect $S' = \frac{1}{2}(S + T) \leftrightarrow T^1$ and $T' = \frac{1}{2}(S - T) \leftrightarrow T^2$. Thus

[14] This is the opposite of blowing up, see footnote 23 in section B.10.

[15] Strictly speaking we should first undo the duality transformation which related the three-dimensional vector multiplets to chiral multiplets since in the F-theory limit the four-dimensional vector multiplets (which have no scalar) have to be recovered.

we obtain

$$e^{-2\Phi^{(4)}_{\rm het}} \sim S + \bar{S} \sim (T^1 + \bar{T}^1) + (T^2 + \bar{T}^2) \sim e^{-2\Phi^{(3)}_{\rm het}} \frac{1}{\sqrt{\beta}} \,, \tag{2.80}$$

$$r^2 \sim T + \bar{T} \sim (T^1 + \bar{T}^1) - (T^2 + \bar{T}^2) \sim e^{-2\Phi^{(3)}_{\rm het}} \sqrt{\beta}. \tag{2.81}$$

The volume of the elliptic fiber is given by

$$J.v = \frac{1}{\sqrt{2\beta}} \,, \tag{2.82}$$

which has to be taken to zero in the F-theory limit [35]. In view of (2.82) this is equivalent to $\beta \to \infty$. On the heterotic side it corresponds to the expected decompactification limit $r \to \infty$ (with $re^{2\Phi^{(3)}_{\rm het}} = e^{2\Phi^{(4)}_{\rm het}}$ fixed) as can be seen from (2.81). Finally, we learn from (2.16) and (2.80) that $\sqrt{\beta} \sim r_s$.

Chapter 3

$D = 2$ effective theories with four supercharges

We now want to extend the analysis of the last chapter to heterotic theories with four unbroken supercharges in $D = 2$ and their duality to Calabi-Yau fourfold compactifications of type IIA string theory.

3.1 Heterotic theories in $D = 2$

In this section we perform a Kaluza-Klein reduction of the generic $D = 4, N = 1$ heterotic effective theory (2.1) on a torus T^2. This results in an effective theory in $D = 2$ with $(2, 2)$ supersymmetry. We again focus only on the vector multiplets and the chiral moduli multiplets in the four-dimensional effective action and ignore all charged matter multiplets. Let us first discuss the spectrum of the resulting theory. The chiral multiplets survive the reduction unaltered and continue to contain a complex scalar field as bosonic component. The vectors on the other hand decompose according to

$$A^a_n dx^n = A^a d\zeta + \bar{A}^a d\bar{\zeta} + A^a_\mu dx^\mu \ , \tag{3.1}$$

where $\mu = 0, 1,\ a = 1, \ldots, \mathrm{dim}G$ and ζ is the complex coordinate on the torus. The vectors A^a_μ do not have any dynamical degrees of freedom and play the role of auxiliary fields. Thus the physical bosonic components of the vector multiplets in $D = 2$ consist of the complex scalars A^a. These scalars transform in the adjoint representation of the gauge group G and thus their vacuum expectation values break G to its maximal Abelian subgroup $U(1)^{\mathrm{rank}(G)}$. As in $D = 3$ and with a slight abuse of notation the index a now only takes the values $a = 1, \ldots, \mathrm{rank}(G)$. On this Coulomb branch each vector occurs in the Lagrangian only via its Abelian field strength. In this case the field strength can be replaced by an auxiliary scalar field [110]. One ends up with the field content of a twisted chiral multiplet [83]. This is one of the two different kinds of matter multiplets, which can occur in two-dimensional theories with $(2, 2)$ supersymmetry, the other one being the chiral multiplet. We later use the fact that under certain conditions they are dual to each other. More precisely, if the Lagrangian is invariant under a Peccei-Quinn shift symmetry of a chiral multiplet one can perform a duality transformation replacing a chiral by a twisted chiral multiplet - and vice versa [83].

Finally, we need to decompose the four-dimensional space-time metric in the Einstein-frame according to

$$g^{(4)}_{mn} dx^m \otimes dx^n = g^{(2)}_{\mu\nu} dx^\mu \otimes dx^\nu + b_{\mu i} dx^\mu \otimes dx^i + h_{ij} dx^i \otimes dx^j \ , \quad i,j = 2,3 \ , \tag{3.2}$$

where $g^{(2)}_{\mu\nu}$ and $b_{\mu i}$ have no dynamical degrees of freedom. The real coordinates x^2, x^3 are related to the complex ζ of (3.1) via $d\zeta = dx^2 + i\tau dx^3$ where τ is the complex structure modulus of the torus given in (B.62).

We omit the details of the Kaluza-Klein compactification on the torus and only present the resulting two-dimensional effective Lagrangian. Inserting (3.1) and (3.2) into (2.1) one derives the following two-dimensional effective Lagrangian

$$\begin{aligned} \mathcal{L}^{(2)}_{\text{het}} &= \sqrt{h}\left(-\frac{1}{2}\partial_\mu\partial^\mu\sigma - \frac{\partial_\mu\tau\partial^\mu\bar\tau}{(\tau+\bar\tau)^2} - G^{(4)}_{I\bar J}\partial_\mu\Phi^I\partial^\mu\bar\Phi^{\bar J}\right) \\ &\quad - (\tau+\bar\tau)^{-1}(\text{Re}f_{ab})D_\mu n^a D^\mu \bar n^b - \frac{i}{2}\epsilon^{\mu\nu}(\tau+\bar\tau)^{-1}(\partial_\mu \text{Im} f_{ab})\left(\bar n^a D_\nu n^b - n^a D_\nu \bar n^b\right), \end{aligned} \tag{3.3}$$

where h is the determinant of h_{ij}, we defined

$$n^a = -i(\tau+\bar\tau)\bar A^a \tag{3.4}$$

and the covariant derivatives of n^a are given by

$$D_\mu n^a \equiv \partial_\mu n^a + (\tau+\bar\tau)^{-1}\partial_\mu\tau\big(n^a + \bar n^a\big) \ . \tag{3.5}$$

Moreover we have chosen the conformal gauge for the two-dimensional metric

$$g^{(2)}_{\mu\nu} = e^\sigma h^{1/4}\eta_{\mu\nu} \ . \tag{3.6}$$

As in equation (2.1) the Φ^I are all moduli of the four-dimensional Lagrangian including the dilaton S. Furthermore we have omitted all vectors because they do not have dynamical degrees of freedom in $D = 2$.

As has been shown in [83] the moduli space of non-linear $(2,2)$ sigma-models in $D = 2$ is in general not Kähler, when chiral and twisted chiral multiplets are present. Nevertheless the Lagrangian can be expressed by second-order partial derivatives of a real function of the moduli, analogous to the Kähler potential of a Kähler manifold. In case of dilaton-supergravity the general form of the Lagrangian has been given in [81]. In our case $\sqrt{h}$ takes the role of the two-dimensional dilaton. In fact $\sqrt{h}$ is the two-dimensional heterotic dilaton because

$$e^{-2\Phi^{(2)}_{\text{het}}} = \text{Re}S\sqrt{h_s} = \sqrt{h} \ , \tag{3.7}$$

where h_s is the determinant of the metric in the string-frame. The last equality is due to the Weyl rescaling relating the four-dimensional Einstein- and string-frame metrics. The Lagrangian (3.3) is the sum of two terms, one multiplied by $\sqrt{h}$, the other one not. Both terms can separately be expressed via a potential.

Defining the two real functions

$$\begin{aligned} K^{\text{het}}_1 &= K^{(4)}(\Phi,\bar\Phi) + \ln(\tau+\bar\tau), \\ K^{\text{het}}_2 &= -\frac{1}{2}(\text{Re}f_{ab}(\Phi))\left[(\tau+\bar\tau)^{-1}(n^a+\bar n^a)(n^b+\bar n^b)\right] \end{aligned} \tag{3.8}$$

and denoting collectively (τ, n^a) as χ^Σ we can express (3.3) in the form

$$\begin{aligned}\mathcal{L}^{(2)}_{\rm het} &= \sqrt{h}\left(-\frac{1}{2}\partial_\mu\partial^\mu\sigma - \left(\partial_{\Phi^I}\bar\partial_{\bar\Phi^{\bar J}}K_1^{\rm het}\right)\partial_\mu\Phi^I\partial^\mu\bar\Phi^{\bar J} + \left(\partial_{\chi^\Sigma}\bar\partial_{\bar\chi^{\bar\Lambda}}K_1^{\rm het}\right)\partial_\mu\chi^\Sigma\partial^\mu\bar\chi^{\bar\Lambda}\right)\\ &\quad + \left(\partial_{\chi^\Sigma}\bar\partial_{\bar\chi^{\bar\Lambda}}K_2^{\rm het}\right)\partial_\mu\chi^\Sigma\partial^\mu\bar\chi^{\bar\Lambda} \\ &\quad - \epsilon^{\mu\nu}\left(\bar\partial_{\bar\Phi^{\bar J}}\partial_{\chi^\Sigma}K_2^{\rm het}\right)\partial_\mu\bar\Phi^{\bar J}\partial_\nu\chi^\Sigma - \epsilon^{\mu\nu}\left(\partial_{\Phi^I}\bar\partial_{\bar\chi^{\bar\Lambda}}K_2^{\rm het}\right)\partial_\mu\Phi^I\partial_\nu\bar\chi^{\bar\Lambda}.\end{aligned} \tag{3.9}$$

This is of the form given in [81, 83]. Obviously the moduli space is not Kähler. As explained in [83] the terms proportional to $\epsilon^{\mu\nu}$ always combine a derivative with respect to a chiral field with a derivative with respect to a twisted chiral field. The Φ^I stem from the four-dimensional chiral moduli and continue to be chiral in $D = 2$. In view of the last line in (3.9) this means that (τ, n^a) reside in twisted chiral multiplets.

To proceed further we insert the tree level form of the gauge kinetic function and the Kähler potential given in (2.4) and (2.5) into the action (3.3). It turns out that in this case the moduli space is actually a Kähler manifold despite the fact that there are both chiral and twisted chiral fields present. However to make this manifest one has to dualize the axion ImS.[1] Using (3.7) one derives

$$\begin{aligned}\mathcal{L}^{(2)}_{\rm het} &= e^{-2\Phi^{(2)}_{\rm het}}\Big[-\frac{1}{2}\partial_\mu\partial^\mu\sigma - \frac{\partial_\mu\tau\partial^\mu\bar\tau}{(\tau+\bar\tau)^2} - \frac{1}{4}h_s^{-1}\partial_\mu\sqrt{h_s}\partial^\mu\sqrt{h_s} - \tilde G^{(4)}_{I\bar J}\partial_\mu\phi^I\partial^\mu\bar\phi^{\bar J} \\ &\quad -(\tau+\bar\tau)^{-1}\sqrt{h_s}^{-1}D_\mu\bar n^a D^\mu n^a - h_s^{-1}\Big(\partial_\mu P + \frac{i}{2}(\tau+\bar\tau)^{-1}[\bar n^a D_\mu n^a - n^a D_\mu\bar n^a]\Big)^2\Big]\ ,\end{aligned} \tag{3.10}$$

where this time we have chosen the conformal gauge for the two-dimensional metric

$$g^{(2)}_{\mu\nu} = e^\sigma e^{2\Phi^{(2)}_{\rm het}}\eta_{\mu\nu}\ . \tag{3.11}$$

Finally, the index I now denotes all moduli except the dilaton and $\tilde G^{(4)}_{I\bar J}$ is the Kähler metric of the Kähler potential $\tilde K^{(4)}_{\rm het}$ given in (2.5).

Defining the complexified Kähler modulus as

$$\rho \equiv 2iP + \sqrt{h_s} + (\tau+\bar\tau)^{-1}n^a(n^a+\bar n^a) \tag{3.12}$$

one verifies that the Lagrangian (3.10) can be written in the form

$$\mathcal{L}^{(2)}_{\rm het} = e^{-2\Phi^{(2)}_{\rm het}}\left[-\frac{1}{2}\partial_\mu\partial^\mu\sigma - G_{\bar\Lambda\Sigma}\partial_\mu\bar Z^{\bar\Lambda}\partial^\mu Z^\Sigma\right]\ , \tag{3.13}$$

where the Z^Σ commonly denote $Z^\Sigma = (\phi^I, \tau, \rho, n^a)$. The Z^Σ are the proper Kähler coordinates in that in these coordinates $G_{\bar\Lambda\Sigma} = \bar\partial_{\bar\Lambda}\partial_\Sigma K^{(2)}_{\rm het}$ with

$$K^{(2)}_{\rm het} = \tilde K^{(4)}_{\rm het}(\phi,\bar\phi) - \ln[(\rho+\bar\rho)(\tau+\bar\tau) - (n^a+\bar n^a)^2]. \tag{3.14}$$

The form (3.13) of the Lagrangian explicitly shows that both $\Phi^{(2)}_{\rm het}$ and σ are non-dynamical degrees of freedom [111] while ρ and τ are propagating degrees of freedom.

[1] More precisely we added the term $\partial_\mu P\epsilon^{\mu\nu}\partial_\nu{\rm Im}S$ to the reduced action and eliminated ImS by its equation of motion. One can also verify that $2P$ is the B_{23} component of the four-dimensional antisymmetric tensor in the compactified directions.

The four physical degrees of freedom in ρ and τ are related to the four-dimensional graviton and dilaton S.

In this parameterization the modular group of the torus $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ is manifest. Its action is a generalization of (B.63). The first $SL(2,\mathbb{Z})$ acts as

$$\tau \to \frac{a\tau - ib}{ic\tau + d}, \quad \rho \to \rho - \frac{ic\, n^a n^a}{ic\tau + d}, \quad n^a \to \frac{n^a}{ic\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{3.15}$$

and the action of the second $SL(2,\mathbb{Z})$ is obtained by exchanging τ and ρ.

τ and ρ both reside in vector or twisted chiral multiplets since in the reduction procedure they come along with the Kaluza-Klein vectors.[2] From (3.14) we learn that the moduli space factorizes into chiral multiplets ϕ^I and twisted chiral multiplets (τ, ρ, n^a) which span the coset space $SO(2,2+r)/SO(2)\times SO(2+r)$, where $r = \mathrm{rank}(G)$. In this case the two-dimensional $(2,2)$ supersymmetric σ-model is known to be Kähler [83], where the Kähler potential is a sum of two terms, one depending on the chiral multiplets and one depending on the twisted chiral multiplets.

Let us mention that there is an alternative way to derive the two-dimensional heterotic effective action. It consists of an S^1-reduction of the effective action derived in section 2.1.2. In this case all scalar fields appear naturally as members of chiral multiplets and as a consequence the metric in these coordinates is always Kähler. The Kähler potential of the two-dimensional effective theory coincides with the Kähler potential of the three-dimensional effective theory. This is summarized in section C.6.1.

3.2 Type IIA theory on Calabi-Yau fourfolds

In this section we discuss the Kaluza-Klein reduction of type IIA string theory on Calabi-Yau fourfolds. Starting point is the low energy effective action in $D = 10$ in the string-frame

$$\begin{aligned} \kappa_{10}^2 \mathcal{S}_{IIA}^{(10)} &= \int d^{10}x\, \sqrt{-g^{(10)}}\, e^{-2\Phi_{\mathrm{IIA}}^{(10)}} \left(\frac{1}{2} R^{(10)} + 2\partial_M \Phi_{\mathrm{IIA}}^{(10)} \partial^M \Phi_{\mathrm{IIA}}^{(10)} - \frac{1}{4}|H_3|^2 \right) \\ &\quad - \frac{1}{4} \int d^{10}x\, \sqrt{-g_{(10)}^{II}} \left(|F_2|^2 + |\tilde{F}_4|^2 \right) - \frac{1}{4} \int B_2 \wedge F_4 \wedge F_4 \ , \end{aligned} \tag{3.16}$$

where $\Phi_{\mathrm{IIA}}^{(10)}$ is the ten-dimensional dilaton, $H_3 = dB_2$ is the field strength of the antisymmetric tensor B_2 and $F_2 = dA_1$ the field strength of the RR vector A_1. $F_4 = dA_3$ is the field strength of the RR 3-form A_3 and we use the abbreviation $\tilde{F}_4 = F_4 - A_1 \wedge H_3$. For further conventions on the notation see appendix A.

In (3.16) only the leading terms of $S_{IIA}^{(10)}$ are displayed and higher derivative couplings are suppressed. In particular the term proportional to $\int B_2 \wedge X_8$, which is related by dimensional reduction to the higher derivative term of M-theory (2.23), imposes a consistency condition on the compactification [74, 76]. The absence of a B_2-tadpole requires again (2.25) where n is now the number of space-time filling strings. In this section we focus on the case $\chi = n = F_4 = 0$ and return to the case of non-trivial F_4 in chapter 4. Furthermore we set $\kappa_{10} \equiv 1$ for the rest of the chapter.

[2]The modulus ρ is the dual of the chiral field S and therefore twisted chiral. For τ we have given a further argument below (3.9).

The spectrum of the two-dimensional theory is determined by the deformations of the Calabi-Yau metric and the expansion of A_1, B_2 and A_3 in terms of the non-trivial forms of Y_4. For the 10-dimensional metric we take the Ansatz

$$g^{(10)}_{MN}(x,y) = \begin{pmatrix} g^{(2)}_{\mu\nu}(x) & 0 \\ 0 & g^{(8)}_{ab}(x,y) \end{pmatrix}, \tag{3.17}$$

The deformations of the metric are the same as in $D=3$, i.e. they comprise $h^{1,1}$ real Kähler deformations $M^A, A = 1,\dots,h^{1,1}$, see (2.28), and $h^{1,3}$ complex deformations $Z^\alpha, \alpha = 1,\dots,h^{1,3}$, of the complex structure, see (2.29). Since vectors contain no physical degree of freedom in $D=2$ and since there are no 1-forms on Y_4 the one-form A_1 does not contribute any massless mode in $D=2$. The antisymmetric tensor B_2 is expanded in terms of the $(1,1)$-forms e_A according to

$$B_{i\bar{\jmath}} = \sum_{A=1}^{h^{1,1}} a^A(x)\, e_{Ai\bar{\jmath}} \tag{3.18}$$

and leads to $h^{1,1}$ real scalar fields a^A while A_3 contributes $h^{1,2}$ complex scalars $N^I, I = 1,\dots,h^{1,2}$, see (2.31). The $(1,1)$-moduli reside in twisted chiral multiplets[3] while all other scalars are members of chiral multiplets.

The dimensional reduction of (3.16) yields

$$\begin{aligned}\frac{\mathcal{L}^{(2)}_{\mathrm{IIA}}}{\sqrt{-g^{(2)}}} &= e^{-2\Phi^{(2)}_{\mathrm{IIA}}}\left(\frac{1}{2}R^{(2)} + 2\partial_\mu\Phi^{(2)}_{\mathrm{IIA}}\partial^\mu\Phi^{(2)}_{\mathrm{IIA}} - G_{A\bar{B}}\partial_\mu t^A\partial^\mu\bar{t}^{\bar{B}} - G_{\alpha\bar{\beta}}\partial_\mu Z^\alpha\partial^\mu\bar{Z}^{\bar{\beta}}\right) \\ &\quad - \left(G_{I\bar{J}}D_\mu N^I D^\mu\bar{N}^{\bar{J}} + \frac{1}{4}d_{AI\bar{J}}\epsilon^{\mu\nu}\partial_\mu a^A\left[N^I D_\nu\bar{N}^{\bar{J}} - \bar{N}^{\bar{J}}D_\nu N^I\right]\right), \end{aligned}\tag{3.19}$$

where we used (2.32), (2.36), (2.38) and (C.32) and the following definitions

$$\begin{aligned} e^{-2\Phi^{(2)}_{\mathrm{IIA}}} &\equiv e^{-2\Phi^{(10)}_{\mathrm{IIA}}}\mathcal{V}\,, \\ t^A &\equiv \frac{1}{\sqrt{2}}(M^A + ia^A)\,. \end{aligned}\tag{3.20}$$

As in the heterotic case there are two terms, one has a factor $e^{-2\Phi^{(2)}_{\mathrm{IIA}}}$ and the other one not. Again both can separately be expressed via a potential. We define

$$\begin{aligned} K^{\mathrm{IIA}}_1 &= -\ln\Big[\int_{Y_4}\Omega\wedge\bar{\Omega}\Big] + \ln\mathcal{V}\,, \qquad (3.21) \\ K^{\mathrm{IIA}}_2 &= -\frac{1}{\sqrt{2}}(t^A+\bar{t}^A)\left(\frac{i}{2}d_{AM\bar{L}}\hat{G}^{-1}_{\bar{J}M}\hat{G}^{-1}_{\bar{L}I}\hat{N}^I\bar{\hat{N}}^{\bar{J}} - \omega_{AIK}\hat{N}^I\hat{N}^K - \bar{\omega}_{A\bar{J}\bar{L}}\bar{\hat{N}}^{\bar{J}}\bar{\hat{N}}^{\bar{L}}\right)\,, \end{aligned}$$

where (2.40), (2.47) and (2.48) have been used.

With the help of K^{IIA}_1 and K^{IIA}_2 and denoting collectively the fields $(Z^\alpha, \hat{N}^I)$ as Φ^Σ the Lagrangian (3.19) can be expressed as

$$\mathcal{L}^{(2)}_{\mathrm{IIA}} = e^{-2\Phi^{(2)}_{\mathrm{IIA}}}\left(-\frac{1}{2}\partial_\mu\partial^\mu\sigma - (\partial_{\Phi^\Sigma}\bar{\partial}_{\bar{\Phi}^{\bar{\Lambda}}}K^{\mathrm{IIA}}_1)\partial_\mu\Phi^\Sigma\partial^\mu\bar{\Phi}^{\bar{\Lambda}} + (\partial_{t^A}\bar{\partial}_{\bar{t}^{\bar{B}}}K^{\mathrm{IIA}}_1)\partial_\mu t^A\partial^\mu\bar{t}^{\bar{B}}\right)$$

[3]Strictly speaking these multiplets are vector multiplets containing in addition the vectors arising from expanding the three-form in terms of the $(1,1)$-forms of Y_4. But as noted in the last section, a vector multiplet is related to a twisted chiral multiplet, differing only in the auxiliary field content.

$$-(\partial_{\Phi^\Sigma}\bar\partial_{\bar\Phi^{\bar\Lambda}}K_2^{\rm IIA})\partial_\mu\Phi^\Sigma\partial^\mu\bar\Phi^{\bar\Lambda} - \epsilon^{\mu\nu}\Big((\bar\partial_{\bar\Phi^{\bar\Lambda}}\partial_{t^A}K_2^{\rm IIA})\partial_\mu\bar\Phi^{\bar\Lambda}\partial_\nu t^A + (\partial_{\Phi^\Lambda}\bar\partial_{\bar t^{\bar A}}K_2^{\rm IIA})\partial_\mu\Phi^\Lambda\partial_\nu\bar t^{\bar A}\Big), \tag{3.22}$$

where the conformal gauge has been chosen

$$g^{(2)}_{\mu\nu} = e^\sigma e^{2\Phi^{(2)}_{\rm IIA}}\eta_{\mu\nu} \ . \tag{3.23}$$

This is of the form given in [81, 83] and the terms proportional to $\epsilon^{\mu\nu}$ display the fact that the Φ^Σ reside in chiral multiplets whereas the t^A are members of twisted chiral multiplets.

To proceed further we restrict to the case where either $h^{2,1}(Y_4) = 0$ or the $(2,1)$-moduli are frozen to some fixed value. This results in

$$\mathcal{L}^{(2)}_{IIA} = \sqrt{-g^{(2)}}\, e^{-2\Phi^{(2)}_{\rm IIA}}\left(\frac{1}{2}R^{(2)} + 2\partial_\mu\Phi^{(2)}_{\rm IIA}\partial^\mu\Phi^{(2)}_{\rm IIA} - G_{A\bar B}\partial_\mu t^A\partial^\mu\bar t^{\bar B} - G_{\bar\alpha\beta}\partial_\mu\bar Z^{\bar\alpha}\partial^\mu Z^\beta\right), \tag{3.24}$$

Now the moduli space factorizes into chiral and twisted chiral multiplets, i.e. it is Kähler although both kinds of multiplets occur with the Kähler potential

$$K^{(2)}_{IIA} = -\ln\big(\int_{Y_4}\Omega\wedge\bar\Omega\big) - \ln\mathcal{V} \ . \tag{3.25}$$

As in the heterotic case there is an alternative way to derive the two-dimensional effective theory making the detour over $D = 3$ and using the results of chapter 2. The details can be found in section C.6.2.

3.3 Heterotic – type IIA duality in D=2

In order to establish the duality relationship between the heterotic and type IIA variables we specify the compactification manifolds as in section 2.3. The Calabi-Yau fourfold is taken to be a $K3$-fibration over a large Hirzebruch surface $\mathbb{F}_n$ with n even while the Calabi-Yau threefold on the heterotic side is an elliptic fibration over the same base. In the large base limit and freezing the $(3,1)$-moduli the type IIA Kähler potential simplifies as discussed in section 2.3

$$K^{(2)}_{IIA} = -\ln\mathcal{V} \ \rightarrow\ -\ln(t^U + \bar t^U) - \ln(t^V + \bar t^V) - \ln\left[\eta_{ij}(t^i + \bar t^i)(t^j + \bar t^j)\right] , \tag{3.26}$$

where t^U and t^V are the moduli of the base while the t^i denote the moduli of the $K3$-fiber except those from reducible bad fibres and η is the intersection matrix of $K3$ as given in section 2.3.

On the heterotic side the Kähler potential (3.14) becomes in the large base limit[4]

$$K^{(2)}_{\rm het} = -\ln(U + \bar U) - \ln(V + \bar V) - \ln[(\tau' + \bar\tau')^2 - (\rho' + \bar\rho')^2 - (n^a + \bar n^a)^2] , \tag{3.27}$$

where U and V are the two base moduli and we have defined $\tau' = \frac{1}{2}(\tau + \rho)$ and $\rho' = \frac{1}{2}(\tau - \rho)$. All other moduli are again frozen at generic values. Comparing (3.26) and (3.27) it is tempting to equate the two expressions. However, this would map the twisted chiral superfields t^U and t^V to chiral superfields U and V. Hence one has to

[4]Strictly speaking this also requires a large volume $\sqrt{h_s}$ of the heterotic torus.

first perform an additional duality transformation on the base moduli U, V so that all heterotic variables are twisted chiral like their type IIA counterparts. This is possible in the large base limit by defining $c_U^\nu = -\epsilon^{\mu\nu}\partial_\mu \mathrm{Im}U$ and $c_V^\nu = -\epsilon^{\mu\nu}\partial_\mu \mathrm{Im}V$ and adding the Lagrange multipliers P_U, P_V via $-c_U^\mu \partial_\mu P_U - c_V^\mu \partial_\mu P_V$ to the action. Using the equations of motion to eliminate c_U^μ and c_V^μ, defining the coordinates

$$u \equiv 2iP_U + e^{-2\Phi_{\mathrm{het}}^{(2)}}(\mathrm{Re}U)^{-1} \ , \qquad v \equiv 2iP_V + e^{-2\Phi_{\mathrm{het}}^{(2)}}(\mathrm{Re}V)^{-1} \tag{3.28}$$

and redefining the two-dimensional metric

$$g_{\mu\nu}^{(2)} = e^{\sigma} e^{4\Phi_{\mathrm{het}}^{(2)}} \mathrm{Re}U \mathrm{Re}V \eta_{\mu\nu} \tag{3.29}$$

one derives the following form of the Kähler potential

$$\hat{K}_{\mathrm{het}}^{(2)} = -\ln[(u+\bar{u})(v+\bar{v})] - \ln[(\tau'+\bar{\tau}')^2 - (\rho'+\bar{\rho}')^2 - (n^a+\bar{n}^a)^2]. \tag{3.30}$$

In this form all the coordinates belong to twisted chiral multiplets. Hence the duality map relates[5]

$$\begin{aligned} (t^i) \ &\leftrightarrow \ (\tau', \rho', n^a), \\ (t^U, t^V) \ &\leftrightarrow \ (u, v). \end{aligned} \tag{3.31}$$

As we discussed in section 3.1 the heterotic theory has an $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ symmetry (3.15) which is just the modular symmetry of the torus. From the map (3.31) one learns that in the type IIA theory this symmetry has to be a property of $K3$-fibered fourfolds in the large base limit. This is precisely the same situation one encounters in the four-dimensional duality relating type IIA compactified on $K3$-fibered threefolds to heterotic vacua compactified on $K3 \times T^2$ [84].

