Moduli spaces of Anti-de Sitter vacua in five-dimensional $\mathcal{N} = 2$ supergravity

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Abstract

In this thesis we study the moduli spaces of maximally supersymmetric Anti-de Sitter (AdS) vacua in gauged, five-dimensional $\mathcal{N}=2$ supergravity. These vacua feature in ten-dimensional compactifications of type IIB supergravity on Sasaki-Einstein manifolds and are an integral part of the AdS/CFT correspondence. In particular, moduli spaces of five-dimensional AdS vacua are related to conformal manifolds of the dual, four-dimensional superconformal field theories via the AdS/CFT correspondence. For a general five-dimensional $\mathcal{N}=2$ supergravity coupled to an arbitrary number of vector, tensor and hypermultiplets, we determine the conditions for AdS vacua in the first part of this thesis and prove that the unbroken gauge group in the vacuum always contains an $U(1)_R$-factor. As a next step, we study the moduli space of the AdS vacuum by varying the scalar fields. We show that this moduli space is a Kähler submanifold of the ambient quaternionic Kähler manifold spanned by the hypermultiplet scalars.

To relate our results to the full ten-dimensional solutions, we consider consistent truncations of type IIB supergravity on Sasaki-Einstein manifolds in the second part. In particular, we study maximally supersymmetric AdS vacua in consistent $\mathcal{N}=2$ truncations on the Sasaki-Einstein manifold $T^{1,1}$. Here we focus on truncations that contain fields coming from the second and third cohomology forms on $T^{1,1}$. There are two possibilities: The Betti-vector truncation contains $\mathcal{N}=2$ supergravity coupled to two vector and two hypermultiplets, while the Betti-hyper truncation contains one vector multiplet and three hypermultiplets. We find that both truncations admit AdS vacua with an unbroken $U(1)_R$-symmetry. Finally, we explicitly determine the moduli spaces and compute their respective metrics.

Zusammenfassung

In dieser Dissertation studieren wir die Moduliräume von maximal-supersymmetrischen Anti-de Sitter (AdS) Vakua in geeichter, fünfdimensionaler $\mathcal{N} = 2$ Supergravitation. Diese Vakua treten in zehndimensionalen Kompaktifizierungen von Typ IIB Supergravitation auf Sasaki-Einstein Mannigfaltigkeiten auf und sind ein essentieller Bestandteil der AdS/CFT Korrespondenz. Die Moduliräume fünfdimensionaler AdS Vakua stehen durch die AdS/CFT Korrespondenz im Zusammenhang mit den konformen Mannigfaltigkeiten der dualen, vierdimensionalen superkonformen Feldtheorien. Wir bestimmen die Bedingungen für AdS Vakua in einer allgemeinen, fünfdimensionalen $\mathcal{N} = 2$ Supergravitation gekoppelt an eine beliebige Anzahl von Vektor-, Tensor- und Hypermultipletts im ersten Teil dieser Arbeit und zeigen, dass die ungebrochene Eichgruppe im Vakuum immer einen $U(1)_R$-Faktor enthält. Als nächsten Schritt studieren wir die Moduliräume der AdS Vakua durch Variationen in den Skalarfeldern. Wir zeigen, dass dieser Moduliraum eine Kähler-Untermannigfaltigkeit der umgebenden quaternionischen Kählermannigfaltigkeit ist, welche von den Hypermultiplettskalaren aufgespannt wird.

Um unsere Ergebnisse mit der vollen zehndimensionalen Lösung in Verbindung zu bringen, betrachten wir im zweiten Teil dieser Arbeit konsistente Trunkierungen von Typ IIB Supergravitation auf Sasaki-Einstein Mannigfaltigkeiten. Insbesondere untersuchen wir maximal-supersymmetrische AdS Vakua in konsistenten $\mathcal{N} = 2$ Trunkierungen auf der Sasaki-Einstein Mannigfaltigkeit $T^{1,1}$. Hier konzentrieren wir uns auf Trunkierungen, die Felder enthalten, welche von der zweiten und dritten Kohomologieform auf $T^{1,1}$ kommen. Es gibt zwei Möglichkeiten: Die Betti-Vektor-Trunkierung enthält $\mathcal{N} = 2$ Supergravitation gekoppelt an zwei Vektor- und zwei Hypermultipletts, während die Betti-Hyper-Trunkierung ein Vektormultiplett und drei Hypermultipletts enthält. Wir finden heraus, dass beide Trunkierungen AdS Vakua mit ungebrochener $U(1)_R$-Symmetrie zulassen. Abschließend bestimmen wir die Moduliräume explizit und berechnen ihre Metriken.

List of publications

This thesis is based on the following publications:

- J. Louis and **C. Muranaka**, "Moduli spaces of AdS_5 vacua in $\mathcal{N} = 2$ supergravity," JHEP **1604** (2016) 178 [arXiv:1601.00482 [hep-th]]
- J. Louis and **C. Muranaka**, "AdS_5 vacua from type IIB supergravity on $T^{1,1}$," submitted to JHEP [arXiv:1611.02982 [hep-th]]

Contents

Chapter 1

Introduction

In the search for a consistent quantum theory of gravity, string theory provides a framework to address many interesting topics in high energy theoretical physics.[1] Its main idea is that all elementary particles are made up of tiny extended one-dimensional strings. Different excitations of these strings then give rise to the different particles that we observe in nature. Interestingly, string theory necessarily contains general relativity and is expected to provide a well-behaved UV-completion of quantum gravity. Due to the extended nature of the fundamental strings, the UV-divergences that appear in attempts to quantize gravity using the usual perturbative approach to quantum field theory do not arise in string theory. Moreover, Yang-Mills gauge theories, such as the standard model of particle physics, also appear naturally in string theory. In this sense, string theory provides a framework for unifying Einstein's theory of general relativity with the principles of quantum field theory. However, up to date no mechanism is known that explains the selection of the standard model field content and gauge group $SU(3) \times SU(2) \times U(1)$ out of all the possible gauge theories provided by string theory.

Concepts for physics beyond the standard model, such as supersymmetry and extra dimensions, can be incorporated consistently in the framework of string theory. However, the theory is also very restrictive; for example, the consistent formulation of fermionic string theories is related to the existence of supersymmetry. Moreover, such theories necessarily require the allowed spacetimes to be ten- or eleven-dimensional. For phenomenological applications one then has to worry about deriving effective four-dimensional theories by compactifying on suitable internal manifolds. Even though the predictions of string theory for phenomenological models in four dimensions have been extensively studied, so far no experimental evidence has been found.

On the other hand, string theory also provides tools to conceptionally understand quantum field theories. The idea is to use the rigid mathematical structure of string theory to derive statements about strongly coupled quantum field theories. This has famously been studied in the context of the gauge/gravity duality that is known as the

[1] See [1–5] for introductions to string theory.

AdS/CFT correspondence[2] and was first conjectured by Maldacena in [6] and formulated more precisely in [7, 8].[3] It states that an AdS_d vacuum preserving q real supercharges is related to a superconformal field theory (SCFT) on the boundary of AdS with $q/2$ supercharges and $q/2$ superconformal charges preserved. In short, we have a conjectured relation

$$\begin{gathered}\mathrm{AdS}_d \text{ vacuum preserving } q \text{ real supercharges} \\ \updownarrow \\ \mathrm{SCFT}_{d-1} \text{ preserving } q/2 \text{ supercharges} + q/2 \text{ superconformal charges}\,.\end{gathered} \tag{1.1}$$

The reduction of supersymmetry comes from the fact that half of the supercharges present in the AdS solution are converted to superconformal charges in the dual field theory. Eventhough many interesting examples of this duality have been studied [10–14], no complete proof for the AdS/CFT correspondence is known up to date. However, one can perform a large amount of non-trivial consistency checks by employing the relations between gravity and field theory provided by the correspondence. For example, the gauge group of the AdS solution is mapped to the global flavour symmetry of the dual SCFT. Moreover, scalar fields in the AdS vacuum are related to the coupling of operators to the SCFT. In particular, one can consider the supersymmetric moduli space of the AdS vacuum, i.e. the space of scalar field variations that leave the vacuum invariant. This moduli space is then mapped via the AdS/CFT correspondence to the conformal manifold, i.e. the space of gauge invariant operators that preserve the superconformal invariance of the field theory.

The AdS/CFT correspondence was thoroughly studied over the last decades. In its original form [6], it was formulated as a duality between $\mathrm{AdS}_5 \times S^5$ solutions of type IIB string theory and an $\mathcal{N} = 4$ SCFT on the boundary. However, in this thesis we will be interested in the case of minimal supersymmetry on the boundary SCFT, i.e. $\mathcal{N} = 1$ or four real supercharges. The dual solution in type IIB is then given by $\mathrm{AdS}_5 \times M_5$ [15–19], where M_5 is a five-dimensional compact manifold. In particular, the structure of M_5 has to be such that this solution preserves the correct amount of eight real supercharges as required by (1.1).

Due to their relevance in the gauge/gravity duality, supersymmetric solutions of type IIB supergravity containing an AdS_5 factor and preserving eight real supercharges have been extensively studied from the ten-dimensional point of view [20–25]. Even though there are different types of compact manifolds that appear in the various versions of the AdS/CFT correspondence, we will focus here on the case of compact Sasaki-Einstein manifolds. Because the AdS spacetime appearing in the AdS/CFT correspondence has only five non-compact dimensions, it would be interesting to learn more about the relationship between the four-dimensional field theory, the supergravity on AdS_5 and the

[2] AdS/CFT stands for Anti de-Sitter/Conformal Field Theory, where Anti-de Sitter space is the maximally-symmetric spacetime with constant negative curvature.

[3] Reference [9] provides a recent review of the gauge/gravity duality.

compact internal manifold. To this end, we want to discuss AdS vacua purely from the supergravity perspective and without relation to any higher-dimensional compactification in the first part of this thesis. Similar investigations have been performed in different dimensions and for different numbers of preserved supercharges in [26–32]. In particular, we will analyze the conditions for preserving supersymmetry and find the restrictions on possible unbroken gauge groups in the AdS vacuum. As explained above, these gauge groups map under the AdS/CFT correspondence to global flavour symmetries of the dual SCFT. This is interesting to study for the case of the global $U(1)_R$-symmetry present in the superconformal algebra for every four-dimensional $\mathcal{N} = 1$ SCFT. In particular, one expects this global $U(1)_R$-symmetry to be appear as an unbroken factor in the gauge group of the five-dimensional AdS vacuum. Hence, the first problem we want to address in this thesis is:

1) *What are the conditions for the existence of maximally supersymmetric AdS_5 vacua in the most general form of gauged five-dimensional $\mathcal{N} = 2$ supergravity?*

In part, this question has already been addressed in the context of a-maximization in [33]. However, only the case of Abelian gauge groups was discussed in this reference, while tensor multiplets were not considered at all. Based on [34], we find that we can formulate the conditions for AdS_5 vacua in terms of isometries and associated moment maps of the scalar fields. In particular, we prove that the $U(1)_R$-symmetry which is always present in the dual four-dimensional $\mathcal{N} = 1$ SCFT always remains unbroken in the AdS vacuum.[4] The unbroken gauge group has to be a direct product of this R-symmetry with an otherwise arbitrary gauge group. Finally, we explain how the spontaneous gauge symmetry breaking arises in AdS backgrounds and identify the Goldstone bosons.

The negative cosmological constant of an AdS vacuum in a gauged supergravity is provided by the vacuum expectation value (VEV) of the scalar potential that was introduced by the gauging. Thus a particular AdS solution is specified by the vacuum values of the scalar fields in the theory. Once we understand the conditions a theory has to satisfy in order to admit a supersymmetric AdS vacuum, it would be interesting to find the parameter space of this solution. In particular, given a configuration of scalar fields with negative cosmological constant, we can ask whether it is possible to deform the scalar fields but keep the cosmological constant while preserving supersymmetry. That is, we want to study the supersymmetric moduli space of the AdS vacuum. For a neighborhood of the vacuum configuration in the target space of the scalar fields, the moduli space is given by all deformations of the scalar fields that preserve supersymmetry and thus leave the vacuum conditions invariant. Thus the second problem we want to study is:

2) *What is the structure of the supersymmetric moduli space of AdS_5 vacua?*

[4] For Abelian gauge groups, this was also shown in [33].

Unfortunately, in general it will be difficult to explicitly determine the moduli space in a model-independent way. However, we will show that the conditions obtained from varying the AdS vacuum are sufficient to analyze the structure of the moduli space. Again, a part of this question has been studied in [33], but we will generalize their results. In particular, we will discuss the presence of Goldstone bosons as directions in the deformation space and explain how to remove these unphysical deformations to obtain the physical moduli space. Finally, we will prove that the moduli space is a Kähler manifold and consists only of deformations in the hypermultiplet scalars. The moduli space of AdS_5 vacua is related to the conformal manifold of the dual SCFT by the AdS/CFT correspondence. It was shown that the conformal manifold of $\mathcal{N} = 1$ SCFTs is a Kähler manifold [35] and we find an agreement between four-dimensional field theory and five-dimensional supergravity. Hence, our proof presents an additional consistency check for the AdS/CFT correspondence.

After obtaining a deeper understanding of the five-dimensional AdS factors relevant for applications in the AdS/CFT correspondence, we would like to study the impact of the compact manifold. So far, we found that the moduli space of supersymmetric AdS_5 vacua has the same structure as the conformal manifold of the dual $\mathcal{N} = 1$ SCFT in four dimensions, i.e. both are Kähler manifolds. However, a priori it is unclear whether mapping the moduli space to the conformal manifold is bijective under AdS/CFT. For example, it could be possible that the conformal manifold has a higher dimension than the AdS_5 moduli space. Then the five-dimensional moduli space should be a subspace of the full ten-dimensional moduli space of the $\mathrm{AdS}_5 \times \mathrm{SE}_5$ solution. To gain a deeper understanding of this issue, we want to study consistent truncations of type IIB supergravity on Sasaki-Einstein manifolds. A consistent truncation is a reduction of the ten-dimensional theory to a five-dimensional one such that every solution of the five-dimensional theory again lifts to a full solution of type IIB supergravity. These truncations have been studied for general Sasaki-Einstein manifolds in [36–43].

In what follows, we will focus on the prominent example of type IIB supergravity compactified on the homogeneous Sasaki-Einstein manifold $T^{1,1} = (SU(2) \times SU(2))/U(1)$ that is underlying the conifold [44, 45]. Since this background is dual to the Klebanov-Witten theory [10], it has been extensively studied in the past. In particular, the moduli space of the ten-dimensional solution and the dual field theory have been shown to be complex five-dimensional and the moduli have been identified [10, 23, 25, 46, 47]. There is one modulus corresponding to the Axion-Dilaton τ and another coming from the vacuum expectation value (VEV) of the complex B-field of type IIB supergravity integrated over the nontrivial two-cycle[5] of $T^{1,1}$. Moreover, the remaining three complex moduli (one of which is the deformation identified in [50]) transform as a triplet under the $SU(2) \times SU(2)$ in the isometry group of $T^{1,1}$.

[5]This is due to the fact that $T^{1,1}$ is diffeomorphic to $S^2 \times S^3$ and thus has non-vanishing second Betti-number [48, 49].

Consistent truncations to gauged $\mathcal{N} = 2$ supergravity have been derived for $T^{1,1}$ compactifications in [43, 48, 49, 51]. In particular, these generalize the situation studied for general Sasaki-Einstein manifolds in [36–43] by taking the non-trivial second and third cohomology classes of $T^{1,1} \cong S^2 \times S^3$ into account. Therefore, the final problem we will study in this thesis is:

3) *Do AdS_5 vacua exist in consistent five-dimensional $\mathcal{N} = 2$ truncations of type IIB supergravity on $T^{1,1}$? If so, can we explicitly compute their moduli spaces?*

This problem is interesting to study for two reasons: First of all, since we only discussed the general structure on the moduli space so far, we would like to determine the explicit moduli space metrics in some examples. This might give some insight into determining the moduli space metrics in a more general setting. Secondly, we can then compare the explicit five-dimensional moduli space to the ten-dimensional one and determine the impact of the compact manifold on their relation. This is related to the question whether the five-dimensional moduli space is dual to the full conformal manifold of the $\mathcal{N} = 1$ SCFT or only a certain submanifold. In answering question 3), we follow [52] and study consistent $\mathcal{N} = 2$ truncations known as the Betti-vector truncation and the Betti-hyper truncation which involve multiplets associated with the topology of $T^{1,1}$.

The Betti-vector truncation contains gravity coupled to two vector multiplets and two hypermultiplets, while the Betti-hyper truncation contains gravity coupled to one vector multiplet and three hypermultiplets. We then apply the methods developed for general AdS_5 backgrounds to these truncations with the following results. Both truncations admit AdS vacua with an unbroken $U(1)_R$-symmetry and moreover have non-trivial supersymmetric moduli spaces. Furthermore, we can explicitly compute the metrics on the moduli spaces. In the case of the Betti-vector truncation, we find that the moduli space $\mathcal{M}^{BV}$ is spanned by the Axion-Dilaton τ and $\mathcal{M}^{BV} = \mathcal{H}$ is the complex upper half plane. For the Betti-hyper truncation we compute a complex two-dimensional moduli space $\mathcal{M}^{BH}$ that is given as a torus bundle with base space parametrized again by τ. In particular, this reproduces the result of [10] that the moduli in question are the Axion-Dilaton and a complex scalar that parametrizes a torus. However, the metric on the moduli space is not a direct product but a non-trivial fibration known as the universal elliptic curve. Thus we find that we cannot detect the full ten-dimensional moduli space in the consistent truncations that we study. This is not surprising, as it was shown in [53] that truncations on $T^{1,1}$ (and more general $U(1)$-bundles over products of copies of $\mathbb{CP}^n$'s) can only retain singlets under the isometry of the compact manifold in order to remain consistent. Including higher representations of the isometry leads to inconsistencies in the five-dimensional equations of motion. Since three of the five complex moduli of the $T^{1,1}$ solution transform as a triplet under $SU(2) \times SU(2)$, these cannot be found in consistent truncations. However, this also shows that we found all moduli of the ten-dimensional solution that are possible to detect in the Betti-hyper truncation.

The main part of this thesis is organized as follows. In Chapter 2 we provide the most general form of gauged five-dimensional $\mathcal{N} = 2$ supergravity and study its supersymmetric AdS vacua. In Chapter 3 we then determine the necessary conditions on the moduli space of these vacua by studying the space of scalar field deformations that preserve the maximally supersymmetric AdS_5 vacuum. Moreover, we briefly discuss in which cases these conditions are not only necessary but sufficient to determine the moduli space. We proceed by using these results to study AdS_5 vacua and their moduli spaces in consistent truncations of type IIB supergravity compactified on $T^{1,1}$ in Chapter 4. Thereafter, we conclude, discuss our findings and give a brief outlook. In Appendix A we provide general facts about Sasaki-Einstein manifolds and discuss the supersymmetry of type IIB supergravity compactified on such manifolds in Appendix B. Furthermore, we explain consistent truncations of type IIB supergravity on Sasaki-Einstein manifolds in Appendix C. Finally, we discuss the absence of AdS vacua for yet another $\mathcal{N} = 2$ truncation of $T^{1,1}$ in Appendix D.

Chapter 2

AdS vacua in gauged five-dimensional $\mathcal{N}$=2 supergravity

In this chapter we lay the foundation for the remainder of this thesis and study the conditions on maximally supersymmetric AdS_5 vacua in a general five-dimensional gauged $\mathcal{N} = 2$ supergravity. To this end, we first introduce the most general form of $\mathcal{N} = 2$ supergravity coupled to an arbitrary number of vector, tensor and hypermultiplets charged under an arbitrary gauge group G.

Before we begin the technical analysis, let us first illustrate the general idea. Consider a given $\mathcal{N} > 1$ supergravity[1] in d spacetime dimensions and described by a Lagrangian $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(\Phi, \Psi, ...)$ depending on a number of scalars Φ, fermions Ψ and possibly some other fields. The scalars are maps from spacetime M to a target space $\mathcal{T}$,

$$\Phi : M \to \mathcal{T} \ . \tag{2.1}$$

$\mathcal{T}$ is endowed with a Riemannian metric g. In general, supersymmetry will restrict the form of this metric, e.g. $(\mathcal{T}, g)$ could be a manifold with special holonomy or even a homogeneous space. Moreover, the dimension of $\mathcal{T}$ is equal to the number of scalars present in the theory.

In general, the Lagrangian $\tilde{\mathcal{L}}$ is invariant under some global symmetry group $\tilde{G}$. Here, we will further be interested in gauged supergravity theories, i.e. theories in which a subgroup $G \subset \tilde{G}$ of the global symmetries can be promoted to local symmetries. In order to preserve the same amount of supersymmetry as before, the Lagrangian $\tilde{\mathcal{L}}$ has to be modified to a new Lagrangian $\mathcal{L}$. The process of selecting the subgroup G and replacing $\tilde{\mathcal{L}} \mapsto \mathcal{L}$ is called the gauging of the supergravity.[2] In particular, the derivatives $\partial_\mu \Phi$ of the scalars have to be replaced by covariant derivatives,

$$\partial_\mu \Phi \to \mathcal{D}_\mu \Phi = \partial_\mu \Phi + g_G K_I A_\mu^I \ . \tag{2.2}$$

[1] We exclude the $\mathcal{N} = 1$ case from our general considerations, since in these theories a scalar potential can appear in terms of the superpotential even if the theory is ungauged.

[2] See [54, 55] for review articles on gauged supergravities.