[5] The fact that it is really u and v which should be mapped to t^U and t^V can heuristically be understood from the duality in $D = 6$ between the type IIA theory on K3 and the heterotic theory on T^4. This duality maps the string-frame metrics according to $g_{\mathrm{het}}^{(6)} = e^{2\Phi_{\mathrm{het}}^{(6)}} g_{\mathrm{IIA}}^{(6)}$ and the dilatons are related via $\Phi_{\mathrm{het}}^{(6)} = -\Phi_{\mathrm{IIA}}^{(6)}$. Fibering the six-dimensional duality over a four-dimensional base manifold B yields $\mathcal{V}_B^{\mathrm{IIA}} = e^{-4\Phi_{\mathrm{het}}^{(6)}} \mathcal{V}_B^{\mathrm{het}} = e^{-2\Phi_{\mathrm{het}}^{(6)}} e^{-2\Phi_{\mathrm{het}}^{(2)}}$ and similarly $\mathcal{V}_B^{\mathrm{het}} = e^{-2\Phi_{\mathrm{IIA}}^{(6)}} e^{-2\Phi_{\mathrm{IIA}}^{(2)}} = e^{2\Phi_{\mathrm{het}}^{(6)}} e^{-2\Phi_{\mathrm{IIA}}^{(2)}}$, where all volumes are those of the base measured in the corresponding string-frame metric. Using (C.54) one verifies immediately $\mathcal{V}_B^{\mathrm{IIA}} = (\mathcal{V}_B^{\mathrm{het}} e^{4\Phi_{\mathrm{het}}^{(2)}})^{-1}$.

Chapter 4

Inclusion of background fluxes

4.1 Compactifications with fluxes

In this chapter we want to generalize the previous analysis to the case $\chi \neq 0$ which requires either space-time filling membranes/strings or a non-trivial four-form flux F_4 on Y_4 and a warped space-time metric [73, 80]. We shall concentrate on the effects of non-trivial fluxes and do not consider space-time filling membranes/strings here. Let us therefore recall some general facts about compactifications with fluxes, where we focus on the conditions for residual supersymmetry.

One of the first occasions showing the importance of compactifications with background fluxes is the equivalence of 11-dimensional supergravity on S^7 and $N = 8$ gauged supergravity in $D = 4$, which is reviewed in [112]. Only if compactifications with non-vanishing internal components of the four-form are taken into account can the equivalence be established. However, in order to keep some residual supersymmetry in $D = 4$ the product Ansatz for the metric has to be generalized to include a warp factor as in (C.1). The first warped metric on S^7 leading to residual $N = 1$ supersymmetry in $D = 4$ has been given in [113]. In the context of 11-dimensional supergravity on S^7 retaining some unbroken supersymmetry requires either both, the warp factor and the four-form flux, to be trivial or both to be non-trivial and we will see that this is a rather general feature of compactifications with fluxes.

The S^7 reduction of 11-dimensional supergravity leads to a four-dimensional AdS vacuum. It has been shown in [114, 115] that a compactification of 11-dimensional supergravity to four-dimensional Minkowski space is only compatible with residual supersymmetry if the warp factor is constant and the internal components of the four-form field strength vanish. This statement is however based on the assumption of a smooth compact internal manifold. In case the manifold is non-compact and the background contains 5-brane sources for the four-form flux a four-dimensional Minkowski space can be realized with a non-trivial warp factor [116]. Also four-dimensional heterotic/M-theory compactifications circumvent the assumptions of [114,115] by taking the internal space to be topologically the product of a six-dimensional manifold with an interval, where sources for the four-form flux are present at the ends of the interval. The supersymmetry transformations in this background have been investigated in [117, 118]. In this case there are different ways to warp the geometry [118]. One possibility is to consider a warp factor which only depends on the eleventh coordinate x^{11}, i.e. the position along the interval. This leads to the strongly coupled heterotic string and the six-dimensional manifold can be chosen to be a Calabi-Yau space with x^{11} dependent volume. Another possibility is to take a warp factor which depends on all internal coordinates except x^{11}.[1] This leads to the weakly coupled heterotic string with torsion, i.e. a non-trivial background for the three-form field strength H, which has first been analyzed in [30,115]. It turns out that the warp factor is given by the heterotic dilaton which now depends on the internal coordinates. Furthermore the internal manifold is not a Kähler manifold, the metric is not even conformal to a Calabi-Yau metric [30], and the embedding of the spin connection into the gauge connection is not possible anymore [119].

Type II compactifications on Calabi-Yau threefolds with fluxes have been discussed in [115,120–124]. One finds that at a generic point in the moduli space $N = 2$ supersymmetry is completely broken. Only for appropriately chosen fluxes supersymmetric ground states can exist. This happens for example if the fluxes are aligned with cycles of the threefold which can degenerate at specific points in the moduli space. These points (or subspaces) then correspond to $N = 2$ supersymmetric ground states. A partial breaking of $N = 2$ to $N = 1$ seems to be only possible if gravity is decoupled and certain fields are taken to be non-dynamical [122].

Similar features arise in compactifications to lower dimensions. In view of the importance for our analysis in the next section let us go into some of the details for the case of M-theory compactified on a Calabi-Yau fourfold [73]. It turns out that a three-dimensional Minkowski space vacuum is consistent with $N = 2$ supersymmetry in the presence of a non-trivial warp factor and four-form flux if higher derivative corrections are taken into account. More specifically supersymmetry can be maintained if the metric is not a direct product metric but instead includes a warp factor Δ

$$\hat{g}_{MN} = \begin{pmatrix} e^{-\Delta} g_{\mu\nu} & 0 \\ 0 & e^{\frac{1}{2}\Delta} g_{ab} \end{pmatrix} , \tag{4.1}$$

where to leading order in κ_{11} g_{ab} is a Ricci-flat Calabi-Yau metric.[2] The warp factor Δ is determined by

$$\nabla_a \partial^a \Delta = -\frac{1}{3} \star (G_4 \wedge G_4) - \frac{4}{3} T_2 \kappa_{11}^2 \star X_8 , \tag{4.2}$$

where X_8 and T_2 have been defined in and below (2.23) and the Laplace operator and the Hodge $\star$-operator are defined with respect to the metric g_{ab}.[3] Due to the maximal symmetry of the three-dimensional space-time the only non-vanishing components of F_4 are $F_{abcd} \equiv G_{abcd}$ and $F_{\mu\nu\rho a}$, where the latter are related via supersymmetry to the warp factor according to $F_{\mu\nu\rho a} = \epsilon_{\mu\nu\rho} \partial_a e^{-\frac{3}{2}\Delta}$. (We use G_4 to denote just the internal background flux in distinction to the whole F_4 background.) Moreover supersymmetry puts the following constraints on the internal components of F_4

$$G_{4,0} = 0 = G_{3,1} , \qquad J \wedge G_4 = 0 , \tag{4.3}$$

where the second condition is just the requirement for G_4 to be primitive, see (B.40).

Since the Hodge decomposition (B.35) depends on the complex structure, see section (B.7), the supersymmetry constraints (4.3) lead to a lifting of the moduli space. They leave only those complex structure deformations in the moduli space for which

[1] The general case of a warp factor depending on all internal coordinates is also discussed in [118].
[2] Note that our notation slightly differs from [73].
[3] In (4.2) we have adapted the formula given in [73] to the conventions we will use in the next section.

G_4 remains a $(2,2)$-form and only those Kähler moduli remain massless which leave G_4 primitive. This has led to the proposal in [67] to implement the supersymmetry constraints (4.3) in the three-dimensional effective action via two superpotentials[4]

$$W = \int_{Y_4} \Omega \wedge G_4 \ , \qquad \tilde{W} = \int_{Y_4} \mathcal{K} \wedge \mathcal{K} \wedge G_4 \ , \tag{4.4}$$

where $\mathcal{K} = T^A e_A$ is a complexified Kähler form. Thus W depends only on the complex structure moduli and $\tilde{W}$ on the Kähler moduli of Y_4. Now the constraints (4.3) translate into the conditions

$$D_\alpha W = 0 = W \ , \qquad \partial_A \tilde{W} = 0 \ . \tag{4.5}$$

To see the equivalence with (4.3) we have to use [94]

$$D_\alpha \Omega \equiv \partial_\alpha \Omega + (\partial_\alpha K_{3,1})\, \Omega = \Phi_\alpha \ , \tag{4.6}$$

where, as in chapter 2, Φ_α is the basis of $H^{3,1}(Y_4)$ and $K_{3,1}$ is the Kähler potential for the $(3,1)$-moduli defined in (2.37). In the next section we derive the potential via a Kaluza-Klein reduction with a non-trivial four-form flux and using the Ansatz (4.1). We find basic agreement with [67] except that one of our superpotentials is the real version of $\tilde{W}$.

The analysis of [73] has been generalized in two ways. First it has been adopted to the compactification of massive type IIA string theory on Calabi-Yau fourfolds in [80].[5] This time the range of possible background fluxes is broader because the IIA theory contains more antisymmetric tensor fields. The conditions for unbroken supersymmetry can be expressed via similar superpotentials as those given in (4.4) but involving also the other possible background fluxes. Again the metric is a warped product with the warp factor related to the dilaton. The precise form of the warped metric seems to be missing in the literature. However, in [71] it has been noted that (4.1) coincides with the metric of a 2-brane in $D = 11$. More generally the metric of a $(d-1)$-brane in a D-dimensional space is given by

$$\hat{g}_{MN} = \begin{pmatrix} e^{-\Delta} g_{\mu\nu} & 0 \\ 0 & e^{b\Delta} g_{ab} \end{pmatrix} \tag{4.7}$$

with $b = d/(D-d-2)$. We expect also the metric of type IIA theory on a Calabi-Yau fourfold with fluxes to be of the form (4.7), with $b = 1/3$.

A second generalization of [73] has been performed in [126], where 11-dimensional supergravity is compactified on a $Spin(7)$ manifold. This leads only to $N = 1$ supersymmetry in $D = 3$ (i.e. two supercharges) and is therefore not relevant for the purpose of our thesis. The warped metric is again given by (4.1), where now g_{ab} is the $Spin(7)$ metric.

[4]Another way to justify them is the following [67]. 5-branes are magnetic charges for the four-form field strength. If they are wrapped over a four-cycle c_4 of Y_4 they appear as domain walls in the three-dimensional space-time. Crossing this domain wall changes the four-form by $\Delta G_4 = \gamma_4$, where γ_4 is the Poincaré dual of c_4. If the four-cycle is a special Lagrangian cycle, which has minimal volume with respect to the measure ReΩ, the domain wall is supersymmetric and its tension is the absolute value of $\int_{c_4} \Omega = \int_{Y_4} \Omega \wedge \Delta G$. Furthermore in a theory with four supercharges the tension of a supersymmetric domain wall is the absolute value of the change in the superpotential. This justifies the definition of W in (4.4). For $\tilde{W}$ a similar argument can be given.

[5]The massive generalization of IIA supergravity has been derived in [125] and contains an additional mass parameter.

4.2 Derivation of the potential in $D = 3$

In this section we compute some of the corrections to the Lagrangian (C.40) which result from higher order terms and non-vanishing background flux. In doing so we fill in some of the steps left out in [89]. For simplicity we consider only the case where the $(2,1)$-scalars are frozen and perform a Kaluza-Klein reduction keeping only the $(1,1)$- and $(3,1)$-modes. We focus on the potential to order $\mathcal{O}(\kappa_{11}^{-2/3})$ and a Chern-Simons term while the corrections to the kinetic terms of (C.40) are not calculated.

As we have reviewed in the last section the inclusion of background flux is consistent with a three-dimensional Minkowski space and residual supersymmetry if higher derivative terms are taken into account, see (4.2). One of them has already been given in (2.23), which leads to the tadpole cancellation condition (2.25). In the absence of space-time filling membranes and reinstating κ_{11} (2.25) reads

$$\frac{1}{4\kappa_{11}^2}\int_{Y_4} G_4\wedge G_4 \;=\; \frac{T_2}{24}\,\chi\ . \tag{4.8}$$

The only other 11-dimensional term we need to consider is[6]

$$\delta\mathcal{S}_2^{(11)} = b_1 T_2\int d^{11}x\sqrt{-g}(J_0-\frac{1}{2}E_8)\ , \tag{4.9}$$

where $b_1^{-1}\equiv(2\pi)^4 3^2 2^{13}$ and

$$\begin{aligned} E_8 &= \frac{1}{3!}\epsilon^{ABCM_1N_1\ldots M_4N_4}\epsilon_{ABCM_1'N_1'\ldots M_4'N_4'}\hat R^{M_1'N_1'}{}_{M_1N_1}\ldots\hat R^{M_4'N_4'}{}_{M_4N_4}\ , \\ J_0 &= t^{M_1N_1\ldots M_4N_4}t_{M_1'N_1'\ldots M_4'N_4'}\hat R^{M_1'N_1'}{}_{M_1N_1}\ldots\hat R^{M_4'N_4'}{}_{M_4N_4}+\frac{1}{4}E_8\ . \end{aligned} \tag{4.10}$$

The tensor t is defined by $t^{M_1\ldots M_8}A_{M_1M_2}\ldots A_{M_7M_8} = 24\mathrm{tr}A^4-6(\mathrm{tr}A^2)^2$ for antisymmetric tensors A.[7] Note that E_8 given in (4.10) is not the eight-dimensional Euler density but an 11-dimensional generalization of it. More generally one can define [127]

$$\begin{aligned} E_n(M_D) \;=\; &\frac{1}{(D-n)!}\,\epsilon_{N_1\ldots N_{D-n}N_{D-n+1}\ldots N_D}\,\epsilon^{N_1\ldots N_{D-n}N'_{D-n+1}\ldots N'_D} \\ &R^{N_{D-n+1}N_{D-n+2}}{}_{N'_{D-n+1}N'_{D-n+2}}\ldots R^{N_{D-1}N_D}{}_{N'_{D-1}N'_D}\ , \end{aligned} \tag{4.11}$$

where D denotes the real dimension of the manifold. Then $E_8(Y_4)$ is proportional to the eight-dimensional Euler density (B.5), i.e.

$$12b_1\int_{Y_4} d^8y\sqrt{g}\,E_8(Y_4) = \chi\ . \tag{4.12}$$

The Kaluza-Klein reduction is a good approximation if the size of the internal Y_4 manifold is large compared to the 11-dimensional Planck length $l_{11}^9=\kappa_{11}^2$ or in other words for $l_Y \gg l_{11}$ where l_Y^8 is the 'average' size of Y_4. From (4.8) we infer that

[6] All other bosonic higher derivative terms which are related via supersymmetry to the ones given in (2.23) and (4.9) are proportional to the Ricci-tensor or contain at least one 4-form field strength [127]. Their contribution to the potential is therefore subleading.

[7] We follow here the conventions of [127] which differ from the tensor t_8 used in [17] in that the ϵ-term is omitted.

$G_4 \sim \mathcal{O}(l_{11}^3/l_Y^4)$ while (4.2) implies $\Delta \sim \mathcal{O}(l_{11}^6/l_Y^6)$ so that in the limit $l_{11}/l_Y \to 0$ the metric (4.1) becomes the unwarped product metric and G_4 vanishes [71].[8]

The Einstein equation in $D = 11$ is given by

$$\hat{R}_{MN} = -\frac{1}{6}|F_4|^2\hat{g}_{MN} + \frac{1}{12}F_{MQ_1\dots Q_3}F_N{}^{Q_1\dots Q_3} + 2b_1T_2\kappa_{11}^2\left[\tilde{W}_{MN} - \frac{1}{9}\hat{g}_{MN}(\tilde{J}_0 + \tilde{W}_K^K)\right] , \tag{4.13}$$

where we have introduced the following notation

$$\tilde{J}_0 = J_0 - \frac{1}{2}E_8 , \qquad \tilde{W}_{MN} = \frac{\delta\tilde{J}_0}{\delta\hat{g}^{MN}} . \tag{4.14}$$

It splits into an external and an internal part

$$\hat{R}_{\mu\nu} = \hat{g}_{\mu\nu}\left(-\frac{3}{4}\partial_c\Delta\partial^c\Delta - \frac{1}{6}|G_4|^2 - \frac{2}{9}b_1T_2\kappa_{11}^2(\tilde{J}_0 + \tilde{W}_K^K)\right) + 2b_1T_2\kappa_{11}^2\tilde{W}_{\mu\nu} , \tag{4.15}$$

$$\begin{aligned}\hat{R}_{ab} = {}& \hat{g}_{ab}\left(\frac{3}{8}\partial_c\Delta\partial^c\Delta - \frac{1}{6}|G_4|^2 - \frac{2}{9}b_1T_2\kappa_{11}^2(\tilde{J}_0 + \tilde{W}_K^K)\right) \\ & - \frac{9}{8}\partial_a\Delta\partial_b\Delta + \frac{1}{12}G_{ac_1\dots c_3}G_b{}^{c_1\dots c_3} + 2b_1T_2\kappa_{11}^2\tilde{W}_{ab} ,\end{aligned} \tag{4.16}$$

where all contractions are performed with the warped metric (4.1). The left hand side of (4.16) is the Ricci tensor for the warped metric which can be expressed through the one of the unwarped metric via[9]

$$\hat{R}_{ab} = R_{ab} - \frac{1}{4}g_{ab}\nabla^c\partial_c\Delta - \frac{3}{8}g_{ab}\partial^c\Delta\partial_c\Delta - \frac{3}{8}\partial_a\Delta\partial_b\Delta . \tag{4.17}$$

In the presence of higher derivative terms the Ricci tensor R_{ab} is non-vanishing but still the first Chern-class vanishes, i.e. introducing complex coordinates like in section 2.2 we have that $\mathcal{R} = R_{i\bar{\jmath}}d\xi^i \wedge d\bar{\xi}^{\bar{\jmath}}$ is exact. This implies that

$$\kappa_{11}^{-2}\int_{Y_4}\sqrt{g^{(8)}}g^{i\bar{\jmath}}R_{i\bar{\jmath}} = 0 . \tag{4.18}$$

To $\mathcal{O}(\kappa_{11}^{-2/3})$ this remains true if one replaces $R_{i\bar{\jmath}}$ with $\hat{R}_{i\bar{\jmath}}$. This is clear for the first correction term in (4.17). The vanishing of the other two terms to $\mathcal{O}(\kappa_{11}^{-2/3})$ follows from

$$\kappa_{11}^{-2}\int_{Y_4}d^8\xi\sqrt{g^{(8)}}\partial_m\Delta\partial^m\Delta = -\kappa_{11}^{-2}\int_{Y_4}d^8\xi\sqrt{g^{(8)}}\Delta\nabla_m\partial^m\Delta , \tag{4.19}$$

which is of higher order.

However, on the right hand side of (4.16) there are terms whose contraction with $g^{i\bar{\jmath}}$ do not have vanishing integral over Y_4 to order $\mathcal{O}(\kappa_{11}^{-2/3})$. This is especially true

[8] Strictly speaking we can not exclude the possibility that Δ has a harmonic part, which might be fixed by the equations of motion. However, this should also vanish in the limit $l_{11}/l_Y \to 0$ in order to ensure that (4.1) becomes the unwarped metric. Thus the warp factor obeys in any case $\Delta \sim \mathcal{O}(\kappa_{11}^{\lambda})$, $\lambda > 0$, which is all we will need.

[9] Note that this does not just correspond to the relation between the Ricci tensors of two metrics differing by a conformal factor which is given e.g. in appendix D of [1]. Rather there is a further contribution coming from $\hat{R}^{(11)\mu}{}_{a\mu b}$ as in (C.27).

for the terms $\sim \int \sqrt{g^{(8)}}|G_4|^2$ and $\sim \int \sqrt{g^{(8)}}E_8$. In case that these terms do not cancel each other one has to generalize the metric ansatz (4.1) and allow for a variation of the internal metric g_{ab} over the $D = 3$ space-time. This leads to additional terms on the left hand side of (4.16) involving space-time derivatives of the varying metric moduli fields. This space-time dependence of the moduli has to ensure that the equations of motion are fulfilled. A similar situation appeared in [128, 129] where the compactification of massive type IIA theory on S^1 respectively K3 is performed. It has been shown in [128] that the mass parameter of massive type IIA theory can be traded for the background flux of a 10-form field strength so that our situation here is comparable to that encountered in [128, 129]. In these cases the volume of the internal manifold has to vary over space-time in order to fulfill the equations of motion.

We thus assume that the effects of the cohomologically non-trivial terms on the right hand side of (4.16) are canceled either among themselves (which is the case in a supersymmetric background) or by an appropriate space-time dependence of the internal metric. In this way the only sources for the potential to order $\mathcal{O}(\kappa_{11}^{-2/3})$ are the kinetic term $\sim |F_4|^2$ and the $\tilde{J}_0$-term of (4.9) and there are no contributions from the reduction of the Einstein Hilbert term.[10]

We now perform the reduction of the action $\mathcal{S} + \delta\mathcal{S}_1 + \delta\mathcal{S}_2$ consisting of the terms given in (2.22), (2.23) and (4.9). The integral $\int_{Y_4} d^8\xi\sqrt{g^{(8)}}J_0$ vanishes for Ricci-flat Kähler manifolds [22, 33]. To order $\mathcal{O}(\kappa_{11}^{-2/3})$ we can assume the internal metric to be the Ricci-flat one in this term so that it does not contribute to the potential to this order.[11] Furthermore on a product space $M_3 \times Y_4$, which we can assume in calculating E_8 to $\mathcal{O}(\kappa_{11}^{-2/3})$, we have

$$E_8(M_3 \times Y_4) = -E_8(Y_4) + 4E_2(M_3)E_6(Y_4) \ , \tag{4.20}$$

where $E_2(M_3) = -2R^{(3)}$. Thus the Einstein term has a non-canonical normalization in the three-dimensional effective action

$$\mathcal{S}^{(3)} = \frac{1}{2\kappa_{11}^2}\int d^3x\sqrt{-g^{(3)}}\Lambda R^{(3)} + \dots \ , \tag{4.21}$$

where

$$\Lambda = \mathcal{V}_\Delta + 8\kappa_{11}^2 b_1 T_2 \int_{Y_4} d^8\xi\sqrt{g^{(8)}}E_6(Y_4) \ , \tag{4.22}$$

and $\mathcal{V}_\Delta = \int_{Y_4} d^8\xi\sqrt{\hat{g}^{(8)}}e^{-\Delta/2}$ denotes a warped Calabi-Yau volume; $\hat{g}^{(8)}$ is the determinant of the internal part of the warped metric (4.1). With the help of a Weyl rescaling $g_{\mu\nu} \to \Lambda^2 g_{\mu\nu}$ the Einstein term can be put into canonical form. At leading order Λ can be replaced by $\mathcal{V}$ and one obtains the Weyl rescaled low energy effective Lagrangian (we set $\kappa_{11} = 1$ henceforth)[12]

$$\mathcal{L}^{(3)} \ = \ \mathcal{L}_0^{(3)} - \sqrt{-g^{(3)}}\left(\frac{1}{2}\epsilon^{\mu\nu\rho}\,\tilde{W}_{AB}A_\mu^A F_{\nu\rho}^B \ + V\right) \ , \tag{4.23}$$

[10]Furthermore note that the components $F_{\mu\nu\rho m}$ just contribute a term $\sim \int_{Y_4} d^8\xi\sqrt{g^{(8)}}\partial_m\Delta\partial^m\Delta$ which is again of higher order, see (4.19).

[11]J_0 is the sum of an internal and an external part. Since J_0 can be expressed through the Weyl-tensor only [130, 131] the external part vanishes because the Weyl tensor vanishes identically in $D = 3$.

[12]As we said before $\mathcal{L}_0^{(3)}$ is also corrected at this order but we did not compute those correction.

where $\mathcal{L}_0^{(3)}$ is given in (C.40), with the $(2,1)$-moduli frozen, and one has

$$\begin{aligned} V &= \frac{1}{4\mathcal{V}^3}\left(\int_{Y_4} d^8\xi\sqrt{g^{(8)}}|G_4|^2 - \frac{1}{6}T_2\,\chi\right) , \\ \tilde{W}_{AB} &= \frac{1}{2}\partial_A\partial_B\tilde{W} , \qquad \tilde{W} = \frac{1}{4}\int_{Y_4} G_4\wedge J\wedge J . \end{aligned} \tag{4.24}$$

In our analysis in chapter 2 with vanishing four-form flux both V and the Chern-Simons terms were absent.[13]

In order to display the relationship of the potential with the two superpotentials of [67] we need to further rewrite V. By definition we have

$$\int_{Y_4} d^8\xi\sqrt{g^{(8)}}|G_4|^2 = \int_{Y_4} G_4\wedge\star G_4 , \tag{4.25}$$

where to leading order $\star G_4$ is the Hodge dual of G_4 with respect to the metric g_{ab}. G_4 can be expanded as the sum $G_4 = G_{4,0}+G_{3,1}+G_{2,2}+G_{1,3}+G_{0,4}$. In order to proceed we need some properties of primitive (p,q)-forms $\omega_{p,q}^{(0)}$ on Y_4, see the definition (B.40). The Hodge dual of a primitive four-form is given by [67]

$$\star\omega_{p,4-p}^{(0)} = (-1)^p\omega_{p,4-p}^{(0)} . \tag{4.26}$$

From (B.43) we see that on Y_4 all components of G_4 except $G_{2,2}$ are primitive and as a consequence their Hodge duals are simply given by

$$\star G_{4,0} = G_{4,0} , \qquad \star G_{3,1} = -G_{3,1} , \qquad \star G_{1,3} = -G_{1,3} , \qquad \star G_{0,4} = G_{0,4} . \tag{4.27}$$

For $G_{2,2}$ one uses the Lefschetz decomposition (B.41) which asserts

$$G_{2,2} \equiv G_{2,2}^{(0)} + J\wedge G_{1,1}^{(0)} + J^2\wedge G_{0,0}^{(0)} . \tag{4.28}$$

By an explicit computation we have shown that

$$\star G_{2,2} = G_{2,2}^{(0)} - J\wedge G_{1,1}^{(0)} + J^2\wedge G_{0,0}^{(0)} = G_{2,2} - 2J\wedge G_{1,1}^{(0)} . \tag{4.29}$$

To derive this formula we have used (B.20) to calculate

$$\star G_{2,2} = G_{2,2} - ig^{i\bar{\jmath}}G_{ik\bar{\jmath}\bar{l}}\,d\xi^k\wedge d\bar{\xi}^{\bar{l}}\wedge J + \frac{1}{4}g^{i\bar{\jmath}}g^{k\bar{l}}G_{ik\bar{\jmath}\bar{l}}\,J\wedge J . \tag{4.30}$$

With the help of (B.18), (B.36) and (B.37) this can be expressed as

$$\star G_{2,2} = G_{2,2} - L\Lambda G_{2,2} + \frac{1}{4}L^2\Lambda^2 G_{2,2} . \tag{4.31}$$

Inserting (4.28) into (4.31) and using (B.38) to permute the Λ's to the right of the L's we find (4.29), where the definition of primitivity (B.40) has to be used again.

Combining (4.27) and (4.29) one arrives at

$$\star G_4 = G_4 - 2G_{3,1} - 2G_{1,3} - 2J\wedge G_{1,1}^{(0)} , \tag{4.32}$$

[13] The presence of the Chern-Simons terms in compactifications with background fluxes has first been noticed in [41] and the fact that the E_8-term contributes to the potential in $D=3$ has first been mentioned in [86].

and hence

$$\int_{Y_4} G_4 \wedge \star G_4 = \int_{Y_4} G_4 \wedge G_4 - 4 \int_{Y_4} G_{3,1} \wedge G_{1,3} - 2 \int_{Y_4} J \wedge G^{(0)}_{1,1} \wedge J \wedge G^{(0)}_{1,1} \ , \quad (4.33)$$

where we have used $J \wedge G^{(0)}_{2,2} = 0$ and $J^3 \wedge G^{(0)}_{1,1} = 0$ in accord with (B.40).

The second term in (4.33) can be further rewritten. With the help of (4.6), (2.36) and (2.37) one derives

$$\int_{Y_4} G_{3,1} \wedge G_{1,3} = -e^{K_{3,1}} G^{-1\alpha\bar{\beta}} D_\alpha W D_{\bar{\beta}} \bar{W} \ , \quad (4.34)$$

where

$$W = \int_{Y_4} \Omega \wedge G_4 \quad (4.35)$$

is precisely the chiral superpotential of [67].

Finally, with the help of (C.38) also the last term in (4.33) can be expressed in terms of the superpotential (4.24). To see this we expand $G^{(0)}_{1,1} \equiv G^A e_A$. The primitivity of $G^{(0)}_{1,1}$ implies $G^A \mathcal{V}_A = 0$, where $\mathcal{V}_A$ is defined in (C.35). Using this and (C.38) we verify that

$$G^{-1AB} \left(\int_{Y_4} G_4 \wedge e_A \wedge J \right) \left(\int_{Y_4} G_4 \wedge e_B \wedge J \right) = (4!)^2 \left(-\frac{1}{6} \mathcal{V} \mathcal{V}_{CD} G^C G^D + \frac{1}{2} \mathcal{V}^2 G^{(0)2}_{0,0} \right) , \quad (4.36)$$

where $\mathcal{V}_{CD}$ is given in (C.36). With the help of

$$4! \, \mathcal{V}_{CD} G^C G^D = \int_{Y_4} J \wedge G^{(0)}_{1,1} \wedge J \wedge G^{(0)}_{1,1} \quad (4.37)$$

and

$$4! \, \mathcal{V} G^{(0)}_{0,0} = \int_{Y_4} G_4 \wedge J \wedge J \quad (4.38)$$

we derive

$$\int_{Y_4} J \wedge G^{(0)}_{1,1} \wedge J \wedge G^{(0)}_{1,1} = -\mathcal{V}^{-1} \left(G^{-1AB} \partial_A \tilde{W} \partial_B \tilde{W} - 2\tilde{W}^2 \right) \ . \quad (4.39)$$

Inserting (4.34), (4.39) into (4.33) and using (4.25) we arrive at

$$\begin{aligned} \int_{Y_4} d^8\xi \sqrt{g^{(8)}} |G_4|^2 \ &= \ \int_{Y_4} G_4 \wedge G_4 + 4 e^{K_{3,1}} G^{-1\alpha\bar{\beta}} D_\alpha W D_{\bar{\beta}} \bar{W} \\ &\qquad + 2 \ \mathcal{V}^{-1} \left(G^{-1AB} \partial_A \tilde{W} \partial_B \tilde{W} - 2\tilde{W}^2 \right) \ . \end{aligned} \quad (4.40)$$

With (4.40) and taking into account the tadpole cancellation condition (4.8) the potential (4.24) becomes

$$V = e^{K^{(3)}} G^{-1\alpha\bar{\beta}} D_\alpha W D_{\bar{\beta}} \bar{W} + \mathcal{V}^{-4} \left(\frac{1}{2} G^{-1AB} \partial_A \tilde{W} \partial_B \tilde{W} - \tilde{W}^2 \right) \ , \quad (4.41)$$

where

$$K^{(3)} = K_{3,1} - 3 \ln \mathcal{V} \ . \quad (4.42)$$

Finally, V and $\mathcal{L}^{(3)}$ can be written in a more canonical form by transforming to new coordinates

$$\hat{M}^A = \mathcal{V}^{-1} M^A \ , \qquad \hat{J} = \hat{M}^A e_A \ . \tag{4.43}$$

From eqs. (2.32), (2.33) and (4.42) we learn

$$\hat{\mathcal{V}} \equiv \frac{1}{4!}\int_{Y_4} \hat{J}^4 = \mathcal{V}^{-3} \ , \qquad \hat{G}_{AB} = \mathcal{V}^2 G_{AB} = -\frac{1}{2}\partial_A \partial_B \ln \hat{\mathcal{V}} \ , \qquad K^{(3)} = K_{3,1} + \ln \hat{\mathcal{V}} \ , \tag{4.44}$$

where the derivatives ∂_A are now with respect to $\hat{M}^A$. If we furthermore introduce

$$\hat{W} = \frac{1}{4}\int_{Y_4} G_4 \wedge \hat{J} \wedge \hat{J} = \mathcal{V}^{-2}\tilde{W} \tag{4.45}$$

and insert (4.44) and (4.45) into (4.23) using (C.40) we arrive at

$$\begin{aligned} \mathcal{L}^{(3)} \ = \ & \sqrt{-g^{(3)}} \Big(\frac{1}{2} R^{(3)} - G_{\alpha\bar{\beta}} \partial_\mu Z^\alpha \partial^\mu \bar{Z}^{\bar{\beta}} - \frac{1}{2}\hat{G}_{AB}\partial_\mu \hat{M}^A \partial^\mu \hat{M}^B - \frac{1}{4}\hat{G}_{AB} F^A_{\mu\nu} F^{B\mu\nu} \\ & - \frac{1}{2}\epsilon^{\mu\nu\rho}\, \hat{W}_{AB} A^A_\mu F^B_{\nu\rho} \ - V \Big) \ , \end{aligned} \tag{4.46}$$

where the explicit dependence on $\hat{\mathcal{V}}$ has disappeared from the Lagrangian. The potential in the new variables is given by

$$V = e^{K^{(3)}} G^{-1\alpha\bar{\beta}} D_\alpha W D_{\bar{\beta}} \bar{W} + \frac{1}{2}\hat{G}^{-1AB}\partial_A \hat{W} \partial_B \hat{W} - \hat{W}^2 \ , \tag{4.47}$$

where the derivatives $\partial_A \hat{W}$ are again taken with respect to the new variables $\hat{M}^A$.

We see that the potential is entirely expressed in terms of two superpotentials W and $\hat{W}$. The W given in (4.35) is precisely the chiral superpotential of [67] while $\hat{W}$ given in (4.45) is the real version of the superpotential of [67], see (4.4). This is related to the fact that the presence of the Chern-Simons terms no longer allows a duality transformation from vector to chiral multiplets. As a consequence $\hat{W}$ can not be complexified as there are only real scalars $\hat{M}^A$ in the vector multiplets. However, upon further S^1 reduction $\hat{W}$ should become complex and coincide with the two-dimensional superpotential of [67,80]. We come back to this case in the next section.

Let us now compare the potential (4.47) with the potentials of $D = 3, N = 2$ supergravity. Unfortunately, the relevant potential for chiral and vector multiplets with Chern-Simons terms coupled to $D = 3, N = 2$ supergravity is not available in the literature and here we just derive part of it by an S^1 compactification of a corresponding four-dimensional supergravity.

The four-dimensional theory we need to consider has to contain both chiral and linear multiplets. A linear multiplet in $D = 4$ consists of an antisymmetric tensor and a real scalar L as bosonic components. The four-dimensional Lagrangian is determined by two functions, the holomorphic superpotential $W(\phi^\alpha)$ and the real function $K^{(4)} = K_\phi(\phi^\alpha, \bar{\phi}^{\bar{\alpha}}) + K_L(L^{\tilde{A}})$, where ϕ^α denote the scalars of the chiral multiplets. K_ϕ is the Kähler potential of the chiral fields and the second derivative of K_L determines the σ-model metric of $L^{\tilde{A}}$ [132,133].[14] In this theory the scalar potential is given by[15]

$$V^{(4)} = e^{K^{(4)}} \Big(G^{-1\alpha\bar{\beta}} D_\alpha W D_{\bar{\beta}} \bar{W} + (L^{\tilde{A}} \partial_{\tilde{A}} K^{(4)} - 3)|W|^2 \Big) \ , \tag{4.48}$$

[14] We have chosen a theory where $K^{(4)}$ is the sum of two terms since this matches the situation we have in $D = 3$.

[15] We thank R. Grimm for discussions on this point.

where $D_\alpha W = \partial_\alpha W + (\partial_\alpha K^{(4)})W$. This form of the potential can be derived for example from the four-dimensional duality between an antisymmetric tensor and a scalar. At the level of superfields this results in the duality between a linear multiplet L and a chiral multiplet S with $S + \bar{S} \sim L^{-1}$. The Kähler potential of S is given by $K = -\ln(S + \bar{S})$ and the superpotential continues to be a function of only the ϕ^α. In this dual description with only chiral multiplets $V^{(4)}$ takes the standard form $V^{(4)} = e^{K^{(4)}}\left(G^{-1I\bar{J}} D_I W D_{\bar{J}} \bar{W} - 3|W|^2\right)$, where the index I now runs over all chiral multiplets $(\phi^\alpha, S^{\tilde{A}})$.

Reducing the theory on a circle leaves the chiral multiplets unaltered. The linear multiplets, however, become vector multiplets and an additional vector multiplet containing the radius r and the Kaluza-Klein vector of the circle as its bosonic components appears in the spectrum. Let us define $L^0 = r^{-2}$. A straightforward S^1-reduction shows that after an appropriate Weyl rescaling the three-dimensional potential is given by

$$V^{(3)} = e^{K^{(3)}}\left(G^{-1\alpha\bar{\beta}} D_\alpha W D_{\bar{\beta}} \bar{W} + (L^A \partial_A K^{(3)} - 4)|W|^2\right) , \tag{4.49}$$

where $K^{(3)} = K^{(4)} + \ln L^0$ and the index A now includes 0. This form of the potential is indeed consistent with the first term of (4.47) if one identifies $L^A = \hat{M}^A$, uses (4.44) and the identity $\hat{M}^A \partial_A \ln \hat{\mathcal{V}} = 4$.

The second term in (4.47) is precisely the D-term of a pure $D = 3, N = 2$ Chern-Simons theory coupled to supergravity. As was noted in [134] supersymmetrization of the Chern-Simons terms requires a D-term potential which coincides with the second term in (4.47). The last term of (4.47) should arise when not only a pure Chern-Simons theory but a more general gauge theory including the standard kinetic term and the Chern-Simons term is coupled to $N = 2$ supergravity.

Curiously, the potential for $\hat{W}$ as displayed in (4.47) is the three-dimensional version of a general formula for the scalar potential in arbitrary dimension D. The requirement of the stability of AdS backgrounds leads to [135]

$$V^{(D)} = \frac{1}{2}(D-2)(D-1)\left(\frac{D-2}{D-1} G^{-1AB} \partial_A \hat{W} \partial_B \hat{W} - \hat{W}^2\right) , \tag{4.50}$$

which for $D = 3$ indeed gives $V^{(3)} = \frac{1}{2} G^{-1AB} \partial_A \hat{W} \partial_B \hat{W} - \hat{W}^2$.

Next, let us discuss the conditions for unbroken supersymmetry. From our derivation of the three-dimensional potential it is clear that for the chiral multiplets unbroken supersymmetry requires $D_\alpha W = 0$. As discussed in section 4.1 this implies $G_{3,1} = G_{1,3} = 0$. Furthermore, one would expect that for the vector multiplets the conditions of unbroken supersymmetry are[16]

$$(\partial_A K^{(3)})\, W \;=\; 0 \;=\; \partial_A \hat{W} , \tag{4.51}$$

while the vacuum energy is determined by $\hat{W}^2$ only. The first condition is the precise analog of the four-dimensional situation when the superpotential does not depend on a chiral scalar, in which case $D_S W = (\partial_S K)W$ holds and supersymmetry forces $W = 0$. As we have seen the vector multiplets in $D = 3$ are closely related to the linear multiplets in $D = 4$ which precisely have the feature $D_S W = (\partial_S K)W$. $(\partial_A K^{(3)})W = 0$ can in general only be fulfilled if $W = 0$ which implies $G_{4,0} = G_{0,4} = 0$. Finally, using (4.45)

[16] A rigorous derivation would require an application of the Noether procedure.

$\partial_A \hat{W} = 0$ implies $\hat{J} \wedge G_4 = 0$, i.e. G_4 has to be primitive. This last condition implies not only $\partial_A \hat{W} = 0$ but also $\hat{W} = 0$. Thus the cosmological constant always vanishes in a supersymmetric minimum and no supersymmetric AdS_3 solution exists.[17] Moreover, a non-vanishing $G_{4,0}$ breaks supersymmetry spontaneously without introducing a vacuum energy.

To summarize, we have reproduced the supersymmetry constraints (4.3) found in [73] by different methods. By performing a Kaluza-Klein reduction on a Calabi-Yau fourfold with four-form flux turned on we have verified that to order $\mathcal{O}(\kappa_{11}^{-2/3})$ the potential can be expressed in terms of the superpotentials (4.35), (4.45) which are closely related to those proposed in [67], see (4.4).

Finally, some comments are in order here. There is another way to generate a superpotential in $D = 3$ by wrapping 5-branes over certain six-cycles of Y_4 [96]. This can however not occur if there is a non-vanishing four-form flux localized on a four-dimensional submanifold of the six-cycle [136]. We have therefore not considered any such contributions to the superpotential here.

Moreover, it is not clear how to extend the duality between the heterotic string on $Y_3 \times S^1$ and M-theory on Y_4 to the case with non-trivial four-form flux. It has been noted in [137] that it is indeed possible to generate three-dimensional Chern-Simons terms in heterotic string compactifications on $Y_3 \times S^1$ if one works with the dual formulation in which the NS two-form is dualized in $D = 10$ to an antisymmetric six-form potential. However the resulting Chern-Simons terms seem to be of a very restricted form. For example they only depend on one single flux parameter whereas in the M-theory case there are generically $h^{2,2}(Y_4)$ parameters specifying the Chern-Simons terms, see (4.24). It is well conceivable that the duality between the heterotic string on $Y_3 \times S^1$ and M-theory on Y_4 is not valid anymore in the presence of background fluxes. This would match the observation that also the duality between the heterotic string on T^3 and M-theoy on K3 does not survive the inclusion of background fluxes [138].

4.3 Some comments about $D = 2$

In this section we make some remarks about a generalization of the preceding analysis to the reduction of type IIA on Y_4 with background fluxes. As we have mentioned in section 4.1 there are more possibilities to turn on fluxes in this case. Besides the four-form F_4 the (massive) type IIA theory has a two-form F_2 and a zero-form F_0. Backgrounds of F_2 and F_4 with two indices in the non-compact directions can be dualized to yield an eight-form F_8 and a six-form F_6 which have only internal indices. (In this way we do not have to introduce G_4 for the internal part of the four-form background because no ambiguity exists.) In [80] the conditions for unbroken supersymmetry have been analyzed (again for fixed $(2,1)$-moduli) and shown to be derivable from two superpotentials

$$W = \int_{Y_4} \Omega \wedge F_4 \ , \qquad \tilde{W} = \int_{Y_4} \left(F_0 \mathcal{K}^4 + F_2 \wedge \mathcal{K}^3 + F_4 \wedge \mathcal{K}^2 + F_6 \wedge \mathcal{K} + F_8 \right) \ . \quad (4.52)$$

The complexified Kähler class is given by $\mathcal{K} = t^A e_A$, where the t^A have been defined in (3.20). W only depends on the chiral complex structure moduli Z^α whereas $\tilde{W}$ depends

[17] Recently this has also been noticed in the revised version of [67].

on the twisted chiral Kähler moduli t^A.[18] Furthermore, the potential V corresponding to these superpotentials has been given in [81]

$$V \sim e^{K^{(2)}_{IIA}} \left(G^{-1A\bar{B}} D_A \tilde{W} D_{\bar{B}} \bar{\tilde{W}} + G^{-1\alpha\bar{\beta}} D_\alpha W D_{\bar{\beta}} \bar{W} - |W|^2 - |\tilde{W}|^2 \right) , \tag{4.53}$$

where $K^{(2)}_{IIA}$ is the Kähler potential (3.25) and the Kähler covariant derivatives are defined as

$$D_A \tilde{W} = \partial_A \tilde{W} + \tilde{W} \partial_A K^{(2)}_{IIA} , \qquad D_\alpha W = \partial_\alpha W + W \partial_\alpha K^{(2)}_{IIA} . \tag{4.54}$$

A derivation of this potential via a Kaluza-Klein reduction would again require the consideration of higher order terms in the ten-dimensional effective action. The relevant ones are

$$\delta \mathcal{S}^{(10)} = \int d^{10}x \sqrt{-g^{(10)}} J_0 F(\Phi^{(10)}_{\mathrm{IIA}}) - \frac{b_1}{\pi\alpha'} \int d^{10}x \sqrt{-g^{(10)}}\, \mathcal{I}_2 , \tag{4.55}$$

where we again use the notation of [127], i.e. J_0 is given in (4.10) where the definition of E_8 this time uses (4.11) with $D = 10$, b_1 is defined above (4.10) and

$$F(\Phi^{(10)}_{\mathrm{IIA}}) = \frac{\alpha'^3 \zeta(3)}{3 \cdot 2^{12} \kappa^2_{10}} e^{-2\Phi^{(10)}_{\mathrm{IIA}}} + \frac{b_1}{2\pi\alpha'} + \dots , \qquad \mathcal{I}_2 = \frac{1}{4} E_8 + 6\epsilon_{10} B_2 [\mathrm{tr} R^4 - \frac{1}{4} (\mathrm{tr} R^2)^2]. \tag{4.56}$$

To leading order in the reduction $F(\Phi^{(10)}_{\mathrm{IIA}})$ can be considered as constant and the integral over J_0 vanishes as in the previous section. The correction term $\sim E_8$ is related via supersymmetry to the anomaly-cancelling term $\sim \epsilon_{10} B_2 [\mathrm{tr} R^4 - \frac{1}{4}(\mathrm{tr} R^2)^2]$, see [127]. Therefore its coefficient does not receive any loop corrections unlike the one of J_0 [139]. If we would just reduce $\mathcal{S}^{(10)} + \delta\mathcal{S}^{(10)}$ with non-vanishing four-form flux, and with $\mathcal{S}^{(10)}$ given in (3.16), we would get a similar potential as in $D = 3$ with the difference that the superpotential in the term $\sim G^{-1AB}$ in (4.41) would be complexified, i.e. it would be given by (4.52) with only F_4 non-vanishing. This comes about as the reduction of $B_2 \wedge F_4 \wedge F_4$ leads to a term $\sim \tilde{W}_{AB} a^A F^B_{\mu\nu} \epsilon^{\mu\nu}$, where (3.18), (4.24) and $F_{\mu\nu i\bar{j}} = F^B_{\mu\nu} e_{Bi\bar{j}}$ have been used. Although a vector in $D = 2$ has no dynamical degree of freedom such a term contributes to the potential upon eliminating the $F^B_{\mu\nu}$ via their equations of motion. This provides the necessary terms to render $\tilde{W}$ complex in the term $\sim G^{-1AB}$ in (4.41). However, in order to reproduce the full potential (4.53) we would have to include all possible fluxes and start with the effective action of *massive* type IIA theory in $D = 10$ [125]. This contains more interaction terms. Some of them give a further contribution to the B_2-tadpole in the Y_4 reduction. Therefore it is well conceivable that the cancellation condition (2.25) is generalized in this case.[19]

Finally, we would like to mention that it is not clear how to map the type IIA fluxes to the heterotic string. There at least some of them should correspond to electric or magnetic fields in the internal directions. This has first been conjectured in [120]. See also [88] for some remarks on this issue.

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[18] Thus $\tilde{W}$ gets world sheet instanton corrections which are in principle calculable via mirror symmetry. It asserts $W(Y_4) = \tilde{W}(\tilde{Y}_4)$ and $\tilde{W}(Y_4) = W(\tilde{Y}_4)$ where $\tilde{Y}_4$ is the mirror manifold. This has been exploited in [88] to calculate the full $\tilde{W}(Y_4)$ for a special kind of Calabi-Yau fourfolds.

[19] We thank S. Gukov for drawing our attention to this possibility.

Chapter 5

Conclusion

Let us finally summarize our main results and give an outlook on the steps which could follow our analysis in a natural way.

F-theory compactifications on Calabi-Yau fourfolds Y_4 are of considerable interest because they capture some of the non-perturbative features of four-dimensional heterotic theories with $N = 1$ supersymmetry which serve as promising candidates to make contact between string theory and the (supersymmetric) standard model of particle physics. Closely related but much simpler to handle are their circle respectively torus reductions to three and two dimensions. In these cases the dual theory is M- respectively type IIA string theory compactified on Y_4.

The aim of our thesis was the investigation of the low energy effective theories arising in Calabi-Yau fourfold compactifications of 11-dimensional and type IIA supergravity. We have focused on the moduli and analyzed the duality to heterotic theories with four supercharges in $D = 3$ respectively $D = 2$. More specifically our results are as follows.

- In $D = 3$ the moduli space is a Kähler manifold. We have derived the corresponding heterotic Kähler potential and expressed it in terms of the four-dimensional Kähler potential and gauge kinetic functions (2.12). Furthermore we have performed a Kaluza-Klein reduction of 11-dimensional supergravity on a Calabi-Yau fourfold Y_4 and obtained the corresponding Kähler potential for all the moduli including those stemming from an expansion of the three-form potential in terms of the $(2,1)$-forms of Y_4, see (2.49) and (2.50). This required to determine the correct Kähler coordinates given in (2.46) and (2.47).

- In $D = 2$ there are two different kinds of scalar multiplets relevant for the effective theories – the chiral and the twisted chiral multiplet. If both of them are present the moduli space is in general not Kähler. Nevertheless the effective action can still be expressed via two 'generalized Kähler potentials' which we specified for the heterotic theory in (3.8) and for the type IIA case in (3.21). However, if one restricts on the heterotic side to weak (four-dimensional) coupling and freezes the $(2,1)$-moduli on the type IIA side the respective moduli spaces become Kähler although both kinds of multiplets are present.

- By specifying Y_4 to be a K3 fibered Calabi-Yau manifold with a Hirzebruch surface as the base we have shown in sections 2.3 and 3.3 that the respective Kähler potentials for the perturbative heterotic theories and M/IIA-theory on Y_4 coincide in the large base space limit and at weak (four-dimensional) heterotic coupling. In doing so we have frozen on the M/IIA-theory side all except the Kähler moduli at generic values and established a map between them and the corresponding heterotic fields, see (2.75) and (3.31).

- We have generalized the compactification of 11-dimensional supergravity on Calabi-Yau fourfolds by the inclusion of non-trivial four-form background flux. This leads to a potential for the 'moduli' fields (4.47) which can be expressed via the two superpotentials (4.35) and (4.45). The first of them coincides with the one proposed in [67] while the second came out to be a real version of the corresponding proposition in [67]. Furthermore we have reproduced the constraints on the four-form flux for unbroken supersymmetry derived in [73] by different methods.

A lot of open problems remain, some of which have already been mentioned at the end of chapter 4. A natural question is what happens to the duality between the heterotic theory in $D = 3$ and M-theory on Y_4 in the presence of the background flux. Furthermore, it would be very interesting to extend the analysis of section 4.2 to the case of type IIA theory on Y_4 with fluxes. Some comments have already been made in section 4.3 but the investigation is still at an early stage. Also in this case is the fate of the duality to the heterotic string an open question. Finally one could try to extend the results of chapter 2 in the following way. In section 2.3 we have considered a special fibration structure of the Calabi-Yau fourfold and verified that this leads to perturbative heterotic duals. However, it would be even more interesting to find the *necessary* constraints that Y_4 has to fulfill in order to encode perturbative heterotic physics. This might require further knowledge of higher derivative couplings in the three-dimensional effective theory as the example of [85] shows. Furthermore, one could try to use the duality in order to extract some information about (non-perturbative) properties of the four-dimensional heterotic string. For example, the Kähler potential of the heterotic string in $D = 3$ contains the (real part of) the four-dimensional gauge kinetic functions. It might be possible to determine some of its quantum corrections by considering the M-theory Kähler potential in the limit in which the modulus corresponding to the four-dimensional heterotic dilaton becomes small. Thus the volume of some six-cycle in Y_4 has to shrink. However, our analysis is based upon 11-dimensional supergravity and therefore requires the 'average radius' of Y_4 to be large. It is therefore not obvious that this limit can be consistently taken within our framework.

Appendix A

Notation and conventions

The signature of the space-time metric is $(-+\ldots+)$. The Levi-Civita symbol is defined to transform as a tensor, i.e.

$$\epsilon^{1\ldots D} = (\pm g^{(D)})^{-1/2} \qquad \text{and} \qquad \epsilon_{1\ldots D} = \pm(\pm g^{(D)})^{1/2} \ , \tag{A.1}$$

where the $+$ sign corresponds to Euclidean and the $-$ sign to Minkowskian signature.

Our conventions for the Riemann curvature tensor are

$$R^{\mu}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\sigma\nu} - \partial_\sigma \Gamma^{\mu}_{\rho\nu} + \Gamma^{\omega}_{\sigma\nu}\Gamma^{\mu}_{\rho\omega} - \Gamma^{\omega}_{\rho\nu}\Gamma^{\mu}_{\sigma\omega} \ , \tag{A.2}$$

where we use the following definition of the Christoffel symbols:

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}) \ . \tag{A.3}$$

The Ricci tensor is defined as

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} \ . \tag{A.4}$$

We are thus using the (+++) conventions of [140].

A p-form A_p can be expanded as

$$A_p = \frac{1}{p!} A_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \ . \tag{A.5}$$

The exterior derivative is defined as

$$dA_p = \frac{1}{p!} \partial_\mu A_{\mu_1\ldots\mu_p} dx^{\mu} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \tag{A.6}$$

and its adjoint is given by

$$d^\dagger A_p = -\frac{1}{(p-1)!} \nabla^\nu A_{\nu\mu_2\ldots\mu_p} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p} \ . \tag{A.7}$$

The Hodge $\star$-operator for a p-form on a (real) D-dimensional manifold is defined as

$$\star A_p = \frac{1}{p!(D-p)!} A_{\mu_1\ldots\mu_p} \epsilon^{\mu_1\ldots\mu_p}{}_{\nu_{p+1}\ldots\nu_D} dx^{\nu_{p+1}} \wedge \ldots \wedge dx^{\nu_D} \ . \tag{A.8}$$

Due to (A.6) the field strength of a p-form is

$$F_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\ldots\mu_{p+1}]} \tag{A.9}$$

and the action for a p-form potential A_p is given by

$$-\frac{1}{4}\int d^D x \sqrt{-g^{(D)}} |F_{p+1}|^2 = -\frac{1}{4}\int d^D x \frac{\sqrt{-g^{(D)}}}{(p+1)!} F_{\mu_1\ldots\mu_{p+1}} F^{\mu_1\ldots\mu_{p+1}} \ . \tag{A.10}$$

Appendix B

Calabi-Yau manifolds

In this appendix we give a short introduction to the mathematics of complex manifolds and especially Calabi-Yau manifolds.[1] It requires familiarity with the basic notions of Riemannian differential geometry and differential calculus as can be found e.g. in [141]. The presentation is partly inspired by [141, 142] and the second chapter of [25]. The mathematically more inclined reader is referred e.g. to [143–148].

B.1 (Co)homology primer

Before we start with our introduction into complex manifolds let us briefly review some basic facts from homology and cohomology. We follow closely the discussion in [142]. For a mathematically more precise treatment see e.g. [149].

Let M be a smooth orientable Riemannian manifold. A **p-chain** is a sum $a_p = \sum_k c_k N_k$, where the N_k are smooth p-dimensional oriented submanifolds of M. The chain is called integral, real or complex depending on whether the coefficients c_k are integral, real or complex. The operation ∂, which associates to a manifold N its oriented boundary and which squares to zero (i.e. $\partial\partial N = 0$), can be extended to chains by linearity. A **p-cycle** is a p-chain with vanishing boundary, i.e. $\partial a_p = 0$.[2] If we denote by Z_p the set of all p-cycles and by B_p the set of all p-dimensional boundaries, i.e. $B_p = \{a_p | a_p = \partial a_{p+1}\}$, we can define the p-th integral/real/complex **homology group** of M as

$$H_p(M, \mathbb{S}) \equiv Z_p/B_p \ , \tag{B.1}$$

where $\mathbb{S} = \mathbb{Z}/\mathbb{R}/\mathbb{C}$ depending on whether the chains are integral/real/complex. Thus two p-cycles in H_p are considered equivalent if they just differ by a boundary. The real and complex homology groups are vector spaces (over $\mathbb{R}$ resp. $\mathbb{C}$).

A similar construction can be made for p-forms. Let us recall that a p-form on M is a totally skew symmetric covariant tensor field of rank p, see (A.5). The exterior derivative of a p-form is defined in (A.6). Like the boundary operator it squares to zero and thus allows the following definition. If we denote by Z^p the set of closed p-forms, $Z^p = \{\omega_p | d\omega_p = 0\}$, and by B^p the set of exact p-forms, $B^p = \{\omega_p | \omega_p = d\omega_{p-1}\}$, we can define the p-th **de Rham cohomology group**

$$H^p(M, \mathbb{R}) \equiv Z^p/B^p \ , \tag{B.2}$$

[1] We assume throughout that all manifolds are compact, connected and without boundary.