Here g_G is the gauge coupling constant for G, A_μ^I are the gauge fields and $K_I(\Phi)$ the associated Killing vectors that span the Lie algebra of the gauge group. Moreover, in order for the theory to preserve the same amount of supersymmetry after the gauging, the supersymmetry transformations of the fields receive corrections and one has to introduce a scalar potential $V(\Phi)$ into the Lagrangian. Thus $\mathcal{L}$ is of the general form

$$\mathcal{L} = \tfrac{1}{2}\mathcal{R} - V(\Phi) + g(\mathcal{D}_\mu\Phi, \mathcal{D}^\mu\Phi) + \dots \ , \tag{2.3}$$

where $\mathcal{R}$ denotes the Ricci scalar of the spacetime metric.

In this thesis we study maximally supersymmetric AdS vacua of gauged supergravities. For a vacuum to be AdS, it must be invariant under the respective symmetry group $SO(2,d)$ and in particular the Lorentz group $SO(1,d) \subset SO(2,d)$. In other words, the vacuum cannot have any distinguished directions, i.e. only the scalar fields can acquire a non-trivial vacuum expectation value (VEV) $\langle\Phi\rangle \neq 0$. Moreover, the supersymmetry variations of all the fields in the spectrum have to vanish. Because supersymmetry relates scalars Φ and fermions Ψ, this reads

$$\langle\delta\Psi\rangle = \langle F(\Phi)\rangle = 0 \ , \tag{2.4}$$

for some function F depending on the explicit form of the supergravity. This condition restricts the possible background values of the scalar fields $\langle\Phi\rangle$.

In a given supergravity, the precise form of the function F is determined by supersymmetry. Moreover, $\langle F\rangle$ fixes the value of the scalar potential $\langle V\rangle$ in the background, which can then be interpreted as a cosmological constant. Since we are interested in AdS backgrounds, we will study solutions with $\langle V\rangle < 0$ and exclude the Minkowskian case $\langle V\rangle = 0$. The restrictions posed by (2.4) can then be used to derive properties of the vacuum, for example its preserved symmetries and the spontaneous gauge symmetry breaking.

In what follows we explicitly carry out this analysis in the case of five-dimensional gauged supergravity preserving eight real supercharges.

2.1 General gauged $\mathcal{N} = 2$, $d = 5$ supergravity

In this section we review the most general form of five-dimensional gauged $\mathcal{N} = 2$ supergravity following [56–58].[3] The theory consists of the gravity multiplet,

$$\{g_{\mu\nu}, \psi_\mu^{\mathcal{A}}, A_\mu^0\} \ , \quad \mu,\nu = 0,\dots,4 \ , \quad \mathcal{A} = 1,2 \ , \tag{2.5}$$

where $g_{\mu\nu}$ is the metric of spacetime, $\psi_\mu^{\mathcal{A}}$ is an $SU(2)_R$-doublet of symplectic Majorana gravitini[4] and A_μ^0 is a vector field called the graviphoton. In this thesis we consider

[3] The most general gauged $\mathcal{N} = 2$ supergravity in five dimensions was constructed in [58].

[4] See Appendix B for our spinor conventions in five-dimensional supergravity.

theories that are additionally coupled to n_V vector multiplets, n_T tensor multiplets and n_H hypermultiplets.

A vector multiplet $\{A_\mu, \lambda^{\mathcal{A}}, \phi\}$ transforms in the adjoint representation of the gauge group G and contains a vector A_μ, a doublet of symplectic Majorana gauginos $\lambda^{\mathcal{A}}$ and a real scalar ϕ. A special feature of five-dimensional theories is the fact that a vector field is Poincaré dual to an antisymmetric tensor field $B_{\mu\nu}$. One can show [59] that vector fields carrying an arbitrary representation of G other than the adjoint representation have to be dualized to tensor fields for the theory to be consistent. This gives rise to tensor multiplets $\{B_{\mu\nu}, \lambda^{\mathcal{A}}, \phi\}$, which have the same field content as vector multiplets but with a two-form instead of a vector. However, since vector and tensor multiplets mix in the Lagrangian, we label their scalars ϕ^i by a common index $i, j = 1, ..., n_V + n_T$. Moreover, we label the vector fields (including the graviphoton) by $I, J = 0, 1, ..., n_V$, the tensor fields by $M, N = n_V + 1, ..., n_V + n_T$ and also introduce a combined index $\tilde{I} = (I, M)$. Finally, the n_H hypermultiplets,

$$\{q^u, \zeta^\alpha\}, \quad u = 1, 2, ..., 4n_H\ , \quad \alpha = 1, 2, ..., 2n_H\ , \tag{2.6}$$

contain $4n_H$ real scalars q^u and $2n_H$ symplectic Majorana hyperini ζ^α.

The dynamics of these fields are described by the Lagrangian $\mathcal{L}_5$ of $\mathcal{N}=2$ gauged supergravity in five dimensions [58]. Since we are interested in a situation where only the scalar fields have non-trivial backgrounds values, we give here only the bosonic parts of the Lagrangian[5],

$$\begin{aligned} e^{-1}\mathcal{L}_5 = {} & \tfrac{1}{2}\mathcal{R} - \tfrac{1}{4}a_{\tilde{I}\tilde{J}}H^{\tilde{I}}_{\mu\nu}H^{\tilde{J}\mu\nu} - \tfrac{1}{2}g_{ij}\mathcal{D}_\mu\phi^i\mathcal{D}^\mu\phi^j - \tfrac{1}{2}G_{uv}\mathcal{D}_\mu q^u\mathcal{D}^\mu q^v - g^2V(\phi,q) \\ & + \tfrac{1}{16g}e^{-1}\epsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}B^M_{\mu\nu}\left(\partial_\rho B^N_{\sigma\tau} + 2gt^N_{IJ}A^I_\rho F^J_{\sigma\tau} + gt^N_{IP}A^I_\rho B^P_{\sigma\tau}\right) \\ & + \tfrac{1}{12}\sqrt{\tfrac{2}{3}}e^{-1}\epsilon^{\mu\nu\rho\sigma\tau}C_{IJK}A^I_\mu\left[F^J_{\nu\rho}F_{\sigma\tau} + f^J_{FG}A^F_\nu A^G_\rho\left(-\tfrac{1}{2}F^K_{\sigma\tau} + \tfrac{g^2}{10}f^K_{HL}A^H_\sigma A^L_\tau\right)\right] \\ & - \tfrac{1}{8}e^{-1}\epsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}t^M_{IK}t^N_{FG}A^I_\mu A^F_\nu A^G_\rho\left(-\tfrac{g}{2}F^K_{\sigma\tau} + \tfrac{g^2}{10}f^K_{HL}A^H_\sigma A^L_\tau\right)\ . \end{aligned} \tag{2.7}$$

Here $\mathcal{R}$ is the Ricci-scalar of the spacetime metric and $H^{\tilde{I}}_{\mu\nu} = (F^I_{\mu\nu}, B^M_{\mu\nu})$, where we denote by $F^I_{\mu\nu} = 2\partial_{[\mu}A^I_{\nu]} + gf^I_{JK}A^J_\mu A^K_\nu$ the field strengths of the vector fields A^I_μ, f^K_{IJ} are the structure constants of the Lie algebra of G and g_G is the gauge coupling constant. Moreover, $V(\phi, q)$ is the scalar potential arising in the gauging and $\mathcal{D}_\mu\phi^i$ and $\mathcal{D}_\mu q^u$ are the covariant derivatives of the vector/tensor scalars and hypermultiplet scalars with respective metrics g_{ij} and G_{uv}. For the topological terms of the Lagrangian we furthermore need a constant, completely symmetric tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ and an antisymmetric invertible matrix Ω_{MN}. Finally, the matrices $t^{\tilde{K}}_{I\tilde{J}}$ describe the action of the gauge group on vector and tensor multiplets. Now we recall the various quantities that are introduced in this Lagrangian in more detail.

[5]Note that we set the gravitational constant $\kappa = 1$ in this thesis.

As outlined above, the scalar fields in supergravity theories can be interpreted as maps from spacetime M_5 to a target space T,

$$\phi^i \otimes q^u : M_5 \longrightarrow T \ . \tag{2.8}$$

Due to the fact that we have two different types of scalar fields in the theory, T is a product manifold,

$$T = M_V \times M_H \ . \tag{2.9}$$

The first factor M_V is spanned by the scalars ϕ^i from the vector and tensor multiplets, while the manifold M_H is parametrized by the scalars q^u in the hypermultiplets. In particular, both spaces are endowed with Riemannian metrics $g = g_{ij} d\phi^i d\phi^j$ on M_V and $G = G_{uv} dq^u dq^v$ on M_H, respectively. Supersymmetry requires these metrics to have certain properties; the manifold (M_V, g) is a projective special real manifold of real dimension $n_V + n_T$, while (M_H, G) is a quaternionic Kähler manifold of real dimension $4n_H$. Since these geometries play a key role in the remainder of this thesis, we now review their definitions and relevant properties.

Vector/tensor multiplet geometry: projective special real manifolds

Projective special real manifolds[6] were first introduced in the context of five-dimensional supergravity in [60]. Here we follow the presentation in [58,61]. Let H be an $n_V + n_T + 1$-dimensional real manifold and $\mathcal{P}$ a cubic homogeneous polynomial. For local coordinates $h^{\tilde{I}}$ on H, this translates to

$$\mathcal{P} = C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} \ , \tag{2.10}$$

where $C_{\tilde{I}\tilde{J}\tilde{K}}$ is a constant, completely symmetric trilinear form. We can define conjugate coordinates by lowering the index $\tilde{I}$ on $h^{\tilde{I}}$ via

$$h_{\tilde{I}} := C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{J}} h^{\tilde{K}} \ . \tag{2.11}$$

Then the matrix of gauge couplings $a_{\tilde{I}\tilde{J}}$ in (2.7) defines a positive definite metric $a = a_{\tilde{I}\tilde{J}} dh^{\tilde{I}} dh^{\tilde{J}}$ on H by

$$a = -\tfrac{1}{2} \partial^2 \log(\mathcal{P}) \ , \tag{2.12}$$

where $\partial^2 \log(\mathcal{P})$ denotes the Hessian of $\log(\mathcal{P})$, i.e. $a_{\tilde{I}\tilde{J}} = -\frac{1}{2} \partial_{\tilde{I}} \partial_{\tilde{J}} \log(\mathcal{P})$. Then the components of a can be given in terms of the coordinates $h^{\tilde{I}}$ and the symmetric tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ as

$$a_{\tilde{I}\tilde{J}} = -2 C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{K}} + 3 h_{\tilde{I}} h_{\tilde{J}} \ . \tag{2.13}$$

Thus (H, a) is a Riemannian manifold. From this, a projective special real manifold (M_T, g) is defined as the hypersurface in H given by $\mathcal{P} \equiv 1$, i.e.

$$M_V = \{\mathcal{P} \equiv 1\} \subset H \ . \tag{2.14}$$

[6] In the supergravity literature, these manifolds are often referred to as very special real manifolds, while they are called projective special real manifolds in the mathematics literature.

By solving the condition $\mathcal{P}\equiv 1$ for

$$\mathcal{P}(\phi)=C_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{I}}(\phi)h^{\tilde{J}}(\phi)h^{\tilde{K}}(\phi)\ , \tag{2.15}$$

we can define local coordinates ϕ^i on M_V. Then M_V carries a natural Riemannian metric $g=g_{ij}d\phi^i d\phi^j$ by pulling back the metric a onto the hypersurface $\{\mathcal{P}=1\}$,

$$g_{ij}=h_i^{\tilde{I}}h_j^{\tilde{J}}a_{\tilde{I}\tilde{J}}\ . \tag{2.16}$$

Here we denote

$$h_i^{\tilde{I}}:=-\sqrt{\tfrac{3}{2}}\partial_i h^{\tilde{I}}(\phi)\ ,\quad h_{i\tilde{I}}:=\sqrt{\tfrac{3}{2}}\partial_i h_{\tilde{I}}(\phi)\ , \tag{2.17}$$

where the index $\tilde{I}$ was lowered with respect to the metric a.

Let us now prove some important properties of these manifolds. Due to the fact that $\mathcal{P}(\phi)=1$ on M_V, we immediately see that $h^I h_I=1$. Moreover, since $\partial_i\mathcal{P}(\phi)=0$ on M_V, we find

$$0=\partial_i\mathcal{P}=3C_{\tilde{I}\tilde{J}\tilde{K}}(\partial_i h^{\tilde{I}})h^{\tilde{J}}h^{\tilde{K}}=3(\partial_i h^{\tilde{I}})h_{\tilde{I}}\ , \tag{2.18}$$

where we used the fact that $C_{\tilde{I}\tilde{J}\tilde{K}}$ is completely symmetric and constant. Thus

$$h^{\tilde{I}}h_{\tilde{I}}=1,\qquad h_i^{\tilde{I}}h_{\tilde{I}}=0\ . \tag{2.19}$$

Combining (2.19) with the definitions of a and g, (2.13) and (2.16), we find

$$g_{ij}=h_i^{\tilde{I}}h_j^{\tilde{J}}a_{\tilde{I}\tilde{J}}=-2C_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{K}}\ . \tag{2.20}$$

The above relations can be compactly formulated in matrix form as [58]

$$\begin{pmatrix}h^{\tilde{I}}\\ h_i^{\tilde{I}}\end{pmatrix}a_{\tilde{I}\tilde{J}}\begin{pmatrix}h^{\tilde{J}} & h_j^{\tilde{J}}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & g_{ij}\end{pmatrix}\ . \tag{2.21}$$

Defining $h_{\tilde{I}}^j=g^{ij}h_{i\tilde{I}}$, this implies

$$\begin{pmatrix}h_{\tilde{I}} & h_{\tilde{I}}^i\end{pmatrix}\begin{pmatrix}h^{\tilde{J}}\\ h_i^{\tilde{J}}\end{pmatrix}=\delta_{\tilde{I}}^{\tilde{J}}\ , \tag{2.22}$$

and

$$a_{\tilde{I}\tilde{J}}=h_{\tilde{I}}h_{\tilde{J}}+h_{\tilde{I}}^i h_{i\tilde{J}}\ . \tag{2.23}$$

The fact that the matrix $(h^{\tilde{I}},h_i^{\tilde{I}})$ is invertible will be crucial in the analysis of the AdS vacuum conditions later.

Finally, let us introduce a covariant derivative on M_V. This can be defined on the tangent vectors $h_j^{\tilde{I}}$ of M_V as

$$D_i h_j^{\tilde{I}}:=-\sqrt{\tfrac{2}{3}}\left(h^{\tilde{I}}g_{ij}+T_{ijk}h^{k\tilde{I}}\right)\ . \tag{2.24}$$

Here $T_{ijk}=C_{\tilde{I}\tilde{J}\tilde{K}}h_i^{\tilde{I}}h_j^{\tilde{J}}h_k^{\tilde{K}}$ is a symmetric tensor.

Hypermultiplet geometry: quaternionic Kähler manifolds

In the following we introduce the notion of quaternionic Kähler manifolds and study some of their properties. For an extensive introduction, see [61–63]. To begin with, let (M_H, G) be a Riemannian manifold of dimension $4n_H$ with local coordinates q^u and denote by TM_H the tangent bundle of M_H. A quaternionic structure Q on M_H is a ∇^G-invariant rank three subbundle of the endomorphism bundle of the tangent bundle, $Q \subset \mathrm{End}(TM_H)$, such that it is locally spanned by a triplet of almost complex structures,

$$J^n : TM_H \to TM_H, \quad n = 1,2,3 \ . \tag{2.25}$$

These satisfy $J^1 J^2 = J^3$ and $(J^n)^2 = -\mathrm{Id}$. Let us describe this important definition in some more detail. Component wise, the almost complex structures satisfy

$$(J^m)_u^{\ w} (J^n)_w^{\ v} = -\delta^{mn} \delta_u^v + \varepsilon^{mnp} (J^p)_u^{\ v} \ . \tag{2.26}$$

Here δ^{mn} is the Kronecker delta and ε^{mnp} denotes the completely antisymmetric Levi-Civita symbol in three dimensions, normalized to $\varepsilon^{123} = 1$. Moreover, the metric G is Hermitian with respect to all three J^n, i.e. for vector fields[7] $X, Y \in \Gamma(TM_H)$,

$$G(J^n X, J^n Y) = G(X, Y) \quad \forall n \ . \tag{2.27}$$

One defines an associated triplet of two-forms $\omega^n = \omega^n_{uv} dq^u \wedge dq^v$ by

$$\omega^n_{uv} = G_{uw} (J^n)_v^{\ w} \ . \tag{2.28}$$

The invariance of Q under the the action of ∇^G, i.e. $\nabla^G Q \subset Q$, implies for the local basis J^n that the Levi-Civita connection of G rotates the almost complex structures J^n in Q, i.e.

$$\nabla^G J^n = \varepsilon^{npq} \Theta^p J^q \ . \tag{2.29}$$

Here Θ^n is a triplet of $SU(2)$-connection one-forms. Thus a quaternionic structure is specifically not a Kähler structure, since none of the almost complex structures are covariantly constant with respect to the Levi-Civita connection of the metric G. However, we can introduce a new connection ∇ by

$$\nabla J^n := \nabla^G J^n - \epsilon^{npq} \Theta^p J^q \ , \tag{2.30}$$

with the property that $\nabla J^n = 0$. Note that ∇ differs from the Levi-Civita connection by an $SU(2)$-connection corresponding to Θ^n.

We call a Riemannian manifold (M_H, G, Q) a quaternionic Kähler manifold if it admits a quaternionic structure.[8] One can show that a Riemannian manifold is quaternionic Kähler if and only if its holonomy group is contained in $SU(2) \times Sp(n_H)$. Thus

[7] We denote by $\Gamma(TM_H)$ the set of sections of the tangent bundle on M_H.

[8] For $n_H = 1$, i.e. dim $M_H = 4$, this definition is satisfied by every oriented Riemannian manifold [61]. Thus one additionally requires that the sections J^n annihilate the Riemann tensor R of G,

$$(J^n)_u^{\ s} R_{svwt} + (J^n)_v^{\ s} R_{uswt} + (J^n)_w^{\ s} R_{uvst} + (J^n)_t^{\ s} R_{uvws} = 0 \ , \tag{2.31}$$

for all $n = 1, 2, 3$.

one can express G in terms of local vielbeins $\mathcal{U}_u^{\alpha\mathcal{A}}$ as [62, 63]

$$G_{uv} = C_{\alpha\beta}\varepsilon_{\mathcal{AB}}\mathcal{U}_u^{\alpha\mathcal{A}}\mathcal{U}_v^{\beta\mathcal{B}} \ , \tag{2.32}$$

where $C_{\alpha\beta}$ denotes the flat metric on $Sp(n_H)$ and the $SU(2)$-indices $\mathcal{A}$,$\mathcal{B}$ are raised and lowered with the $SU(2)$-invariant metric $\varepsilon_{\mathcal{AB}}$. One can show that quaternionic Kähler manifolds are always Einstein, $\mathrm{Ric}_G = \nu(n+2)G$, where ν is proportional to the scalar curvature.[9] From now on, we will take the physically relevant value for ν which is fixed by supersymmetry to $\nu = -1$.[10]

Isometric action of the gauge group

Let us now discuss how the gauge group acts on the scalar target spaces in vector, tensor and hypermultiplets. The gauge group G is specified by the generators of its Lie algebra $\mathfrak{g}$ and the structure constants f_{IJ}^K. These satisfy the usual relation[11]

$$[t_I, t_J] = -f_{IJ}^K t_K \ . \tag{2.33}$$

Note that indices in this equation only run over the vector multiplets (and the graviphoton). As previously explained, the vector fields must transform in the adjoint representation of the gauge group, i.e. $t_{IJ}^K = f_{IJ}^K$. On the other hand, the tensor multiplets can transform in an arbitrary representation of G. It was shown in [57] that the most general representation for n_V vector multiplets and n_T tensor multiplets is given by

$$t_{I\tilde{J}}^{\tilde{K}} = \begin{pmatrix} f_{IJ}^K & t_{IJ}^N \\ 0 & t_{IM}^N \end{pmatrix} \ . \tag{2.34}$$

We immediately realize that the block matrix t_{IJ}^N introduces a mixing between vector and tensor fields, for example in the interaction terms of the Lagrangian (2.7). This is why we introduced a combined index $\tilde{I} = (I, M)$ in the first place.

However, the matrix t_{IJ}^N is only nonzero if the chosen representation of the gauge group is not completely reducible. This never occurs for compact gauge groups, but there exist non-compact gauge groups containing an Abelian ideal that admits representations of this type, see [57]. There it is also shown that the construction of a generalized Chern-Simons term in the action for vector and tensor multiplets requires the existence of an invertible and antisymmetric matrix Ω_{MN}. In particular, the components of the representation acting on tensor multiplets are of the form

$$t_{I\tilde{J}}^N = C_{I\tilde{J}P}\Omega^{PN} \ . \tag{2.35}$$

[9] The case $\nu = 0$ corresponds to locally hyper Kähler manifolds and is usually excluded.

[10] Quaternionic Kähler manifolds with negative scalar curvature are called negative quaternionic Kähler manifolds.

[11] Note the minus sign in the defining relation of the Lie algebra.