[2] For an example see figure B.1 in section B.8, where the two basic one-cycles of the torus are depicted.

which is an $\mathbb{R}$-vector space like $H_p(M,\mathbb{R})$. It has been proven by de Rham that $H^p(M,\mathbb{R})$ and $H_p(M,\mathbb{R})$ are isomorphic. More precisely for any basis $\{a_p^i\}$ of $H_p(M,\mathbb{R})$ there exists a dual basis $\{\omega_p^i\}$ for $H^p(M,\mathbb{R})$ such that

$$\int_{a_p^i} \omega_p^j = \delta_i^j \ , \tag{B.3}$$

where the integral over a general real chain is defined as $\int_{\sum c_k N_k} = \sum c_k \int_{N_k}$. If the basis $\{a_p^i\}$ is actually a basis for $H_p(M,\mathbb{Z})$ the dual basis generates the integral cohomology group $H^p(M,\mathbb{Z})$ via linear combinations with integral coefficients.[3] Furthermore if we allow for complex coefficients in the linear combinations of the ω_p^j we generate the complex de Rham cohomology group $H^p(M,\mathbb{C})$.

The dimensions of the de Rham cohomology groups, $h^p(M) \equiv \mathrm{dim} H^p(M,\mathbb{R})$, are called **Betti numbers** and can be used to define the **Euler number**

$$\chi(M) \equiv \sum_{p=0}^{\mathrm{dim}M} (-1)^p h^p(M) \ . \tag{B.4}$$

For $2n$-dimensional manifolds the Euler number is given as an integral over the **Euler class**

$$E_{2n}(M) \equiv \frac{1}{(4\pi)^n n!} \epsilon^{a_1 \ldots a_{2n}} \mathbf{R}_{a_1 a_2} \wedge \ldots \wedge \mathbf{R}_{a_{2n-1} a_{2n}} \ , \tag{B.5}$$

where

$$\mathbf{R}_{ab} \equiv \frac{1}{2} R_{abcd} dx^c \wedge dx^d \tag{B.6}$$

is the curvature two-form. For odd-dimensional manifolds the Euler number vanishes. This comes about as follows. According to a theorem by Hodge every p-form has a unique decomposition into a harmonic, exact and co-exact piece, $\omega_p = \alpha_p + d\beta_{p-1} + d^\dagger \gamma_{p+1}$, where we have used (A.7) and $\Delta \alpha_p = 0$ with $\Delta = dd^\dagger + d^\dagger d$ the Laplace operator. An exact form can therefore uniquely be written as $\omega_p = \alpha_p + d\beta_{p-1}$ showing that every cohomology class has exactly one harmonic representative. The definition of the Laplace operator implies that the notion of a harmonic form depends on the metric whereas the overall number of independent harmonic p-forms is a topological invariant given by $h^p(M)$. Now one can verify that a p-form ω_p is harmonic if and only if $\star\omega_p$ is harmonic which implies $h^p(M) = h^{D-p}(M)$, where D denotes the dimension of M. Using this in (B.4) leads to a vanishing Euler number for odd values of D.

Besides the dual of a p-cycle defined in (B.3), which is a p-form, there is also another notion of duality leading to a $(D-p)$-form. The **Poincaré dual** α_{D-p} of a p-cycle a_p is defined by

$$\int_{a_p} \omega_p = \int_M \alpha_{D-p} \wedge \omega_p \tag{B.7}$$

for any closed p-form ω_p. Poincaré duality is useful in the calculation of **intersection numbers** of cycles. Let us define the intersection number here only for the simplest case, i.e. for two cycles a_p and b_{D-p} whose dimensions add up to the dimension D of M and which intersect transversely. This means that at each point q, which they have in common, the tangent spaces $T_q a$ and $T_q b$ span the whole tangent space $T_q M$. Thus by introducing an oriented basis $\{u_1, \ldots, u_p\}$ for $T_q a$ and $\{v_1, \ldots, v_{D-p}\}$ for $T_q b$ the

[3] We assume here that $H_p(M,\mathbb{Z})$ and $H^p(M,\mathbb{Z})$ have no torsion.

union $\{u_1, \ldots, u_p, v_1, \ldots, v_{D-p}\}$ is a basis for T_qM. If this has positive orientation one defines the intersection of a and b at q as $i_q(a, b) = 1$. Otherwise it is set to -1. Now the intersection number of a and b is defined as

$$a \cdot b \equiv \sum_{q \in a \cap b} i_q(a, b) \ . \tag{B.8}$$

Let α and β be the Poincaré duals of a and b. Then the intersection number can also be expressed as

$$a \cdot b = \int_M \alpha \wedge \beta \ . \tag{B.9}$$

The definition of intersection numbers can be generalized to non-transversely intersecting cycles and to cycles whose dimensions do not add up to D. In this case the intersection locus is not a discrete point set. If the sum of the two dimensions is smaller than D the two cycles do not intersect generically. If however $\dim a + \dim b = D + d$ the two cycles intersect in a d-dimensional sub-cycle. This instance can be used to define the intersection number of three cycles because one can consider the intersection of this sub-cycle with a further cycle of complementary dimension. However, one one can also iterate this procedure and define the intersection number for more than three cycles. It is for example easily verified that on an eight-dimensional manifold four six-dimensional cycles intersect in a set of points. Therefore one can generalize (B.9) to define their intersection number. This is used in compactifications on Calabi-Yau fourfolds, see (2.35). A detailed introduction into intersection theory is given in [150] but the main ideas can also been found in the appendix A of [151].

B.2 Complex manifolds

We now leave the realm of real manifolds and come to the introduction of complex manifolds. In contrast to a real manifold which is locally homeomorphic to $\mathbb{R}^n$ a complex manifold is locally homeomorphic to $\mathbb{C}^n$. More precisely we have the following

Definition B.2.1. A **complex manifold** M of (complex) dimension D is a topological space endowed with a holomorphic atlas, i.e. a family (U_i, ϕ_i) of open sets U_i covering M and homeomorphisms ϕ_i from U_i to an open subset of $\mathbb{C}^D$ (introducing local coordinates $\xi \equiv \phi_i$ on M), such that the maps $\psi_{ji} = \phi_j \phi_i^{-1}$ from $\phi_i(U_i \cap U_j)$ to $\phi_j(U_i \cap U_j)$ are holomorphic for $U_i \cap U_j \neq \emptyset$.

A holomorphic atlas (U_i, ϕ_i) equips M with a **complex structure**. This means that it determines what the holomorphic functions $f : M \to \mathbb{C}$ are. The holomorphicity of a general map $f : M \to N$ between two complex manifolds M and N of (complex) dimension m respectively n is defined in the following way. Let p be a point in the chart $(U \subset M, \phi)$ and $f(p)$ in $(V \subset N, \psi)$ then f is defined to be holomorphic at p if the map $\psi f \phi^{-1} : \mathbb{C}^m \to \mathbb{C}^n$ is holomorphic at $\phi(p)$ in the usual sense. Obviously $\mathbb{C}$ is a complex manifold so that the definition also applies to functions $f : M \to \mathbb{C}$. Which functions are holomorphic depends on the holomorphic atlas and two atlases (U_i, ϕ_i) and (V_j, ψ_j) define the same complex structure, i.e. the same notion of holomorphic functions, only if their union is again a holomorphic atlas. Furthermore two complex manifolds M and N are considered to be equivalent if there is a diffeomorphism $F : M \to N$ which is biholomorphic with respect to the complex structures of M and N (i.e. both F and F^{-1} are holomorphic).

All complex manifolds are even dimensional differentiable manifolds if considered as real manifolds. The reverse is not true however. For example S^{2n} is only a complex manifold for $n = 1$, whereas $S^{2p+1} \times S^{2q+1}$ always is a complex manifold for $p, q \in \mathbb{N}$ [142]. The case $p = q = 0$ is the torus T^2, which is in fact the simplest example of a Calabi-Yau manifold.

If a manifold admits a complex structure it is not necessarily unique. Alternatively stated two manifolds can be different as complex manifolds although they are diffeomorphic and thus equivalent as real manifolds. In this case the inequivalent complex structures can be characterized by a set of parameters, whose number depends on the real manifold under consideration. These parameters are called **complex structure moduli**. The set of all moduli leading to inequivalent complex structures is called the moduli space of complex structures and can be discrete (as is the case for S^2, which has like any projective space a unique complex structure) or continuous (like for Calabi-Yau spaces). Sometimes the moduli space is at least locally itself a complex manifold as is the case for Calabi-Yau manifolds. We will come back to complex structure moduli spaces in section B.7.

On complex manifolds it is useful to introduce the notion of a complexified tangent space. This is obtained from the real tangent space by allowing for complex coefficients, i.e. $T_pM^{\mathbb{C}} \equiv T_pM \otimes \mathbb{C}$. A convenient basis is the coordinate basis induced by the complex coordinates ξ^i on M, $T_pM^{\mathbb{C}} = \text{span}\left\{\partial_{\xi^i}|_p, \partial_{\bar{\xi}^{\bar{\imath}}}|_p\right\}$. The dual space (i.e. the complexified cotangent space) is accordingly given by $T_p^{\star}M^{\mathbb{C}} = \text{span}\left\{d\xi^i|_p, d\bar{\xi}^{\bar{\imath}}|_p\right\}$. Both have an obvious decomposition into a holomorphic and an antiholomorphic subspace. The (co)tangent spaces for different $p \in M$ span the complexified (co)tangent bundle.

Similarly the notion of tensors and r-forms can be complexified. To this end they are first extended to arguments from the complexified tangent or cotangent spaces. If for example $u, v, w, x \in T_pM$ we get complexified vectors via $u + iv, w + ix \in T_pM^{\mathbb{C}}$. A $(0, 2)$-tensor t_p is then extended to the complexified tangent space by linearity, i.e.

$$t_p(u + iv, w + ix) \equiv t_p(u, v) - t_p(v, x) + i[t_p(u, x) + t_p(v, w)] \ . \tag{B.10}$$

It is obvious how to generalize this to tensors of other ranks and to tensor fields. A complexified tensor field is defined by $t = t_1 + it_2$, where t_1 and t_2 are extended tensor fields of the same kind. This definition includes the complexification of r-forms which are totally skew-symmetric $(0, r)$-tensor fields. A form which can be expressed in local coordinates as

$$\omega_{p,q} = \frac{1}{p!q!}\omega_{i_1\ldots i_p\bar{\imath}_1\ldots\bar{\imath}_q} d\xi^{i_1} \wedge \ldots \wedge d\xi^{i_p} \wedge d\bar{\xi}^{\bar{\imath}_1} \wedge \ldots \wedge d\bar{\xi}^{\bar{\imath}_q} \tag{B.11}$$

is called a (p, q)-form. A general r-form is a sum of (p, q)-forms with $p+q = r$. The set of all (p, q)-forms is denoted by $A^{p,q}(M)$. In addition to the usual exterior derivative there are two further derivative operators on a complex manifold, the **Dolbeault operators** ∂ and $\bar{\partial}$. They are defined as

$$\bar{\partial}\omega_{p,q} \equiv \frac{1}{p!q!}\partial_{\bar{\xi}^{\bar{\jmath}}}\omega_{i_1\ldots i_p\bar{\imath}_1\ldots\bar{\imath}_q} d\bar{\xi}^{\bar{\jmath}} \wedge d\xi^{i_1} \wedge \ldots \wedge d\xi^{i_p} \wedge d\bar{\xi}^{\bar{\imath}_1} \wedge \ldots \wedge d\bar{\xi}^{\bar{\imath}_q} \in A^{p,q+1}(M) \tag{B.12}$$

and analogously for ∂, so that $d = \partial + \bar{\partial}$. Like the exterior derivative d the Dolbeault operators square to zero and can therefore be used to define cohomology groups as in section B.1. The cohomology groups $H^{p,q}(M)$ represented by $\bar{\partial}$-closed (p,q)-forms are called **Dolbeault cohomology groups** and their complex dimensions $h^{p,q}(M)$ are termed **Hodge numbers**. In general they are not subgroups of the complex de Rham cohomology groups $H^r(M,\mathbb{C})$ with $p+q=r$, which are represented by d-closed forms. This is however true for Kähler manifolds which we will define in section B.4.

The complex structure of a complex manifold can be completely specified by a certain tensor field. To understand this we introduce the following

Definition B.2.2. A differentiable manifold M which admits a globally defined tensor field J fulfilling

$$J^2 = -\mathbb{1} \tag{B.13}$$

is an **almost complex manifold**. The tensor field J is called an **almost complex structure**.

A complex manifold M induces an almost complex structure on its underlying differentiable manifold in the following way. Let $\xi^j = x^j + iy^j$ be the coordinates of a point $p \in M$ in a chart (U,ϕ). The real tangent space T_pM of M at p is spanned by $\{\partial_{x^j}, \partial_{y^j}\}$. One can define a linear map $J_p : T_pM \to T_pM$ by

$$J_p(\partial_{x^j}) \equiv \partial_{y^j} \ , \qquad J_p(\partial_{y^j}) \equiv -\partial_{x^j} \ . \tag{B.14}$$

It satisfies $J_p^2 = -\mathbb{1}$. The definition (B.14) can be made independent of the chart which can be shown using the holomorphicity of the coordinate change maps. Thus for all $p \in M$ the components of J_p are constant and given with respect to the local coordinates by

$$J_p = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ . \tag{B.15}$$

They can therefore be used to define a smooth tensor field J whose components at p are (B.15). This is the almost complex structure induced by the complex structure of M. J can be extended to the complexified tangent space. With respect to the coordinates ξ^j it has the components

$$J_i{}^j = i\delta_i^j \ , \qquad J_{\bar{\imath}}{}^{\bar{\jmath}} = -i\delta_{\bar{\imath}}^{\bar{\jmath}} \ , \qquad J_i{}^{\bar{\jmath}} = 0 \ , \qquad J_{\bar{\imath}}{}^j = 0 \ . \tag{B.16}$$

Whether an almost complex structure J is induced by a complex structure and therefore allows the introduction of a holomorphic coordinate system on the almost complex manifold M (which is then actually a complex manifold) can be answered by examining the **Nijenhuis tensor**

$$N_{mn}{}^k \equiv \partial_{[n}J_{m]}{}^k - J_{[m}{}^p J_{n]}{}^q \, \partial_q J_p{}^k \ . \tag{B.17}$$

If this vanishes a theorem by Newlander and Nirenberg [152] ensures that there is a unique complex structure on M for which the induced almost complex structure coincides with J. Thus on complex manifolds there is a one-to-one correspondence between complex structures and almost complex structures with vanishing Nijenhuis tensor.[4]

Not all almost complex manifolds are actually complex; a counter example is S^6. Almost complex manifolds owe their name to the fact that the presence of the almost

[4] This is the reason why such almost complex structures on a complex manifold are also called complex structures, in slight abuse of denotation.

complex structure allows for a decomposition of the complexified tangent spaces into holomorphic and antiholomorphic parts in a similar way as on complex manifolds. Also the definition of (p,q)-forms is possible although they can not be expressed in the form of (B.11) which requires the existence of complex coordinates. See e.g. [142, 147] for further details.

B.3 Hermitian manifolds

The definition of a complex manifold M does not involve any metric on M. Considered as a real manifold M can be endowed with a Riemannian metric g. As this is a symmetric tensor of rank $(0,2)$ it can be extended to arguments from the complexified tangent space like in (B.10).

Definition B.3.1. A complex manifold is called **Hermitian** if it is endowed with a Hermitian metric, i.e. the only non-vanishing components of the metric are $g_{i\bar{j}} = g(\partial_{\xi^i}, \partial_{\bar{\xi}^{\bar{j}}})$, whereas $g_{ij} = g(\partial_{\xi^i}, \partial_{\xi^j}) = 0$ and $g_{\bar{i}\bar{j}} = g(\partial_{\bar{\xi}^{\bar{i}}}\partial_{\bar{\xi}^{\bar{j}}}) = 0$.

Hermiticity is a restriction only on the metric and not on the manifold. In fact every complex manifold admits a Hermitian metric. On the other hand, whether a given metric is Hermitian or not depends on the complex structure of M.

A Hermitian metric satisfies $g_{i\bar{j}} = J_i{}^m J_{\bar{j}}{}^{\bar{n}} g_{m\bar{n}}$, where J is the induced almost complex structure (B.16). Multiplying this equation with $J_{\bar{k}}{}^{\bar{j}}$ shows that $J_{i\bar{k}} = -J_{\bar{k}i}$, where $J_{i\bar{k}} = J_i{}^m g_{m\bar{k}}$ and $J_{\bar{k}i} = J_{\bar{k}}{}^{\bar{m}} g_{i\bar{m}}$. This means that there is a natural $(1,1)$-form on every Hermitian manifold, $J = J_{i\bar{j}} d\xi^i \wedge d\bar{\xi}^{\bar{j}}$.[5] Furthermore equation (B.16) shows that

$$J_{i\bar{j}} = i g_{i\bar{j}} \ . \tag{B.18}$$

One verifies that

$$\begin{aligned} \overbrace{J\wedge\ldots\wedge J}^{\text{D factors}} &= i^D D!\,(\det g_{i\bar{j}})\, d\xi^1\wedge d\bar{\xi}^1\wedge\ldots\wedge d\xi^D\wedge d\bar{\xi}^D \\ &= i^D(-1)^{D(D-1)/2} D!\,(\det g_{i\bar{j}})\, d^{2D}\xi \\ &= D!\sqrt{g}\, d^{2D}y \end{aligned} \tag{B.19}$$

is proportional to the volume element on the manifold.[6] In (B.19) the real coordinates y^a and the complex coordinates ξ^i are related by $\xi^i = \frac{1}{\sqrt{2}}(y^{2i-1} + iy^{2i})$. Furthermore we have used the definition $d^{2D}\xi \equiv d\xi^1\wedge\ldots\wedge d\xi^D\wedge d\bar{\xi}^1\wedge\ldots\wedge d\bar{\xi}^D$ and the fact that $\det(g_{i\bar{j}})$ is equal to the square root $\sqrt{g}$ of the determinant of the original Riemannian metric.

On Hermitian manifolds it is possible to define an inner product on the space of (p,q)-forms. One has two different ϵ-symbols $\epsilon^{i_1\ldots i_D}$ and $\epsilon^{\bar{\imath}_1\ldots\bar{\imath}_D}$. With their help one can generalize the definition of the **Hodge $\star$-operator** to (p,q)-forms according to

$$\begin{aligned} \star\omega_{p,q} \equiv \frac{(-1)^{D(D-1)/2+Dp} i^D}{p!q!(D-p)!(D-q)!}\, &\omega_{i_1\ldots i_p\bar{\imath}_1\ldots\bar{\imath}_q}\epsilon^{i_1\ldots i_p}{}_{\bar{\jmath}_{p+1}\ldots\bar{\jmath}_D}\epsilon^{\bar{\imath}_1\ldots\bar{\imath}_q}{}_{j_{q+1}\ldots j_D} \\ &\times d\xi^{j_{q+1}}\wedge\ldots\wedge d\xi^{j_D}\wedge d\bar{\xi}^{\bar{\jmath}_{p+1}}\wedge\ldots\wedge d\bar{\xi}^{\bar{\jmath}_D} \ . \end{aligned} \tag{B.20}$$

[5] As common in the literature we denote both, the complex structure and the natural $(1,1)$-form with the same letter J. The index structure or the context allow to distinguish them.

[6] From now on D always denotes the complex dimension of M.

Because of the special index structure of the metric the components of the ϵ-symbols appearing in (B.20) are the only ones that can occur on a Hermitian manifold. Obviously $\star\omega_{p,q} \in A^{D-q,D-p}(M)$. In fact the Hodge $\star$ is an isomorphism between $A^{p,q}(M)$ and $A^{D-q,D-p}(M)$. However in order to define the inner product we actually need an operator that maps a (p,q)-form to a $(D-p, D-q)$-form. This can be achieved by taking the complex conjugate in addition to acting with the Hodge $\star$. We are thus led to define the Hodge $\bar{\star}$-operator

$$\begin{aligned}\bar{\star}\omega_{p,q} \equiv \frac{(-1)^{D(D+1)/2} i^D}{p!q!(D-p)!(D-q)!}\, \bar{\omega}_{i_1\ldots i_q \bar{\imath}_1\ldots\bar{\imath}_p}\, \epsilon^{i_1\ldots i_q}{}_{\bar{\jmath}_{q+1}\ldots\bar{\jmath}_D}\, \epsilon^{\bar{\imath}_1\ldots\bar{\imath}_p}{}_{j_{p+1}\ldots j_D} \\ \times d\xi^{j_{p+1}} \wedge \ldots \wedge d\xi^{j_D} \wedge d\bar{\xi}^{\bar{\jmath}_{q+1}} \wedge \ldots \wedge d\bar{\xi}^{\bar{\jmath}_D}\ .\end{aligned} \tag{B.21}$$

With the help of $\bar{\omega}_{i_1\ldots i_q\bar{\imath}_1\ldots\bar{\imath}_p} = (-1)^{pq}\overline{\omega_{i_1\ldots i_p\bar{\imath}_1\ldots\bar{\imath}_q}}$ one verifies that $\bar{\star}\omega_{p,q} = \overline{\star\omega_{p,q}} = \star\bar{\omega}_{p,q}$. Furthermore the signs work out to give $\bar{\star}\bar{\star}\omega_{p,q} = (-1)^{p+q}\omega_{p,q}$. The inner product on the space of (p,q)-forms can now be defined as

$$\begin{aligned}(\omega_{p,q}, \eta_{p,q}) &\equiv \int \omega_{p,q} \wedge \bar{\star}\eta_{p,q} \\ &= \frac{(-1)^{D(D-1)/2} i^D}{p!q!} \int d^{2D}\xi\, \sqrt{g}\omega_{i_1\ldots i_p\bar{\imath}_1\ldots\bar{\imath}_q}\bar{\eta}^{i_1\ldots i_p\bar{\imath}_1\ldots\bar{\imath}_q}\ ,\end{aligned} \tag{B.22}$$

where one has to take into account that $\bar{\eta}^{i_1\ldots i_p\bar{\imath}_1\ldots\bar{\imath}_q} = (-1)^{pq}\bar{\eta}^{\bar{\imath}_1\ldots\bar{\imath}_q i_1\ldots i_p}$.

With the help of the inner product one can define the adjoint operators $\partial^\dagger$ and $\bar{\partial}^\dagger$ of the Dolbeault operators ∂ and $\bar{\partial}$ by

$$(\omega, \partial\eta) = (\partial^\dagger\omega, \eta)\ , \qquad (\omega, \bar{\partial}\eta) = (\bar{\partial}^\dagger\omega, \eta)\ , \tag{B.23}$$

which can be expressed as $\partial^\dagger = -\star\bar{\partial}\star = -\bar{\star}\partial\bar{\star}$ and $\bar{\partial}^\dagger = -\star\partial\star = -\bar{\star}\bar{\partial}\bar{\star}$. This shows that $\partial^\dagger\omega_{p,q} \in A^{p-1,q}(M)$ and $\bar{\partial}^\dagger\omega_{p,q} \in A^{p,q-1}(M)$. On Hermitian manifolds one therefore has two additional Laplacians, $\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial$ and $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$. Furthermore there is a complex version of Hodge's theorem, i.e. $A^{p,q}(M)$ has a unique orthogonal decomposition

$$A^{p,q}(M) = \bar{\partial}A^{p,q-1}(M) \oplus \bar{\partial}^\dagger A^{p,q+1}(M) \oplus \mathrm{Harm}^{p,q}_{\bar{\partial}}(M)\ , \tag{B.24}$$

where $\mathrm{Harm}^{p,q}_{\bar{\partial}}(M) = \{\Delta_{\bar{\partial}}\omega_{p,q} = 0\}$ are the **$\bar{\partial}$-harmonic** (p,q)-forms. In particular every element of $H^{p,q}(M)$ has a unique $\bar{\partial}$-harmonic representative. Furthermore we notice that $\bar{\star}\Delta_{\bar{\partial}} = \Delta_{\bar{\partial}}\bar{\star}$, which implies

$$h^{p,q}(M) = h^{D-p,D-q}(M)\ . \tag{B.25}$$

However the usual Hodge $\star$ does *not* commute with the $\bar{\partial}$-Laplacian. In fact one has $\star\Delta_{\bar{\partial}} = \Delta_\partial\star$, so that on a general Hermitian manifold the Hodge numbers $h^{p,q}(M)$ and $h^{D-q,D-p}(M)$ are completely independent. This is not the case anymore on Kähler manifolds as we shall see in the next section.

The Levi-Civita connection of a differentiable manifold is uniquely determined by requiring metric compatibility and symmetry in its lower indices. On a Hermitian manifold it is more appropriate to replace the second condition by demanding that the decomposition of the tangent space into a holomorphic and an antiholomorphic subspace is respected by parallel transport, i.e. a holomorphic vector at a point $p \in M$

should remain holomorphic if it is parallel transported to another point q. This is guaranteed if the connection has only pure indices so that Γ^i_{jk} and $\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} = \overline{\Gamma^i_{jk}}$ are the only non-vanishing components. Demanding in addition that the metric be covariantly constant fixes the connection uniquely and the coefficients are given by

$$\Gamma^i_{jk} = g^{\bar{l}i}\partial_j g_{k\bar{l}} \ , \qquad \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} = g^{\bar{\imath}l}\partial_{\bar{\jmath}} g_{l\bar{k}} \ . \tag{B.26}$$

This connection is called the **Hermitian connection**. It is in general not torsion free and has the important property that the almost complex structure is covariantly constant with respect to it. The corresponding covariant derivative can also be used to specify the components of the adjoints of the Dolbeault operators, e.g.

$$\bar{\partial}^\dagger \omega_{p,q} = -\frac{1}{p!(q-1)!}\nabla^{\bar{\jmath}}\omega_{\bar{\jmath}i_1\ldots i_p\bar{\imath}_2\ldots\bar{\imath}_q} d\xi^{i_1}\wedge\ldots\wedge d\xi^{i_p}\wedge d\bar{\xi}^{\bar{\imath}_2}\wedge\ldots\wedge d\bar{\xi}^{\bar{\imath}_q} \ . \tag{B.27}$$

The simple form of (B.26) leads to some simplifications for the Riemann tensor. It turns out that the only independent non-vanishing components are

$$R^i_{jk\bar{l}} = -\partial_{\bar{l}}\Gamma^i_{kj} = -\partial_{\bar{l}}\left(g^{\bar{m}i}\partial_k g_{j\bar{m}}\right) \ . \tag{B.28}$$

All other components are determined by complex conjugation $R^{\bar{\imath}}_{\bar{\jmath}\bar{k}l} = \overline{R^i_{jk\bar{l}}}$ or using the symmetry $R^i_{j\bar{l}k} = -R^i_{jk\bar{l}}$. Related to the Riemann tensor is the **Ricci form**

$$\begin{aligned}
\mathcal{R} &= i\mathcal{R}_{i\bar{\jmath}} d\xi^i\wedge d\bar{\xi}^{\bar{\jmath}} = iR^k_{ki\bar{\jmath}} d\xi^i\wedge d\bar{\xi}^{\bar{\jmath}} \\
&= -i\partial_{\bar{\jmath}}(g^{\bar{l}k}\partial_i g_{k\bar{l}}) d\xi^i\wedge d\bar{\xi}^{\bar{\jmath}} = -i\partial_{\bar{\jmath}}\partial_i(\ln\sqrt{g})\, d\xi^i\wedge d\bar{\xi}^{\bar{\jmath}} = -i\partial\bar{\partial}\ln\sqrt{g} \ .
\end{aligned} \tag{B.29}$$

It should be noted that despite of the name the components $\mathcal{R}_{i\bar{\jmath}}$ are in general *not* the components of the Ricci tensor. The identity $\partial\bar{\partial} = -\frac{1}{2}d(\partial-\bar{\partial})$ shows that $\mathcal{R}$ is closed. It is however in general not exact because $(\partial-\bar{\partial})\ln\sqrt{g}$ is not necessarily globally defined. Therefore $\mathcal{R}$ defines a generically non-trivial element of $H^2(M,\mathbb{C})$ called the **first Chern class** $c_1(M) = \left[\frac{\mathcal{R}}{2\pi}\right]$.

B.4 Kähler manifolds

We have seen in the last section that on every Hermitian manifold there is a natural two-form J. This allows for the following

Definition B.4.1. A **Kähler manifold** is a Hermitian manifold whose natural two-form is closed, i.e. $dJ = 0$. In this case J is called the **Kähler form** and the Hermitian metric is called **Kähler metric**.[7]

Whereas every complex manifold admits a Hermitian metric the demand for the existence of a Kähler metric is actually a constraint on the manifold. An example of a complex manifold not admitting a Kähler metric is $S^{2p+1}\times S^{2q+1}$ for $q>0$ to exclude the torus. However every complex manifold of complex dimension 1 admits a Kähler metric as does any compact complex manifold that can be embedded in a complex projective space $\mathbb{P}^n$.