Here Ω^{MN} is antisymmetric and invertible,

$$\Omega_{MP}\Omega^{MN} = \delta_P^N \ . \tag{2.36}$$

Since the gauge group is the local symmetry group of the Lagrangian (2.7), it has to be realized by isometries on the scalar target spaces. In particular, the vector/tensor and hypermultiplet scalars transform under G as

$$\delta_G \phi^i = -g_G \Lambda_G^I \xi_I^i(\phi) \ , \quad \delta_G q^u = -g_G \Lambda_G^I k_I^u(q) \ , \tag{2.37}$$

where Λ_G^I are the parameters of the gauge transformation. For the vector and tensor multiplets, this action is provided by Killing vector fields $\xi_I = \xi_I^i \partial_i$ that satisfy the Lie algebra $\mathfrak{g}$ of G,

$$[\xi_I, \xi_J]^i = \xi_I^j \partial_j \xi_J^i - \xi_J^j \partial_j \xi_I^i = -f_{IJ}^K \xi_K^i \ . \tag{2.38}$$

The invariance of the Lagrangian (2.7) under the transformations (2.37) determines the explicit form of the components ξ_I^i in terms of the functions $h^{\tilde{I}}$, their derivatives $h_i^{\tilde{I}}$ and the representation $t_{I\tilde{J}}^{\tilde{K}}$ to be [64]

$$\xi_I^i = -\sqrt{\tfrac{3}{2}} t_{I\tilde{J}}^{\tilde{K}} h^{\tilde{J}} h_{\tilde{K}}^i = -\sqrt{\tfrac{3}{2}} t_{I\tilde{J}}^{\tilde{K}} h^{i\tilde{J}} h_{\tilde{K}} \ . \tag{2.39}$$

The second equality is due to the fact that [64]

$$t_{I\tilde{J}}^{\tilde{K}} h^{\tilde{J}} h_{\tilde{K}} = 0 \ , \tag{2.40}$$

and thus

$$0 = \partial_i (t_{I\tilde{J}}^{\tilde{K}} h^{\tilde{J}} h_{\tilde{K}}) = t_{I\tilde{J}}^{\tilde{K}} h^{\tilde{J}} \partial_i h_{\tilde{K}} + t_{I\tilde{J}}^{\tilde{K}} (\partial_i h^{\tilde{J}}) h_{\tilde{K}} \ , \tag{2.41}$$

which implies

$$t_{I\tilde{J}}^{\tilde{K}} h^{\tilde{J}} h_{\tilde{K}}^i = t_{I\tilde{J}}^{\tilde{K}} h^{\tilde{J}i} h_{\tilde{K}} \ . \tag{2.42}$$

The situation is more involved in the hypermultiplet case. Again, the gauge group is realized on the quaternionic Kähler target space M_H by Killing vector fields $k_I = k_I^u \partial_u$ satisfying the Lie algebra $\mathfrak{g}$,

$$[k_I, k_J]^u = k_I^v \partial_v k_J^u - k_J^v \partial_v k_I^u = -f_{IJ}^K k_K^u \ , \tag{2.43}$$

and the Killing equation

$$\nabla_u k_{vI} + \nabla_v k_{uI} = 0 \ . \tag{2.44}$$

Moreover, the k_I have to be triholomorphic, i.e. they must respect the quaternionic structure defined by (J^n, Θ^n) [61, 65],

$$\mathcal{L}_I J^n = \varepsilon^{npq} J^p W_I^q \ , \quad \mathcal{L}_I \Theta^n = \nabla W_I^n \ . \tag{2.45}$$

Here $\mathcal{L}_I := \mathcal{L}_{k_I}$ is the Lie derivative in direction of the Killing vector k_I and $W_I^n(q)$ is a triplet of real, field dependent functions called the $SU(2)$-compensator. Moreover, the compensator W_I^n satisfies the cocycle condition [65]

$$\mathcal{L}_I W_J^n - \mathcal{L}_J W_I^n + \varepsilon^{npq} W_I^p W_J^q = -f_{IJ}^K W_K^n \ . \tag{2.46}$$

To every Killing vector k_I we can then associate a triplet of real functions μ_I^n satisfying

$$\nabla_u \mu_I^n = -\tfrac{1}{2} \omega_{uv}^n k_I^v \ , \tag{2.47}$$

which can be related to W_I^n by

$$\mu_I^n = \Theta^n(k_I) - W_I^n \ . \tag{2.48}$$

In particular, the μ_I^n define a section $\mu_I \in \Gamma(Q)$ on M_H given by $\mu_I := \mu_I^n J_n$ and satisfying

$$\nabla^G \mu_I = \tfrac{1}{2} \omega^n(k_I, \cdot) J_n \ . \tag{2.49}$$

By introducing the moment maps μ_I^n, we have associated a triplet of real functions to every generator k_I of the Lie algebra of the gauge group. Furthermore, it is possible to realize a Lie algebra structure on the moment maps. To this end, we define the triholomorphic Poisson bracket as

$$\{\mu_I, \mu_J\}^n := \tfrac{1}{2} \omega^n(k_I, k_J) - 2\varepsilon^{npq} \mu_I^p \mu_J^q \ . \tag{2.50}$$

A technical calculation [65] then reveals that this bracket indeed realizes the Lie algebra on the moment maps, i.e.

$$\{\mu_I, \mu_J\} = f_{IJ}^K \mu_K \ . \tag{2.51}$$

In components, this condition can be written as [62, 65]

$$f_{IJ}^K \mu_K^n = \tfrac{1}{2} \omega_{uv}^n k_I^u k_J^v - 2\varepsilon^{npq} \mu_I^p \mu_J^q \ , \tag{2.52}$$

and is usually referred to as the equivariance condition. We will use this equation several times when studying moduli spaces of AdS vacua in Chapter 3.

Now consider the operator ∇k_I and note that $\nabla_u k_v^I$ is antisymmetric due to the fact that the k_I satisfy the Killing equation (2.44). Thus we can expand ∇k_I in terms of two-forms on M_H. Since the three-dimensional subbundle Q in $\mathrm{End}(TM_H)$ gives rise to a triplet of canonical two-forms $\omega^n = g \circ J^n$, we can decompose ∇k_I into a part proportional to ω^n and a part related to endomorphisms orthogonal to Q in $\mathrm{End}(TM_H)$. This second part is then given by antisymmetric operators L_I that commute with ω^n and are related to the hyperino mass matrix [61, 63]. Explicitly, this reads

$$2\nabla_u k_{Iv} = \omega_{uv}^n \mu_{nI} + L_{Iuv} \ . \tag{2.53}$$

One refers to the operators $\omega^n \mu_{nI}$ and L_I as the $\mathfrak{su}(2)$-part and $\mathfrak{sp}(n_H)$-part of ∇k^I, respectively [61]. For later use we define the combinations

$$S_I^n := L_I \circ J^n \ , \quad L := h^I L_I \ , \quad S^n := h^I S_I^n \ , \tag{2.54}$$

where the S_I^n are symmetric operators with components $S_{Iuv}^n = L_{Iuw}(J^n)^w_{\ v}$ [61]. Moreover, using the decomposition (2.53) we can compute the commutator of ∇k_I with the quaternionic structure J^n, i.e.

$$\nabla_u k_w^I (J^n)^w_{\ v} - (J^n)^w_{\ u} \nabla_v k_w^I = 2\varepsilon^{npq} \omega_{uv}^p \mu_I^q \ . \tag{2.55}$$

Finally, the covariant derivatives of the scalars in the Lagrangian (2.7) are given by

$$\mathcal{D}_\mu \phi^i = \partial_\mu \phi^i + g_G A_\mu^I \xi_I^i(\phi) \ , \qquad \mathcal{D}_\mu q^u = \partial_\mu q^u + g_G A_\mu^I k_I^u(q) \ . \tag{2.56}$$

If the Killing vectors k_I and ξ_I have non-trivial background values, the kinetic terms for the scalar fields contain mass terms for the vector fields A_μ^I,

$$g_{ij} \mathcal{D}_\mu \phi^i \mathcal{D}^\mu \phi^j = g_G^2 g_{ij} \xi_I^i \xi_J^j A_\mu^I A^{J\mu} + \dots \ , \tag{2.57}$$

and

$$G_{uv} \mathcal{D}_\mu q^u \mathcal{D}^\mu q^v = g_G^2 G_{uv} k_I^u k_J^v A_\mu^I A^{J\mu} + \dots \ . \tag{2.58}$$

These terms introduce a breaking of the gauge symmetry in the vacuum, which we will study in more detail later in this chapter.

Before we proceed, let us note that for $n_H = 0$, i.e. in the case when there are no hypermultiplets present, constant Fayet-Iliopoulos (FI) terms can exist which satisfy the equivariance condition (2.52). In this case the first term on the right hand side of (2.52) vanishes since there are no Killing vectors in the hypermultiplets, i.e. $k_I = 0$ for all I. This implies that there are only two possible solutions [58]: If the gauge group contains an $SU(2)$-factor, the FI-terms have to be of the form

$$\mu_I^n = c e_I^n \ , \quad c \in \mathbb{R} \ , \tag{2.59}$$

where the e_I^n are nonzero constant vectors for indices $I = 1, 2, 3$ of the $SU(2)$-factor. These vectors satisfy

$$\varepsilon^{mnp} e_I^m e_J^n = f_{IJ}^K e_K^p \ . \tag{2.60}$$

The second solution has $U(1)$-factors in the gauge group and thus the respective structure constants f_{IJ}^K vanish. Due to the equivariance condition (2.52) in this case, the constant moment maps are given by

$$\mu_I^n = c_I e^n \ , \quad c_I \in \mathbb{R} \ , \tag{2.61}$$

where e^n is a constant $SU(2)$-vector and I labels the $U(1)$-factors.

Scalar potential and SUSY variations

Using the structures defined previously we are now in the position to write down the scalar potential that appears in the Lagrangian (2.7). To do so, let us first define the useful combinations

$$\mu^n := h^I \mu_I^n \ , \quad \partial_i \mu^n := h_i^I \mu_I^n \ , \quad K := h^I k_I \ , \quad \Xi := h^I \xi_I \ . \tag{2.62}$$

The moment map μ^n is sometimes referred to as "dressed moment map" [61]. Note that the functions h^M corresponding to tensor multiplets do not appear explicitly. However, the combinations defined in (2.62) can implicitly depend on the scalars in the tensor multiplets as they might feature in the $h^I(\phi)$ after solving (2.15).

To make contact between the scalar potential and the fermionic supersymmetry variations, one usually also defines the following couplings

$$\begin{aligned} \mathcal{S}^{\mathcal{AB}} &:= \mu^n \sigma_n^{\mathcal{AB}} \ , \qquad & \mathcal{W}_i^{\mathcal{AB}} &:= \partial_i \mu^n \sigma_n^{\mathcal{AB}} \ , \\ \mathcal{K}^i &:= \tfrac{\sqrt{6}}{4} \Xi^i \ , \qquad & \mathcal{N}^{\alpha\mathcal{A}} &:= \tfrac{\sqrt{6}}{4} K^u \mathcal{U}_u^{\alpha\mathcal{A}} \ . \end{aligned} \tag{2.63}$$

Here $\sigma^n_{\mathcal{AB}}$ are the Pauli matrices with an index lowered by $\varepsilon_{\mathcal{AB}}$, i.e.

$$\sigma^1_{\mathcal{AB}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad \sigma^2_{\mathcal{AB}} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \ , \quad \sigma^3_{\mathcal{AB}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \ . \tag{2.64}$$

With these definitions the scalar potential is given by

$$V = 2 g_{ij} \mathcal{W}^{i\mathcal{AB}} \mathcal{W}^j_{\mathcal{AB}} + 2 g_{ij} \mathcal{K}^i \mathcal{K}^j + 2 \mathcal{N}^\alpha_{\mathcal{A}} \mathcal{N}^{\mathcal{A}}_\alpha - 4 \mathcal{S}_{\mathcal{AB}} \mathcal{S}^{\mathcal{AB}} \ . \tag{2.65}$$

In the following we will also make use of the scalar parts of the fermionic supersymmetry variations. Like the scalar potential, the supersymmetry variations are given in terms of the couplings (2.63),

$$\begin{aligned} \delta_\epsilon \psi_\mu^{\mathcal{A}} &= \nabla_\mu \epsilon^{\mathcal{A}} - \tfrac{i g_G}{\sqrt{6}} \mathcal{S}^{\mathcal{AB}} \gamma_\mu \epsilon_{\mathcal{B}} + \dots \ , \\ \delta_\epsilon \lambda^{i\mathcal{A}} &= g_G \mathcal{K}^i \epsilon^{\mathcal{A}} - g_G \mathcal{W}^{i\mathcal{AB}} \epsilon_{\mathcal{B}} + \dots \ , \\ \delta_\epsilon \zeta^\alpha &= g_G \mathcal{N}^\alpha_{\mathcal{A}} \epsilon^{\mathcal{A}} + \dots \ . \end{aligned} \tag{2.66}$$

Here $\epsilon^{\mathcal{A}}$ denotes the supersymmetry parameter, ∇_μ is the five-dimensional covariant derivative on spinors and γ_μ are the gamma matrices of $Spin(1,4)$, see Appendix B. Moreover, the ellipses correspond to higher spin contributions. This concludes our brief review of gauged five-dimensional $\mathcal{N}=2$ supergravity and we now turn to the study of its supersymmetric AdS backgrounds.

2.2 Supersymmetric AdS vacua

In this section we determine the conditions for AdS$_5$ vacua that preserve all eight supercharges. Note that some of these results were already obtained in [27, 33, 61]. However, we extend their considerations and discuss here the most general case of gauged $\mathcal{N} = 2$ supergravity coupled to vector, tensor and hypermultiplets, charged under an arbitrary gauge group G. This section is based on [34].

As usual, we denote the background value of a quantity with brackets $\langle\ \rangle$. We already discussed in the introduction of this chapter that a vacuum is maximally supersymmetric if the supersymmetry variations of all the fields vanish. Since we are interested in $SO(2, d)$-invariant vacua, the only nonzero supersymmetry variations are those of the fermions (2.66). Thus we set

$$\langle \delta_\epsilon \psi_\mu^{\mathcal{A}} \rangle = \langle \delta_\epsilon \lambda^{i\mathcal{A}} \rangle = \langle \delta_\epsilon \zeta^\alpha \rangle = 0 \ . \tag{2.67}$$

We immediately realize that the vanishing of the hyperini variation $\delta_\epsilon \zeta^\alpha$ implies

$$\langle \mathcal{N}^{\alpha\mathcal{A}} \rangle = \frac{\sqrt{6}}{4} \langle K^u \mathcal{U}_u^{\alpha\mathcal{A}} \rangle = 0 \ . \tag{2.68}$$

Now recall that the vielbeins $\mathcal{U}_u^{\alpha\mathcal{A}}$ are invertible in the sense that [62]

$$\mathcal{U}_u^{\alpha\mathcal{A}} \mathcal{U}_{\alpha\mathcal{A}}^v = \delta_u^v \ . \tag{2.69}$$

Thus we can multiply (2.68) with $\mathcal{U}_{\alpha\mathcal{A}}^v$ from the right to obtain

$$\langle K^u \rangle = \langle h^I k_I^u \rangle = 0 \ . \tag{2.70}$$

Similarly, the vanishing of the gaugini variation $\delta_\epsilon \lambda^{i\mathcal{A}}$ implies

$$\langle \mathcal{K}^i \rangle \epsilon^{\mathcal{A}} - \langle \mathcal{W}^{i\mathcal{A}\mathcal{B}} \rangle \epsilon_{\mathcal{B}} = 0 \ . \tag{2.71}$$

The action of the couplings $\mathcal{W}^{i\mathcal{A}\mathcal{B}}$ on the supersymmetry parameter $\epsilon_{\mathcal{B}}$ is given by the Pauli matrices defined in (2.64) while $\mathcal{K}^i$ acts as $\varepsilon^{\mathcal{A}\mathcal{B}}$ on $\epsilon_{\mathcal{B}}$. Since the matrices $\{\varepsilon_{\mathcal{A}\mathcal{B}}, \sigma^n_{\mathcal{A}\mathcal{B}}\}$ are linearly independent, (2.71) implies the vanishing of the individual couplings

$$\langle \mathcal{K}^i \rangle = \langle \mathcal{W}_i^{\mathcal{A}\mathcal{B}} \rangle = 0 \ . \tag{2.72}$$

Inserting the definitions (2.63), we find from the first equation that

$$\langle \Xi^i \rangle = \langle h^I \xi_I^i \rangle = 0 \ , \tag{2.73}$$

while the second equations gives

$$\langle \partial_i \mu^n \rangle = \langle h_i^I \mu_I^n \rangle = 0 \ . \tag{2.74}$$

Finally, let us study the gravitino variation $\delta_\epsilon \psi_\mu^{\mathcal{A}}$. In a supersymmetric background it gives rise to a Killing spinor equation for the supersymmetry parameter $\epsilon^{\mathcal{A}}$,

$$\langle \nabla_\mu \epsilon^{\mathcal{A}} \rangle = \frac{i g_G}{\sqrt{6}} \langle \mathcal{S}^{\mathcal{A}\mathcal{B}} \rangle \gamma_\mu \epsilon_{\mathcal{B}} \ . \tag{2.75}$$

One can show [66] that Killing spinor equations for symplectic Majorana fermions in AdS_5 are always of the following form,

$$\nabla_\mu \epsilon^{\mathcal{A}} = \frac{ia}{2} U^{\mathcal{AB}} \gamma_\mu \epsilon_{\mathcal{B}} \ , \tag{2.76}$$

for $a \in \mathbb{R}$ a constant and $U^{\mathcal{AB}} = v^n \sigma_n^{\mathcal{AB}}$ an $SU(2)$-matrix. Here $v \in S^2$ is a unit vector. Thus the vanishing of the gravitino variation in the backgrounds implies

$$\langle \mathcal{S}^{\mathcal{AB}} \rangle \epsilon_{\mathcal{B}} = \lambda U^{\mathcal{AB}} \epsilon_{\mathcal{B}} \ , \tag{2.77}$$

where $\lambda \in \mathbb{R}$ is related to the cosmological constant Λ by $\lambda = \frac{1}{2}\sqrt{|\Lambda|}$. Comparing with (2.63), we find that

$$\langle \mu^n \rangle = \langle h^I \mu_I^n \rangle = \lambda v^n \ . \tag{2.78}$$

Note that the case $\lambda = 0$ corresponds to Minkowski solutions and thus we assume $\lambda \neq 0$ in the following.

To summarize, we have shown that a supersymmetric background has to satisfy the following equations,

$$\begin{aligned} \langle \mu^n \rangle &= \langle h^I \mu_I^n \rangle = \lambda v^n \ , \\ \langle \partial_i \mu^n \rangle &= \langle h_i^I \mu_I^n \rangle = 0 \ , \\ \langle K^u \rangle &= \langle h^I k_I^u \rangle = 0 \ , \\ \langle \Xi^i \rangle &= \langle h^I \xi_I^i \rangle = 0 \ . \end{aligned} \tag{2.79}$$

Note that due to (2.15) and (2.16) we must have $\langle h^I \rangle \neq 0$ for some I and $\langle h_i^I \rangle \neq 0$ for every i and some I. In particular, this can also hold for $\langle \phi^i \rangle = 0$, i.e. at the origin of the scalar field space. If we insert the equations (2.79) into the definition of the scalar potential (2.65), we find that $\langle V \rangle < 0$ contributes as negative cosmological constant and thus the backgrounds we study are indeed AdS.

Moment map conditions

Let us try to obtain a deeper understanding of the conditions for supersymmetric AdS backgrounds we derived. To begin with, we combine the first two equations of (2.79) as

$$\langle \begin{pmatrix} h^I \\ h_i^I \end{pmatrix} \mu_I^n \rangle = \begin{pmatrix} \lambda v^n \\ 0 \end{pmatrix} . \tag{2.80}$$

Let us enlarge these equations to the tensor multiplet indices by introducing $\mu_{\tilde{I}}^n$ where we keep in mind that $\mu_N^n \equiv 0$. Then we use the fact that the matrix $(h^{\tilde{I}}, h_i^{\tilde{I}})$ is invertible in special real geometry (2.22) and multiply (2.80) with $(h^{\tilde{I}}, h_i^{\tilde{I}})^{-1}$ to obtain a solution for both equations, given by

$$\langle \mu_{\tilde{I}}^n \rangle = \lambda v^n \langle h_{\tilde{I}} \rangle \ . \tag{2.81}$$

This implies that the moment maps for all $\tilde{I}$ point in the same direction in $SU(2)$-space. In particular, using the $SU(2)_R$-symmetry we can rotate the vector v such that $v^n = \delta^{3n}$

points into the z-direction. Thus only $\langle\mu_I\rangle := \langle\mu_I^3\rangle \neq 0$ for all I in the above equation. Moreover, since by definition $\mu_N^n = 0$, we find

$$\langle\mu_I\rangle = \lambda\langle h_I\rangle \ , \quad \langle h_N\rangle = 0 \ . \tag{2.82}$$

Because the h_N corresponding to the tensor multiplets have to vanish in the background, the equations (2.19) simplify to

$$\langle h^I h_I\rangle = 1 \ , \quad \langle h_I h_i^I\rangle = 0 \ . \tag{2.83}$$

Moreover, due to the explicit form of the moment maps in (2.82), the equivariance condition (2.52) in the AdS background translates to

$$f_{IJ}^K\langle\mu_K\rangle = \tfrac{1}{2}\langle\omega_{uv}^3 k_I^u k_J^v\rangle \ . \tag{2.84}$$

Since (2.15) must hold in the vacuum, $\langle h^I\rangle \neq 0$ for some I and thus there are always nonzero moment maps present in the supersymmetric background due to (2.82). This implies that part of the R-symmetry is gauged, which can be seen from the covariant derivatives of the fermions as they always contain a term of the form $A_\mu^I\langle\mu_I^3\rangle$ [58]. More precisely, this combination gauges a $U(1)_R$ subgroup of the global $SU(2)_R$-symmetry. If the $SU(2)_R$ is generated by the Pauli matrices σ^n and we choose $v^n = \delta^{3n}$ as above, then the gauged $U(1)_R \subset SU(2)_R$ is the one generated by σ^3. Furthermore, from (2.82) we find $A_\mu^I\langle\mu_I^3\rangle = \lambda A_\mu^I\langle h_I\rangle$ which can be identified with the graviphoton [64].

Spontaneous gauge symmetry breaking

We now focus on the last two equations characterizing the AdS vacuum (2.79). Let us first prove that the third equation $\langle h^I k_I^u\rangle = 0$ implies the fourth $\langle h^I\xi_I^i\rangle = 0$. This can be done by expressing $\langle\xi_I^i\rangle$ in terms of $\langle k_I^u\rangle$ via the modified equivariance condition of the vacuum (2.84). Observe that the background value of the Killing vectors on the projective special real manifold is given by (2.39)

$$\begin{aligned}\langle\xi_I^i\rangle &= -\sqrt{\tfrac{3}{2}}\,\langle t_{I\tilde{J}}^{\tilde{K}} h^{\tilde{J}i} h_{\tilde{K}}\rangle \\ &= -\sqrt{\tfrac{3}{2}}\,\langle f_{IJ}^K h^{Ji} h_K + t_{IJ}^N h^{Ji} h_N\rangle \\ &= -\sqrt{\tfrac{3}{2}}\,\langle f_{IJ}^K h^{Ji} h_K\rangle \ ,\end{aligned} \tag{2.85}$$

where we used (2.34) and (2.82). Inserting (2.82), (2.84) into (2.85), one indeed computes

$$\langle\xi_I^i\rangle = -\sqrt{\tfrac{3}{2}}\tfrac{1}{2\lambda}\,\langle h_i^J\omega_{uv}^3 k_I^u k_J^v\rangle \ . \tag{2.86}$$

But then $\langle h^I k_I^u\rangle = 0$ always implies

$$\langle h^I\xi_I^i\rangle = -\sqrt{\tfrac{3}{2}}\tfrac{1}{2\lambda}\langle h_i^J\omega_{uv}^3 h^I k_I^u k_J^v\rangle = 0 \ . \tag{2.87}$$

Moreover, this shows that $\langle \xi_I^i \rangle \neq 0$ is only possible for $\langle k_I^u \rangle \neq 0$. Note that the reverse is not true in general as can be seen from (2.85). We are thus left with analyzing the third condition in (2.79).

Before we proceed, let us briefly comment on the situation where there are no hypermultiplets in the spectrum ($n_H = 0$). In this case there is no quaternionic Kähler manifold and thus also no Killing vectors, i.e. $k_I^u \equiv 0$. Then the third condition of (2.79) is automatically satisfied without restricting the h^I. However, we discussed in Section (2.1) that there can be non-trivial solutions for the moment maps μ_I^n given by constant FI-terms. Comparing the possible FI-terms (2.59) and (2.61) with (2.82), we realize that only Abelian FI-terms are allowed in a supersymmetric AdS background. Thus for theories without hypermultiplets, the possible gauge groups in the vacuum are those with only Abelian factors.

Now fix $n_H \neq 0$ and let us analyze the third equation $\langle h^I k_I^u \rangle = 0$ from (2.79). This equation has two possible solutions:

$$\begin{aligned} &1) \quad \langle k_I^u \rangle = 0 \ , \quad \text{for all } I \\ &2) \quad \langle k_I^u \rangle \neq 0 \ , \quad \text{for some } I \text{ with } \langle h^I \rangle \text{ chosen accordingly.} \end{aligned} \tag{2.88}$$

We already observed in (2.57) and (2.58) that the covariant derivatives of the scalar fields introduce a mass term for the gauge fields A_μ^I if the vacuum value of the corresponding Killing vector is nonzero. This implies that the gauge group is unbroken in case 1), while it is broken by the non-vanishing Killing vectors $\langle k_I^u \rangle$ in case 2). To be more precise, using the relation (2.86) between $\langle \xi_I \rangle$ and $\langle k_I \rangle$, we can compute the mass matrix M_{IJ} of the gauge fields to be

$$M_{IJ} = \langle G_{uv} k_I^u k_J^v \rangle + \langle g_{ij} \xi_I^i \xi_J^j \rangle = \langle K_{uv} k_I^u k_J^v \rangle \ . \tag{2.89}$$

Here K_{uv} is an invertible matrix which is defined in terms of the metric G_{uv} and the symmetric operator S^3_{uv} (2.54). Explicitly, we find

$$K_{uv} = \langle \tfrac{5}{8} G_{uv} - \tfrac{6}{8\lambda} S^3_{uv} \rangle \ . \tag{2.90}$$

Since $\langle h^I k_I^u \rangle = 0$ in the background, the mass matrix M_{IJ} has a zero eigenvector given by $\langle h^I \rangle$. Thus the graviphoton $\langle h^I \rangle A_I^\mu$ always remains massless in a supersymmetric AdS vacuum.

Moreover, the $U(1)_R$-symmetry $h^I k_I$ gauged by the graviphoton commutes with every other symmetry $\langle k_J \rangle = 0$ of the vacuum. This can be seen by computing

$$\langle [h^I k_I, k_J]^u \rangle = \langle h^I (k_I^v \partial_v k_J^u - k_J^v \partial_v k_I^u) \rangle = -\langle h^I k_J^v \partial_v k_I^u \rangle = 0 \ , \tag{2.91}$$

where we used (2.79) in the second step. Thus the unbroken gauge group of an AdS background is always of the form $U(1)_R \times H$ for some subgroup $H \subset G$. Note that a priori $U(1)_R \subset G$ can be a subgroup even though $G \neq U(1)_R \times H$. However, gauge groups of this form always have to be broken in the background, $G \to U(1)_R \times H$.

Let us now analyze the possible gauge groups in case 1) where $\langle k_I \rangle = 0$ for all I. No spontaneous gauge symmetry breaking appears in this case, since the mass matrix (2.89) is trivial. Moreover, the equivariance condition (2.52) then implies

$$f_{IJ}^{K} \langle \mu_K \rangle = 0 \ . \tag{2.92}$$

This requires the adjoint representation of the Lie algebra $\mathfrak{g}$ to admit a non-trivial eigenvector with eigenvalue 0. Thus the center of G must be non-trivial and continuous [67]. This holds for all gauge groups with an Abelian factor, however, semisimple gauge groups have to be broken in the vacuum and are thus not allowed in case 1). Note that for example $SU(2)$ as a gauge group would have to be broken in case 1), but gauge groups of the form $U(1) \times SU(2)$ would still be possible.

The situation is quite different in case 2), as spontaneous gauge symmetry breaking appears due to nonzero background values for some of the isometries k_I. First of all, assume that only Abelian factors in G are broken with $\langle k_I^u \rangle \neq 0$. Then $f_{IJ}^K = 0$ for these factors and (2.39) implies $\langle \xi_I^i \rangle = 0$. Thus the spontaneous symmetry breaking originates from the hypermultiplet sector and the associated Goldstone bosons necessarily reside in these hypermultiplets. A vector multiplet corresponding to a broken Abelian factor in the gauge group then becomes massive by “eating” an entire hypermultiplet. The resulting multiplet is a “long” massive vector multiplet that contains the massive vector field, four gauginos and four scalars.

For non-Abelian factors of G, we can also consider spontaneous symmetry breaking. In this case the Killing vectors from the vector multiplets (2.39) can have nonzero vacuum values $\langle \xi_I^i \rangle \neq 0$, but $\langle \xi_I^i \rangle = 0$ is also still possible. Since the ξ_I^i are functions of the k_I^u due to (2.86), the spontaneous symmetry breaking is essentially unchanged to the Abelian case. Again, entire hypermultiplets are eaten and the massive vector fields reside in long vector multiplets.

Before we summarize this section, let us remark that the number of broken generators of the gauge group G is determined by the number of linearly independent Killing vectors $\langle k_I \rangle$ in the AdS background. In particular, this coincides with the number n_G of Goldstone bosons, because the $\langle k_I^u \rangle$ form a basis in the space $\mathcal{G}$ of Goldstone bosons. We have $\mathcal{G} = \mathrm{span}_{\mathbb{R}}\{\langle k_I \rangle\}$ with $\dim \mathcal{G} = \mathrm{rk}\, \langle k_I^u \rangle = n_G$ as outlined above.

Let us conclude with a short summary of the conditions for maximally supersymmetric AdS backgrounds in five-dimensional gauged $\mathcal{N} = 2$ supergravity. We have shown that these conditions can be formulated in terms of moment maps and Killing vectors on the scalar target spaces as

$$\langle \mu_I \rangle = \lambda \langle h_I \rangle \ , \quad \langle h_M \rangle = 0 \ , \quad \langle h^I k_I \rangle = \langle h^I \xi_I \rangle = 0 \ . \tag{2.93}$$

Note that the tensor multiplets enter the final result only implicitly since the h^I and their derivatives are functions of all scalars ϕ^i in vector and tensor multiplets.

The first equation shows that in a supersymmetric AdS background a $U(1)_R$-symmetry is always gauged by the graviphoton while the last equation implies that the unbroken gauge group of the vacuum is of the form $U(1)_R \times H$. Here H denotes the unbroken part of the gauge group other than $U(1)_R$, which is generated by the vanishing Killing vectors $\langle k_I \rangle = 0$ other than $\langle h^I k_I \rangle$. This reproduces the known result from [33] that the $U(1)_R$-symmetry has to be unbroken and gauged in a maximally supersymmetric AdS_5 background. In the dual four-dimensional SCFT this $U(1)_R$ is given by a-maximization [68–70]. Furthermore, we described the Higgs mechanism in the case of spontaneous gauge symmetry breaking by nonzero background values of Killing vectors k_I. The associated gauge fields become massive by "eating" an entire hypermultiplet and henceforth reside in long vector multiplets. Finally, we discussed the space of Goldstone bosons and showed that $\mathcal{G} = \mathrm{span}_{\mathbb{R}}\{\langle k_I \rangle\}$.

Chapter 3

Moduli spaces of AdS vacua

The next step in our analysis of AdS_5 vacua in gauged $\mathcal{N} = 2$ supergravity will be to determine their supersymmetric moduli spaces. As previously explained, the intuition from the AdS/CFT correspondence tells us that these spaces should be Kähler manifolds [35]. We explicitly verify this by studying the variations of the AdS conditions (2.79) with respect to the scalar fields and show that the moduli space is given as a subset of the hypermultiplet scalars. This subset admits a natural Kähler structure coming from the quaternionic Kähler structure of the ambient space.

Let us begin again by sketching the general idea in an arbitrary supergravity theory containing scalars Φ and fermions Ψ. In Chapter 2 we studied the conditions $\langle F(\Phi)\rangle = 0$ for a vacuum to be AdS. Now we are interested in the deformation space of these vacua. To this end, we expand the scalar fields

$$\Phi \to \langle \Phi \rangle + \delta\Phi \ , \tag{3.1}$$

where $\delta\Phi$ denotes a small variation around the vacuum point $\langle\Phi\rangle$. The variation of the functions $F(\Phi)$ then reads

$$\delta F(\Phi) = \frac{\partial F}{\partial \Phi}\, \delta\Phi \ . \tag{3.2}$$

Hence, the deformation space $\mathcal{D}$ of an AdS_5 vacuum is given by all $\delta\Phi$ such that $\langle \delta F\rangle = 0$, i.e. all variations of the scalar fields that leave the conditions $\langle F(\Phi)\rangle = 0$ invariant.[1] That is, we are looking for directions in the scalar field space that preserve supersymmetry.[2]

However, not all of these deformations correspond to physical moduli. If the gauge group is spontaneously broken, i.e. if there exist Killing vectors K_I on the target space T that acquire a non-vanishing background value $\langle K_I\rangle \neq 0$ for some I, then we always have $C^I\langle K_I\rangle \in \mathcal{D}$ for some constants $C^I \in \mathbb{R}$. The resulting deformation then only transforms

[1] Because we are only considering the first order variations of the scalar fields, the resulting conditions on the moduli are only necessary but not sufficient conditions. We will comment on this in more detail later.

[2] These deformations are really only tangent vectors to the AdS vacuum point in the scalar manifold. However, we can locally identify them with coordinate directions in the manifold.

the vacuum via an isometry of the scalar field space and hence is considered unphysical. Such deformations correspond to the Goldstone bosons of the vacuum and should not be counted as moduli of the AdS vacuum, i.e. we are only interested in the deformation space $\mathcal{D}$ up to isometries of T. If we denote by $\mathcal{G}$ the space of Goldstone bosons $C^I\langle K_I\rangle$, then the supersymmetric moduli space is given by the quotient $\mathcal{M} = \mathcal{D}/\mathcal{G}$.

Variation of the AdS vacuum conditions

Let us now compute the moduli space $\mathcal{M}$ of the maximally supersymmetric AdS$_5$ vacua (2.79) determined in the previous chapter. This discussion is largely based on reference [34]. To begin with, we expand the scalar fields in vector, tensor and hypermultiplets around the AdS vacuum,

$$\phi^i \to \langle\phi^i\rangle + \delta\phi^i \ , \quad q^u \to \langle q^u\rangle + \delta q^u \ . \tag{3.3}$$

We then vary the first condition in (2.79) to find

$$\langle\delta(h^I\mu_I^n)\rangle = \langle\partial_i h^I\mu_I^n\rangle\delta\phi^i + \langle h^I\nabla_u\mu_I^n\rangle\delta q^u = -\tfrac{1}{2}\langle\omega_{uv}^n h^I k_I^v\rangle\delta q^u \equiv 0 \ , \tag{3.4}$$

where we used (2.47) and (2.79). No conditions are imposed on the scalar field deformations since this variation vanishes identically in a supersymmetric background.

The variation of the second equation in (2.79) is given by

$$\begin{aligned}\langle\delta(h_i^I\mu_I^n)\rangle &= \langle D_j h_i^I\mu_I^n\rangle\delta\phi^j + \langle h_i^I\nabla_u\mu_I^n\rangle\,\delta q^u \\ &= -\sqrt{\tfrac{2}{3}}\langle\mu_I^n(h^I g_{ij} + h^{Ik}T_{ijk})\rangle\delta\phi^j - \tfrac{1}{2}\langle h_i^I\omega_{uv}^n k_I^v\rangle\delta q^u \\ &= -\sqrt{\tfrac{2}{3}}\lambda\delta^{n3}\delta\phi_i - \tfrac{1}{2}\langle h_i^I\omega_{uv}^n k_I^v\rangle\delta q^u = 0 \ ,\end{aligned} \tag{3.5}$$

where we used (2.24) and (2.47) in the second step, while in the third we used the vacuum conditions (2.79). For $n = 1, 2$ this simplifies to

$$\langle h_i^I\omega_{uv}^{1,2}k_I^v\rangle\delta q^u = 0 \ . \tag{3.6}$$

A priori, this appears to restrict $2n_G = 2\cdot\mathrm{rk}\,\langle k_I^u\rangle$ of the deformations δq^u in the hypermultiplets due to the fact that the two-forms $\omega^{1,2}$ are non-degenerate. However, we show later that the moduli space has a complex structure given by $\frac{1}{\lambda}\mu_3 J^3$ that maps the two equations (3.6) into each other. Thus only n_G independent deformations are fixed by (3.6). For $n = 3$, we can solve (3.5) for the deformations in the vector/tensor multiplet scalars. Thus the $\delta\phi^i$ can always be given in terms of the variations δq^u in the hypermultiplets as

$$\delta\phi_i = -\sqrt{\tfrac{3}{2}}\tfrac{1}{2\lambda}\langle h_i^I\omega_{uv}^3 k_I^v\rangle\delta q^u \ . \tag{3.7}$$

With this we have shown that all deformations $\delta\phi^i$ are fixed and the space of deformations is spanned by scalar fields in the hypermultiplets. Note the similarity to the relation between the Killing vectors ξ_I and k_I given in (2.86). This is consistent with the fact that, as discussed in Section 2.2, also all Goldstone bosons reside in hypermultiplets.

Because we have shown above that $\langle h^I k_I \rangle = 0$ already implies $\langle h^I \xi_I \rangle = 0$, all we have left to do is to vary the third equation in (2.79). We find

$$\langle \delta(h^I k_I^u) \rangle = \langle \partial_i h^I k_I^u \rangle \delta\phi^i + \langle h^I \nabla_v k_{uI} \rangle \delta q^v = 0 \ . \tag{3.8}$$

Substituting $\delta\phi^i$ with (3.7), using (2.23) and the background conditions (2.79), we can formulate this condition solely as an equation for the hypermultiplet scalars,

$$\left(\tfrac{1}{2\lambda} \langle k^{Iu} \omega^3_{vw} k_I^w \rangle + \langle h^I \nabla_v k_I^u \rangle \right) \delta q^v = 0 \ . \tag{3.9}$$

We have thus shown that the deformation space $\mathcal{D}$ is characterized by the two equations (3.6) and (3.9) that restrict the variations δq^u. For a generic supergravity we will not solve these equations here in general. However, we will discuss some examples coming from type IIB supergravity in Chapter 4. In these cases we can explicitly compute the moduli space of AdS vacua. For now, we will show that the conditions determining $\mathcal{D}$ alone suffice to prove that the supersymmetric moduli space $\mathcal{M}$ has a Kähler structure given by the restriction of the section $\frac{1}{\lambda}\mu_3 J^3$.[3]

Kähler structure of the moduli space

Let us begin by showing that the Goldstone bosons $c^I \langle k_I^u \rangle$, $c^I \in \mathbb{R}$, indeed satisfy the conditions (3.6) and (3.9). We immediately see that (3.6) is satisfied due to

$$c^I \langle h_i^J \omega^{1,2}_{uv} k_I^u k_J^v \rangle = 2 c^I \langle h_i^J f_{IJ}^K \mu_K^{1,2} \rangle = 0 \ , \tag{3.10}$$

where we used the equivariance condition (2.84) and the fact that $\langle \mu_I^{1,2} \rangle = 0$ due to (2.82). Turning to (3.9), we first note that in the AdS background

$$\langle h^I (\nabla_v k_I^u) k_J^v \rangle = \langle h^I (\partial_v k_I^u) k_J^v - h^I (\partial_v k_J^u) k_I^v \rangle = -\langle h^I [k_I, k_J]^u \rangle = \langle f_{IJ}^K h^I k_K^u \rangle \ , \tag{3.11}$$

where we used (2.79) in the first step, added a term which vanishes in the vacuum and then used (2.43) in the second step. Additionally, we have

$$\langle f_{IJ}^K h^I k_K^u \rangle = \langle f_{IJ}^K h_K k^{uI} \rangle \ . \tag{3.12}$$

To see this, using (2.23) and $\langle h^I k_I^u \rangle = 0$ we find

$$\langle f_{IJ}^K h^I k_K^u \rangle = \langle f_{IJ}^K h^I k^{uL} a_{KL} \rangle = \langle f_{IJ}^K h^I k^{uL} h_K^i h_{iL} \rangle \ . \tag{3.13}$$

Now note that evaluating (2.42) in the vacuum gives

$$\langle f_{IJ}^K h^J h_K^i \rangle = \langle f_{IJ}^K h^{iJ} h_K \rangle \ . \tag{3.14}$$

[3]The fact that the complex structure will be given by $\frac{1}{\lambda}\mu_3 J^3$ relates to the choice of direction in $SU(2)$-space for the moment maps μ_I^n that we made in (2.82). For an arbitrary direction $v \in S^2$, the complex structure would be given by the linear combination $v_n J^n$.

Inserting (3.14) into (3.13) and using again (2.23), we obtain

$$\langle f_{IJ}^{K} h^{I} k_{K}^{u} \rangle = \langle f_{IJ}^{K} h^{iI} k^{uL} h_{K} h_{iL} \rangle = \langle f_{IJ}^{K} h_{K} k^{uL} \delta_{L}^{I} \rangle = \langle f_{IJ}^{K} h_{K} k^{uI} \rangle \ , \tag{3.15}$$

which proves the claim (3.12). Turning back to (3.9), we insert $\delta q^{u} = c^{I} \langle k_{I}^{u} \rangle$ and use (2.84) and (3.11) to find

$$\tfrac{1}{2\lambda} c^{I} \langle k^{uJ} \omega_{vw}^{3} k_{J}^{w} k_{I}^{v} \rangle + c^{I} \langle h^{J} (\nabla_{v} k_{J}^{u}) k_{I}^{v} \rangle = \tfrac{1}{\lambda} c^{I} \langle k^{uJ} f_{IJ}^{K} \mu_{K} \rangle + c^{I} \langle f_{IJ}^{K} h^{J} k_{K}^{u} \rangle \ . \tag{3.16}$$

Since in the background $\langle \mu_{I} \rangle = \lambda \langle h_{I} \rangle$, applying (3.12) then shows

$$\tfrac{1}{\lambda} c^{I} \langle k^{uJ} f_{IJ}^{K} \mu_{K} \rangle + c^{I} \langle f_{IJ}^{K} h^{J} k_{K}^{u} \rangle = (f_{JI}^{K} + f_{IJ}^{K}) c^{I} \langle h^{J} k_{K}^{u} \rangle = 0 \ . \tag{3.17}$$

In conclusion, we have shown that the Goldstone directions $c^{I} \langle k_{I}^{u} \rangle$ satisfy the conditions (3.6) and (3.9) on the deformation space of the AdS vacuum and hence $\mathcal{G} \subset \mathcal{D}$.