[7] There is an alternative and more geometrical definition of Kählerness [144]. A metric on M is Kähler if and only if at each point $p\in M$ there are holomorphic coordinates for which $g_{i\bar{\jmath}} = \delta_{i\bar{\jmath}} + \tilde{g}_{i\bar{\jmath}}$, where $\tilde{g}_{i\bar{\jmath}}$ vanishes to second order at p.

The fact that J is closed implies

$$\partial_i g_{k\bar{\jmath}} = \partial_k g_{i\bar{\jmath}} , \qquad \partial_{\bar{\imath}} g_{k\bar{\jmath}} = \partial_{\bar{\jmath}} g_{k\bar{\imath}} . \tag{B.30}$$

This means however that on each coordinate neighborhood U_k there exists a real function K_k such that

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K_k \tag{B.31}$$

on U_k. This function is called the **Kähler potential**. As we consider only compact manifolds the Kähler potential can not be defined globally.[8] Rather on the intersection of two coordinate neighborhoods $U_k \cap U_l$ the two Kähler potentials differ by the real part of a holomorphic function, $K_k = K_l + f_{kl}(\xi) + \overline{f_{kl}(\xi)}$. In view of (B.31) the metric is thus globally defined as it should be.

The property (B.30) has some influence on the differential geometry of a Kähler manifold. First one sees from (B.26) that the connection coefficients are now symmetric in their lower indices, i.e. the Hermitian connection is the Levi Civita connection of the manifold. This provides in fact a third possible definition of Kählerness. A Hermitian metric is Kähler (with respect to a given complex structure) if and only if the complex structure is covariantly constant with respect to the Levi Civita connection. It follows that the Riemann tensor on a Kähler manifold has the usual symmetries known from general relativity

$$R_{i\bar{\jmath}k\bar{l}} = R_{k\bar{l}i\bar{\jmath}} , \qquad R_{i[\bar{\jmath}k\bar{l}]} = 0 , \qquad R_{\bar{\jmath}[i\bar{l}k]} = 0 . \tag{B.32}$$

The last equation of (B.32) implies $R^{j}_{ik\bar{l}} = R^{j}_{ki\bar{l}}$, which can also be easily seen from (B.28) and (B.30). This additional symmetry ensures that the components of the Ricci form are indeed given by the Ricci tensor.

Moreover the fact that on a Kähler manifold the Hermitian connection is the Levi Civita connection further implies that the Kähler form is not only closed but even harmonic. We have mentioned in the last section that on any Hermitian manifold the coefficients $J_i{}^j$ are covariantly constant with respect to the Hermitian connection. Due to the metric compatibility of the Hermitian connection the same is true for the components of the Kähler form $J_{i\bar{\jmath}}$. On a Kähler manifold this implies $d^\dagger J = 0$, see (A.7). Taken together with $dJ = 0$ this is equivalent to the harmonicity of J.

We have seen in the last section that there are three Laplacians on a Hermitian manifold which are generically totally independent. This is however not the case for Kähler manifolds. In fact the second characterization of a Kähler metric given in footnote 7 can be used to prove

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}} . \tag{B.33}$$

This has some immediate consequences. Now also the usual Hodge $\star$ commutes with $\Delta_{\bar{\partial}}$ and we therefore have $h^{p,q}(M) = h^{D-q,D-p}(M)$. In combination with (B.25) this implies

$$h^{p,q}(M) = h^{q,p}(M) . \tag{B.34}$$

Furthermore the fact that the notions of a harmonic and a $\bar{\partial}$-harmonic form coincide on Kähler manifolds is at the heart of the **Hodge decomposition**

$$H^r(M, \mathbb{C}) = \oplus_{p+q=r} H^{p,q}(M) . \tag{B.35}$$

[8] Otherwise the Kähler-form $J = -\frac{i}{2} d(\partial - \bar{\partial})K$ would be exact in conflict with (B.19).

This ensures the relation $h^r(M) = \sum_{p+q=r} h^{p,q}(M)$ between the Betti numbers and the Hodge numbers which is not valid on an arbitrary Hermitian manifold.

There is another kind of decomposition of the complex de Rham cohomology on a Kähler manifold called the **Lefschetz decomposition**. In order to define it we have to introduce some notation. Associated to the Kähler form there is a map $L : A^{p,q}(M) \to A^{p+1,q+1}(M)$ defined by

$$L(\omega_{p,q}) \equiv J \wedge \omega_{p,q} \ , \tag{B.36}$$

with the adjoint $\Lambda : A^{p,q}(M) \to A^{p-1,q-1}(M)$ given by

$$\Lambda(\omega_{p,q}) = \frac{(-1)^p}{(p-1)!(q-1)!} J^{i\bar{j}} \omega_{ik_1\ldots k_{p-1}\bar{j}\bar{l}_1\ldots\bar{l}_{q-1}} d\xi^{k_1} \wedge \ldots \wedge d\xi^{k_{p-1}} \wedge d\bar{\xi}^{\bar{l}_1} \wedge \ldots \wedge d\bar{\xi}^{\bar{l}_{q-1}} \tag{B.37}$$

for $p, q \geq 1$ and zero otherwise. One can proof the relation [144]

$$[L, \Lambda] = p + q - D \tag{B.38}$$

on $A^{p,q}(M)$ and also verify

$$[L, \Delta_d] = [\Lambda, \Delta_d] = 0 \ . \tag{B.39}$$

Therefore L and Λ can be considered as maps acting between cohomology groups. The **primitive cohomology** is defined as

$$P^q(M) \equiv \mathrm{ker}\Lambda \cap H^q(M, \mathbb{C}) = \mathrm{ker}L^{D-q+1} \cap H^q(M, \mathbb{C}) \ , \tag{B.40}$$

where the proof of the last equality can again be found in [144]. The Lefschetz decomposition is now given by

$$H^q(M, \mathbb{C}) = \oplus_{k \leq \frac{q}{2}} L^k P^{q-2k}(M) \ . \tag{B.41}$$

It is compatible with the Hodge decomposition, i.e. $P^r(M) = \oplus_{p+q=r} P^{p,q}(M)$, where $P^{p,q}(M) \equiv \mathrm{ker}\Lambda \cap H^{p,q}(M)$. A general non-primitive cohomology class ω_r is of the form

$$\omega_r = \omega_r^{(0)} + J \wedge \omega_{r-2}^{(0)} + \ldots + J^n \wedge \omega_{r-2n}^{(0)} \ , \tag{B.42}$$

where the $\omega_i^{(0)}$ are primitive cohomology classes and some $\omega_i^{(0)} \neq 0$ for $i < r$. Furthermore if we denote by $h_0^{p,q}(M)$ the dimension of $P^{p,q}(M)$ then one has [145]

$$h_0^{p,q}(M) = h^{p,q}(M) - h^{p-1,q-1}(M) \ , \qquad \text{for} \quad p + q \leq D \ . \tag{B.43}$$

Finally let us remark that from (B.33) and the harmonicity of J one infers that $h^{p,p}(M) > 0$ for Kähler manifolds because the p-fold wedge product of J with itself is a non-trivial element of $H^{p,p}(M)$.

B.5 Holonomy groups

A useful concept for a Riemannian manifold is that of its holonomy group.

Definition B.5.1. For $p \in M$ consider the set of all closed loops $\{c(t)|t \in [0, 1], c(0) = c(1) = p\}$. If a vector $v \in T_p(M)$ is parallel transported along such a curve $c(t)$ (using the Levi Civita connection) it generically turns back to a different vector $v' \in T_p(M)$ which is related to the original v via a linear map. This map depends on the curve $c(t)$ but not on v. The set of all maps that one gets by all the different closed loops is endowed with a group structure and is called the **holonomy group** at p.

As we are only considering connected manifolds the holonomy group $\mathcal{H}(M)$ does not depend on the point $p \in M$. It contains information about the curvature of the manifold [143, 147]. On a flat manifold the holonomy group is finite.[9] As the Levi Civita connection is metric compatible the length of the vector is conserved during the parallel transport so that $\mathcal{H}(M) \subseteq O(n)$. On orientable manifolds this is further reduced to $\mathcal{H}(M) \subseteq SO(n)$. For manifolds with even dimension we have the following results.

1. $\mathcal{H}(M) \subseteq U(n/2)$ if and only if M is Kähler ,

2. $\mathcal{H}(M) \subseteq SU(n/2)$ if and only if M is Kähler and Ricci-flat.

This property can be used to give an alternative definition of Kähler manifolds.[10] Similarly the holonomy group can be used to define two other classes of manifolds which also play an important role in physical applications.

Definition B.5.2. A manifold with $\mathcal{H}(M) \subseteq Sp(n/4)$ is called a **hyperkähler manifold** and one with $\mathcal{H}(M) \subseteq Sp(n/4) \otimes Sp(1)$ is a **quaternionic Kähler manifold**.

Obviously the real dimension of such manifolds has to be divisible by 4. As there is an embedding $Sp(n/4) \subset SU(n/2)$ every hyperkähler manifold is Ricci-flat and Kähler. However, a quaternionic Kähler manifold is despite its name generically not Kähler and even not a complex manifold. More concretely, whereas on a hyperkähler manifold three different complex structures $\{J_1, J_2, J_3\}$ exist which obey the algebra of the quaternions, i.e. $(J_1)_i{}^j (J_2)_j{}^k = (J_3)_i{}^k$, $(J_1)_{\bar{\imath}}{}^{\bar{\jmath}} (J_2)_{\bar{\jmath}}{}^{\bar{k}} = (J_3)_{\bar{\imath}}{}^{\bar{k}}$ and cyclic relations, such tensor fields only exist locally on a quaternionic Kähler manifold. Moreover, on a hyperkähler manifold any linear combination $aJ_1 + bJ_2 + cJ_3$ can be chosen as complex structure if $a^2 + b^2 + c^2 = 1$. Thus for any hyperkähler structure we have an S^2 of complex structures. An example of a 4-dimensional hyperkähler manifold is the K3 surface with its Ricci-flat metric. We come back to it in section B.9.

An important consequence of a restricted holonomy group is the existence of covariantly constant tensors or spinors. We have already seen that the Kähler form is covariantly constant on a Kähler manifold. Similarly the three complex structures of a hyperkähler manifold are covariantly constant. Further examples will appear during the discussion of Calabi-Yau spaces to which we now turn.

B.6 Calabi-Yau manifolds

We have seen in section B.4 that on Kähler manifolds the components of the Ricci form are given by the Ricci tensor. As the first Chern class is a topological invariant, i.e. it does not change under smooth changes of the metric, a necessary condition for a Kähler manifold to admit a Ricci-flat metric is that its first Chern class vanishes. The fact that the vanishing of the first Chern class is even sufficient has been conjectured by Calabi and proved by Yau. More concretely one has the following

[9] If one only considers contractible loops in the definition B.5.1 one gets the so called **restricted holonomy group**. This is always trivial on flat manifolds. The same does not hold for the full holonomy group. For example the Klein bottle with a flat metric has $\mathcal{H} = \mathbb{Z}_2$, see [147].

[10] Let us remind that the Levi Civita connection coincides with the Hermitian connection if and only if the manifold is Kähler. For a general Hermitian manifold the Levi Civita connection is therefore not pure in its indices and the holonomy group is not a subgroup of $U(n/2)$.

Theorem 1. On a Kähler manifold with $c_1 = 0$ and Kähler form J there exists a *unique* Ricci-flat metric whose Kähler form is in the same cohomology class as J.

This result justifies the

Definition B.6.1. A **Calabi-Yau manifold** is a compact Kähler manifold with $c_1 = 0$.

In this section we collect some of the most important facts about Calabi-Yau manifolds. The vanishing of the first Chern class implies that the Ricci form is exact, i.e. $\mathcal{R} = dA$ for a globally defined 1-form $A = A_i d\xi^i + A_{\bar{\imath}} d\bar{\xi}^{\bar{\imath}}$. This amounts to

$$\mathcal{R} = \partial(A_{\bar{\imath}} d\bar{\xi}^{\bar{\imath}}) + \bar{\partial}(A_i d\xi^i) , \qquad \partial(A_i d\xi^i) = \bar{\partial}(A_{\bar{\imath}} d\bar{\xi}^{\bar{\imath}}) = 0 . \tag{B.44}$$

An important property of Calabi-Yau manifolds is the following

Theorem 2. On a Calabi-Yau manifold there is a pair of spinors $\zeta, \bar{\zeta}$ of opposite chirality which are related by complex conjugation and which are gauge covariantly constant[11]

$$(\nabla_a - \frac{i}{2} A_a)\zeta = 0 , \qquad (\nabla_a + \frac{i}{2} A_a)\bar{\zeta} = 0 , \tag{B.45}$$

where for notational simplicity we have expressed (B.45) in terms of real coordinates, i.e. $a = 1, \ldots, 2D$.

The holonomy of a Calabi-Yau manifold is contained in $SU(D) \times U(1)$. This means that the spin connection implicit in the covariant derivatives in (B.45) is a gauge connection with respect to that group. The $U(1)$ part of this gauge connection is canceled by A_a so that $\nabla_a - \frac{i}{2} A_a$ is the $SU(D)$ gauge covariant derivative and ζ is an $SU(D)$ singlet [142, 154]. If the metric is the Ricci-flat one, which exists due to Yau's theorem 1, the holonomy is contained just in $SU(D)$ as has been discussed in the last section. In this case the spinors from theorem 2 can be chosen to be covariantly constant if $h^1(M) = 0$. This can be seen as follows. Ricci-flatness implies $\mathcal{R} = dA = 0$. Since we assume $h^1(M) = 0$ and A is globally defined it must be exact $A = da$. Redefining $\zeta \to \zeta e^{-\frac{i}{2} a}$ and using (B.45) shows that this redefined spinor is covariantly constant.

The spinor from theorem 2 is useful in various circumstances. It can be used to give expressions for the complex structure and arbitrary spinors on the manifold [142]. With the help of the Gamma matrices, which obey the Dirac algebra

$$\{\gamma^i, \gamma^j\} = \{\gamma^{\bar{\imath}}, \gamma^{\bar{\jmath}}\} = 0 , \qquad \{\gamma^i, \gamma^{\bar{\jmath}}\} = 2g^{i\bar{\jmath}} , \tag{B.46}$$

one defines

$$J_i^{\ j} \equiv -i\zeta^\dagger \gamma_i^{\ j} \zeta , \tag{B.47}$$

where $\gamma_{ij} \equiv \gamma_{[i}\gamma_{j]}$. This can be shown to square to minus the identity and to be covariantly constant so that it is indeed the complex structure.

Furthermore an arbitrary spinor on the manifold can be decomposed as [154]

$$\eta = \omega\zeta + \omega_{\bar{\jmath}} \gamma^{\bar{\jmath}} \zeta + \ldots + \omega_{\bar{\jmath}_1 \ldots \bar{\jmath}_D} \gamma^{\bar{\jmath}_1 \ldots \bar{\jmath}_D} \zeta , \tag{B.48}$$

[11] In fact the statement of the theorem even holds for a generic Kähler manifold with the difference that in this case the vector A is not necessarily globally defined, see e.g. [153].

where the coefficients $\omega_{\bar{\jmath}_1\ldots\bar{\jmath}_p}$ transform under coordinate transformations as components of $(0,p)$-forms. If one assumes the Ricci-flat metric and uses the covariantly constant spinor ζ in (B.48) it can be shown that η is a zero mode of the Dirac operator $\gamma^i\partial_i+\gamma^{\bar{\imath}}\partial_{\bar{\imath}}$ if and only if the corresponding forms $\omega_{0,p}$ are harmonic. One can generalize this formalism to spinors with values in some holomorphic vector bundle V, i.e. $\eta^A = \omega^A\zeta+\omega^A_{\bar{\jmath}}\gamma^{\bar{\jmath}}\zeta+\ldots+\omega^A_{\bar{\jmath}_1\ldots\bar{\jmath}_D}\gamma^{\bar{\jmath}_1\ldots\bar{\jmath}_D}\zeta$, where the range of A depends on the representation of the structure group of V under which η^A transforms. In the Ricci-flat case η^A is now a zero mode of the Dirac operator if the $\omega^A_{\bar{\jmath}_1\ldots\bar{\jmath}_p}$ are elements of $H^{0,p}(M,V)$.[12]

Finally, the existence of the gauge covariantly constant spinor can be used to prove the following

Theorem 3. A compact Kähler manifold has $c_1=0$ if and only if there exists a unique nowhere vanishing $(D,0)$-form

$$\Omega = \frac{1}{D!}\Omega_{i_1\ldots i_D}(\xi)d\xi^{i_1}\wedge\ldots\wedge d\xi^{i_D} \tag{B.49}$$

with holomorphic coefficients $\Omega_{i_1\ldots i_D}(\xi)$ and the following properties

1. Ω is harmonic ,

2. Ω is covariantly constant with respect to the Ricci-flat metric.

The proof can be found e.g. in [154], where also an explicit expression for Ω is given in terms of the gauge covariantly constant spinor ζ for Calabi-Yau manifolds with $h^1(M)=0$. In fact the last equality of eq. (B.44) in connection with $h^1(M)=0$ shows that $A_{\bar{\imath}}d\bar{\xi}^{\bar{\imath}}$ is $\bar{\partial}$-exact, i.e. $A_{\bar{\imath}}d\bar{\xi}^{\bar{\imath}}=\bar{\partial}\bar{\alpha}$. The coefficients of the holomorphic $(D,0)$-form can then be given as

$$\Omega_{i_1\ldots i_D} = e^{-i\bar{\alpha}}\zeta^T\gamma_{i_1\ldots i_D}\zeta\ . \tag{B.50}$$

For the Ricci-flat metric α can be set to zero. The spinor ζ and the $(D,0)$-form Ω are two further examples for the existence of covariantly constant spinors or forms on manifolds with a restricted holonomy which we anticipated at the end of the last section.

The presence of the holomorphic $(D,0)$-form leads to an additional symmetry of the Hodge numbers on a Calabi-Yau manifold which is not valid for a general Kähler manifold. One has

$$h^{p,0}(M) = h^{0,D-p}(M). \tag{B.51}$$

The isomorphism between $H^{p,0}(M)$ and $H^{0,D-p}(M)$ is given by

$$\omega_{\bar{\jmath}_1\ldots\bar{\jmath}_{D-p}} \sim \bar{\Omega}_{\bar{\jmath}_1\ldots\bar{\jmath}_{D-p}}{}^{i_1\ldots i_p}\omega_{i_1\ldots i_p}\ . \tag{B.52}$$

For Calabi-Yau spaces with $h^{1,0}(M)=0$ we can conclude that also $h^{D-1,0}(M)=0$, where (B.34) has been used. In the physics literature one usually demands that the holonomy group of the Ricci-flat metric is exactly $SU(D)$ and not a proper subgroup thereof. This excludes for example all product manifolds like $T^2\times$ K3, $T^2\times Y_3$, with Y_3 a Calabi-Yau threefold, K3 $\times$ K3, etc. For Calabi-Yau manifolds with exact $SU(D)$ holonomy one can show that $h^{p,0}(M)=0$ for all $p\neq 0,D$ [46].[13] Henceforth we will

[12] For a thorough definition of $\bar{\partial}$-cohomology groups with coefficients in a holomorphic vector bundle we refer to [146].

[13] Note that there are hyperkähler fourfolds with $h^{2,0}\neq 0$ which are not of a simple product type. However they are excluded by demanding that the holonomy is exactly $SU(4)$ [46].

also assume this. Furthermore, the $(D,0)$-form also provides an isomorphism between the cohomology groups $H^{0,1}(M,T_M)$ and $H^{D-1,1}(M)$, where T_M is the tangent bundle of M. It is given by

$$\omega_{j_1\dots j_{D-1}\bar{\imath}} \sim \Omega_{j_1\dots j_D}\omega_{\bar{\imath}}{}^{j_D} \ . \tag{B.53}$$

Let us finally make some comments on the moduli space of Ricci-flat metrics on a Calabi-Yau manifold Y_D. We know from theorem 1 that a Ricci-flat metric on Y_D is uniquely determined by specifying its complex structure and Kähler class. Thus there are two types of deformation, one can vary the complex structure of Y_D or the choice of the cohomology class of J. If $h^{2,0}(Y_D) = 0$ the two deformations are independent of each other and the moduli space factorizes, at least locally, into a direct product with one component describing the complex structure deformations and the other one the Kähler deformations. As the Kähler form is of type $(1,1)$ it can be expanded in a complete set of $(1,1)$-forms e_A according to (2.34). If $h^{2,0}(Y_D) = 0$ the Kähler form J and the basis e^A can be chosen independent of the complex structure. Otherwise an element of $H^2(Y_D)$ which is of type $(1,1)$ for a given complex structure might get contributions of type $(2,0)$ or $(0,2)$ when the complex structure is varied. In this case there are complex structure deformations which enforce a variation of the Kähler class because the original J is not of type $(1,1)$ for the new complex structure anymore. Thus for $h^{2,0}(Y_D) \neq 0$ the moduli space loses its local product structure. However, the only case for which this happens is $D = 2$, i.e. for K3 surfaces. We will come back to them in section B.9. Let us for the moment assume $D > 2$ and follow closely the discussion in [92]. In this case the M^A are valid coordinates on the Kähler moduli space. However, not any real values in (2.34) lead to a Kähler form. Rather J has to determine a positive definite metric according to (B.18). Such $(1,1)$-forms are called positive. If J_1 and J_2 are two positive $(1,1)$-forms then also $aJ_1 + bJ_2$ is positive, if $a, b > 0$. Therefore the Kähler moduli span a cone in $\mathbb{R}^{h^{1,1}}$, the so called **Kähler cone**.[14] Now even for $D > 2$ the moduli space is in general only locally a direct product. The Kähler cone might vary inside $\mathbb{R}^{h^{1,1}}$ when the complex structure is deformed. Although the (p,q)-type of a $(1,1)$-form is independent of the complex structure this does not hold for its positivity in general. However, at a point deep inside the Kähler cone these global considerations do not play any role. Furthermore in many Calabi-Yau manifolds the Kähler cone is actually independent of the complex structure.[15]

B.7 Complex structure deformations

Before we proceed to have a closer look at the physically most relevant Calabi-Yau D-folds, which have $D = 1, \dots, 4$, we would like to make some remarks about deformations of complex structure. Although this could have been discussed at an earlier stage some of the most interesting statements can only be made for Kähler manifolds. Furthermore special results are valid for Calabi-Yau spaces. Some of the basic ideas can be found

[14] Another way of seeing that the M^A are restricted is the following. They can be interpreted as the volumes of the 2-cycles dual to e_A (in the sense of (B.3)) and thus have to be positive. Further restrictions come from the requirement that also the volumes of $2n$-cycles c_{2n} for $n > 1$ have to be positive. They are proportional to $\int_{c_{2n}} J^n$.

[15] This is for example the case for so called *complete intersection Calabi-Yau manifolds* which can be realized as the vanishing locus of a system of polynomials in a product of projective spaces. We refer to [92] for some details.

in [145]. A very detailed reference is [146] and a good review including recent results is [148].

We have already remarked in section B.2 that the space of all complex structures of a manifold M is in many cases at least locally again a complex manifold $\mathcal{M}^{\rm cs}$. This is in particular true for Calabi-Yau manifolds M. We denote by M_{Z^α} the manifold M equipped with the complex structure corresponding to the point $Z^\alpha \in \mathcal{M}^{\rm cs}$, where $\alpha = 1, \ldots, \dim_{\mathbb{C}}\mathcal{M}^{\rm cs}$. Of course all M_{Z^α} are diffeomorphic as real manifolds as they only differ in their complex structure. One can combine them in a family $f : \mathcal{X} \to \mathcal{M}^{\rm cs}$ of complex manifolds so that M_{Z^α} is the fiber over Z^α, i.e. $f^{-1}(Z^\alpha) = M_{Z^\alpha}$. For every $Z^\alpha \in \mathcal{M}^{\rm cs}$ there is a map

$$\rho_{Z^\alpha} : T_{Z^\alpha}(\mathcal{M}^{\rm cs}) \to H^{0,1}(M, T_M) \tag{B.54}$$

called the Kodaira-Spencer map [146]. For Calabi-Yau manifolds this is an isomorphism [155]. Thus for them infinitesimal deformations of the complex structure are in one-to-one correspondence with the elements of $H^{0,1}(M, T_M) \cong H^{D-1,1}(M)$.

In principle there is no reason why the Hodge numbers $h^{p,q}(M_{Z^\alpha})$ should be independent of the complex structure. If however all M_{Z^α} are Kähler manifolds and if $\mathcal{M}^{\rm cs}$ is connected this is in fact true [145]. Thus it makes sense to speak about the Hodge numbers of a Kähler manifold M without specifying its complex structure. However, although the Hodge numbers themselves do not depend on the complex structure the Hodge decomposition (B.35) indeed does.[16] If the complex structure of M varies the subspace $H^{p,q}(M) \subset H^r(M, \mathbb{C})$ also varies inside the fixed de Rham group $H^r(M, \mathbb{C})$. This fact is known as a **variation of the Hodge structure**. Infinitesimally, the holomorphic differentials $d\xi^i$ mix linearly with the antiholomorphic differentials

$$\partial_{Z^\alpha} d\xi^i = \mu_{\alpha\bar{\jmath}}{}^i d\bar{\xi}^{\bar{\jmath}} + \nu_{\alpha j}{}^i d\xi^j \ , \tag{B.55}$$

where $\mu_\alpha \in H^{0,1}(M, T_M)$. Thus infinitesimally a (p,q)-form mixes with $(p+1, q-1)$- and $(p-1, q+1)$-forms only [94].

In many cases the variation of the Hodge structure is a complete measure for variations of the complex structure. This has been made more precise by Griffiths through the introduction of the **period map** [156]. Let us outline the basic idea. For a Kähler manifold M we define

$$F^s(M) \equiv H^{r,0}(M) + \ldots + H^{r-s,s}(M) \ , \qquad s \leq r \ , \tag{B.56}$$

The set $\{F^s(M)\}_{s=0,\ldots,r}$ is called the **Hodge filtration** of $H^r(M, \mathbb{C})$. If we denote by $\sigma \equiv [(r-1)/2]$ the greatest integer $\leq (r-1)/2$ we can consider the sequence of subspaces $F^0(M) \subset \ldots \subset F^\sigma(M) \subset H^r(M, \mathbb{C})$, which is called a **flag**. The set of all flags forms a manifold, the so called flag manifold.[17] Griffiths defines now the period map

$$\Phi(Z^\alpha) \equiv [F^0(M_{Z^\alpha}) \subset \ldots \subset F^\sigma(M_{Z^\alpha}) \subset H^r(M, \mathbb{C})] \tag{B.57}$$

and shows that it is holomorphic. The question whether the variation of the Hodge structure can provide a complete description of deformations of the complex struc-

[16] This is in contrast to the Lefschetz decomposition which is independent of the complex structure [144].

[17] Its construction is analogous to that of a Grassmannian manifold but we omit the details here.

ture for a given manifold, i.e. whether the period map is injective, is called a **Torelli problem**.[18]

For Calabi-Yau manifolds it is known that the variation of the Hodge structure indeed describes the variation of the complex structure completely. To make this description precise it is possible to make use of an easier version of the period map which is called the classical period map in [157]. It can be obtained from (B.57) with $r = D$ by combining it with the projection

$$\pi([F^0(M_{Z^\alpha}) \subset \ldots \subset F^\sigma(M_{Z^\alpha}) \subset H^D(M,\mathbb{C})]) \equiv [F^0(M_{Z^\alpha}) \subset H^D(M,\mathbb{C})] \; . \tag{B.58}$$

The classical period map $\tilde{\Phi} = \pi \circ \Phi$ is also holomorphic. It describes the variation of the $(D,0)$-form Ω of the Calabi-Yau manifold (B.49) under deformations of the complex structure. In fact this variation is always nonzero. For $D \geq 3$ this can be seen from

$$\partial_{Z^\alpha}\Omega = k_\alpha \Omega + \Phi_\alpha \; , \tag{B.59}$$

where k_α is a function of the Z^β and Φ_α is a member of a basis for the $(D-1,1)$-forms [93,94,155]. One can even be more precise. The complex structure moduli space turns out to be itself a Kähler manifold with Kähler potential K^{cs} and k_α is actually given by $k_\alpha = -\partial_{Z^\alpha} K^{\mathrm{cs}}$. Furthermore the holomorphicity of $\tilde{\Phi}$ implies $\partial_{\bar{Z}^{\bar{\alpha}}}\Omega = 0$.