Let us now consider the moduli space $\mathcal{M} = \mathcal{D}/\mathcal{G}$ and prove that it admits a natural Kähler structure descending from the quaternionic Kähler structure of the ambient space. To this end, we first prove that the almost complex structure $J^{3} \in Q$ combined with the dressed moment map μ^{3} from (2.62) restricts to an almost complex structure $\mathcal{J} := \frac{1}{\lambda} \mu^{3} J_{3}$ on $\mathcal{M}$.[4] We immediately find that $\mathcal{J}^{2} = -\mathrm{Id}$. To prove that $\mathcal{J}$ is well-defined on $\mathcal{M}$, we will show that the equation (3.6) and (3.9) are $\mathcal{J}$-invariant. Let us begin by examining the first term in equation (3.9) and define

$$B_{v}^{u} := k_{I}^{u} \omega_{vw}^{3} k^{wI} \ . \tag{3.18}$$

Considering this as a linear map on $\mathcal{D}$, we find

$$\mathrm{rk} \, \langle B_{v}^{u} \rangle \leq \mathrm{rk} \, \langle k_{I}^{u} \rangle = n_{G} \ , \tag{3.19}$$

since ω^{3} is non-degenerate and thus has full rank. In other words, the image of $\langle B_{v}^{u} \rangle$ is at most n_{G}-dimensional. However, we already saw in (3.16) that $\langle B_{v}^{u} \rangle$ is non-vanishing on Killing vectors $\langle k_{J}^{v} \rangle$,

$$\langle B_{v}^{u} k_{I}^{v} \rangle = 2 \langle k^{uJ} f_{IJ}^{K} \mu_{K} \rangle \neq 0 \ , \tag{3.20}$$

and thus the Goldstone bosons have a non-trivial image under $\langle B_{v}^{u} \rangle$. In particular, this proves $\mathrm{rk} \, \langle B_{v}^{u} \rangle = n_{G}$ because the rank of $\langle B_{v}^{u} \rangle$ is bounded by (3.19). The rank-nullity Theorem from linear algebra,

$$\dim \, \mathcal{D} = \mathrm{rk} \, \langle B_{v}^{u} \rangle + \dim \, \ker \langle B_{v}^{u} \rangle \ , \tag{3.21}$$

then implies that all physical moduli must lie in the kernel of $\langle B_{v}^{u} \rangle$, i.e. $\langle B_{v}^{u} \rangle \delta q^{v} = 0$ for all $\delta q^{u} \in \mathcal{M}/\mathcal{G}$. In conclusion, $\langle B_{v}^{u} \rangle|_{\mathcal{M}} = 0$ and (3.9) restricted to $\mathcal{M}$ reads

$$\langle h^{I} \nabla_{v} k_{I}^{u} \rangle|_{\mathcal{M}} = 0 \ . \tag{3.22}$$

[4] Due to our choice $v^{n} = \delta^{3n}$ in (2.82), we could just use the restriction of J^{3} as an almost complex structure on $\mathcal{M}$. However, in a setting for general v^{n}, the definition for the almost complex structure on $\mathcal{M}$ is $\mathcal{J} := \frac{1}{\lambda} \mu^{n} J_{n}$ and thus we use $\frac{1}{\lambda} \mu^{3} J_{3}$ here.

Since only $\langle \mu_I^3 \rangle \neq 0$ in the vacuum, the covariant derivative of the killing vectors k_I commutes with J^3 in the background due to (2.55), i.e.

$$\langle \nabla_u k_w^I (J^n)_v^w - (J^n)_u^w \nabla_v k_w^I \rangle = 2\epsilon^{npq} \langle \omega_{uv}^p \mu^{Iq} \rangle = 0 \ . \tag{3.23}$$

Thus the equation (3.9) is J^3-invariant on $\mathcal{M}$. For the two equations (3.6) the J^3-invariance follows from the fact that J^3 interchanges both equations. This can be seen by substituting $\delta q'^u = (J^3)_v^u \delta q^v$ and using that $J^1 J^2 = J^3$ on a quaternionic Kähler manifold. In conclusion, we have shown that the equations (3.6) and (3.9) defining the moduli space $\mathcal{M}$ are invariant under the action of $\mathcal{J}$. Thus we have shown that $\mathcal{J}$ defines an almost complex structure on the supersymmetric moduli space $\mathcal{M}$.

In what follows we want to use Theorem 1.12 of [71]:

Theorem 1 *An almost Hermitian submanifold $(\mathcal{M}, \check{G}, \mathcal{J})$ of a quaternionic Kähler manifold (M, G, Q) is Kähler if and only if it is totally complex, i.e. if there exists a section $\mathcal{I}$ of Q that anticommutes with $\mathcal{J}$ and satisfies*

$$\mathcal{I}(T_p\mathcal{M}) \perp T_p\mathcal{M} \quad \forall p \in \mathcal{M} \ . \tag{3.24}$$

In particular, the condition (3.24) can be written for all tangent vectors $V, W \in T_p\mathcal{M}$ as

$$\omega_{\mathcal{I}}(X, Y) = \check{G}(X, \mathcal{I}Y) = 0 \ , \tag{3.25}$$

where $\omega_{\mathcal{I}} = \check{G} \circ \mathcal{I}$ is the fundamental two-form associated to $\mathcal{I}$. Thus (3.24) is satisfied if and only if $\omega_{\mathcal{I}}|_{\mathcal{M}}$ vanishes.

Now let us prove that the supersymmetric moduli space $\mathcal{M}$ of AdS vacua actually is totally complex and hence Kähler. To do this, we want to apply Theorem 1 to the case of $\mathcal{I} = J^1$. Using (2.53) and (2.54), we find that in the AdS vacuum (2.93) $\langle \omega_{uv}^3 \rangle$ is given by

$$\langle \omega_{uv}^3 \rangle = \tfrac{2}{\lambda} \langle h^I \nabla_u k_{Iv} - L_{uv} \rangle \ . \tag{3.26}$$

Since $\langle \omega_{uv}^1 \rangle = -\langle \omega_{uw}^3 (J^2)_v^w \rangle$, we can multiply (3.26) with $-(J^2)_v^w$ from the right and obtain

$$\langle \omega_{uv}^1 \rangle = \tfrac{2}{\lambda} \langle S_{uv}^2 - h^I \nabla_u k_{wI} (J^2)_v^w \rangle \ . \tag{3.27}$$

Here we used the definitions (2.54). The second term in this expression for $\langle \omega_{uv}^1 \rangle$ vanishes on $\mathcal{M}$ due to (3.22),

$$\langle \omega_{uv}^1 \rangle|_{\mathcal{M}} = \tfrac{2}{\lambda} \langle S_{uv}^2 \rangle|_{\mathcal{M}} \ . \tag{3.28}$$

However, $S_{uv}^2 = L_{uw}(J^2)_v^w$ is symmetric in u, v while ω_{uv}^1 is antisymmetric. Thus we must have $\langle \omega_{uv}^1 \rangle|_{\mathcal{M}} = \langle S_{uv}^2 \rangle|_{\mathcal{M}} = 0$ on the moduli space. By (3.25) this implies that $(\mathcal{M}, \check{G}, \mathcal{J})$ is totally complex for $\check{G} := G|_{\mathcal{M}}$. Thus we can apply Theorem 1 and find that $\mathcal{J}$ is a

Kähler structure[5] on $\mathcal{M}$ and $(\mathcal{M}, \check{G}, \mathcal{J})$ is a Kähler subspace of the hypermultiplet target space M_H.[6]

It was also proved in [71] that a Kähler submanifold can have at most half the dimension of the ambient quaternionic Kähler manifold, i.e. $\dim(\mathcal{M}) \leq 2n_H$.[7] Note that in the case of an unbroken gauge group we have $\mathcal{G} = \{\emptyset\}$ and thus $\mathcal{D} = \mathcal{M}$. In this case the moduli space can have the maximal possible dimension in the following sense: The dimension of $\mathcal{M}$ is restricted by the dimension of the ambient quaternionic Kähler manifold to be smaller or equal to $2n_H$. A spontaneous symmetry breaking then gives rise to Goldstone bosons which additionally have to be removed from the moduli space, i.e. lowering the dimension of $\mathcal{M}$ even further. Thus for a given dimension $4n_H$ of the quaternionic Kähler target space, the dimension of the moduli space can be at most $2n_H$ which can only occur in the case of an unbroken gauge group. Thus the dimension of the moduli space is least restricted for an unbroken gauge group and in this sense maximal. If the gauge group is spontaneously broken then additional scalars are fixed by (3.6). Since $\mathcal{M}$ is J^3-invariant, every $\delta q^u \in \mathcal{M}$ can be written as $\delta q^u = (J^3)^u_v \delta q'^v$ for some $\delta q'^u \in \mathcal{M}$. Combined with the fact that $J^1 J^2 = J^3$ this implies that the two conditions in (3.6) are equivalent on $\mathcal{M}$. Furthermore, we have $\mathrm{rk}\,\langle h^I_i \omega^1_{uv} k^v_I \rangle = \mathrm{rk}\,\langle k^I_u \rangle = n_G$ and thus n_G scalars are fixed by (3.6). In conclusion, we have shown that

$$\dim(\mathcal{M}) = \dim(\mathcal{D}) - \dim(\mathcal{G}) \leq (2n_H - n_G) - n_G \ , \tag{3.30}$$

so the moduli space has at most real dimension $2n_H - 2n_G$.

Sufficient conditions for the moduli space

In this chapter we computed the variations of the AdS vacuum conditions (2.79) to first order in the scalar fields. However, this only gives the necessary conditions on the supersymmetric moduli space of AdS_5 vacua. For instance, consider schematically the second variation of a vacuum condition $\langle F(\Phi) \rangle = 0$,

$$\delta^2 F = \frac{\partial^2 F}{\partial \Phi \partial \Phi'} \delta\Phi \delta\Phi' \ . \tag{3.31}$$

Then the condition $\langle \delta^2 F \rangle = 0$ can give rise to equations that are not automatically satisfied by solutions to the first order equations and thus further restrict the moduli

[5] A different way to see that $\mathcal{J}$ defines a Kähler structure on $\mathcal{M}$ is to compute the variation of $\mathcal{J}$ under a shift in the scalar fields. Considering for a moment the more general case $\mathcal{J} = \frac{1}{\lambda}\mu^n J_n$, this gives

$$\delta \mathcal{J}^v_u = \tfrac{1}{\lambda} \langle \delta(h^I \mu^n_I)(J_n)^v_u \rangle = 0 \tag{3.29}$$

where we used (3.4) in the second step. Inserting the vacuum conditions (2.93) into $\mathcal{J} = \frac{1}{\lambda}\mu^n J_n$ then leads us back to our choice $\mathcal{J} = \frac{1}{\lambda}\mu^3 J_3$ for the Kähler structure on $\mathcal{M}$.

[6] As we have constructed the moduli space out of deformations around the AdS vacuum (2.93), we actually prove that a subspace of the tangent space of M_H at the vacuum point is a Kähler subspace up to first order in the scalar fields. Since we can identify the a manifold locally with its tangent space, this holds for a neighborhood of the vacuum point.

[7] Applying the same method as in $d = 4$, $\mathcal{N} = 2$ this can be checked explicitly [28].

space of the vacuum $\langle F \rangle = 0$. Let us briefly discuss this for the case of five-dimensional AdS vacua in gauged $\mathcal{N} = 2$ supergravity.

The vacuum conditions were given in terms of moment maps and Killing vectors in (2.79). If we consider for the moment the case of supergravity coupled to only hypermultiplets, i.e. $n_V = n_T = 0$, then the special real manifold in (2.9) is trivial. The gauge group is $G = U(1)$ and unbroken in the vacuum, as we have shown in Chapter 2 that an AdS vacuum always has an unbroken $U(1)_R$-symmetry. Moreover, only one Killing vector $K = h^0 k_0$, one moment map $\mu^n = h^0 \mu_0^n$ and one $h = h^0$ exist on T. Due to (2.19), we have $h = 1$ and in particular all $\partial_i h = 0$ as there are no scalars ϕ in the theory. The AdS vacuum conditions (2.93) reduce to

$$\langle \mu^n \rangle = \lambda v^n \ , \quad \langle K^u \rangle = 0 \ . \tag{3.32}$$

Furthermore, the condition (3.6) on the moduli space is trivial while (3.9) reduces to

$$\langle \delta K^u \rangle = \langle \nabla_v K^u \rangle \delta q^v = 0 \ , \tag{3.33}$$

which is simply the condition that K has to be covariantly constant on the moduli space. The second variation is then just the second covariant derivative, i.e.

$$\langle \delta^2 K^u \rangle = \langle \nabla_w \nabla_v K^u \rangle \delta q^v \delta q'^w \ . \tag{3.34}$$

Since K is Killing, the second covariant derivative can be expressed in terms of the Riemann tensor R to the metric G and the Killing vector K itself [61],

$$\nabla_w \nabla_v K^u = R^u_{vwr} K^r \ . \tag{3.35}$$

Hence, the second variation is

$$\langle \delta^2 K^u \rangle = \langle \nabla_w \nabla_v K^u \rangle \delta q^v \delta q'^w = \langle R^u_{vwr} K^r \rangle \delta q^v \delta q'^w \equiv 0 \ , \tag{3.36}$$

where we used the vacuum condition (3.32). Similarly, we find

$$\langle \delta^2 \mu^n \rangle = -\tfrac{1}{2} \langle \omega^n_{uv} \nabla_w K^v \rangle \delta q^u \delta q'^w \equiv 0 \ , \tag{3.37}$$

where we used (3.33). Thus the second order variations impose no further restrictions on the scalar field variations. Furthermore, due to (3.35) every consecutive variation will only contain terms proportional to K^u or $\nabla_v K^u$ and thus vanish identically. In conclusion, we have shown that the condition $\langle \delta K^u \rangle = 0$ is necessary and sufficient for the supersymmetric moduli space of the AdS vacuum given by (3.32). Thus our computation of the moduli spaces extends to a globally well-defined Kähler submanifold of the scalar target space.

This result is already known in the mathematics literature and was proven in [72]:

Theorem 2 *Let (M_H, G, Q) be a quaternionic Kähler manifold endowed with an isometric $U(1)$-action and corresponding moment map $\mu \in \Gamma(Q)$, i.e. $\mu = \mu^n J_n$ for J_n a local basis of Q. Then every connected component of the fixed point set not contained in $\mu^{-1}(0)$ is a Kähler submanifold of $M_H \backslash \mu^{-1}(0)$.*

The Kähler structure provided by this theorem is $\mu/||\mu||$ for $||\mu|| = \sqrt{\mu^n \mu_n}$. Theorem 2 can then be related to the AdS vacua (3.32) for supergravity coupled to n_H-hypermultiplets as follows: The notation $\langle\ \rangle$ for the vacuum corresponds to evaluating the respective quantities at a vacuum point p in the scalar field space T. Thus we can rewrite the AdS background (3.32) as

$$\mu^n_p = \lambda v^n \ , \quad K_p = 0 \ . \tag{3.38}$$

Moreover, we can associate a section μ to the functions μ^n via $\mu = \mu^n J_n$, see (2.49). This section evaluated at the point p is then $\mu_p = \lambda v^n J_{n,p}$ and we define the moduli space of the AdS vacuum p as[8]

$$\mathcal{M} := \{p \in M_H \mid \mu_p = \lambda v^n J_{n,p} \ , K_p = 0\} \ . \tag{3.39}$$

We see immediately that $\mathcal{M} \subset M_H \backslash \mu^{-1}(0)$. The second equation in (3.38) then shows that the Killing vector K has a zero at p. Therefore, p is fixed under the $U(1)$-action generated by K. If we denote the set of fixed points in M_H by M^K, then the moduli space is contained in $M_H \backslash \mu^{-1}(0) \cap M^K$ and we can apply Theorem 2 to prove that $\mathcal{M}$ is a Kähler manifold.[9]

Now let us return to the general situation of AdS vacua (2.93) coupled to a number of vector and tensor multiplets. For this case, we cannot simply generalize the computations presented above for the case of supergravity coupled only to hypermultiplets. For example, we also have to take into account the variation of the vector fields $\partial_i \mu^n$ that could impose further restrictions on the deformations of the scalar fields. On the other hand, one would expect from computations in the dual gauge theory that the moduli space is a globally well-defined Kähler manifold. It would be interesting to consider this problem in more detail in the future.

We conclude this section with a short summary of our results. In a first step, we computed the space of variations in the scalar fields that leave the AdS vacuum conditions (2.79) invariant. We then explained that the moduli space is given by these variations after removing the unphysical variations corresponding to the Goldstone bosons in the case of a broken gauge group. In particular, we proved that the moduli space admits a Kähler structure that can naturally be obtained from the quaternionic Kähler structure

[8] For the special case $v^n = \delta^{3n}$ we considered throughout this thesis, the value of the section μ in the vacuum (3.32) is $\mu_p = \lambda J_{3,p}$.

[9] Note that the proof of Theorem 2 relies on the fact that the vanishing of the first covariant derivative ∇K implies that all other derivatives vanish as well.

of the hypermultiplet target space. Finally, we discussed how the variations of the vacuum conditions to first order are sufficient to determine the moduli space to all orders in the case of supergravity coupled to only hypermultiplets.

Chapter 4

AdS vacua from type IIB supergravity on $T^{1,1}$

Much of our motivation to study supersymmetric AdS$_5$ backgrounds in the previous sections comes from the AdS/CFT correspondence [6]. However, the AdS/CFT correspondence relates the four-dimensional gauge theory on the boundary of AdS$_5$ only to the full ten-dimensional solution of type IIB supergravity instead of just its AdS factor. Since we have shown that the moduli space of AdS$_5$ vacua is Kähler and thus has the same property as the conformal manifold of $\mathcal{N} = 1$ SCFTs in four dimensions [19], it would be interesting to study whether there are additional moduli in the ten-dimensional solution. In particular, one wonders whether the compact manifold has an impact on the moduli space that cannot be detected in the five-dimensional setting.

The moduli spaces of ten-dimensional solutions of type IIB supergravity containing an AdS factor have been computed in the framework of exceptional generalized geometry in [23]. Moreover, explicit examples were provided and the moduli of certain Sasaki-Einstein compactifications have been identified [25]. Now the idea of this chapter is to consider consistent truncations of type IIB supergravity on Sasaki-Einstein manifolds that provide a five-dimensional $\mathcal{N} = 2$ truncation. Then we can apply the results of the previous section to find AdS vacua and their moduli spaces in these truncations and compare them with the known results in ten dimensions. In this thesis, we will focus on the type IIB background of the form

$$\mathrm{AdS}_5 \times T^{1,1} \,, \tag{4.1}$$

where $T^{1,1} = \frac{SU(2)\times SU(2)}{U(1)}$ is a Sasaki-Einstein manifold (see Appendix A). In particular, the moduli space of this background was computed in [10, 25, 46, 47] and found to be complex five-dimensional.

Before we proceed, let us briefly discuss the consistent truncations of type IIB supergravity on Sasaki-Einstein manifolds in general and $T^{1,1}$ in particular.[1] In contrast

[1] We provide a more detailed introduction in Appendix C.

to the case of compactifications on Calabi-Yau manifolds, Sasaki-Einstein manifolds usually have non-trivial isometry groups and thus consistent truncations on them are more difficult to find. References [36–43] constructed consistent truncations on five-dimensional Sasaki-Einstein manifolds SE_5 by expanding the type IIB fields in terms of the forms defining the Sasaki-Einstein structure, i.e. the contact form η, its derivative $d\eta$ and a three-form Ω related to the holomorphic $(4,0)$-form on the Calabi-Yau cone over SE_5. The resulting five-dimensional theory was then shown to be an $\mathcal{N}=4$ supergravity coupled to two vector multiplets. However, these truncations do not take non-trivial cohomology classes of the Sasaki-Einstein manifold into account. Hence, references [43, 48, 49] generalized the known Sasaki-Einstein truncations in the case of type IIB supergravity on $T^{1,1}$ to include the non-trivial second and third cohomology groups. Since $T^{1,1} \cong S^2 \times S^3$, there exists a closed non-trivial two form $\mathcal{Y} \in H^2(T^{1,1}, \mathbb{R})$ in the second de-Rham cohomology. Adding this form to the truncation ansatz, i.e. expanding the type IIB fields in the set $(\eta, d\eta, \Omega, \mathcal{Y})$ then gives rise to a consistent truncation of type IIB supergravity on $T^{1,1}$. The resulting five-dimensional theory can be shown to be $\mathcal{N}=4$ supergravity coupled to three vector multiplets. Since the additional $\mathcal{N}=4$ vector multiplet appears in the spectrum do to the fact that the second Betti number $b_2(T^{1,1}) = 1$, it is referred to as the $\mathcal{N}=4$ Betti-vector multiplet.

Because we are interested in studying AdS vacua preserving $\mathcal{N}=2$ supersymmetry, we want to consider the consistent $\mathcal{N}=2$ subtruncations that were studied in [51]. To begin with, let us discuss the supersymmetry breaking from $\mathcal{N}=4$ to $\mathcal{N}=2$ in five dimensions. The $\mathcal{N}=4$ gravity multiplet decomposes into an $\mathcal{N}=2$ gravity multiplet, an $\mathcal{N}=2$ gravitino multiplet and an $\mathcal{N}=2$ vector multiplet. Moreover, each $\mathcal{N}=4$ vector multiplet decomposes into an $\mathcal{N}=2$ vector multiplet and a hypermultiplet, giving in total four vector multiplets and three hypermultiplets for the truncations on $T^{1,1}$ discussed in Appendix C. In particular, the $\mathcal{N}=4$ Betti-vector multiplet decomposes into the $\mathcal{N}=2$ Betti-vector multiplet and the Betti-hyper multiplet. The truncation to $\mathcal{N}=2$ supergravity is then done by removing the massive $\mathcal{N}=2$ gravitino multiplet[2] from the spectrum [48]. However, by examining the resulting equations of motion one finds that this alone would not lead to a consistent truncation. In fact, one can show [48] that for the truncation to be consistent, one additionally has to either truncate the $\mathcal{N}=2$ Betti-vector multiplet or the Betti-hypermultiplet. Keeping the $\mathcal{N}=2$ Betti-vector multiplet in this setting leads to the Betti-vector truncation with field content given by

$$\begin{aligned} \text{gravity multiplet}: &\quad \{g_{\mu\nu}, A^0_\mu\} \\ \text{2 vector multiplets}: &\quad \{A^1_\mu, A^2_\mu, u_2, u_3\} \\ \text{2 hypermultiplets}: &\quad \{u_1, \sigma, \tau, \bar{\tau}, b^a, \bar{b}^a\}\ . \end{aligned} \tag{4.2}$$

[2] A massive semi-long gravitino multiplet in five dimensions consists of two gravitini, two vector fields, two complex tensor fields and four spin-1/2 fermions [73]. For more details on the truncation from $\mathcal{N}=4$ to $\mathcal{N}=2$ supersymmetry in Sasaki-Einstein truncations, see [37, 48].