The name period map for (B.57) stems from the fact that it is a generalization of the classical period map which can indeed be represented by period integrals. To be more concrete let $\{\gamma^a\}_{a=1,\ldots,h^D(M)}$ be a basis for the integral homology $H_D(M,\mathbb{Z})$, which is independent of the complex structure. Then the periods

$$\Pi_a \equiv \int_{\gamma^a} \Omega \tag{B.60}$$

vary with the complex structure because of the variation of Ω. These period integrals for a given choice of D-cycles provide a realization of the classical period map $\tilde{\Phi}$ for Calabi-Yau manifolds M. Furthermore they determine the complex structure of M completely. This will become clear in the following as we consider the physically relevant Calabi-Yau spaces in turn.

B.8 Torus

The torus is the simplest Calabi-Yau space. As many general features of Calabi-Yau manifolds can be demonstrated in this simple setting we want to go through the analysis in some detail. The torus has the Hodge numbers

$$h^{0,0} = h^{1,1} = h^{1,0} = h^{0,1} = 1 \; . \tag{B.61}$$

Thus it has vanishing Euler number $\chi = 0$. As for two dimensional manifolds c_1 is the Euler class (B.5), see [158], it is clear that the torus has vanishing first Chern class. This is in contrast to all other Riemann surfaces which have $\chi \neq 0$.

[18] In fact in the mathematical literature one usually considers a slightly different flag manifold which is defined by restricting to primitive cohomology in (B.56). Consequently the flags are subsets of $P^r(M)$. A certain subspace of this flag manifold is known as the **classifying space** or **Griffiths domain** and is used in investigations of Torelli problems. The reason is that the whole cohomology is in general 'too big' and gives redundant information. See e.g. [148] for the details.

A torus can be defined by taking the quotient of the complex plane $\mathbb{C}$ with a lattice $L(\omega_1,\omega_2) = \{\omega_1 m + \omega_2 n | m,n \in \mathbb{Z}\}$, i.e. points $z_1, z_2 \in \mathbb{C}/L(\omega_1,\omega_2)$ are identified if $z_1 - z_2 = \omega_1 m + \omega_2 n$ for some $m,n \in \mathbb{Z}$ and ω_1 and ω_2 are two non-vanishing complex numbers such that $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$. Then $\mathbb{C}/L(\omega_1,\omega_2)$ is homeomorphic to the torus as is indicated in figure B.1. Furthermore, it inherits a complex structure from $\mathbb{C}$ which

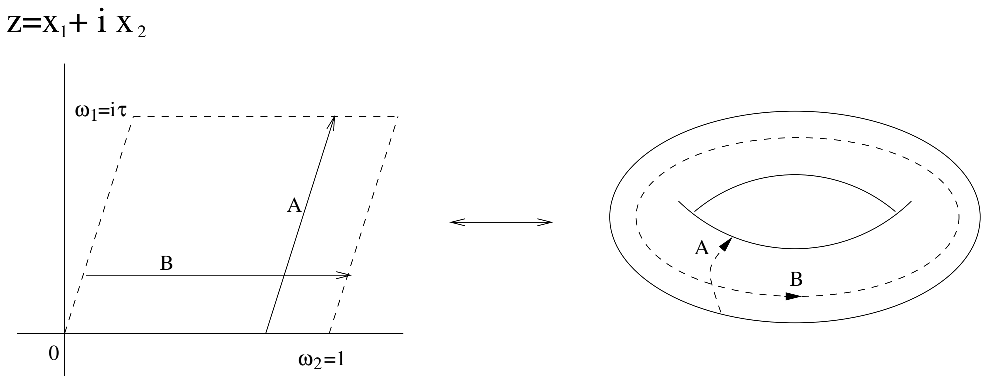

Figure B.1: Two different representations of the torus. A and B are the basic representatives of the two one-cycles.

however depends on the choice of the pair (ω_1,ω_2). It can be shown that the complex structure actually only depends on the **modular parameter**

$$\tau \equiv -i\frac{\omega_1}{\omega_2} \ , \tag{B.62}$$

so that one could define the lattice with $\omega_2 = 1$, a value which has already been chosen in figure B.1.[19] Moreover τ can be chosen to have $\mathrm{Im}(i\tau) > 0$ and τ and τ' define the same complex structure if

$$\tau' = \frac{a\tau - ib}{ic\tau + d} \qquad \text{for} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \ . \tag{B.63}$$

Thus the moduli space of independent complex structures on the torus is given by the quotient space $H/SL(2,\mathbb{Z})$, where H is the upper half plane. This so called **fundamental region** can be represented in many different ways. The standard one is depicted in figure B.2. This is a common feature that the moduli space can be represented as a quotient of a manifold $\mathcal{T}$ by some discrete group G. Then the manifold $\mathcal{T}$ is called **Teichmüller space** and the group G is denoted **modular group**. Modding out the modular group takes care of the fact that there are 'large' diffeomorphisms of the torus, which are not continuously connected to the identity but leave the defining lattice $L(\omega_1,\omega_2)$ and therefore the complex structure of the torus invariant. For the details we refer to [141] and references therein.

Here we just give an example taken from [142] which makes plausible that different values for τ may lead to different complex structures. Consider the two tori

$$T_1 \quad = \quad \{(x_1,x_2)|(x_1,x_2) \cong (x_1+m, x_2+n)\} \ ,$$

[19] This definition of τ differs by a factor of $-i$ from that found in many text books like [141]. It is however the one we use in chapter 3.

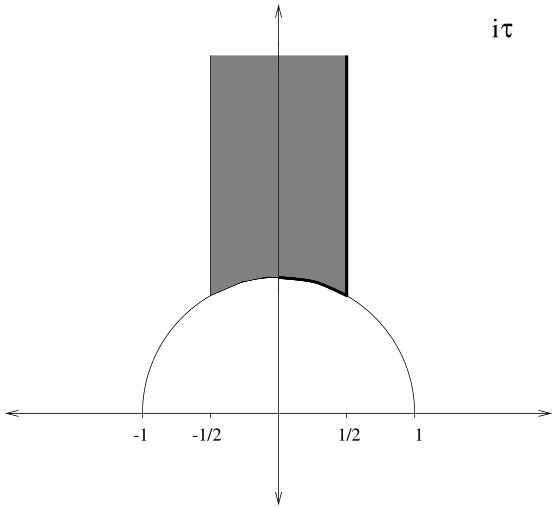

Figure B.2: Complex structure moduli space of the torus. The thick part of the boundary is included in the moduli space.

$$T_2 \quad = \quad \{(y_1, y_2)|(y_1, y_2) \cong (y_1 + m, y_2 + 2n)\} \ , \tag{B.64}$$

which have $\tau_1 = 1$ and $\tau_2 = 2$. The map $(y_1, y_2) = (x_1, 2x_2)$ shows that the two manifolds are diffeomorphic. However the complex coordinates $\zeta = x_1 + ix_2$ and $\xi = y_1 + iy_2$ are related in a non-holomorphic way via

$$\xi = \frac{3}{2}\zeta - \frac{1}{2}\bar{\zeta} \ , \tag{B.65}$$

showing that T_1 and T_2 have different complex structures.

The torus inherits the natural flat metric of $\mathbb{C}$ through the definition we have given above, i.e. $ds^2 = dz \otimes d\bar{z}$, independently of the complex structure. The different complex structures in this way only occur in the different periodicity conditions on the coordinates $z \cong z + (m + ni\tau)$. One can however introduce new coordinates $\zeta = x_1 + i\tau x_2$ in which the periodicity condition is $(x_1, x_2) \cong (x_1, x_2) + (m, n)$ and the complex structure shows up in different metrics, i.e. $ds^2 = d\zeta \otimes d\bar{\zeta}$. The differential $d\zeta$ is the unique holomorphic $(1,0)$-form of the torus which exists according to theorem 3. The modular parameter can be expressed through period integrals of $d\zeta$ over the cycles A and B in figure B.1. In fact we have[20]

$$\tau = -i \frac{\int_A d\zeta}{\int_B d\zeta} \ , \tag{B.66}$$

see e.g. [27]. This is the first example of how the period integrals of the holomorphic $(D,0)$-form completely determine the complex structure of Calabi-Yau manifolds. We will see further examples in the following sections.

Of course not every metric on the torus is flat and of the form $ds^2 = d\zeta \otimes d\bar{\zeta}$, with ζ as above. However, according to a theorem from Riemann surface theory any

[20]The period integrals are coordinate independent. One could have used the z-coordinates instead. In this case the integrals depend on τ through the cycles A and B, see figure B.1. However the situation in which the τ dependence resides in the differential $d\zeta$ is more adapted to the discussion of the last section.

metric on the torus is conformal to a flat metric of the form $ds^2 = d\zeta \otimes d\bar{\zeta}$ modulo diffeomorphisms, see e.g. [159]. Thus the moduli space of flat metrics is spanned by the complex structure moduli and a conformal factor, the Kähler modulus, which is a measure for the overall size of the torus. As we have discussed in section 1.3 the Kähler modulus gets complexified in string compactifications. It turns out that not all complexified Kähler moduli lead to different space-time physics but only those which span exactly the same fundamental region as depicted in figure B.2 for the complex structure. Furthermore the physics is invariant under an exchange of the complex structure and the complexified Kähler moduli, thus providing the simplest example of mirror symmetry [160].

B.9 K3

All Calabi-Yau twofolds are diffeomorphic as real manifolds. They are called **K3 surfaces** and are in many ways special. As they play an important role in our discussion of dualities in chapter 2 and 3 we want to go into some of the details. An introduction into the mathematics of K3 surfaces can be found in [161], its physical applications are covered to a great extend in [106].

The Hodge numbers can be collected in a **Hodge diamond** and take the values

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} . \tag{B.67}$$

Thus the Euler number is $\chi = 24$.

By a theorem of Torelli the Teichmüller space for complex structures is again given by the space of possible periods

$$\varpi_i = \int_{\gamma^i} \Omega \ , \tag{B.68}$$

where $\{\gamma^i\}$ is a basis of integral two-cycles, see [161]. For a K3 surface Y_2 the integral homology group $H_2(Y_2, \mathbb{Z})$ forms an even selfdual lattice of signature $(3, 19)$ on which the inner product is defined via the intersection number. By Poincaré duality the same holds for $H^2(Y_2, \mathbb{Z})$ with integration over the wedge product being the inner product. Such a lattice is unique up to isometries and denoted by $\Gamma_{3,19}$. $H_2(Y_2, \mathbb{Z})$ is independent of the complex structure. If the complex structure changes the periods vary through their dependence on the holomorphic two-form Ω. It is straightforward to show [106] that $\mathrm{Re}\Omega$ and $\mathrm{Im}\Omega$ span an oriented space-like two-plane in $H^2(Y_2, \mathbb{R}) \cong \mathbb{R}^{3,19}$ which we will also denoted by Ω, again following [106]. As the lattice $H^2(Y_2, \mathbb{Z})$ is embedded into $H^2(Y_2, \mathbb{R})$ we get the following picture: The choice of a complex structure of Y_2 determines an oriented space-like two-plane in $\mathbb{R}^{3,19}$ and a change of the complex structure results in a variation of this two-plane with respect to the fixed lattice $\Gamma_{3,19} \subset \mathbb{R}^{3,19}$. Thus the Teichmüller space of complex structures is given by the Grassmannian of oriented two-planes in $\mathbb{R}^{3,19}$, i.e. $O^+(3, 19)/(O(2) \times O(1, 19))^+$, where the '+' indicates restriction to orientation preserving rotations. As for the torus there are however diffeomorphisms of Y_2 which leave the complex structure unchanged. The modular group in this case is $O^+(\Gamma_{3,19})$, the group of orientation preserving isometries of $\Gamma_{3,19}$. This is also analogous to the torus, where the modular group leaves the lattice $L(\omega_1, \omega_2)$ invariant. Thus the moduli space of complex structures on a K3 surface is

$$\mathcal{M}^{\rm cs} = O^+(\Gamma_{3,19}) \setminus O^+(3,19) \; / \, (O(2) \times O(1,19))^+ \; . \tag{B.69}$$

More important for applications in string theory is the moduli space of Ricci-flat Kähler metrics. From theorem 1 we know that for a given complex structure and Kähler class there exists a unique Ricci-flat metric. The converse is not true for K3 surfaces. This is due to the fact that Y_2 endowed with a Ricci-flat metric is not only Calabi-Yau but also hyperkähler as we have already anticipated in section B.5. Thus the given metric is Kähler with respect to a whole sphere of complex structures, i.e. a whole sphere of complex structures is covariantly constant with respect to the metric's Levi Civita connection. Depending on which complex structure one actually chooses out of this sphere one ends up with a different Kähler form $J_{i\bar{\jmath}} = J_i{}^k g_{k\bar{\jmath}}$. In any case this Kähler form is a space-like direction in $H^2(Y_2, \mathbb{R})$ as $\int J \wedge J \sim \mathrm{vol}(Y_2) > 0$. Furthermore it is perpendicular to the two-plane Ω because the Kähler form is of type $(1,1)$ with respect to the chosen complex structure. Thus J and Ω span an oriented space-like three-plane Σ in $\mathbb{R}^{3,19}$. Choosing a different complex structure out of the sphere of complex structures amounts to a rotation within Σ. This leaves the Ricci-flat metric unchanged but changes what we consider to be the Kähler form or the complex structure. Only rotations of Σ with respect to the lattice $\Gamma_{3,19}$ lead to variations of the Ricci-flat metric. Thus the moduli space of unit volume Ricci-flat metrics is the Grassmannian of oriented three-planes Σ in $\mathbb{R}^{3,19}$, again modded out by the modular group $O^+(\Gamma_{3,19})$. Altogether the moduli space of Ricci-flat metrics can be shown to be the 58-dimensional space

$$\mathcal{M}^{\rm rf} = O(\Gamma_{3,19}) \setminus O(3,19) \; / \, (O(3) \times O(19)) \times \mathbb{R}_+ \; , \tag{B.70}$$

where the $\mathbb{R}_+$ factor determines the overall volume [106].

Let us make two remarks here. One distinction between K3 surfaces and Calabi-Yau spaces of $D \neq 2$ is that the moduli space is not even locally a product of complex structure and Kähler moduli spaces. This is due to the non-trivial Dolbeault group $H^{2,0}(Y_2)$ as we have remarked at the end of section B.6. Here we have seen in another way how the dependence of the Kähler form on the complex structure comes about. The Kähler form has to vary when the split between J and Ω is changed by rotating Ω inside Σ. As the moduli space of Ricci-flat metrics on Y_2 has no product structure it is not clear how to define mirror symmetry on a general K3 surface. We will come back to this in a moment.

Secondly we want to mention that on a K3 surface the second cohomology decomposes into a self-dual and an anti-self-dual subspace $H^2(Y_2, \mathbb{R}) = \mathcal{H}^+ \oplus \mathcal{H}^-$ with the dimensions $\mathrm{dim}\mathcal{H}^+ = 3$ and $\mathrm{dim}\mathcal{H}^- = 19$. Actually $\mathcal{H}^+$ is identical to the three-plane $\Sigma \subset H^2(Y_2, \mathbb{R})$. Furthermore we have already mentioned that the three self-dual forms are also covariantly constant. This is not the case for the anti-self-dual forms.

In string theory one is more interested in the moduli space of conformally invariant non-linear sigma models with K3 target space. We therefore have to take into account the moduli coming from expanding the NS B-field into the 22 harmonic forms. The conformal field theory based on a K3 background has $N = (4,4)$ supersymmetry. From this and the fact that it has dimension 80 one can uniquely deduce the form of the Teichmüller space [162, 163]

$$\mathcal{T}^{\rm cf} = \frac{O(4,20)}{O(4) \times O(20)} \cong \frac{O(3,19)}{O(3) \times O(19)} \times \mathbb{R}^{22} \times \mathbb{R}_+ \; , \tag{B.71}$$

which again has to be modded out by an appropriate modular group in order to get the moduli space, see [106] for details.

Finally we want to make some remarks about mirror symmetry in the context of K3 surfaces although it is slightly out of the line. It is however a good opportunity to introduce the Picard group which we need in section 2.3. We have already remarked that it is not clear how to define mirror symmetry in the moduli space of a general K3 surface. It can however be sensibly discussed in the context of **algebraic K3** manifolds Y_2, i.e. those which are holomorphically embedded in a projective space $\mathbb{P}^n$ for some n. In this case the K3 inherits the Kähler form from the embedding $\mathbb{P}^n$. This is a (space-like) element of the so called **Picard lattice** or **Picard group**

$$\mathrm{Pic}(Y_2) = H^2(Y_2, \mathbb{Z}) \cap H^{1,1}(Y_2) \tag{B.72}$$

whose rank $\rho(Y_2)$ is called the **Picard number**. As the Picard lattice is orthogonal to $\Omega \subset \mathbb{R}^{3,19}$ its signature is $(1, \rho-1)$. The Picard group depends on the complex structure because $H^{1,1}(Y_2)$ depends on it as we have discussed at the end of section B.6. Thus a generic K3 has vanishing Picard number. However, an algebraic K3 surface has at least $\rho(Y_2) = 1$ and its Kähler form is integral. To be more precise the embedding into $\mathbb{P}^n$ implies the existence of one or more holomorphically embedded curves in Y_2. Via Poincaré duality they correspond to elements of $H^2(Y_2, \mathbb{Z})$ and due to the fact that the curves are holomorphically embedded these two-forms are actually of type $(1,1)$ and therefore elements of the Picard group [144]. Thus if there are k holomorphically embedded curves in an algebraic K3 surface Y_2 its Picard number is generically $\rho(Y_2) = k$. Consequently its complex structure moduli space is constrained to the subspace of (B.69) which does not change the (p,q)-type of the two-forms Poincaré dual to the holomorphically embedded curves.

It can be shown now that the Teichmüller space of conformally invariant sigma models with an algebraic K3 surface with k holomorphically embedded curves as target space is given by

$$\mathcal{T}^{\mathrm{cf,alg}} = \frac{O(2, 20-k)}{O(2) \times O(20-k)} \times \frac{O(2,k)}{O(2) \times O(k)}\ . \tag{B.73}$$

In this case one actually has a product structure of the Teichmüller space into one factor for the complex structure (the first one) and one for the Kähler form and B-field. Now mirror symmetry has its usual effect of exchanging the two factors. For further details see [106, 164].

B.10 Calabi-Yau threefolds

Calabi-Yau threefolds have two independent non-trivial Hodge numbers. The Hodge diamond is given by

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}\ . \tag{B.74}$$

In contrast to the previous cases, Calabi-Yau manifolds with $D \geq 3$ are not all diffeomorphic. In fact a complete classification of Calabi-Yau D-folds is not achieved so far, not even for $D = 3$. However there are large lists available of Calabi-Yau three- and fourfolds which can be constructed as hypersurfaces in toric varieties or weighted projective spaces [165]. We will not go into the details of those constructions. The Euler number $\chi = 2(h^{1,1} - h^{2,1})$ is not fixed and can become negative.

Also for Calabi-Yau threefolds a Torelli theorem holds [166]. If one introduces a canonical basis for $H_3(Y_3, \mathbb{Z})$, which we denote by $\{\gamma^\alpha, \delta_\alpha\}, \alpha = 0, \ldots, h^{2,1}$, with intersection numbers $\gamma^\alpha \cdot \gamma^\beta = 0,\ \delta_\alpha \cdot \delta_\beta = 0$ and $\gamma^\alpha \cdot \delta_\beta = \delta^\alpha_\beta$, the periods

$$A^\alpha = \int_{\gamma^\alpha} \Omega \ , \qquad B_\alpha = \int_{\delta_\alpha} \Omega \tag{B.75}$$

locally completely determine the complex structure. One can be more precise. The dimension of the complex structure moduli space is given by that of $H^{0,1}(Y_3, T_{Y_3}) \cong H^{2,1}(Y_3)$. Therefore the periods (B.75) form a redundant set of parameters. Rather it suffices to take only one group of periods, e.g. A^α. They form projective coordinates for the complex structure moduli space. In fact Ω is only defined up to a normalization factor and different normalizations lead to a rescaling of the periods (B.75) so that one has to identify $A^\alpha \cong \lambda A^\alpha \ , \lambda \neq 0$. The other periods B_α are functions of the A^α [93, 94]. In the physics literature one usually denotes the A^α as Z^α and the B_α as $\mathcal{F}_\alpha$. Furthermore one can show that $\mathcal{F}_\alpha = \partial_{Z^\alpha} \mathcal{F}$ for a homogeneous function of degree two, $\mathcal{F}(\lambda Z) = \lambda^2 \mathcal{F}(Z)$, called the **prepotential**.

As we have discussed in section B.6 the fact that $h^{2,0} = 0$ for Calabi-Yau threefolds implies that the moduli space for Ricci-flat metrics is the product of the moduli space of complex structures with that of the Kähler class. In string theory the Kähler class is complexified through the moduli coming from the expansion of the NS B-field

$$J + iB = (M^A + iB^A) e_A \equiv t^A e_A \ . \tag{B.76}$$

Thus the moduli space of conformally invariant non-linear sigma models with Calabi-Yau threefold target space is locally given by a factor for the complex structure and one for the complexified Kähler moduli which are exchanged under mirror symmetry. An important observation is that both factors are themselves Kähler manifolds with the Kähler potentials given in eqs. (2.7) and (2.8). In fact the two factors of the moduli space are not only Kähler but even **special Kähler manifolds**, i.e. their Kähler potentials can be expressed via prepotentials

$$K^{\rm cs} = -\ln\left(i\bar{Z}^{\bar\alpha} \partial_{Z^\alpha} \mathcal{F} - i Z^\alpha \partial_{\bar{Z}^{\bar\alpha}} \bar{\mathcal{F}}\right) \ , \ K^{\rm k} = -\ln\left(i\bar{X}^{\bar{A}} \partial_{X^A} \mathcal{G} - i X^A \partial_{\bar{X}^{\bar{A}}} \mathcal{G}\right) \ . \tag{B.77}$$

Now the index A includes the value 0, because we have reinstated projective coordinates on the Kähler moduli space by introducing the additional coordinate X^0, i.e. $t^A = \frac{X^A}{X^0}$ for $A = 1, \ldots, h^{1,1}(Y_3)$. Furthermore $\mathcal{G} = \frac{1}{3!} d_{ABC} \frac{X^A X^B X^C}{X^0}$ is like $\mathcal{F}$ homogeneous of degree two.[21]

Finally we want to make some remarks about Calabi-Yau manifolds with a certain fibration structure. The conditions for a Calabi-Yau threefold to admit an elliptic (i.e.

[21] Strictly speaking this form for $\mathcal{G}$ is only valid in the large radius limit of Y_3 and gets corrections from world-sheet instantons. Only if these corrections are taken into account are the two Kähler potentials in (B.77) consistent with mirror symmetry.

torus) or a K3 fibering have been investigated in [167]. They can be stated in the following way [107]. A Calabi-Yau threefold admits an elliptic fibration if it has a **divisor**[22] D such that $D \cdot \Gamma \geq 0$ for all curves Γ, $D^3 = 0$ and $D^2 \cdot F \neq 0$ for some divisor F, where the dot denotes intersection. The base must be a rational surface, i.e. an algebraic surface isomorphic to $\mathbb{P}^2$. Thus it has to be either $\mathbb{P}^2$ itself, a Hirzebruch surface $\mathbb{F}_n$ or a **blow up** of one of them [144].[23] For a K3 fibration there must be a divisor $\hat{D}$ with $\hat{D} \cdot \Gamma \geq 0$ for all curves and $\hat{D}^2 \cdot F = 0$ for all divisors F. The base has to be a $\mathbb{P}^1$. The two fibration structures are compatible with each other, i.e. the K3 is itself elliptically fibered, if $D^2 \cdot \hat{D} = 0$. In the case of a K3 fibration the divisor $\hat{D}$ is the generic fiber and the fact that $\hat{D}^2 \cdot F = 0$ shows that there are no obstructions to move it over the base $\mathbb{P}^1$.

B.11 Calabi-Yau fourfolds

The last Calabi-Yau spaces which have found application in physics so far are the fourfolds. They have the following Hodge diamond

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & h^{1,1} & & 0 & & \\ & 0 & & h^{2,1} & & h^{1,2} & & 0 & \\ 1 & & h^{3,1} & & h^{2,2} & & h^{1,3} & & 1 \\ & 0 & & h^{3,2} & & h^{2,3} & & 0 & \\ & & 0 & & h^{3,3} & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array} \quad . \tag{B.78}$$

The symmetries discussed in sections B.3 and B.4 reduce the independent Hodge numbers to four $h^{1,1} = h^{3,3}$, $h^{2,1} = h^{1,2} = h^{3,2} = h^{2,3}$, $h^{3,1} = h^{1,3}$ and $h^{2,2}$. However there is a further relation between the Hodge numbers on a Calabi-Yau fourfold [46]

$$h^{2,2} = 2(22 + 2h^{1,1} + 2h^{1,3} - h^{1,2}) \ , \tag{B.79}$$

so that in fact only three of them can be varied independently. The middle cohomology splits into a selfdual subspace $B_+(Y_4)$ ($\star\omega = \omega$) and an anti-selfdual subspace $B_-(Y_4)$ ($\star\omega = -\omega$). The action of the Hodge $\star$ operator (B.20) on a four-form of Y_4 has been given in (4.26) and (4.29). Using (B.43) one easily establishes the number of selfdual and anti-selfdual four-forms

$$\mathrm{dim} B_+(Y_4) = 3 + h^{2,2} - h^{1,1} \ , \qquad \mathrm{dim} B_-(Y_4) = 2h^{3,1} + h^{1,1} - 1 \ . \tag{B.80}$$

With the help of (B.79) it can straightforwardly be shown that this is compatible with the Hirzebruch signature [46]

$$\tau(Y_4) = \mathrm{dim} B_+(Y_4) - \mathrm{dim} B_-(Y_4) = \frac{\chi}{3} + 32 \ . \tag{B.81}$$

[22] A divisor in a complex manifold is a holomorphically embedded submanifold of (complex) codimension one. According to Poincaré duality and a theorem by Lefschetz there is a one-to-one correspondence between divisors and integral $(1,1)$-forms [144]. This has already been used in the discussion of the Picard group of an algebraic K3 in the last section.

[23] For a thorough definition of a blow up we refer to [144]. Heuristically, to blow up a manifold at a point one takes out this point and replaces it with a whole sphere.

Furthermore, the symmetric quadratic form $Q(\omega_1, \omega_2) = \int_{Y_4} \omega_1 \wedge \omega_2$ is positive definite on $B_+(Y_4)$ and negative definite on $B_-(Y_4)$ [46].

The discussion of the moduli space follows the same lines as for threefolds. The complex structure and complexified Kähler moduli are again independent of each other, at least locally. It has been speculated in [94] that the periods of the $(4,0)$-form Ω over an integral basis for the middle homology might provide some highly redundant coordinates for the complex structure moduli space. The two factors of the moduli space of conformal sigma models with Y_4 target space are again Kähler manifolds with the Kähler potentials for the complex structure and complexified Kähler moduli given by the first respectively second summand in (3.25) [81,94], where $\mathcal{V}$ is the volume of Y_4 defined in (2.33). The two components of the moduli space are exchanged by mirror symmetry.[24] Moreover if $\tilde{Y}_4$ denotes the mirror fourfold the following relations hold between the respective Hodge numbers [46]

$$\begin{aligned} h^{3,1}(Y_4) &= h^{1,1}(\tilde{Y}_4) \ , \qquad & h^{1,1}(Y_4) &= h^{3,1}(\tilde{Y}_4) \\ h^{2,1}(Y_4) &= h^{2,1}(\tilde{Y}_4) \ , \qquad & h^{2,2}(Y_4) &= h^{2,2}(\tilde{Y}_4) \end{aligned} \tag{B.82}$$

and the cohomology $H^{2,2}(Y_4)$ decomposes according to

$$H^{2,2}(Y_4) = P^{2,2}(Y_4) \oplus P^{2,2}(\tilde{Y}_4) \ , \tag{B.83}$$

where $P^{2,2}(Y_4)$ denotes the primitive cohomology as defined below (B.41). There is another decomposition of the Dolbeault cohomology group $H^{2,2}(Y_4)$. It splits into a so called vertical and horizontal part

$$H^{2,2}(Y_4) = H^{2,2}_V(Y_4) \oplus H^{2,2}_H(Y_4) \ . \tag{B.84}$$

The vertical part consists of all wedge products $\omega^v_{2,2} \sim e_A \wedge e_B$.[25] The horizontal part is defined as follows. We have seen in (B.59) that the Kähler covariant derivative of the $(4,0)$-form Ω (4.6) generates all of $H^{3,1}(Y_4)$. In contrast the second Kähler covariant derivatives of Ω generate only a subspace of $H^{2,2}(Y_4)$, the horizontal part. $H^{2,2}_V(Y_4)$ can be used to define the **vertical primary subspace** of the cohomology

$$H^{0,0}(Y_4) \oplus H^{1,1}(Y_4) \oplus H^{2,2}_V(Y_4) \oplus H^{3,3}(Y_4) \oplus H^{4,4}(Y_4) \tag{B.85}$$

and the **horizontal primary subspace**

$$H^{4,0}(Y_4) \oplus H^{3,1}(Y_4) \oplus H^{2,2}_H(Y_4) \oplus H^{1,3}(Y_4) \oplus H^{0,4}(Y_4) \ . \tag{B.86}$$

They are exchanged under mirror symmetry. We have seen the necessity for a split in $H^{D/2,D/2}(Y_D)$, with D even, already in the discussion of mirror symmetry for K3 surfaces. It is a general property of Calabi-Yau manifolds with even dimension because for them $H^{D/2,D/2}(Y_D)$ belongs to both, the vertical and the horizontal cohomology, which are exchanged under mirror symmetry.[26] The details and further aspects of mirror symmetry for Calabi-Yau fourfolds are discussed in [41,46,168].

[24] As in the threefold case one has to take into account the effects of world-sheet instantons in order to render (3.25) mirror symmetric.

[25] Note that not all combinations of e_A and e_B lead to independent forms $\omega^v_{2,2}$.

[26] Let us recall that in general the vertical cohomology is defined as $\sum_{p=0}^{D} H^{p,p}(Y_D)$ and the horizontal cohomology as $\sum_{p=0}^{D} H^{D-p,p}(Y_D)$.

Finally let us remark that the conditions for Calabi-Yau fourfolds to have special fibration structures are much less studied as in the threefold case. A general classification in the spirit of [167] has not been done so far. It is also not even known what the allowed base manifolds are in case of K3 fibrations [169]. The Hirzebruch surfaces considered in sections 2.3 and 3.3 are only one possible choice.

Appendix C

Kaluza Klein reduction

The general idea behind Kaluza-Klein theories and their application in string theory has been outlined in the introduction. Here we want to give a brief survey of the method of Kaluza-Klein reduction and review the problem of consistency arising in this context. We largely follow the discussion presented in [112]. Another thorough introduction into Kaluza-Klein reduction is [170].

C.1 The Kaluza-Klein recipe

We start with a theory in D dimensions describing gravity coupled to some matter fields, which we collectively denote by Φ suppressing all space-time or internal indices.[1] In view of the application in string theory we also allow for the possibility that there are already gauge fields in the D-dimensional theory. We then look for a stable ground state solution $<g_{MN}>$ and $<\Phi>$ of the equations of motion, such that the metric $<g_{MN}>$ describes a product space $M_d \times M_{D-d}$. This is known as *spontaneous compactification*. The space M_d is a d-dimensional space-time with Lorentz signature whereas M_{D-d} is a $(D-d)$-dimensional compact space with Euclidean signature. Usually one is interested in the case $d = 4$ but in view of the applications in this thesis we let d unspecified. If we demand maximal symmetry for the d-dimensional space-time and denote the coordinates on M_d by x^μ and those on M_{D-d} by y^a the most general Ansatz for the ground state metric is a *warped product*

$$<g_{MN}(x,y)> = \begin{pmatrix} f(y)\ \hat{g}_{\mu\nu}(x) & 0 \\ 0 & \hat{g}_{ab}(y) \end{pmatrix} , \tag{C.1}$$

where the function $f(y)$ is called the *warp factor*. In supersymmetric theories the warp factor is essential to get an unbroken supersymmetry in the presence of background fluxes as we have discussed in chapter 4. It has also played an important role in establishing the equivalence of four-dimensional gauged $N = 8$ supergravity and the Kaluza-Klein reduction of 11-dimensional supergravity on S^7. Here however we restrict our attention to the case $f \equiv 1$.

In order to determine the spectrum of the d-dimensional theory we consider small fluctuations of the D-dimensional fields about their ground-state values

$$g_{MN}(x,y) = <g_{MN}(x,y)> + \delta g_{MN}(x,y) , \qquad \Phi(x,y) = <\Phi(x,y)> + \delta\Phi(x,y) \tag{C.2}$$

[1] These comprise scalars, spinors and antisymmetric tensor fields.

and insert them into the D-dimensional equations of motion. Retaining only terms linear in the fluctuations allows to determine the relevant mass operators in the d-dimensional theory and to expand the fluctuations in terms of a set of eigenfunctions of these mass operators

$$\begin{aligned} \delta\Phi(x,y) &= \sum_n \phi^{(n)}(x)\hat{Y}^{(n)}(y) \ , \qquad \delta g_{\mu\nu}(x,y) = \sum_n h^{(n)}_{\mu\nu}(x)Y^{(n)}(y) \ , \\ \delta g_{\mu a}(x,y) &= \sum_n A^{(n)}_\mu(x)Y^{(n)}_a(y) \ , \qquad \delta g_{ab}(x,y) = \sum_n X^{(n)}(x)Y^{(n)}_{ab}(y) \ . \end{aligned} \tag{C.3}$$

The x-dependent coefficient functions appear as fields in the d-dimensional theory whose masses are given by the eigenvalues of the eigenfunctions $\hat{Y}$ and Y. Thus we obtain a spectrum which consists of a finite number of massless states and an infinite tower of massive states with masses quantized in units of a fundamental mass $m \sim r^{-1}$, where r is the 'typical length scale' of the internal manifold. The relevant mass operators depend on the ground state metric $<g_{MN}>$. If this is a product of d-dimensional Minkowski space with a Ricci-flat internal manifold M_{D-d} they are given by the Laplace operator (with respect to $\hat{g}_{ab}$) for $Y^{(n)}(y)$ and $Y^{(n)}_a(y)$, by the Laplace respectively Dirac operator for $\hat{Y}^{(n)}(y)$, depending on whether Φ is bosonic or fermionic, and by the Lichnerowicz operator for $Y^{(n)}_{ab}(y)$.[2] The Lichnerowicz operator is defined as

$$\Delta_L Y_{ab} = -\Box Y_{ab} - 2R_{acbd}Y^{cd} + 2R_{(a}{}^c Y_{b)c} \tag{C.4}$$

and its transverse traceless zero modes describe variations of the internal metric leaving the Ricci tensor invariant to linear order. To be more precise we have

$$R_{ab}(\hat{g}+\delta g) = R_{ab}(\hat{g}) + \frac{1}{2}\Delta_L \delta g_{ab} + \nabla_{(a}\nabla^c \delta g_{b)c} - \frac{1}{2}\nabla_a\nabla_b \delta g^c{}_c + \mathcal{O}(\delta g^2) \ . \tag{C.5}$$

Thus on a Ricci-flat manifold the moduli M of the metric appear as massless modes in the effective theory and the Ansatz $g_{ab} = \hat{g}_{ab}(<M>)+\delta g_{ab}(\delta M(x))$ can be interpreted as a variation of the moduli around their background values which determine the ground state solution. However, this is in general only true for Ricci-flat manifolds. For example in the case of Freund-Rubin reductions[3] of 11-dimensional supergravity on seven-dimensional non-Ricci-flat manifolds like S^7 the massless modes coming from the expansion of δg_{ab} do not correspond to zero modes of the Lichnerowicz operator and thus not to parameters of the metric.

One of the major motivations for Kaluza-Klein theories is the possibility to get (non-Abelian) Yang-Mills gauge fields in the d-dimensional theory without putting them in by hand in the D-dimensional theory. Let us briefly outline how this comes about. Suppose the internal metric $\hat{g}_{ab}$ of M_{D-d} has a group of isometries G, i.e. it admits Killing vectors K^i_a, $i = 1, \ldots, \mathrm{dim}G$, which fulfill

$$[K^{ia}\partial_{y^a}, K^{jb}\partial_{y^b}] = f^{ij}{}_k K^{kc}\partial_{y^c} \ , \tag{C.6}$$

where $f^{ij}{}_k$ are the structure constants of G. Then the massless modes coming from the off-diagonal metric fluctuations are given by

$$\delta g_{\mu a}(x,y) = A^i_\mu(x)K^i_a(y) \ . \tag{C.7}$$

[2] Strictly speaking these statements require a choice of gauge. For example δg_{ab} should be transverse and traceless.

[3] These involve an Ansatz $F_{\mu\nu\rho\sigma} \sim \epsilon_{\mu\nu\rho\sigma}$ for the four-form field strength.

Under a general coordinate transformation $z^M \to z^M - \xi^M$ the (full) metric transforms according to $g_{MN} \to g_{MN} + \nabla_M \xi_N + \nabla_N \xi_M$. For the special choice $\xi^M = (0, \lambda^i(x) K^{ia}(y))$ this implies the following transformation behavior for $A^i_\mu(x)$

$$A^i_\mu(x) \to A^i_\mu(x) + \partial_\mu \lambda^i(x) - f^i{}_{jk} A^j_\mu(x) \lambda^k(x) \ , \tag{C.8}$$

which is just the transformation law of Yang-Mills fields. Their gauge group is given by the subgroup of the D-dimensional general coordinate group that leaves the background metric $< g_{MN} >$ invariant. A Calabi-Yau manifold has no isometries and therefore no Kaluza-Klein vectors arise from the metric. In this case in order to have a non-Abelian gauge group in the d-dimensional theory it has to be present already in the D-dimensional theory. Abelian gauge fields can however originate in the expansion of antisymmetric tensor gauge fields as in (2.31).

C.2 An example

So far we have only analyzed the d-dimensional spectrum and gauge group and not said anything about the interactions of the d-dimensional fields. Furthermore one is usually interested in only keeping a finite subset of the spectrum in the effective theory, e.g. only the massless fields. We will come back to these points in the next section. First we want to clarify the ideas presented in the last section with the simple example of pure five-dimensional gravity compactified on a circle. This also sets the stage for our discussion in the next section of the consistency issues arising in truncating the d-dimensional spectrum to the massless fields. The form of our presentation again follows [112].

Starting point is the five-dimensional Einstein-Hilbert action

$$S = \frac{1}{2\pi} \int d^4x d\theta \sqrt{-g} R \ , \tag{C.9}$$

where $0 \leq \theta \leq 2\pi$ is a periodic variable. We change the field variables to[4]

$$g_{MN} = X^{-1/3} \begin{pmatrix} g_{\mu\nu} + X A_\mu A_\nu & X A_\mu \\ X A_\nu & X \end{pmatrix} \ . \tag{C.10}$$

Because of the periodicity of θ the corresponding fields can be expanded as

$$g_{\mu\nu} = \sum_{n=-\infty}^{\infty} g^{(n)}_{\mu\nu}(x) e^{in\theta} \ , \quad A_\mu = \sum_{n=-\infty}^{\infty} A^{(n)}_\mu(x) e^{in\theta} \ , \quad X = \sum_{n=-\infty}^{\infty} X^{(n)}(x) e^{in\theta} \tag{C.11}$$

with $\bar{g}^{(n)}_{\mu\nu} = g^{(-n)}_{\mu\nu}$, etc. The ground state is determined by

$$<g^{(0)}_{\mu\nu}> = \eta_{\mu\nu} \ , \qquad <A^{(0)}_\mu> = 0 \ , \qquad <X^{(0)}> = 1 \tag{C.12}$$

and all other vacuum expectation values vanishing. If we retain only the $n = 0$ modes, insert them into (C.9) and integrate over θ the resulting action describes four-dimensional gravity coupled to an Abelian vector and a scalar

$$S = \int d^4x \sqrt{-g^{(0)}} \left(R(g^{(0)}) - \frac{1}{4} X^{(0)} F^{(0)}_{\mu\nu} F^{(0)\mu\nu} - \frac{1}{6(X^{(0)})^2} \partial_\mu X^{(0)} \partial^\mu X^{(0)} \right) \ . \tag{C.13}$$

[4] This is similar to (2.9). The extra factor $X^{-1/3}$ ensures that the four-dimensional action directly comes out with the right normalization of the Einstein-Hilbert term.

What we have discarded in (C.13) are the massive modes whose masses are quantized in units of the inverse radius of the circle, which we have set to 1 in (C.12). If we are only interested in effects which involve energies much below the mass of the lightest massive states the action (C.13) should be a good approximation.

Let us now come to an analysis of the symmetries of the four-dimensional theory. The original action (C.9) is invariant under five-dimensional general coordinate transformations with parameters $\xi^M(x,\theta)$ whereas (C.13) is invariant under general coordinate transformations with parameters $\xi^\mu_{(0)}(x)$ and gauge transformations with parameter $\xi^5_{(0)}(x)$. This notation already indicates that the four-dimensional parameters are just the zero modes of an expansion of the five-dimensional parameters

$$\xi^\mu(x,\theta) = \sum_{n=-\infty}^{\infty} \xi^\mu_{(n)}(x) e^{in\theta} \ , \qquad \xi^5(x,\theta) \sum_{n=-\infty}^{\infty} \xi^5_{(n)}(x) e^{in\theta} \ . \tag{C.14}$$

If we retain all the modes of the expansion (C.11) in (C.13) the four-dimensional action has an infinite-dimensional symmetry algebra with the gauge parameters given by the $\xi^M_{(n)}(x)$. However, the vacuum (C.12) is only invariant under Poincaré and global $U(1)$-transformations. Therefore all symmetries of the action corresponding to $n \neq 0$ in (C.14) are spontaneously broken. The corresponding Goldstone bosons $A^{(n)}_\mu$ and $X^{(n)}$ are absorbed by the gauge fields $g^{(n)}_{\mu\nu}$ leaving a massive spectrum purely consisting of spin 2 particles. The fact that the massive gauge fields gain their mass through a Higgs mechanism puts strong restrictions on their interactions. This plays an important role in the discussion of a consistent truncation to the massless sector to which we now turn.

C.3 Consistency

We derived the four-dimensional action (C.13) for the massless modes by setting all the massive fields to zero and inserting the massless Ansatz, the $n = 0$ terms of (C.11), into the action (C.9). However, this procedure works only in very few cases. In general one has to make sure that setting the massive fields to zero is consistent with the equations of motion. That is to say a truncation to the massless Ansatz is consistent if its insertion into the D-dimensional equations of motion either leads to y-independent equations or the y-dependence has to factorize into a common factor on both sides of the equations. The same holds for plugging the Ansatz into the supersymmetry transformations in a supersymmetric theory. Otherwise the massive modes transform into the massless ones under a supersymmetry transformation and vice versa. Again following [112] we would like to illustrate this source of inconsistency with the example of the field equations of pure gravity with a positive cosmological constant Λ in $D = 4+k$ dimensions $R_{MN} = \Lambda g_{MN}$. This has a ground state solution $dS_4 \times M_k$, where dS_4 is four-dimensional de-Sitter space and M_k is a k-dimensional compact manifold. Introducing a $(4+k)$-dimensional generalization of (C.10) into the Einstein equation leads to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{1}{2}(F^i_{\mu\rho} F^{j\rho}_\nu - \frac{1}{4} g_{\mu\nu} F^i_{\rho\sigma} F^{j\rho\sigma}) K^i_a K^{ja} \ , \tag{C.15}$$

where K^{ja} are the Killing vectors of M_k corresponding to an isometry group G. Obviously the left-hand side of (C.15) is independent of y, whereas the right-hand side is in general not. To cure this inconsistency one generically has to restrict the Killing vectors to a subset $K^{j'a}$ corresponding to a subgroup $G' \subset G$ for which $K^{i'}_a K^{j'a}$ is y-independent. Thus in order to guarantee consistency we generally have to restrict the four-dimensional gauge group to a subgroup of the isometry group of M_k.[5]

It turns out that even in relatively 'simple' cases like the S^7-reduction of 11-dimensional supergravity the truncation of the linear Ansatz (C.3) to the zero modes is not consistent. The solution in this case is a nonlinear modification of the massless Ansatz. For example the correct massless Ansatz for the internal metric is not given by $g_{ab} = \hat{g}_{ab} + X^{(0)} Y^{(0)}_{ab}$ but instead it is $g_{ab} = \hat{g}_{ab} + f_{ab}(X^{(0)}, \phi^{(0)})$, where $f_{ab}(X^{(0)}, \phi^{(0)})$ is a non-linear function of the massless modes $X^{(0)}$ and the scalars $\phi^{(0)}$ coming from the three-form potential A_3 which reduces to $X^{(0)} Y^{(0)}_{ab}$ in the linear approximation. The consistent massless Ansätze and the proof that their insertion into the 11-dimensional equations of motion and supersymmetry transformations leads to a factorization of the y-dependence into a common factor are given in [171].

Obviously such a procedure is out of reach for Calabi-Yau manifolds. No explicit Ricci-flat metric is known and therefore a non-linear consistent massless Ansatz for the metric in terms of the Kähler and complex structure moduli (and perhaps other moduli of the theory) is inconceivable. In such a situation the only way to proceed is to insert the Ansatz (C.3) into the D-dimensional action and to integrate over the extra dimensions as we have done in the last section. But unlike in the example discussed there it is in general not consistent to set all the massive modes to zero. This time the inconsistency arises through terms $\sim HL^n$, $n \geq 2$, in the d-dimensional Lagrangian, where we contrary to section C.1 denote an arbitrary massive mode by H and a massless one by L. Such a term leads schematically to an equation of motion

$$\Box H + m^2 H \sim L^n \ , \tag{C.16}$$

which does not allow to set the massive mode to zero. Only in very simple cases is the absence of such terms guaranteed. This happens for example if the internal space is a torus T^{D-d}. In this case the d-dimensional theory has a $U(1)^{D-d}$ gauge symmetry and it turns out that all the massless states are neutral under this gauge group whereas the massive states are all charged. This suffices to ensure the absence of terms $\sim HL^n$. The special case of T^1 has been treated in the last section. More general conditions ensuring consistency of a massless Ansatz are discussed in [172]. One possibility, also applicable in the Calabi-Yau case, is to consider only covariantly constant zero modes in the expansion (C.3). This is always a consistent truncation.

However, if one does not want to restrict to the covariantly constant zero modes in a Calabi-Yau compactification, in principle one has to keep all the massive modes in the reduction and integrate them out via their equations of motion. This is however not a practical solution. Apart from other difficulties the derivation of the relevant interaction terms in the d-dimensional action requires a knowledge of the internal metric. Thus they are incalculable for Calabi-Yau manifolds and one has to argue in a different manner that integrating out the massive modes does not change at least the low energy effective action in a Calabi-Yau compactification. This has been done in [173, 174]. The key point is that the massive fields with spin ≥ 1 originate from higher-dimensional massless gauge fields and acquire their masses through a Higgs mechanism as we have discussed in the example of the last section. Thus their interactions are still constrained by the

[5] In certain cases the full isometry group can be realized as the four-dimensional gauge group. This is for example possible for the Freund-Rubin reduction of 11-dimensional supergravity on S^7.

higher-dimensional gauge invariance. In particular the coupling of a single massive gauge field to the massless fields must proceed via a conserved current. If the massless fields are gauge fields, to lowest order (in derivatives) the current has to be a bilinear expression in the corresponding field strengths in order to ensure d-dimensional gauge invariance. In supersymmetric theories the current is promoted to a superfield which is bilinear in the field strength multiplets. Thus also the scalars of the gauge multiplet couple to lowest order through terms bilinear in derivatives. If there are only massless multiplets with maximum spin ≥ 1 the equations of motion for the massive fields are

$$\Box H + m^2 H \sim (\partial L)^2 \; . \tag{C.17}$$

At low energies the kinetic term $\Box H$ can be neglected and the massive field is given by $H \sim m^{-2}(\partial L)^2$. Substituting this into the d-dimensional low energy effective action leads to higher derivative terms and does not modify the terms with up to two derivatives. This argument ensures for example that there are no corrections to the low energy effective action from integrating out the massive modes in the reduction of 11-dimensional supergravity on K3. In this case all the massive multiplets have maximum spin 2 and the massless spectrum consists of the supergravity multiplet and 19 vector multiplets. However, this reasoning is not sufficient anymore for type II compactifications on Calabi-Yau threefolds. In this case the massive multiplets still have maximum spin ≥ 1 but the massless spectrum contains a number of hypermultiplets. They do not involve any gauge fields and the above argument does not apply. In fact the relevant current for the hypermultiplets does not involve any derivatives and it has been shown in [174] that its coupling to the massive vector multiplets has in principle the potential to lead to a modification of the hypermultiplets' kinetic terms. However, the correction would be proportional to m^{-2} and in string theory this scale is set by the vacuum expectation value of the 'breathing mode', which is a modulus sitting in a vector multiplet in the type IIA theory and in a hypermultiplet in the type IIB case. The fact that the moduli space of the vector- and hypermultiplets is a direct product ensures the absence of such corrections for the type IIA string. The (perturbative) Peccei-Quinn (PQ) symmetry of the hypermultiplets is important to argue for their absence in the type IIB case, at least in the large volume limit where the PQ symmetry is not broken by world-sheet instantons. In view of the PQ symmetry of the Kähler moduli also in Calabi-Yau fourfold compactifications, see e.g. (2.56), it is feasible that the arguments of [174] can be extended to this case. A rigorous investigation has however not been done.

Finally, as a caveat we would like to mention a subtle point in the foregoing discussion which has been put forward in a similar way in [175]. The argument that integrating out the massive modes only modifies higher derivative terms in the d-dimensional action relies on neglecting the term $\Box H$ in (C.17). This is in general justified in the low energy limit because the mass m is assumed to be large. In a Kaluza-Klein reduction m is given by the inverse of the 'radius' of the internal manifold so that a large m implies a small volume. If the higher-dimensional action is the low energy effective action of a string or M-theory it gets higher derivative corrections in the small volume limit as we have discussed in the introduction. Furthermore in string theory the effects of world-sheet instantons can only be neglected in the large volume limit. Thus there are two conflicting sources of corrections which are suppressed either in the small or the large volume limit. We therefore have to specify more carefully what we mean by small respectively large volume. In our analysis of Calabi-Yau compactifications we have to assume that there is an intermediate regime in which both kinds of corrections are negligible. More specifically we have to consider Calabi-Yau manifolds whose 'average radius' l_Y fulfills $1/p \gg l_Y \gg l_{11}, \sqrt{\alpha'}$, where p is the characteristic momentum of the lower-dimensional fields and $l_{11}, \sqrt{\alpha'}$ are the 11-dimensional Planck scale respectively the string scale depending on whether we are compactifying 11-dimensional respectively type IIA supergravity.

C.4 S^1 compactification of $D = 4$ supergravity

Here we give some details of the S^1-reduction of four-dimensional supergravity which follows rather closely [95]. Inserting (2.9) into (2.1) and performing a Weyl rescaling $g^{(3)}_{\mu\nu} \to r^2 g^{(3)}_{\mu\nu}$ one arrives at

$$\begin{aligned}\mathcal{L}^{(3)} = \sqrt{-g}\Big(&\frac{1}{2}R^{(3)} - \frac{1}{r^2}\partial_\mu r\partial^\mu r - G_{\bar{I}J}\partial_\mu\bar{\Phi}^{\bar{I}}\partial^\mu\Phi^J + \frac{r^4}{4}H_\mu H^\mu - \frac{1}{2r^2}(\mathrm{Re}f)_{ab}\partial_\mu\zeta^a\partial^\mu\zeta^b \\ &+ \frac{r^2}{2}(\mathrm{Re}f)_{ab}(F^a_\mu + \zeta^a H_\mu)(F^{b\mu} + \zeta^b H^\mu) + (\mathrm{Im}f)_{ab}\partial_\mu\zeta^a(F^{b\mu} + \zeta^b H^\mu)\Big)\,, \end{aligned} \tag{C.18}$$

where the following abbreviations are used:

$$H^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}H_{\nu\rho} = \frac{1}{2}\epsilon^{\mu\nu\rho}(\partial_\nu B_\rho - \partial_\rho B_\nu), \quad F^{a\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho}F^a_{\nu\rho} = \frac{1}{2}\epsilon^{\mu\nu\rho}(\partial_\nu A^a_\rho - \partial_\rho A^a_\nu)\ . \tag{C.19}$$

The vectors can be dualized to scalars by adding r + 1 Lagrange multipliers C^a and b to the Lagrangian (C.18)

$$\mathcal{L}^{(3)} \to \mathcal{L}^{(3)} + \sqrt{-g}\left(\frac{1}{2}H_\mu\partial^\mu(b - \zeta^a C^a) - F^a_\mu\partial^\mu C^a\right)\ , \tag{C.20}$$

and eliminate the fields F^a_μ and H_μ via their equations of motion. This results in

$$\begin{aligned}\mathcal{L}^{(3)} \ =\ & \sqrt{-g}\Big(\frac{1}{2}R^{(3)} - \frac{1}{r^2}\partial_\mu r\partial^\mu r - G_{\bar{I}J}\partial_\mu\bar{\Phi}^{\bar{I}}\partial^\mu\Phi^J - \frac{1}{2r^2}(\mathrm{Re}f)_{ab}\partial_\mu\zeta^a\partial^\mu\zeta^b \\ &-\frac{1}{4r^4}(\partial_\mu b + \zeta^a \overleftrightarrow{\partial}_\mu C^a)^2 - \frac{1}{2r^2}(\partial_\mu C^a - (\mathrm{Im}f)_{ac}\partial_\mu\zeta^c)(\mathrm{Re}f)^{-1}_{ab}(\partial^\mu C^b - (\mathrm{Im}f)_{bd}\partial^\mu\zeta^d)\Big)\ . \end{aligned} \tag{C.21}$$

Expressed in the Kähler coordinates (2.10) $\mathcal{L}^{(3)}$ takes the form

$$\begin{aligned}\mathcal{L}^{(3)} \ =\ & \sqrt{-g}\Big(\frac{1}{2}R^{(3)} - G_{\bar{I}J}\partial_\mu\bar{\Phi}^{\bar{I}}\partial^\mu\Phi^J \\ &- \frac{\left|\partial_\mu T - (D+\bar{D})^a(\mathrm{Re}f)^{-1}_{ab}\partial_\mu D^b + \frac{1}{4}(D+\bar{D})^a(\mathrm{Re}f)^{-1}_{ac}\partial_\mu f^{cd}(\mathrm{Re}f)^{-1}_{db}(D+\bar{D})^b\right|^2}{\left[T + \bar{T} - \frac{1}{2}(D+\bar{D})^a(\mathrm{Re}f)^{-1}_{ab}(D+\bar{D})^b\right]^2} \\ &- \frac{(\partial_\mu D^a - \frac{1}{2}\partial_\mu f^{ac}(\mathrm{Re}f)^{-1}_{cd}(D+\bar{D})^d)(\mathrm{Re}f)^{-1}_{ab}(\partial^\mu\bar{D}^b - \frac{1}{2}\partial^\mu\bar{f}^{bc}(\mathrm{Re}f)^{-1}_{cd}(D+\bar{D})^d)}{\left[T + \bar{T} - \frac{1}{2}(D+\bar{D})^a(\mathrm{Re}f)^{-1}_{ab}(D+\bar{D})^b\right]}\Big)\,. \end{aligned} \tag{C.22}$$

With this form of the Lagrangian one verifies (2.11) and (2.12).[6]

[6] It is essential for this to work that f_{ab} depends holomorphically on the moduli fields, ensuring the identity $\partial_{\Phi^I} f_{ab} = 2\partial_{\Phi^I}(\mathrm{Re}f)_{ab}$.

For completeness let us also give the three-dimensional Lagrangian of the heterotic string in the string frame[7]

$$\frac{\mathcal{L}_s^{(3)}}{\sqrt{-g_s}} = e^{-2\Phi^{(3)}_{\text{het}}}\left(\frac{1}{2}R_s^{(3)} - G^{(4)}_{\bar{\imath}j}\partial_\mu\bar{\phi}^{\bar{\imath}}\partial^\mu\phi^j + \partial_\mu\Phi^{(3)}_{\text{het}}\partial^\mu\Phi^{(3)}_{\text{het}} - \frac{1}{2r_s^2}\partial_\mu r_s\partial^\mu r_s + \frac{r_s^2}{4}H_\mu H^\mu \right.$$
$$\left. -\frac{1}{2r_s^2}\partial_\mu\zeta^a\partial^\mu\zeta^a + \frac{1}{2}(F^a_\mu + \zeta^a H_\mu)^2\right) - \frac{e^{2\Phi^{(3)}_{\text{het}}}}{4}r_s^2\partial_\mu a\partial^\mu a + a\partial_\mu\zeta^a(F^{a\mu} + \zeta^a H^\mu). \quad \text{(C.23)}$$

The fact, that we do not get an overall factor $e^{-2\Phi^{(3)}_{\text{het}}}$ is an artefact of the dualization of the antisymmetric tensor in $D = 4$. We see that the perturbation series is governed by the three-dimensional dilaton (2.18) which also determines the three-dimensional gauge couplings. Notice however, that the gauge coupling for the Kaluza-Klein vector B_μ also depends on the fields r_s and ζ^a. Dualizing the vectors again yields

$$\frac{\mathcal{L}_s^{(3)}}{\sqrt{-g_s}} = \frac{e^{-2\Phi^{(3)}_{\text{het}}}}{2}\Big(R_s^{(3)} - 2G^{(4)}_{\bar{\imath}j}\partial_\mu\bar{\phi}^{\bar{\imath}}\partial^\mu\phi^j + 4\partial_\mu\Phi^{(3)}_{\text{het}}\partial^\mu\Phi^{(3)}_{\text{het}} - \frac{1}{r_s^2}(\partial_\mu r_s\partial^\mu r_s + \partial_\mu\zeta^a\partial^\mu\zeta^a)\Big)$$
$$-e^{2\Phi^{(3)}_{\text{het}}}\left(r_s^2\partial_\mu a\partial^\mu a + \frac{1}{4r_s^2}(\partial_\mu b + \zeta^a \overset{\leftrightarrow}{\partial}_\mu C^a)^2 + \frac{1}{2}(\partial_\mu C^a - a\partial_\mu\zeta^a)^2\right). \quad \text{(C.24)}$$

C.5 $D = 11$ supergravity on Calabi-Yau fourfolds

In this section we present some of the technical details connected to the dimensional reduction of the 11-dimensional supergravity Lagrangian (2.22) on Calabi-Yau fourfolds.