On the other hand, keeping the Betti-hypermultiplet leads to the Betti-hyper truncation with field content

$$\begin{aligned} \text{gravity multiplet}: &\quad \{g_{\mu\nu}, A^0_\mu\} \\ \text{1 vector multiplet}: &\quad \{A^1_\mu + A^2_\mu, u_3\} \\ \text{2 hypermultiplets}: &\quad \{u_1, \sigma, \tau, \bar{\tau}, b^a, \bar{b}^a, e^a, v, \bar{v}\}\ . \end{aligned} \tag{4.3}$$

This truncation can also be obtained from the $\mathcal{N}=4$ truncation on $T^{1,1}$ by keeping only those fields that are invariant under the symmetry $\mathcal{I} = \mathcal{I}_1 \cdot \mathcal{I}_2$, where [49, 51]

$$\begin{aligned} \mathcal{I}_1: &\quad (\hat{g}, \hat{\phi}, \hat{B}_2, \hat{C}_0, \hat{C}_2, \hat{C}_4) \to (\hat{g}, \hat{\phi}, -\hat{B}_2, \hat{C}_0, -\hat{C}_2, \hat{C}_4)\ , \\ \mathcal{I}_2: &\quad (\theta, \varphi, \omega, \nu) \to (\omega, \nu, \theta, \varphi)\ , \end{aligned} \tag{4.4}$$

for θ, φ, ω, and ν coordinates on $T^{1,1}$ as introduced in Appendix A and $\mathcal{I}_2$ acts on the type IIB fields. The $\mathcal{I}$-symmetry can be related [49] to a $\mathbb{Z}_2$-symmetry that appears in the Klebanov-Strassler gauge theories [74].

Additionally, reference [51] studies a truncation that arises by truncating type IIB supergravity to the NS-sector before following the consistent truncation procedure outlined in Appendix C. The resulting truncation is called NS-sector truncation and contains two vector multiplets and two hypermultiplets with field content

$$\begin{aligned} \text{gravity multiplet}: &\quad \{g_{\mu\nu}, A^0_\mu\} \\ \text{2 vector multiplets}: &\quad \{b^2_1, b^2_2, \phi + 4u_1, u_3\} \\ \text{2 hypermultiplets}: &\quad \{\phi - 4u_1, u_2, c^2, e^2, b^2, \bar{b}^2, v, \bar{v}\}\ . \end{aligned} \tag{4.5}$$

Note that even though the number of multiplets is the same, this truncation is different from the Betti-vector truncation. However, the NS-sector truncation does not admit supersymmetric AdS vacua. This is intuitively clear from the fact that it does not contain the Axion-Dilaton, which is always a modulus of such backgrounds [25]. Nonetheless, we will explicitly prove this in Appendix D by using the results from Chapter 2.

These $\mathcal{N}=2$ truncations of type IIB supergravity on $T^{1,1}$ were extensively studied in [51] and the data on isometries of the scalar target spaces we used in Chapter 2 to determine the conditions on AdS vacua in a given supergravity were explicitly computed. In the rest of this chapter, we will use this to compute the conditions on AdS vacua and their moduli spaces in the Betti-vector truncation and the Betti-hyper truncation.

4.1 The Betti-vector truncation

We first discuss the Betti-vector truncation, i.e. the case where the Betti-hypermultiplet is truncated out of the spectrum. Up to minor changes, this section is taken from [52].

Truncation data

The Betti-vector truncation leads to an $\mathcal{N}=2$ theory that contains gravity coupled to two vector multiplets and two hypermultiplets [48]. In total, 10 scalars

$$\{u_1, u_2, u_3, \sigma, \tau, \bar{\tau}, b^a, \bar{b}^a\} \ , \tag{4.6}$$

from the $\mathcal{N}=4$ truncation (C.14) are kept. Here $u_{1,2,3}$ and σ are real while all others are complex. The vector in the gravity multiplet is A^0_μ while the other vectors are given by $A^{1,2}_\mu$ with associated one-forms A_I, $I=0,1,2$. The scalars $\{u_2, u_3\}$ of the vector multiplets parametrize the projective special real manifold [51]

$$M_V^{BV} = SO(1,1) \times SO(1,1) \ . \tag{4.7}$$

The coordinates in (2.10) of the ambient space H are given by [51]

$$h^0 = e^{4u_3} \quad h^1 = e^{2u_2-2u_3}, \quad h^2 = e^{-2u_2-2u_3} \ , \tag{4.8}$$

with $C_{012} = \frac{1}{6}$ and all others zero.[3] Lowering the index I according to (2.11) one finds

$$h_0 = \tfrac{1}{3}e^{-4u_3} \ , \quad h_1 = \tfrac{1}{3}e^{-2u_2+2u_3} \ , \quad h_2 = \tfrac{1}{3}e^{2u_2+2u_3} \ . \tag{4.9}$$

The hypermultiplet scalars $\{u_1, \sigma, \tau, \bar{\tau}, b^a, \bar{b}^a\}$ for $a=1,2$ can be shown to span the quaternionic Kähler manifold [48]

$$M_H^{BV} = \frac{SO(4,2)}{SO(4)\times SO(2)} \ . \tag{4.10}$$

In particular, $\tau = C_0 + ie^{-\phi}$ is the reduction of the Axion-Dilaton of type IIB supergravity. The metric on M_H^{BV} can be read off from the kinetic terms of the hypermultiplets in the Lagrangian [51],

$$\begin{aligned}\mathcal{L}^{BV}_{Hyper} = &- 4e^{-4u_1+\phi} M_{ab} Db^a \wedge *D\bar{b}^b - 8du_1 \wedge *du_1 \\ &- \tfrac{1}{2}e^{-8u_1}\Sigma \wedge *\Sigma - \tfrac{1}{2}d\phi \wedge *d\phi - \tfrac{1}{2}e^{2\phi}dC_0 \wedge *dC_0 \ ,\end{aligned} \tag{4.11}$$

where

$$\begin{aligned} Db^a &= db^a - 3ib^a A_0 \ , \\ \Sigma &= D\sigma + 2\varepsilon_{ab}[b^a D\bar{b}^b + \bar{b}^a Db^b] \ , \\ D\sigma &= d\sigma - qA_0 - 2A_1 - 2A_2 \ , \end{aligned} \tag{4.12}$$

for a constant $q \in \mathbb{R}^+$ and the torus metric

$$M_{ab} = e^{\phi}\begin{pmatrix} a^2 + e^{-2\phi} & -a \\ -a & 1 \end{pmatrix} = \frac{1}{\operatorname{Im}\tau}\begin{pmatrix} |\tau|^2 & -\operatorname{Re}\tau \\ -\operatorname{Re}\tau & 1\end{pmatrix} \ . \tag{4.13}$$

[3] Note that we use conventions for the special real geometry which are different from [51], see [58] for details.

The gauge group G^{BV} is realized on the hypermultiplet scalar manifold via the Killing vectors [51]

$$\begin{aligned} k_0 &= -3ib^a\partial_a + 3i\bar{b}^a\bar{\partial}_a - q\partial_\sigma \ , \\ k_1 &= 2\partial_\sigma \ , \\ k_2 &= 2\partial_\sigma \ , \end{aligned} \tag{4.14}$$

where we defined $\partial_a := \partial_{b^a}$ and $\bar{\partial}_a := \partial_{\bar{b}^a}$. The associated moment maps [51] are[4]

$$\begin{aligned} \vec{\mu}_0 &= 6e^{-2u_1} f_a b^a \vec{e}_1 - 6e^{-2u_1} \bar{f}_a \bar{b}^a \vec{e}_2 + (\tfrac{1}{2}e^{-4u_1}e^Z - 3)\vec{e}_3 \ , \\ \vec{\mu}_1 &= -e^{-4u_1}\vec{e}_3 \ , \\ \vec{\mu}_2 &= -e^{-4u_1}\vec{e}_3 \ , \end{aligned} \tag{4.15}$$

where $e^Z = q - 6i\varepsilon_{ab}(b^a\bar{b}^b - \bar{b}^a b^b)$ and f_a is defined via $f_a b^a = \frac{1}{\sqrt{\operatorname{Im}\tau}}(b^2 - \tau b^1)$. From this we can immediately compute the Lie algebra $\mathfrak{g}^{BV}$ spanned by the k_I. Since $k_1 = k_2$, we only have to compute the respective Lie brackets with k_0. We find

$$[k_0, k_1] = [k_0, k_2] = 0 \ , \tag{4.16}$$

since the vectors $\{\partial_a, \bar{\partial}_a, \partial_\sigma\}$ are linearly independent and q is a constant. Thus by observing that the Killing vectors (4.14) have compact and non-compact parts, the gauge group is $G^{BV} = U(1) \times U(1) \times \mathbb{R}$. However, we can see from (4.14) that the scalars are only charged under an $U(1) \times \mathbb{R}$ subgroup of G^{BV} with associated gauge fields A_0 and $qA_0 - 2A_1 - 2A_2$.

AdS vacua and their moduli space

To find the AdS vacua of the five-dimensional theory coming from the Betti-vector truncation we have to solve the equations (2.93).[5] For the third equation we use (4.9), (4.14) and find

$$\langle h^I k_I \rangle = -(3i\langle b^a\rangle\partial_a - 3i\langle\bar{b}^a\rangle\bar{\partial}_a)e^{4\langle u_3\rangle} + 2e^{2\langle u_3\rangle}(e^{2\langle u_2\rangle} + e^{-2\langle u_2\rangle} - \tfrac{q}{2}e^{6\langle u_3\rangle})\partial_\sigma = 0 \ . \tag{4.17}$$

Due to the linear independence of the coordinate vector fields $\{\partial_a, \bar{\partial}_a, \partial_\sigma\}$, this implies $\langle b^a\rangle = \langle\bar{b}^a\rangle = 0$ and

$$e^{2\langle u_2\rangle} + e^{-2\langle u_2\rangle} = \tfrac{q}{2}e^{6\langle u_3\rangle} \ . \tag{4.18}$$

In particular, the background values of the vector multiplet scalars are not independent of each other. Inserting these results into the moment maps (4.15), we find that only the third component $\langle\mu_I\rangle := \langle\mu_I^3\rangle$ is nonzero in the background. These components are given by

$$\langle\mu_0\rangle = \tfrac{q}{2}e^{-4\langle u_1\rangle} - 3 \ , \quad \langle\mu_1\rangle = \langle\mu_2\rangle = -e^{-4\langle u_1\rangle} \ , \tag{4.19}$$

[4] We converted the moment maps according to $(\mu_I)_{\mathcal{B}}^{\mathcal{A}} = i\vec{\mu}_I(\vec{\sigma}_{\mathcal{B}}^{\mathcal{A}})$ where $(\vec{\sigma})_{\mathcal{B}}^{\mathcal{A}}$ are the Pauli matrices, see [58].

[5] Since no tensor multiplets are present in the Betti-vector truncation, the second equation in (2.93) is trivially satisfied.

where we used the fact that $\langle e^Z \rangle = q$ in the AdS background.

We are thus left with solving the first equation in (2.93). Since $\langle \mu_1 \rangle = \langle \mu_2 \rangle$, (2.93) implies that $\langle h_1 \rangle = \langle h_2 \rangle$ and thus using (4.31) we find

$$e^{-2\langle u_2 \rangle + 2\langle u_3 \rangle} = e^{2\langle u_2 \rangle + 2\langle u_3 \rangle} \ . \tag{4.20}$$

This fixes the background value of the vector multiplet scalar u_2 to $\langle u_2 \rangle = 0$. Also, by using (4.18) we find $\langle u_3 \rangle = \frac{1}{6} \log \frac{4}{q}$. Note that this fixes the scalar $\langle u_3 \rangle$ in the AdS vacuum, since q is a constant. Now consider the zero-component $\langle \mu_0 \rangle$. Using again (4.9) and (4.15) and inserting this into the first equation of (2.93), we find

$$\tfrac{q}{2} e^{-\langle u_1 \rangle} - 3 = \tfrac{\lambda}{3} e^{-4\langle u_3 \rangle} = \tfrac{q^{2/3}\lambda}{6\sqrt[3]{2}} \ , \tag{4.21}$$

fixing the background value of the scalar $\langle u_1 \rangle = -\log(\frac{\lambda}{3\sqrt[3]{2}q^{1/3}} + \frac{6}{q})$. To summarize, the conditions for AdS_5 vacua from the Betti-vector truncation fix all the scalars $\{u_2, u_3\}$ from the vector multiplets and moreover the scalars $\{u_1, b^a, \bar{b}^a\}$ from the hypermultiplets.

Let us now turn to the moduli space $\mathcal{M}^{BV}$ described in Chapter 3. From the above analysis we know that the hypermultiplet scalars $\{\sigma, \tau, \bar{\tau}\}$ are not constrained by the conditions (2.93) for the AdS background. However, the moduli space was proven to be a Kähler manifold and thus has to be even-dimensional. This can be understood by considering the background values of the Killing vectors (4.14),

$$\langle k_0 \rangle = -q\partial_\sigma \ , \quad \langle k_1 \rangle = \langle k_2 \rangle = 2\partial_\sigma \ . \tag{4.22}$$

From this we see that the space of Goldstone bosons is one-dimensional and spanned by ∂_σ. The respective scalar σ then gets eaten by the gauge field $qA_0 - 2A_1 - 2A_2$, which becomes massive as a result of the symmetry breaking. In particular, we explained in Chapter 2 that the $U(1)_R$-symmetry always remains unbroken in the AdS vacuum and is gauged by the graviphoton

$$\lambda \langle h_I \rangle A^I_\mu = \tfrac{\lambda q^{2/3}}{6\sqrt[3]{2}} A^0_\mu + \tfrac{2^{2/3}\lambda}{3\sqrt[3]{q}} (A^1_\mu + A^2_\mu) \ . \tag{4.23}$$

Thus we find that the gauge group in the supersymmetric AdS background is broken according to $U(1) \times U(1) \times \mathbb{R} \longrightarrow U(1)_R \times U(1)$.

We have shown that the moduli space of the AdS vacuum is two-dimensional and spanned by the Axion-Dilaton $\{\tau, \bar{\tau}\}$. This agrees with the bound (3.30) on the dimension of the moduli space for $n_H = 2$, $n_G = 1$,

$$\dim \mathcal{M}^{BV} \leq 2 \cdot 2 - 2 \cdot 1 = 2 \ , \tag{4.24}$$

i.e. the moduli space is of maximal dimension. To compute the metric g^{BV} on the moduli space, note that the coordinate one-forms of the fixed scalars vanish on $\mathcal{M}^{BV}$, i.e.

$$db^a|_{\mathcal{M}^{BV}} = d\bar{b}^a|_{\mathcal{M}^{BV}} = du_1|_{\mathcal{M}^{BV}} = d\sigma|_{\mathcal{M}^{BV}} = 0 \ . \tag{4.25}$$

Using this fact we can read off the metric from the Lagrangian (4.11),

$$g^{BV} = d\phi^2 + e^{2\phi} dC_0^2 = \tfrac{1}{\mathrm{Im}\,\tau^2} d\tau d\bar{\tau} \ . \tag{4.26}$$

This is precisely the metric on the upper half plane $\mathcal{H}$ which is well-known to be a Kähler manifold. Thus the moduli space of the AdS_5 vacua in the Betti-vector truncation is

$$\mathcal{M}^{BV} = \mathcal{H} \ . \tag{4.27}$$

4.2 The Betti-hyper truncation

Let us now turn to the Betti-hyper truncation from which one obtains a five-dimensional $\mathcal{N} = 2$ theory that contains gravity coupled to one vector multiplet and three hypermultiplets [48]. Up to minor changes, this section is taken from [52].

Truncation data

In the Betti-hyper truncation, 13 scalars

$$\{u_1, u_3, \sigma, e^a, \tau, \bar{\tau}, v, \bar{v}, b^a, \bar{b}^a\} \tag{4.28}$$

are kept from the $\mathcal{N} = 4$ truncation (C.14). While the gauge field in the $\mathcal{N} = 2$ gravity multiplet is still A^0_μ, the gauge field in the vector multiplet is given as the combination $\frac{1}{2}(A^1_\mu + A^2_\mu)$. Moreover, the single scalar u_3 in the vector multiplet parametrizes the projective special real manifold [51]

$$M_V^{BH} = SO(1,1) \ . \tag{4.29}$$

The local coordinates of the ambient space H are given by [51]

$$h^0 = e^{4u_3} \ , \quad h^1 = e^{-2u_3} \ , \tag{4.30}$$

with $C_{011} = \frac{1}{3}$ and all others zero. Moreover we obtain

$$h_0 = \tfrac{1}{3} e^{-4u_3} \ , \quad h_1 = \tfrac{2}{3} e^{2u_3} \ , \tag{4.31}$$

by lowering the index according to (2.11).

In the hypermultiplets, the scalars $\{u_1, \sigma, e^a, b^a, \bar{b}^a, \tau, \bar{\tau}, v, \bar{v}\}$ span the quaternionic Kähler manifold [48]

$$M_H^{BH} = \frac{SO(4,3)}{SO(4) \times SO(3)} \ . \tag{4.32}$$

Here b^a and v are complex while all others are real. The hypermultiplet Lagrangian is given by [51]

$$\begin{aligned} \mathcal{L}^{BH}_{Hyper} = & - e^{-4u_1} M_{ab} [\tfrac{1}{2} De^a \wedge *De^b + \tfrac{1}{2} E^a \wedge *E^b + 2(B^a \wedge *\bar{B}^b + \bar{B}^a \wedge *B^b)] \\ & - 8 du_1 \wedge *du_1 - d|v| \wedge *d|v| + |v|^2 D\vartheta \wedge *D\vartheta \\ & - \tfrac{1}{2} e^{-8u_1} \Sigma \wedge *\Sigma - \tfrac{1}{2} d\phi \wedge *d\phi - \tfrac{1}{2} e^{2\phi} dC_0 \wedge *dC_0 \ , \end{aligned} \tag{4.33}$$

where

$$
\begin{aligned}
E^a &= (1+|v|^2)De^a - 4\mathrm{Im}\,(vDb^a)\ , \\
B^a &= Db^a - \tfrac{i}{2}\bar{v}De^a\ , \\
De^a &= de^a - j^a A_0\ , \\
D\vartheta &= d\vartheta + 3A_0\ ,
\end{aligned} \tag{4.34}
$$

j^a are constant charges and $\vartheta = \vartheta(v,\bar{v})$ is a function of v and $\bar{v}$ whose explicit form is not important for our purposes. Moreover Db^a, $D\sigma$, Σ and M_{ab} are defined in (4.12) and (4.13), respectively. From this Lagrangian one can read off the metric for the hypermultiplet scalars.

The gauge group G^{BH} acts on M_H^{BH} via Killing vectors [51]

$$
\begin{aligned}
k_0 &= -(q+\varepsilon_{ab}j^a e^b)\partial_\sigma - 3ib^a\partial_a + 3i\bar{b}^a\bar{\partial}_a + \tfrac{3}{2}(1+\rho^2)\partial_\rho + \tfrac{3}{2}(1+\bar{\rho}^2)\partial_{\bar{\rho}} - j^a\partial_{e^a}\ , \\
k_1 &= 4\partial_\sigma\ ,
\end{aligned} \tag{4.35}
$$

where we introduced a complex scalar $\rho \in \mathcal{H}$ in the upper half plane that is related to v via

$$
v = -\frac{(i-\rho)(i-\bar{\rho})}{1+|\rho|^2}\ . \tag{4.36}
$$

The associated moment maps are then given by [51]

$$
\begin{aligned}
\vec{\mu}_0 &= (\tfrac{1}{2}e^{\tilde{Z}}e^{-4u_1} - \tfrac{3}{2\mathrm{Im}\,\rho}(1+|\rho|^2))\vec{e}_3 \\
&\quad - \tfrac{1}{2\rho}e^{-2u_1}f_a(-3i(\bar{\rho}-i)^2\bar{b}^a - 3i(\bar{\rho}+i)^2 b^a + i(1-i\bar{\rho})(1+i\bar{\rho})j^a)\vec{e_1} \\
&\quad + \tfrac{1}{2\rho}e^{-2u_1}\bar{f}_a(3i(\rho-i)^2 b^a + 3i(\rho+i)^2\bar{b}^a + i(1-i\rho)(1+i\rho)j^a)\vec{e_2}\ , \\
\vec{\mu}_1 &= -2e^{-4u_1}\vec{e}_3\ ,
\end{aligned} \tag{4.37}
$$

where $e^{\tilde{Z}} = e^Z + \varepsilon_{ab}(j^a e^b - e^a j^b) = q - 6i\varepsilon_{ab}(b^a\bar{b}^b - \bar{b}^a b^b) + \varepsilon_{ab}(j^a e^b - j^b e^a)$. Since $[k_0, k_1] = 0$, the gauge group in this case again contains a non-compact part and is given by $G^{BH} = U(1)\times\mathbb{R}$.