We start with a reduction of the Einstein-Hilbert action which leads to the kinetic terms of the moduli stemming from the Calabi-Yau metric. What we are eventually interested in is the metric on the whole moduli space (including the scalars coming from the three-form potential) because this metric appears in the low energy effective action in the kinetic terms of the moduli [162]. As it is sufficient to know the metric at an arbitrary point in the moduli space (determined by the ground state solution) we can restrict ourselves to terms quadratic in (derivatives of) the moduli fluctuations when we reduce the Einstein-Hilbert action. In this leading order reduction we can make use of the metric deformations given in (2.28) and (2.29). The only non-vanishing Christoffel symbols apart from $\Gamma^\rho_{\mu\nu}$ and the ones with only internal indices are (in complex coordinates)

$$\begin{aligned}
\Gamma^j_{\mu i} &= -\frac{i}{2}g^{\bar{k}j}\partial_\mu M^A e_{Ai\bar{k}} + \frac{1}{2}g^{jk}\partial_\mu\bar{Z}^{\bar{\alpha}}b_{\bar{\alpha}ki} \ , \\
\Gamma^{\bar{\jmath}}_{\mu i} &= \frac{1}{2}g^{\bar{\jmath}k}\partial_\mu\bar{Z}^{\bar{\alpha}}b_{\bar{\alpha}ki} - \left(\frac{i}{2}g^{\bar{k}\bar{\jmath}}\partial_\mu M^A e_{Ai\bar{k}}\right) , \\
\Gamma^i_{\mu\bar{\jmath}} &= \frac{1}{2}g^{i\bar{k}}\partial_\mu Z^\alpha b_{\alpha\bar{k}\bar{\jmath}} - \left(\frac{i}{2}g^{ik}\partial_\mu M^A e_{Ak\bar{\jmath}}\right) = \overline{\Gamma^{\bar{\imath}}_{\mu j}} \ , \\
\Gamma^{\bar{\imath}}_{\mu\bar{\jmath}} &= -\frac{i}{2}g^{\bar{\imath}k}\partial_\mu M^A e_{Ak\bar{\jmath}} + \frac{1}{2}g^{\bar{\imath}\bar{k}}\partial_\mu Z^\alpha b_{\alpha\bar{k}\bar{\jmath}} = \overline{\Gamma^i_{\mu j}} \ , \\
\Gamma^\mu_{ij} &= -\frac{1}{2}g^{\mu\nu}\partial_\nu\bar{Z}^{\bar{\alpha}}b_{\bar{\alpha}ij} \ , \\
\Gamma^\mu_{i\bar{\jmath}} &= \frac{i}{2}g^{\mu\nu}\partial_\nu M^A e_{Ai\bar{\jmath}} = \Gamma^\mu_{\bar{\jmath}i} \ ,
\end{aligned}$$

[7] We have used the tree level form of the gauge kinetic functions (2.4) in this formula and inserted $\Phi^I = (S, \phi^i)$.

$$\Gamma^{\mu}_{i\bar{j}} \;=\; -\frac{1}{2}g^{\mu\nu}\partial_\nu Z^\alpha b_{\alpha\bar{\imath}\bar{j}} = \overline{\Gamma^{\mu}_{ij}}. \tag{C.25}$$

The terms in brackets do not contribute to leading order in the expansion of $R^{(11)}$.[8] The 11-dimensional curvature scalar splits into

$$R^{(11)} = 2g^{i\bar{j}}R^{(11)}_{i\bar{j}} + g^{ij}R^{(11)}_{ij} + g^{\bar{\imath}\bar{j}}R^{(11)}_{\bar{\imath}\bar{j}} + g^{\mu\nu}R^{(11)}_{\mu\nu}, \tag{C.26}$$

where we have used the fact that $g^{i\bar{j}}R^{(11)}_{i\bar{j}}$ is real and therefore equal to $g^{\bar{\imath}j}R^{(11)}_{\bar{\imath}j}$. We have[9]

$$\begin{aligned} R^{(11)}{}_{ij} &= R^{(11)\mu}{}_{i\mu j}\ , \\ R^{(11)}{}_{i\bar{j}} &= R^{(11)k}{}_{ik\bar{j}} + R^{(11)\bar{k}}{}_{i\bar{k}\bar{j}} + R^{(11)\mu}{}_{i\mu\bar{j}}\ , \\ R^{(11)}{}_{\mu\nu} &= R^{(11)k}{}_{\mu k\nu} + R^{(11)\bar{k}}{}_{\mu\bar{k}\nu} + R^{(11)\lambda}{}_{\mu\lambda\nu}\ . \end{aligned} \tag{C.27}$$

Furthermore certain components of the 11-dimensional curvature tensor are related to components of the internal and the external curvature tensors:

$$\begin{aligned} R^{(11)\lambda}{}_{\mu\lambda\nu} &= R^{(3)\lambda}{}_{\mu\lambda\nu}\ , \\ R^{(11)k}{}_{ik\bar{j}} &= R^{(8)k}{}_{ik\bar{j}} + \Gamma^{\mu}_{i\bar{j}}\Gamma^{k}_{\mu k} - \Gamma^{\mu}_{ik}\Gamma^{k}_{\mu\bar{j}}, \qquad \text{etc.}\ . \end{aligned} \tag{C.28}$$

Using these relations, Ricci-flatness of the internal metric and the Christoffel symbols of (C.25) one derives to lowest order in the moduli

$$\begin{aligned} \frac{1}{2}\int d^{11}x\sqrt{-g^{(11)}}R^{(11)} = \int d^3x\sqrt{-g^{(3)}}\int d^8\xi\sqrt{\hat{g}}\Big(\frac{1}{2}R^{(3)} - \frac{1}{4}\partial_\mu Z^\alpha\partial^\mu \bar{Z}^{\bar{\beta}} b_{\alpha\bar{j}\bar{m}}\bar{b}_{\bar{\beta}ik}\hat{g}^{i\bar{j}}\hat{g}^{k\bar{m}} \\ -\frac{1}{2}\partial_\mu M^A\partial^\mu M^B e_{Ai\bar{j}}e_{Bk\bar{m}}\hat{g}^{i\bar{j}}\hat{g}^{k\bar{m}} + \frac{1}{4}\partial_\mu M^A\partial^\mu M^B e_{Ai\bar{j}}e_{Bk\bar{m}}\hat{g}^{i\bar{m}}\hat{g}^{k\bar{j}}\Big), \end{aligned} \tag{C.29}$$

where a total derivative has been neglected.

For the reduction of the remaining terms in (2.22) we have to expand the three-form A_3 in terms of the (1,1)-forms e_A and (2,1)-forms Ψ_I:

$$A_3 = A^A_\mu dx^\mu \wedge e_A + N^I\Psi_I + \bar{N}^{\bar{J}}\bar{\Psi}_{\bar{J}}. \tag{C.30}$$

Using (2.42) one derives

$$F_4 = \frac{1}{2}F^A_{\mu\nu}dx^\mu\wedge dx^\nu\wedge e_A + D_\mu N^I dx^\mu\wedge\Psi_I + D_\mu\bar{N}^{\bar{J}}dx^\mu\wedge\bar{\Psi}_{\bar{J}}, \tag{C.31}$$

where we abbreviated

$$D_\mu N^I = \partial_\mu N^I + N^K\sigma_{\alpha K}{}^I\partial_\mu Z^\alpha + \bar{N}^{\bar{L}}\bar{\tau}_{\bar{\beta}\bar{L}}{}^I\partial_\mu\bar{Z}^{\bar{\beta}}, \qquad D_\mu\bar{N}^{\bar{J}} = \overline{D_\mu N^J}. \tag{C.32}$$

Inserting (C.30) and (C.31) into (2.22) one derives to lowest order

$$\int d^{11}x\sqrt{-g^{(11)}}|F_4|^2 = \tag{C.33}$$

[8] Let us note here that the quantities $b_{\bar{\alpha}ij}$ generically depend on the complex structure [93] and therefore implicitly on x^μ. However, in our leading order reduction the x^μ-dependence of $b_{\bar{\alpha}ij}$ can be neglected.

[9] The terms $R^{(11)k}{}_{ikj} = \Gamma^{\mu}_{ij}\Gamma^{k}_{\mu k} - \Gamma^{\mu}_{ik}\Gamma^{k}_{\mu j}$ and $R^{(11)\bar{k}}{}_{i\bar{k}j}$ do not contribute to leading order.

$$\frac{1}{2}\int d^3x\sqrt{-g^{(3)}}\int d^8\xi\sqrt{\hat{g}}\Big(2D_\mu N_I D^\mu \bar{N}_{\bar{J}}\Psi^I_{ij\bar{k}}\bar{\Psi}^{\bar{J}}_{\bar{l}\bar{m}\bar{n}}\hat{g}^{i\bar{m}}\hat{g}^{j\bar{n}}\hat{g}^{l\bar{k}} - F_{A\mu\nu}F_B^{\mu\nu}e^A_{i\bar{j}}e^B_{k\bar{m}}\hat{g}^{i\bar{m}}\hat{g}^{k\bar{j}}\Big)$$

and[10]

$$\begin{aligned} -\frac{1}{12}\int A_3\wedge F_4\wedge F_4 = & \qquad\qquad (C.34)\\ \frac{1}{8}\int d^3x\sqrt{-g^{(3)}}\epsilon^{\mu\nu\rho}A^A_\mu D_\nu N^I D_\rho\bar{N}^{\bar{J}}\int d^8\xi\sqrt{\hat{g}}\epsilon^{ikls}\epsilon^{\bar{j}\bar{m}\bar{n}\bar{r}}e_{Ai\bar{j}}\Psi_{Ikl\bar{m}}\bar{\Psi}_{\bar{J}s\bar{n}\bar{r}}. \end{aligned}$$

Before we proceed let us define (in close analogy with [100])

$$\begin{aligned} \mathcal{V} &\equiv \frac{1}{4!}\int_{Y_4} J\wedge J\wedge J\wedge J\ , \\ \mathcal{V}_A &\equiv \frac{1}{4!}\int_{Y_4} e_A\wedge J\wedge J\wedge J\ , && (C.35)\\ \mathcal{V}_{AB} &\equiv \frac{1}{4!}\int_{Y_4} e_A\wedge e_B\wedge J\wedge J && (C.36)\\ &= \frac{1}{12}\int_{Y_4} d^8\xi\sqrt{g}e_{Ai\bar{j}}e_{Bk\bar{m}}g^{i\bar{m}}g^{k\bar{j}} - \frac{1}{12}\int_{Y_4} d^8\xi\sqrt{g}e_{Ai\bar{j}}e_{Bk\bar{m}}g^{i\bar{j}}g^{k\bar{m}}\ , \end{aligned}$$

where J is the Kähler form defined in eq. (2.34). With the help of (C.35) and (C.36) one derives (again in close analogy with the threefold case [104])

$$\begin{aligned} \star e_A &= \frac{2}{3}\frac{\mathcal{V}_A}{\mathcal{V}}J\wedge J\wedge J - \frac{1}{2}e_A\wedge J\wedge J\ , \\ G_{AB} &= -6\frac{\mathcal{V}_{AB}}{\mathcal{V}} + 8\frac{\mathcal{V}_A\mathcal{V}_B}{\mathcal{V}^2} = -\frac{1}{2}\partial_A\partial_B\ln\mathcal{V} && (C.37)\\ G^{-1}_{AB} &= -\frac{1}{6}\mathcal{V}\mathcal{V}^{-1}_{AB} + \frac{2}{3}M_A M_B && (C.38) \end{aligned}$$

and

$$16\frac{\mathcal{V}_A\mathcal{V}_B}{\mathcal{V}^2}\partial_\mu M^A\partial^\mu M^B = \partial_\mu\ln\mathcal{V}\partial^\mu\ln\mathcal{V}. \qquad (C.39)$$

The integrals over the internal coordinates in (C.29), (C.33) and (C.34) can be performed using the definitions (2.32), (2.36) and (2.38).[11] With the help of (C.37), (C.39) and performing a Weyl rescaling $g^{(3)}_{\mu\nu}\to\mathcal{V}^2 g^{(3)}_{\mu\nu}$ one derives

$$\begin{aligned} \mathcal{L}^{(3)} = &\sqrt{-g^{(3)}}\Big(\frac{1}{2}R^{(3)} - \frac{1}{2}\partial_\mu\ln\mathcal{V}\partial^\mu\ln\mathcal{V} - G_{\alpha\bar{\beta}}\partial_\mu Z^\alpha\partial^\mu\bar{Z}^{\bar{\beta}} - \mathcal{V}^{-1}G_{I\bar{J}}D_\mu N^I D^\mu\bar{N}^{\bar{J}}\\ &- \frac{1}{2}G_{AB}\partial_\mu M^A\partial^\mu M^B - \frac{1}{4}\mathcal{V}^2 G_{AB}F^A_{\mu\nu}F^{B\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho}d_{AI\bar{J}}A^A_\mu D_\nu N^I D_\rho\bar{N}^{\bar{J}}\Big). \qquad (C.40) \end{aligned}$$

[10] In order to derive this form of the reduced action one has to perform a partial integration and use the relation $d_{AI\bar{J}}\tau_{\alpha K}{}^{\bar{J}} = d_{AK\bar{J}}\tau_{\alpha I}{}^{\bar{J}}$, which can be easily derived from $\partial_{Z^\alpha}\int e_A\wedge\Psi_I\wedge\Psi_K = 0$.

[11] Strictly speaking one has to replace the Calabi-Yau metric in these definitions by its background value. This is related to the fact that we perform the Kaluza-Klein reduction to first non-trivial order around a fixed but arbitrary point in the moduli space of metrics. This procedure results in couplings in the effective Lagrangian which depend on the arbitrary background values. However, according to what we have said in the first paragraph of this section it is correct to replace them by full non-linear σ-model type couplings.

The three-dimensional vectors are dualized to scalars by adding the Lagrange multipliers P^A

$$-\sqrt{-g^{(3)}}F^A_\mu\partial^\mu P_A\,, \qquad F^{A\rho} = -\frac{1}{2}\epsilon^{\rho\mu\nu}F^A_{\mu\nu} \tag{C.41}$$

and eliminating the fields F^A_μ via their equations of motion. The result is

$$\begin{aligned}\mathcal{L}^{(3)} \;=\; \sqrt{-g^{(3)}}\Big[&\frac{1}{2}R^{(3)} - G_{\alpha\bar\beta}\partial_\mu Z^\alpha\partial^\mu\bar Z^{\bar\beta} - \mathcal{V}^{-1}G_{I\bar J}D_\mu N^I D^\mu\bar N^{\bar J}\\ &-\frac{1}{2}\partial_\mu\ln\mathcal{V}\partial^\mu\ln\mathcal{V} - \frac{1}{2}G_{AB}\partial_\mu M^A\partial^\mu M^B\\ &-\frac{1}{2\mathcal{V}^2}\Big(\partial_\mu P^A + \frac{1}{4}d^A{}_{K\bar L}\left(N^K D_\mu\bar N^{\bar L} - D_\mu N^K\bar N^{\bar L}\right)\Big)\\ &G^{-1}_{AB}\Big(\partial^\mu P^B + \frac{1}{4}d^B{}_{I\bar J}\left(N^I D^\mu\bar N^{\bar J} - D^\mu N^I\bar N^{\bar J}\right)\Big)\Big]\,.\end{aligned} \tag{C.42}$$

In order to find the Kähler potential for the scalars in (C.42) one introduces the coordinates (2.46), (2.47). With this redefinition the derivatives (C.32) take the form

$$\begin{aligned}D_\mu\bar N^{\bar J} &= \hat G^{-1\bar J}{}_I\left[\partial_\mu\hat N^I + \partial_\mu Z^\alpha\left(\hat G^{-1K}_{\bar M}\hat{\bar N}^{\bar M}\tau_{\alpha K}{}^{\bar L}\hat G^I{}_{\bar L} - \sigma_\alpha{}^I{}_K\hat N^K\right)\right] \equiv \hat G^{-1\bar J}{}_I D_\mu\hat N^I\,,\\ D_\mu N^I &= \hat G^{-1I}_{\bar J}\left[\partial_\mu\hat{\bar N}^{\bar J} + \partial_\mu\bar Z^{\bar\beta}\left(\hat G^{-1\bar M}{}_K\hat N^K\bar\tau_{\bar\beta\bar M}{}^L\hat G_L{}^{\bar J} - \bar\sigma_{\bar\beta}{}^{\bar J}{}_{\bar M}\hat{\bar N}^{\bar M}\right)\right] \equiv \hat G^{-1I}_{\bar J}D_\mu\hat{\bar N}^{\bar J}.\end{aligned} \tag{C.43}$$

The Lagrangian (C.42) becomes

$$\begin{aligned}\mathcal{L}^{(3)} \;=\; \sqrt{-g^{(3)}}\Big[&\frac{1}{2}R^{(3)} - G_{\alpha\bar\beta}\partial_\mu Z^\alpha\partial^\mu\bar Z^{\bar\beta} - \mathcal{V}^{-1}G_{I\bar J}\hat G^{-1}{}_{\bar L}{}^I\hat G^{-1\bar J}{}_M D_\mu\hat{\bar N}^{\bar L}D^\mu\hat N^M\\ &-\frac{1}{2}\partial_\mu\ln\mathcal{V}\partial^\mu\ln\mathcal{V} - \frac{1}{2}G_{AB}\partial_\mu M^A\partial^\mu M^B\\ &-\frac{1}{2\mathcal{V}^2}\Big(\partial_\mu P^A + \frac{1}{4}d^A{}_{K\bar L}\hat G^{-1}{}_{\bar N}{}^K\hat G^{-1\bar L}{}_M\left(\hat{\bar N}^{\bar N}D_\mu\hat N^M - D_\mu\hat{\bar N}^{\bar N}\hat N^M\right)\Big)\\ &G^{-1}_{AB}\Big(\partial^\mu P^B + \frac{1}{4}d^B{}_{I\bar J}\hat G^{-1}{}_{\bar P}{}^I\hat G^{-1\bar J}{}_Q\left(\hat{\bar N}^{\bar P}D^\mu\hat N^Q - D^\mu\hat{\bar N}^{\bar P}\hat N^Q\right)\Big)\Big]\,.\end{aligned} \tag{C.44}$$

Using (2.46)-(2.50) one verifies that $\mathcal{L}^{(3)}$ given in (C.44) coincides with the Lagrangian of (2.45).

C.6 Alternative way to $D = 2$

There is an alternative derivation of the two-dimensional effective actions for the heterotic respectively type IIA theory, which we want to present now.

C.6.1 The heterotic case

The alternative derivation of the two-dimensional heterotic effective action reduces the three-dimensional effective action obtained in chapter 2 on a further circle. For this purpose we make the Ansatz

$$g^{(3)}_{mn} = \begin{pmatrix} g^{(2)}_{\mu\nu} & 0\\ 0 & r^2\end{pmatrix}\,, \tag{C.45}$$

where $\mu, \nu = 0, 1$ and r is the radius of the S^1 measured in the three-dimensional Einstein-frame metric. There are no new Kaluza-Klein gauge bosons in this reduction since they contain no physical degree of freedom. Inserting (C.45) into (2.11) results in

$$\mathcal{L}^{(2)}_{\text{het}} = \sqrt{-g^{(2)}}\, r[\frac{1}{2}R^{(2)} - G_{\bar{\Lambda}\Sigma}\partial_\mu \bar{Z}^{\bar{\Lambda}}\partial^\mu Z^\Sigma] \; . \tag{C.46}$$

Choosing the conformal gauge

$$g^{(2)}_{\mu\nu} = e^\sigma \eta_{\mu\nu} \; , \tag{C.47}$$

and using the relation $r = e^{-2\Phi^{(2)}_{\text{het}}}$ between the radius of the circle measured in the three-dimensional Einstein-frame metric and the two-dimensional heterotic dilaton defined in equation (3.7) one derives

$$\mathcal{L}^{(2)}_{\text{het}} = e^{-2\Phi^{(2)}_{\text{het}}} \left[-\frac{1}{2}\partial_\mu\partial^\mu\sigma - G_{\bar{\Lambda}\Sigma}\partial_\mu \bar{Z}^{\bar{\Lambda}}\partial^\mu Z^\Sigma \right] \; . \tag{C.48}$$

The physical degrees of freedom are exactly the same as in $D = 3$. Also the fact that all scalars Z^Σ are members of chiral multiplets is inherited from $D = 3$. Thus in contrast to equation (3.9) no twisted chiral multiplets occur and the moduli space is therefore a Kähler manifold. In fact the sigma-model geometry is unchanged in the reduction from $D = 3$ to $D = 2$, that is $G_{\bar{\Lambda}\Sigma}$ is the same Kähler metric with the same Kähler potential as in $D = 3$.

In order to verify that the resulting theory is indeed equivalent to the one described by (3.9) one would have to dualize the twisted chiral multiplets τ, n^a in (3.3) and show that the resulting theory has for suitably chosen coordinates the Kähler potential given in (2.12). We have not done this explicitly but do not see any reason why it should not work out correctly. In addition we will see in the next section that this procedure is successful in the case of type IIA theory.

C.6.2 The type IIA case

As in the heterotic case we can derive the two-dimensional effective action by reducing the three-dimensional action (2.45) on a circle. One can again show that the radius in the Einstein-frame coincides with the two-dimensional dilaton (3.20). Choosing again the conformal gauge (C.47) leads to

$$\mathcal{L}^{(2)}_{\text{IIA}} = e^{-2\Phi^{(2)}_{\text{IIA}}} \left[-\frac{1}{2}\partial_\mu\partial^\mu\sigma - G_{\bar{\Lambda}\Sigma}\partial_\mu \bar{Z}^{\bar{\Lambda}}\partial^\mu Z^\Sigma \right] \; , \tag{C.49}$$

where the metric on the moduli space is Kähler with Kähler potential (2.49). This is indeed equivalent to the action given in (3.19). To verify this we have to express the two-dimensional effective action (3.19) by chiral multiplets only. For this purpose it is necessary to dualize the scalars a^A. This is possible because they only appear via their 'field strength' $\partial_\mu a^A$ in (3.19). One adds $-F^A_\mu \partial^\mu P_A$ to (3.19), where P^A is a Lagrange multiplier and $F^{A\rho} \equiv \epsilon^{\rho\mu}\partial_\mu a^A$. Then one eliminates F^A via its equation of motion in favor of P^A. Rescaling the M^A according to[12]

$$\check{M}^A = e^{-2/3\Phi^{(10)}_{\text{IIA}}} M^A \tag{C.50}$$

[12] The $\check{M}^A$ are precisely the Kähler moduli of M-theory used in section 2.2.

rescales the couplings as follows

$$
\begin{aligned}
\check{G}_{AB} &= G_{AB} e^{4/3\Phi^{(10)}_{\mathrm{IIA}}} \ , \\
\check{G}_{I\bar{J}} &= G_{I\bar{J}} e^{-2/3\Phi^{(10)}_{\mathrm{IIA}}} \ , \\
\check{\mathcal{V}} &= e^{-8/3\Phi^{(10)}_{\mathrm{IIA}}} \mathcal{V} = e^{-2/3\Phi^{(10)}_{\mathrm{IIA}}} e^{-2\Phi^{(2)}_{\mathrm{IIA}}} \ .
\end{aligned}
\tag{C.51}
$$

Finally, we choose a different conformal gauge for the two-dimensional metric

$$
g^{(2)}_{\mu\nu} = e^{2\Phi^{(10)}_{\mathrm{IIA}}} e^{4\Phi^{(2)}_{\mathrm{IIA}}} e^{\sigma} \eta_{\mu\nu} \ . \tag{C.52}
$$

Inserting these field redefinitions into (3.19) and performing the duality transformation yields

$$
\begin{aligned}
\mathcal{L}^{(2)}_{IIA} = e^{-2\Phi^{(2)}_{\mathrm{IIA}}} \Big[& -\frac{1}{2}\partial_\mu \partial^\mu \sigma - G_{\alpha\bar{\beta}} \partial_\mu Z^\alpha \partial^\mu \bar{Z}^{\bar{\beta}} - \check{\mathcal{V}}^{-1} \check{G}_{I\bar{J}} D_\mu N^I D^\mu \bar{N}^{\bar{J}} \\
& - \frac{1}{2}\partial_\mu \ln \check{\mathcal{V}} \partial^\mu \ln \check{\mathcal{V}} - \frac{1}{2}\check{G}_{AB} \partial_\mu \check{M}^A \partial^\mu \check{M}^B \\
& - \frac{1}{2\check{\mathcal{V}}^2} \Big(\partial_\mu P^A + \frac{1}{4} d^A{}_{K\bar{L}} \left(N^K D_\mu \bar{N}^{\bar{L}} - D_\mu N^K \bar{N}^{\bar{L}} \right) \Big) \\
& \check{G}^{-1}_{AB} \Big(\partial^\mu P^B + \frac{1}{4} d^B{}_{I\bar{J}} \left(N^I D^\mu \bar{N}^{\bar{J}} - D^\mu N^I \bar{N}^{\bar{J}} \right) \Big) \Big] \ .
\end{aligned}
\tag{C.53}
$$

In view of (C.42) and the discussion of section 2.2 this coincides with (C.49).

Thus in the coordinates Z^Σ the moduli space in $D = 2$ is Kähler and has the same Kähler potential as in $D = 3$. Furthermore the discussion of duality between the heterotic and type IIA theory in $D = 2$ using the variables of (C.48) and (C.49) proceeds exactly along the same lines as in section 2.3.

Finally comparing (C.49) with (C.48) shows

$$
e^{-2\Phi^{(2)}_{\mathrm{het}}} = e^{-2\Phi^{(2)}_{\mathrm{IIA}}} \ . \tag{C.54}
$$

This can also heuristically be derived from the duality between the heterotic and type IIA theory in $D = 6$. Starting from the effective action in $D = 6$ and compactifying further on a four-dimensional manifold B one derives

$$
\begin{aligned}
S &= \int d^6x \sqrt{-g^{(6)}_{\mathrm{het}}} e^{-2\Phi^{(6)}_{\mathrm{het}}} (R^{(6)}_{\mathrm{het}} + \dots) \\
&= \int d^2x \sqrt{-g^{(2)}_{\mathrm{het}}} e^{-2\Phi^{(6)}_{\mathrm{het}}} \mathcal{V}^{\mathrm{het}}_B (R^{(2)}_{\mathrm{het}} + \dots) \\
&= \int d^2x \sqrt{-g^{(2)}_{\mathrm{het}}} e^{-2\Phi^{(2)}_{\mathrm{het}}} (R^{(2)}_{\mathrm{het}} + \dots).
\end{aligned}
\tag{C.55}
$$

On the other hand the duality relations, which are reviewed in footnote 5 in chapter 3, yield $e^{-2\Phi^{(6)}_{\mathrm{het}}} \mathcal{V}^{\mathrm{het}}_B = e^{2\Phi^{(6)}_{\mathrm{het}}} \mathcal{V}^{\mathrm{IIA}}_B = e^{-2\Phi^{(6)}_{\mathrm{IIA}}} \mathcal{V}^{\mathrm{IIA}}_B = e^{-2\Phi^{(2)}_{\mathrm{IIA}}}$, where all volumes are measured in the respective string-frame metrics.

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt, und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

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