AdS vacua and their moduli space

Let us now solve (2.93) in this consistent truncation. To do this, we insert (4.35) and (4.30) into the third equation of (2.93),

$$
\begin{aligned}
0 = \langle h^I k_I\rangle &= (4e^{-6\langle u_3\rangle} - q - \varepsilon_{ab}j^a\langle e^b\rangle)e^{4\langle u_3\rangle}\partial_\sigma - (3i\langle b^a\rangle\partial_a - 3i\langle\bar{b}^a\rangle\bar{\partial}_a)e^{4\langle u_3\rangle} \\
&\quad + \tfrac{3}{2}(1+\langle\rho^2\rangle)e^{4\langle u_3\rangle}\partial_\rho + \tfrac{3}{2}(1+\langle\bar{\rho}^2\rangle)e^{4\langle u_3\rangle}\partial_{\bar{\rho}}\ .
\end{aligned} \tag{4.38}
$$

Using the linear independence of the basis vectors $\{\partial_a, \bar{\partial}_a, \partial_\sigma, \partial_\rho, \partial_{\bar{\rho}}\}$ we immediately find $\langle b^a\rangle = \langle\bar{b}^a\rangle = j^a = 0$.[6] Moreover,

$$
\langle\rho^2\rangle = \langle\bar{\rho}^2\rangle = -1\ , \quad q = 4e^{-6\langle u_3\rangle}\ , \tag{4.39}
$$

[6] The vanishing of the topological charges j^a shows that the backgrounds we are discussing are indeed related to the Klebanov-Witten theory [10] and not the Klebanov-Strassler solutions [75].

where we used $j^a = 0$. In particular, we find $\langle e^{\tilde{Z}} \rangle = q$ and $\langle u_3 \rangle = -\frac{1}{6} \log \frac{q}{4}$. The first equation in (4.39) is solved by $\langle \rho \rangle = i$ which implies $\langle v \rangle = \langle \bar{v} \rangle = 0$ and

$$\langle |\rho|^2 \rangle = 1 \ . \tag{4.40}$$

Thus the vector multiplet scalar u_3 and the hypermultiplet scalars $\{b^a, \bar{b}^a, v, \bar{v}\}$ are fixed by the third equation in (2.93). Using the above results, we find that the moment maps (4.37) are only nontrivial in the $\vec{e}_3$-direction and read

$$\langle \mu_0 \rangle = \tfrac{q}{2} e^{-4\langle u_1 \rangle} - 3 \ , \quad \langle \mu_1 \rangle = -2 e^{-4\langle u_1 \rangle} \ . \tag{4.41}$$

Inserting these expressions into the first equation of (2.93) and using (4.31), we find

$$(\tfrac{q}{2} e^{-4\langle u_1 \rangle} - 3) = 3\lambda e^{4\langle u_3 \rangle} \ , \quad e^{-4\langle u_1 \rangle} = \tfrac{2}{3} \lambda e^{-2\langle u_3 \rangle} \ . \tag{4.42}$$

These equations fix the hypermultiplet scalar $\langle u_1 \rangle = -\frac{1}{4} \log(\frac{2^{5/3}\lambda}{3\sqrt[3]{q}})$ in the background. In conclusion, we have shown that the AdS$_5$ conditions for the Betti-hyper truncation fix the vector multiplet scalar u_3 and the hypermultiplet scalars $\{b^a, \bar{b}^a, v, \bar{v}, u_1\}$.

To compute the moduli space, we observe that the scalars $\{k, e^a, \tau, \bar{\tau}\}$ in the hypermultiplets are not restricted by the AdS conditions (2.93) and thus their associated deformations leave the vacuum invariant. However, as before we find that the Killing vectors have non-trivial background values,

$$\langle k_0 \rangle = -q\partial_\sigma \ , \quad \langle k_1 \rangle = 4\partial_\sigma \ , \tag{4.43}$$

and thus the space of Goldstone bosons is again one-dimensional. The gauge group is broken, $U(1) \times \mathbb{R} \longrightarrow U(1)_R$, and the vector field $qA_0 - 2(A_1 + A_2)$ becomes massive by "eating" the scalar σ. Note that, as discussed in Section 2.2, the $U(1)_R$ symmetry of the background is still present after the spontaneous symmetry breaking. It is gauged by the graviphoton[7]

$$\lambda \langle h_0 \rangle A^0_\mu + \tfrac{1}{2} \langle h_1 \rangle (A^1_\mu + A^2_\mu) = \tfrac{\lambda q^{2/3}}{6\sqrt[3]{2}} A^0_\mu + \tfrac{2^{2/3}\lambda}{3\sqrt[3]{q}} (A^1_\mu + A^2_\mu) \ . \tag{4.44}$$

The moduli space is thus four-dimensional and spanned by the hypermultiplet scalars $\{e^a, \tau, \bar{\tau}\}$. This is in agreement with the bound (3.30) for $n_H = 3$, $n_G = 1$,

$$\dim \mathcal{M}^{BH} \leq 2 \cdot 3 - 2 \cdot 1 = 4 \ . \tag{4.45}$$

Note that $\dim \mathcal{M}^{BH} = 4$ is again the maximal dimension possible for the given number of hypermultiplets and Goldstone bosons.

The moduli space is thus spanned by the reduction of the Axion-Dilaton τ and a doublet of real scalars e^a coming from the Betti-hypermultiplet. To compute the metric,

[7] We find that the graviphoton of the AdS$_5$ background is the same for both Betti truncations on $T^{1,1}$.

we use the fact that the coordinate one-forms of the fixed scalars vanish on $\mathcal{M}^{BH}$. Inserting this into the Lagrangian (4.33) for the hypermultiplets, we find

$$g^{BH} = \gamma M_{ab} de^a de^b + \frac{1}{\operatorname{Im}\tau^2} d\tau d\bar{\tau} \ , \tag{4.46}$$

where $\gamma = 2e^{-4\langle u_1\rangle} = \frac{2}{3}(2Q)^{1/3}\lambda$ and M_{ab} is defined in (4.13). We immediately recognize the second term in (4.46) as the metric of the Axion-Dilaton τ on the upper half plane (4.26). Let us first discuss the isometries of the metric (4.46). Clearly, (4.46) is invariant under shifts in the scalars e^a, i.e. $e^a \mapsto e^a + w^a$ for some constants w^a. Moreover, the metric has an $SL(2,\mathbb{R})$-isometry induced by the global $SL(2,\mathbb{R})$-symmetry of type IIB supergravity [2]. The term $\frac{1}{\operatorname{Im}\tau^2} d\tau d\bar{\tau}$ is the metric on the upper half plane (see (4.27)), which is known to have an $SL(2,\mathbb{R})$-isometry given by (B.7). For the term $M_{ab}de^a de^b$, this follows from the fact that the transformations (B.6) and (B.8) exactly cancel each other. Thus the metric (4.46) has an $\mathbb{R}^2 \times SL(2,\mathbb{R})$ isometry group.

We already discussed that the moduli space of AdS_5 vacua should be Kähler and in particular complex. To this end, let us define a complex structure for the scalars $\{e^a, a, \phi\}$ and construct the Kähler potential associated to the metric g^{BH}, i.e. a real function Z such that $(g^{BH})_{i\bar{j}} = \partial_i \partial_{\bar{j}} Z$ for $i, j = 1, 2$ complex indices on $\mathcal{M}^{BH}$. For the scalars $\{a, \phi\}$, the complex structure is naturally given by the Axion-Dilaton $\tau = a + ie^{-\phi}$. To define a complex structure on the scalars e^a, recall that the three-forms $\hat{F}_3^a$ in type IIB supergravity can be combined into a complex three-form[8],

$$\hat{G}_3 := \hat{F}_3^2 - \tau F_3^1 \ . \tag{4.47}$$

Translating this to the scalars e^a, we may define a complex scalar z by

$$z := e^2 - \tau e^1 \ , \quad \bar{z} := e^2 - \bar{\tau} e^1 \ . \tag{4.48}$$

In particular, this implies $e^1 = -\frac{\operatorname{Im} z}{\operatorname{Im}\tau}$. The associated coordinate one-forms then introduce a twist between z and τ,

$$dz = de^2 - \tau de^1 - e^1 d\tau \ , \quad d\bar{z} = de^2 - \bar{\tau} de^1 - e^1 d\bar{\tau} \ . \tag{4.49}$$

Using these we can rewrite the metric of the e^a in terms of z and $\bar{z}$,

$$M_{ab} de^a de^b = \frac{\operatorname{Im} z^2}{\operatorname{Im}\tau^3} d\tau d\bar{\tau} + \frac{1}{\operatorname{Im}\tau} dz d\bar{z} - \frac{\operatorname{Im} z}{\operatorname{Im}\tau^2}(d\tau d\bar{z} + d\bar{\tau} dz) \ . \tag{4.50}$$

Thus the full complex metric reads

$$g^{BH} = \left(\frac{1}{\operatorname{Im}\tau^2} + \gamma \frac{\operatorname{Im} z^2}{\operatorname{Im}\tau^3}\right) d\tau d\bar{\tau} - \gamma \frac{\operatorname{Im} z}{\operatorname{Im}\tau^2}(d\tau d\bar{z} + d\bar{\tau} dz) + \frac{\gamma}{\operatorname{Im}\tau} dz d\bar{z} \ . \tag{4.51}$$

This metric is derived from the Kähler potential

$$Z = -4\log(\tau - \bar{\tau}) - i\gamma \tfrac{(z-\bar{z})^2}{\tau - \bar{\tau}} \ , \tag{4.52}$$

[8] See Appendix B, equation (B.4).

and the associated Kähler form $\frac{i}{2}\partial\bar{\partial}Z$ is closed. Thus the moduli space is a Kähler manifold with Kähler structure defined by Z.

Let us now show that (4.51) extends to a globally well-defined metric and identify the manifold $\mathcal{M}^{BH}$. Examining the metric (4.46), we already observed that the second part is the metric $\frac{1}{\operatorname{Im}\tau^2}d\tau d\bar{\tau}$ on the upper half plane $\mathcal{H}$. The first term in (4.46) is the metric of a torus $\mathbb{C}/\Lambda_\tau$ with complex structure parameter τ. Here $\Lambda_\tau = \mathbb{Z}\oplus\tau\mathbb{Z}$ is a lattice spanned by $(1,\tau)$. However, this description only holds locally. Globally, the moduli space is not a direct product of a complex torus with the upper half plane, since the complex structure (4.48) on the torus varies with τ. Thus the global metric is the metric on the total space of a complex torus bundle over the upper half plane, i.e.

$$\mathbb{C}/\Lambda_\tau \hookrightarrow \mathcal{M}^{BH} \longrightarrow \mathcal{H} \,, \tag{4.53}$$

Note that this agrees with the results stated in [10]; the moduli are the Axion-Dilaton and a complex scalar parametrizing a torus.

However, it turns out that the metric is in general not a product metric but the metric of a non-trivial fibration. To identify the total space of the fibration (4.53), consider the universal elliptic curve $\mathcal{E}$ over the upper half plane.[9] This is defined as the quotient

$$\mathcal{E} = (\mathbb{C}\times\mathcal{H})\,/\mathbb{Z}^2 \,, \tag{4.54}$$

where $(m,n)\in\mathbb{Z}^2$ acts as

$$(z,\tau)\mapsto(z+m+n\tau,\tau)\ . \tag{4.55}$$

Since this action is free and proper, the quotient $\mathcal{E}$ is a two-dimensional complex manifold [76]. In particular, the fibers of the projection $\mathcal{E}\to\mathcal{H}$ are precisely the complex tori $\mathbb{C}/\Lambda_\tau$. To see that g^{BH} gives a well-defined metric on $\mathcal{E}$, we have to show that it is compatible with the quotient by the action (4.55). Since τ is fixed by (4.55), only the second term in the Kähler potential (4.52) transforms non-trivially. In particular, we find for the transformation of the Kähler potential,

$$Z\mapsto Z' = Z + 2i\gamma n(z-\bar{z}) + i\gamma n^2(\tau-\bar{\tau})\ , \tag{4.56}$$

which is just a Kähler transformation $K\mapsto Z' = Z + f(\tau,z) + \bar{f}(\bar{\tau},\bar{z})$ for a holomorphic function $f(\tau,z) = 2i\gamma nz + i\gamma n^2\tau$. Thus both potentials give rise to the same Kähler metric and g^{BH} is a well-defined global metric on $\mathcal{E}$.[10] In conclusion, the moduli space of AdS vacua in the Betti-hyper truncation is given by the total space of the universal elliptic curve,

$$\mathcal{M}^{BH} = \mathcal{E} = (\mathbb{C}\times\mathcal{H})/\mathbb{Z}^2\ . \tag{4.57}$$

[9] For an introduction to elliptic curves, their moduli spaces and the universal elliptic curve, see [76].

[10] A different way to see this is to realize the $\mathbb{Z}^2$-action (4.55) via the $\mathbb{R}^2$-isometry of the metric (4.46) by $e^1\mapsto e^1-n$, $e^2\mapsto e^2+m$.

This manifold is in particular a homogeneous space, since it has a transitive group action given by the isometries of the metric (4.46), $\mathbb{R}^2 \times SL(2,\mathbb{R}) \cong \mathbb{C} \times SL(2,\mathbb{R})$. Because the upper half plane $\mathcal{H}$ can be written as the quotient $SL(2,\mathbb{R})/SO(2)$, we find

$$\mathcal{M}^{BH} = [\mathbb{C} \times SL(2,\mathbb{R})/SO(2)]/\mathbb{Z}^2 \ . \tag{4.58}$$

Thus for both truncations with homogeneous scalar target spaces, also the AdS moduli space is a homogeneous space.

Before we conclude this section, let us briefly note the following: consider the Kähler potential $\tilde{Z}$ of $SU(2,1)/U(2)$, given by

$$\tilde{Z} = -\log(\tau - \bar{\tau} + i\epsilon(z-\bar{z})^2) \ , \tag{4.59}$$

where $\epsilon \in \mathbb{R}$ is a constant. To make contact with the potential (4.52), we want to split off a term of the form $\log(\tau - \bar{\tau})$ from $\tilde{Z}$. To this end, we separate a factor $\tau - \bar{\tau}$ inside the logarithm,

$$\tilde{Z} = -\log(\tau - \bar{\tau} + i\epsilon(z-\bar{z})^2) = -\log(\tau - \bar{\tau}) - \log(1 + i\epsilon\tfrac{(z-\bar{z})^2}{\tau-\bar{\tau}}) \ . \tag{4.60}$$

For small ϵ the second term can be expanded,

$$-\log(1 + i\epsilon\tfrac{(z-\bar{z})^2}{\tau-\bar{\tau}}) \simeq -i\epsilon\tfrac{(z-\bar{z})^2}{\tau-\bar{\tau}} + \mathcal{O}(\epsilon^2) \ . \tag{4.61}$$

Thus we can write

$$\tilde{Z} = -\log(\tau - \bar{\tau} + i\epsilon(z-\bar{z})^2) \simeq \tfrac{1}{4}K + \mathcal{O}(\epsilon^2) \ , \tag{4.62}$$

for suitable ϵ and find that the Kähler potential (4.52) of the AdS moduli space appears as a first order term in the ϵ-expansion. To interpret this result we first note the following: we can reinstall the five-dimensional gravitational constant κ into the metric (4.51) by $\gamma \mapsto \kappa^2\gamma$. Thus the ϵ-expansion performed above actually corresponds to an expansion in the gravitational constant κ for $\epsilon = 4\kappa^2\gamma$ and fixed γ. Since the limit $\kappa \to 0$ corresponds to the large N limit[11] of the dual field theory, we can interpret the metric $g^{BH}_{\mathcal{M}}$ on the moduli space as the first order contribution in a large N expansion of the metric on $SU(2,1)/U(2)$.

[11]The AdS/CFT correspondence relates $\kappa \propto 1/N$, see [3, Chapter 23.7] for a review.

Chapter 5

Conclusion and outlook

Let us briefly summarize the results we obtained in this thesis. We started by studying AdS_5 vacua in gauged $\mathcal{N} = 2$ supergravity coupled to an arbitrary number of vector tensor and hypermultiplets following [34]. To find the conditions for supersymmetric AdS backgrounds, we analyzed the vanishing of the fermionic supersymmetry variations. We showed that these conditions can be expressed in terms of Killing vectors and moment maps on the scalar target space and found that an $U(1)_R$-symmetry is always gauged in the vacuum. This condition is equivalent to the AdS vacuum being a fixed point under the $U(1)_R$-symmetry. Moreover, we proved that the associated gauge field is given by the graviphoton $\langle h^I \rangle A_I^\mu$. In the dual four-dimensional SCFT, this $U(1)_R$-symmetry is defined by a-maximization while it can be derived in the type IIB setting from volume minimization on Sasaki-Einstein manifolds. Furthermore, we discussed the spontaneous gauge symmetry breaking and explained how the massive vector multiplets are built out of a massless vector multiplet and a hypermultiplet containing the Goldstone boson.

To obtain the moduli space of AdS_5 vacua, we expanded the scalar fields around their vacuum expectation value and used variational calculus to derive a set of conditions on the variations that characterize the deformation space $\mathcal{D}$ of the AdS background. In particular, we showed that the Goldstone bosons are contained in this deformation space. The physical moduli space is then given by the quotient $\mathcal{M} = \mathcal{D}/\mathcal{G}$ of scalar field deformations modulo Goldstone directions $\mathcal{G}$. Even though we could not solve the equations defining the moduli space explicitly, their structure allowed us to prove that $\mathcal{M}$ is a Kähler submanifold of the quaternionic Kähler target space spanned by the hypermultiplet scalars. This result is in agreement with the expectation from the AdS/CFT correspondence, since it was proven in [35] that the conformal manifold of a four-dimensional $\mathcal{N} = 1$ SCFT also admits a Kähler structure. Moreover, we computed the dimension of the moduli space to be given as $\dim \mathcal{M} \leq 2n_H - 2n_G$, where n_H denotes the number of hypermultiplets while n_G is the number of Goldstone bosons in the background.

Since our computations only considered the first order variations in the scalar fields, the resulting conditions on the moduli space of AdS vacua are only necessary. In the case of supergravity coupled to only hypermultiplets, we explained how the first order variations actually determine the moduli space to all orders. However, when coupling the theory to vector and tensor multiplets, the higher order variations might contribute additional restrictions on the moduli space.

Following [52], in the second part of this thesis we studied explicit examples of AdS_5 vauca obtained from consistent $\mathcal{N}=2$ truncations of type IIB supergravity compactified on $T^{1,1}$. Namely, these are the Betti-vector truncation containing gravity coupled to two vector multiplets and two hypermultiplets as well as the Betti-hyper truncation with gravity coupled to one vector multiplet and three hypermultiplets.[1] We found that both truncations admit supersymmetric AdS vacua with an $U(1)_R$-symmetry that always remains unbroken in the vacuum. This is in agreement with the $U(1)_R$ symmetry coming from the dual field theories predicted by the AdS/CFT correspondence [6,10,18]. Since we studied consistent truncations, these results lift to solutions of the full ten-dimensional supergravity.

The moduli space of the type IIB solution $\mathrm{AdS}_5 \times T^{1,1}$ is known to be complex five-dimensional [10,23,25,46,47]. However, only two of those moduli transform as singlets under the $SU(2) \times SU(2)$-factor in the isometry group of $T^{1,1}$ and are thus accessible via consistent truncations [53]; those are the Axion-Dilaton τ and the complex modulus z related to the topology of $T^{1,1}$. In particular, z comes from the fact that $b_2(T^{1,1}) = 1$ and thus the second and third cohomology classes of $T^{1,1}$ are non-trivial. In [23,25] the moduli spaces for type IIB solutions of the form $\mathrm{AdS}_5 \times \mathrm{SE}_5$ were computed from generalized geometry. In particular, it was shown that the Axion-Dilaton is always a modulus, independent from the topology of the Sasaki-Einstein manifold used for the compactification. Moreover, we have shown in Chapter 3 that the moduli of five-dimensional AdS backgrounds must always be recruited out of the hypermultiplets. Our present results agree with these predictions; the moduli space of the Betti-vector truncation is spanned only by the Axion-Dilaton residing in one of the hypermultiplets and the metric is the expected one on the upper half-plane. This can be understood as follows: the fact that $T^{1,1}$ has nontrivial second cohomology leads to the presence of an additional two-form in the reduction ansatz [43,48,49]. This additional two-form gives rise to an additional $\mathcal{N}=4$ vector multiplet which splits into an $\mathcal{N}=2$ vector multiplet and an $\mathcal{N}=2$ hypermultiplet. However, only one of the two can be retained in a consistent $\mathcal{N}=2$ truncation [48]. Thus in the Betti-vector truncation the only hypermultiplet related to the topology of $T^{1,1} \cong S^2 \times S^3$ was removed and one would not expect to find the modulus z in this truncation in the first place.

In the case of the Betti-hyper truncation the situation is different. Here the topology of $T^{1,1}$ contributes to the five-dimensional hypermultiplets of the truncation and gives

[1] In Appendix D we additionally analyzed the conditions on AdS vacua in the NS-sector truncation. However, we found that this truncation does not admit AdS_5 vacua preserving $\mathcal{N}=2$ supersymmetry.

rise to an additional complex modulus z. This modulus parametrizes a complex torus with complex structure parameter given by the Axion-Dilaton τ. Thus the complex two-dimensional moduli space $\mathcal{M}^{BH}$ of the Betti-hyper truncation contains all the moduli of $\mathrm{AdS}_5 \times T^{1,1}$ that are detectable in a consistent truncation. However, even though the Axion-Dilaton is completely unrelated to the geometry of the compact Sasaki-Einstein manifold, it turns out that the metric on the moduli space $\mathcal{M}^{BH}$ is not a product metric with the Axion-Dilaton split off from the moduli corresponding to the geometry. Indeed, we find that $\mathcal{M}^{BH}$ is a torus bundle with base space given by the upper half-plane $\mathcal{H}$. We identified this bundle as the universal elliptic curve with total space $(\mathbb{C} \times \mathcal{H})/\mathbb{Z}^2$. However, it turns out that the metric on the moduli space is also related to the metric on $SU(2,1)/U(2)$ by performing a large N expansion of the Kähler potential (4.52). Thus since the Betti-hyper truncation is consistent and lifts to the full ten-dimensional supergravity, the AdS/CFT correspondence relates this result to the metric on a submanifold of the conformal manifold of the Klebanov-Witten theory in four dimensions.[2]

Finally, let us briefly comment on possible future directions. As we computed the moduli space only to first order in the scalar field variations, it would be interesting to extend our calculation and determine the moduli space to all orders. However, from the results on conformal manifolds in the dual $\mathcal{N} = 1$ SCFTs provided by the AdS/CFT correspondence, one would expect that any further conditions on $\mathcal{M}$ are compatible with the Kähler structure we constructed in Chapter 3.

Turning to the Betti truncations of type II supergravity on $T^{1,1}$, it would be interesting to verify our results in the dual field theory. Since the explicit metrics on the moduli spaces in Betti-vector truncation and Betti-hyper-truncation should be related to metrics on submanifolds of the conformal manifold in Klebanov-Witten theory, it would be a non-trivial check of the AdS/CFT correspondence to obtain these metrics in the field theory. Moreover, it would be interesting to develop a method to incorporate the three remaining complex moduli into a five-dimensional description despite the inconsistencies analyzed in [53].

Because the manifold $T^{1,1}$ is related to the infinite family of Sasaki-Einstein manifolds called $Y^{p,q}$ [11], a natural question would be to extend our computations to these manifolds. On first glance, this might seem easily possible; $Y^{p,q} \cong S^2 \times S^3$ have the same topology as $T^{1,1}$ and a reduced isometry algebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$. Moreover, the moduli of the ten-dimensional solutions $\mathrm{AdS}_5 \times Y^{p,q}$ also contain the Axion-Dilaton and the modulus z from the VEV of the B-field integrated over the nontrivial two-cycle on $S^2 \times S^3$. Additionally, there only exists one other complex modulus transforming in a triplet under the $SU(2)$ in the isometry group, making the full moduli space complex three-dimensional [23,25,46,47]. However, the $Y^{p,q}$ manifolds are in general not homoge-

[2] Consistency of the truncation is related to closure of the associated operators under the operator product expansion in the dual CFT. This then defines the submanifold of the conformal manifold dual to the moduli spaces we discuss here.

neous spaces but only admit a cohomgeneity-one action of the isometry group. Since the techniques used in [48, 49] rely heavily on the transitivity of the $SU(2) \times SU(2)$ action on the $T^{1,1}$ coset, a similar reduction for the $Y^{p,q}$ manifolds might not be consistent.

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Appendix A

Sasaki-Einstein manifolds

Sasaki-Einstein manifolds appear in solutions of type IIB supergravity containing an AdS_5 factor and thus prominently feature in the AdS/CFT correspondence. In this appendix we provide the basic definitions of Sasaki-Einstein manifolds and briefly review classifications of five-dimensional Sasaki-Einstein manifolds with certain properties. We will mostly follow [77, 78], where [77] is a mathematical introduction while [78] focuses on the physically relevant cases.

A.1 Basic notions of Sasaki-Einstein geometry

We start by recalling the notion of contact geometry. For this, let (M, η) be a smooth manifold of odd dimension $2n+1$ and η a one-form on M. If

$$\eta \wedge d\eta^n \neq 0 \ , \tag{A.1}$$

we call (M, η) a contact manifold. One can prove that every contact manifold admits a canonical vector field R, called the Reeb vector field or characteristic vector field. This is uniquely defined by the following conditions,

$$\eta(R) = 1 \ , \quad i_R d\eta = 0 \ . \tag{A.2}$$

In particular, R induces a foliation $\mathcal{F}_R$ on M. Now denote by L_R the line bundle consisting of tangent vectors to the leaves of $\mathcal{F}_R$. Then the tangent bundle of M splits as

$$TM = \mathcal{C} \oplus L_R \ , \tag{A.3}$$

where $\mathcal{C} := \ker \eta$ is a codimension one subbundle of TM. Then (A.1) implies that $d\eta$ restricts to a symplectic form on $\mathcal{C}$.

Since we will be interested in Einstein metrics later, let us now introduce a Riemannian metric g into this setting. To this end, we first define a tensor field J on M such that

$$J^2 = -\mathrm{Id}_{\mathcal{C}} + \eta \otimes R \ , \quad d\eta(JX, JY) = d\eta(X, Y) \ , \tag{A.4}$$

for all vector fields $X, Y \in \Gamma(TM)$ and

$$d\eta(JX, X) \geq 0 \; , \tag{A.5}$$

for all nonzero vector fields $X \in \Gamma(TM)$. In particular, J defines an almost complex structure on $\mathcal{C}$ that is compatible with the symplectic form defined by $d\eta$. We then call a Riemannian contact manifold (M, g, η) a metric contact manifold if

$$g(JX, JY) = g(X, Y) - \eta(X)\eta(Y) \; , \tag{A.6}$$

for all vector fields $X, Y \in \Gamma(TM)$ and

$$g(V, JW) = d\eta(V, W) \tag{A.7}$$

for all $V, W \in \Gamma(\mathcal{C})$. Since $J(\xi) = 0$, these conditions are equivalent to $(g^T, d\eta, J)$ being an almost Kähler structure on the subbundle $\mathcal{C}$. Here g^T denotes the transverse metric that is given by $g^T = g|_{\mathcal{C}}$. For completeness, let us remark that we call a metric contact manifold a K-contact manifold if ξ is a Killing vector field. In particular, every Sasaki manifold is K-contact.

To define Sasaki manifolds, we first have to introduce a notion of integrability of the contact structure. This can be done by considering the metric cone $C(M) = \mathbb{R}^+ \times M$ over M with $\bar{g} = dt^2 + t^2 g$. If we denote the coordinate on $\mathbb{R}^+$ by t, we can define a vector field $t\partial_t$ called the Euler vector field. Then a contact structure on M gives rise to an almost complex structure $\mathcal{Z}$ on $C(M)$ by

$$\mathcal{Z}(X) = J(X) + \eta(X)t\partial_t \; , \quad \mathcal{Z}(t\partial_t) = -R \; , \tag{A.8}$$

for X a vector field on M. One can then prove that, under certain mild assumptions, the reverse is also true (see [77] for more details and the notion of an almost contact structure). For our purposes, we assume that there is a one-to-one correspondence between contact structures on M and almost complex structures on the cone.[1] We then call a contact structure normal if the corresponding almost compelx structure J is integrable. Finally, a Sasaki manifold is defined to be a normal metric contact manifold.[2]

Often in the physics literature another definition of Sasaki manifolds is used: A Riemannian manifold (M, g) is a Sasaki manifold if its metric cone $(C(M), \bar{g})$ is a Kähler manifold. Using the construction (A.8) for an almost complex structure on the cone $C(M)$, one can prove that these two defintions are equivalent [77].

[1] If we additionally have a metric g on M, then the correspondence holds between metric contact structures on M and almost Hermitian structures for $\bar{g} = dt^2 + t^2 g$ on $C(M)$.

[2] We will discuss in Appendix B that simply-connected Sasaki-Einstein manifolds always admit a pair of linearly independent Killing spinors. However, this theorem does not necessary hold in the case where the manifold is not simply-connected. For applications in supergravity compactifications, one thus defines Sasaki-Einstein manifolds to always admit Killing spinors [78].

Let us now include the Einstein condition. Recall that a Riemannian metric is called an Einstein metric if it is proportional to its Ricci-tensor, i.e.

$$\mathrm{Ric}_g = \lambda g \ , \tag{A.9}$$

for $\lambda \in \mathbb{R}$ the Einstein constant. A Sasaki manifold (M, g) is then called Sasaki-Einstein if its metric g is an Einstein metric. In particular, one can prove that for a Sasaki-Einstein manifold of dimension $2n+1$ the Einstein constant is always fixed to be $\lambda = 2n$. There are many equivalent characterizations of Sasaki-Einstein manifolds. The following theorem covers the most important ones [78, Proposition 1.9]

Theorem 3 *Let (M, g) be Sasaki manifold of dimension $2n+1$. Then the following are equivalent,*

1) (M, g) is Sasaki-Einstein with $Ric_g = 2ng$.

2) The cone $(C(M), \bar{g})$ is Kähler and Ricci-flat, i.e. $Ric_{\bar{g}} = 0$.

3) The transverse almost Kähler structure $(g^T, d\eta, J)$ is Kähler-Einstein with $Ric^T = 2(n+1)g^T$.

Note that the same theorem holds if we drop the Einstein and Ricci-flatness conditions on the metrics. In this sense a Sasaki manifold is an odd-dimensional analogue of a Kähler manifold, as it is wedged between a Kähler structure on the cone $C(M)$ and on the space transverse to the foliation $\mathcal{F}_R$. However, the transverse space is not always a manifold. To study this in more detail, we now introduce the notion of regularity in contact geometry.

A.2 Classifications of Sasaki-Einstein manifolds in dimension five

Let (M, g) be a Sasaki manifold with associated contact structure (η, ξ, J) and assume that the leaves $M/\mathcal{F}_R$ are compact. Then the flow of the Reeb vector field R generates a locally free $U(1)$-action on M. This result then motivates the definition of regularity of a Sasaki manifold. A Sasaki manifold is called

1) regular, if the orbits of R are closed and the induced $U(1)$-action is free.

2) quasi-regular, if the orbits of R are closed.

3) irregular, if there exists a non-closed orbit of R.

In the regular case one can prove that the transverse space for every compact contact manifold M is again a manifold [77, Theorem 6.1.26].

Theorem 4 (Boothby-Wang fibration) *Let (M, g, η, R) be a compact regular contact manifold. Then M is the total space of a principal $U(1)$-bundle $\pi : M \to N$ over a symplectic manifold (N, g^T, P) such that $[P] \in H^2(N, \mathbb{Z})$ and $d\eta = \pi^* P$.*

Here $H^2(N, \mathbb{Z})$ denotes the second integral cohomology class of M while $[P]$ denotes the cohomology class of the form P. In particular, if M in Theorem 4 is Sasaki, then the underlying manifold N is Kähler. Note that the converse to Theorem 4 is also true. Moreover, if (M, g) is Einstein, then by Theorem 3 (N, g^T) is Kähler-Einstein. In particular, Theorem 4 tells us that there exists a manifold N whose tangent bundle is the pullback of the subbundle $\mathcal{C} = \ker \eta$ defined in (A.3) and g^T is a globally well-defined metric. In the quasi-regular case a similar theorem provides an underlying orbifold, while no such structure exists in the irregular case.

Due to the correspondence provided by Theorem 4, the classification of regular Sasaki-Einstein manifolds can be reduced to classifying Kähler-Einstein manifolds and is thus a much easier task. For applications in supergravity compactifications, five-dimensional Sasaki-Einstein manifolds are particularly interesting. One can prove the following classification for simply-connected Sasaki-Einstein five-manifolds [77, Theorem 11.4.1].

Theorem 5 *Let M be a simply-connected five-dimensional manifold with regular Sasaki-Einstein structure. Then M is diffeomorphic to $k(S^2 \times S^3)$ for $0 \leq k \leq 8$. Moreover,*

1) for each $k = 0, 1, 3, 4$, there exists up to isometry precisely one regular Sasaki-Einstein structure.

2) for each $5 \leq k \leq 8$, there is a $4(k-4)$ parameter family of inequivalent Sasaki-Einstein structures.

Note that $k(S^2 \times S^3)$ for $k = 0$ is given by S^5. Moreover, the case $k = 2$ does not appear in the above theorem. The regular Sasaki-Einstein metric on $S^2 \times S^3$ is called the Kobayashi-Tanno metric [44] in the mathematics literature and $T^{1,1}$ in the physics literature [45]. Here, $T^{1,1}$ is the homogeneous space $SU(2) \times SU(2)/U(1)$ where the $U(1)$ acts diagonally. This manifold is particularly interesting as it features in the AdS/CFT dual background of the Klebanov-Witten theory [10]. For coordinates $\theta, \omega \in [0, \pi]$, $\varphi, \nu \in [0, 2\pi)$ periodic and $\zeta \in [0, 4\pi)$ periodic on $S^2 \times S^3$ the metric on $T^{1,1}$ can be written as

$$g_{T^{1,1}} = \frac{1}{6}(d\theta^2 + \sin^2 d\varphi^2 + d\omega^2 + \sin^2 \omega d\nu^2) + \frac{1}{9}[d\zeta + \cos\theta d\varphi + \cos\omega d\nu]^2 \ . \tag{A.10}$$

The situation in the quasi-regular and irregular case is more involved. While the existence of quasi-regular Sasaki-Einstein metrics on $k\#(S^2 \times S^3)$ for $k \leq 9$ was proven in [79], no explicit construction was known. Moreover, the existence of irregular Sasaki-Einstein metrics was questioned for a long time [80]. This changed with the discovery of a family of Sasaki-Einstein metrics $Y^{p,q}$ that first appeared in the study of supersymemtric AdS_5 solutions of M-theory [11, 81].

Theorem 6 *There exist countably infinitely many Sasaki-Einstein metrics on $S^2 \times S^3$, labeled by two positive integers p, q such that $gcd(p, q) = 1$ and $q \leq p$. They are explicitly given in local coordinates $\phi, \zeta \in [0, 2\pi)$, $\theta \in [0, \pi]$, $\alpha \in [0, 2\pi\ell)$ and $y \in [y_1, y_2]$ as*

$$\begin{aligned} g_{p,q} = &\frac{1-y}{6}(d\theta^2 + \sin^2\varphi d\varphi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}(d\zeta - \cos\theta d\varphi)^2 \\ &+ w(y)[d\alpha + f(y)(d\zeta - \cos\theta d\varphi)]^2 \ , \end{aligned}$$

where

$$\begin{aligned} w(y) &= \frac{2(a-y^2)}{1-y} \ , \\ q(y) &= \frac{a - 3y^2 + 2y^3}{a - y^2} \ , \\ f(y) &= \frac{a - 2y + y^2}{6(a - y^2)} \ , \end{aligned}$$

for constant

$$\begin{aligned} a = a_{p,q} &= \frac{1}{2} - \tfrac{p^2 - 3q^2}{4p^3}\sqrt{4p^2 - 3q^2} \\ \ell &= \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}} \ , \end{aligned}$$

and y_1, y_2 are the negative and smallest positive root of the function $q(y)$, respectively.

Here $gcd(p, q)$ of two integers p, q denotes the their greatest common divisor. In particular, this family consists of quasi-regular and irregular Sasaki-Einstein metrics. Moreover, one can prove that the set of manifolds is in some sense complete [82]:

Theorem 7 *Let (M, g) be a compact simply-connected Sasaki-Einstein manifold in dimension five for which the isometry group acts with cohomogeneity one, i.e. such that the generic orbit has codimension one. Then (M, g) is isometric to one of the $Y^{p,q}$ manifolds in 6.*

The $Y^{p,q}$ manifolds are of particular interest in physics, since they provide an explicit description of Sasaki-Einstein metrics in local coordinates. With this the dual field theories could be constructed from certain quiver diagrams and thus provide an infinite number of examples to the AdS/CFT correspondence [12–14].

Appendix B

Type IIB supergravity on Sasaki-Einstein manifolds

In this appendix we discuss solutions to type IIB supergravity of the form

$$\mathrm{AdS}_5 \times \mathrm{SE}_5 \ , \tag{B.1}$$

where SE_5 is a five-dimensional Sasaki-Einstein manifold. To this end, we begin by reviewing the field content and Lagrangian description of type IIB supergravity. We proceed by decomposing the ten-dimensional spinors according to (B.1) and show that $\mathcal{N} = 2$ supersymmetry is preserved on AdS_5.

B.1 Type IIB supergravity

The bosonic field content of type IIB supergravity is given by the spacetime metric $\hat{g}$, the Axion-Dilaton $\hat{\tau} = \hat{C}_0 + ie^{-\hat{\phi}}$, a doublet $\hat{B}_2^a$, $a = 1, 2$ of two-forms with fieldstrengths $\hat{F}_3^a = d\hat{B}_2^a$ and the Ramond-Ramond four-form $\hat{C}_4$.[1] Their dynamics are described by the Lagrangian [2, 38]

$$\mathcal{L}_{IIB} = \hat{\mathcal{R}} * 1 - \frac{1}{2\mathrm{Im}\,\hat{\tau}^2} d\hat{\tau} \wedge * d\bar{\hat{\tau}} - \frac{1}{2} M_{ab} \hat{F}_3^a \wedge * \hat{F}_3^b - \frac{1}{4} \hat{F}_5 \wedge * \hat{F}_5 + \frac{1}{4} \epsilon_{ab} \hat{C}_4 \wedge \hat{F}_3^a \wedge \hat{F}_3^b \ , \tag{B.2}$$

where M_{ab} is defined in (4.13). Moreover, we defined $\hat{F}_5 = d\hat{C}_4 - \epsilon_{ab} \frac{1}{2} \hat{B}_2^a \wedge d\hat{B}_2^b$, which has to satisfy a self-duality constraint

$$* \hat{F}_5 = \hat{F}_5 \ . \tag{B.3}$$

Note that this constraint cannot be included in the action and thus has to be introduced by hand. Moreover, for later use we also define a complex three-form by

$$\hat{G}_3 = \hat{F}_3^2 - \hat{\tau} \hat{F}_3^1 \ . \tag{B.4}$$

[1] Here we denote as $\hat{B}_2^1 = \hat{B}_2$ the NS two-form and $\hat{B}_2^2 = \hat{C}_2$ the Ramond-Ramond two-form.

The Lagrangian (B.2) gives rise to a manifestly $SL(2,\mathbb{R})$-invariant action that, combined with the self-duality constraint (B.3), describes the bosonic part of type IIB supergravity. For the discussion of moduli space metrics in Chapter 4, let us explicitly provide the details of the $SL(2,\mathbb{R})$-action (see for example [2] for more details). For a matrix $\Lambda \in SL(2,\mathbb{R})$, we can always write

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{B.5}$$

with $a,b,c,d \in \mathbb{R}$ and $\det \Lambda = ad - bc = 1$. The two-forms $\hat{B}_2^a$ transform as a doublet under $SL(2,\mathbb{R})$,

$$\begin{pmatrix} \hat{B}_2^1 \\ \hat{B}_2^2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \hat{B}_2^1 \\ \hat{B}_2^2 \end{pmatrix} . \tag{B.6}$$

The corresponding three-forms $\hat{F}_3^a = d\hat{B}_2^a$ transform similarly, while the four-form $\hat{C}_4$, the metric $\hat{g}$ in the Einstein-frame and the self-duality constraint (B.3) are invariant. On the other hand, the Axion-Dilaton $\hat{\tau} = \hat{C}_0 + ie^{-\hat{\phi}}$ transforms non-linearly as

$$\hat{\tau} \mapsto \frac{a\hat{\tau} + b}{c\hat{\tau} + d} . \tag{B.7}$$

This transformation translates to a transformation of the matrix M_{ab} in (4.13) as

$$M_{ab} \mapsto (\Lambda^{-1})_a^{\ c} M_{cd} (\Lambda^{-1})_b^{\ d} . \tag{B.8}$$

One can then check that these transformations indeed leave the action constructed from (B.2) invariant and thus the theory has a global $SL(2,\mathbb{R})$-invariance. However, this symmetry does not carry over to the full type IIB string theory, which instead only has a non-perturbative $SL(2,\mathbb{Z})$ symmetry [2].

B.2 Spinor decomposition

Let us now introduce our conventions for spinors in the ten- and five-dimensional setting, see [20, 83, 84]. We start with the ten-dimensional Clifford algebra Cliff$(1,9)$ generated by

$$\{\Gamma^M, \Gamma^N\} = 2\hat{g}^{MN} , \quad M,N = 0,1,...,9 . \tag{B.9}$$

Here Γ^M are the ten-dimensional gamma matrices, $\{\Gamma^M, \Gamma^N\} = \Gamma^M\Gamma^N + \Gamma^N\Gamma^M$ is the anti-commutator and $\hat{g}^{MN}$ denotes the ten-dimensional spacetime metric. For later use, we also define $\Gamma_{11} := \Gamma_0\Gamma_1...\Gamma_9$. In this thesis, we are interested in products (B.1) of five-dimensional AdS spacetime with a compact Sasaki-Einstein manifold SE$_5$. Therefore, we decompose the ten-dimensional Clifford algebra

$$\text{Cliff}(1,9) \to \text{Cliff}(1,4) \times \text{Cliff}(5) \tag{B.10}$$

by [20]

$$\begin{aligned}\Gamma^\mu &= \gamma^\mu \otimes 1 \otimes \sigma^3 \\ \Gamma^m &= 1 \otimes \rho^m \otimes \sigma^1 \ ,\end{aligned} \tag{B.11}$$

where σ^n are the Pauli matrices. Then γ^μ and ρ^m generate the five-dimensional Clifford algebras Cliff$(1,4)$ and Cliff(5), respectively. That is

$$\{\rho^\mu, \rho^\nu\} = 2g^{\mu\nu}_{\mathrm{AdS}_5} \ , \quad \{\gamma^m, \gamma^n\} = 2g^{mn}_{\mathrm{SE}_5} \ . \tag{B.12}$$

In particular, the chirality operator Γ_{11} decomposes under this split as [20]

$$\Gamma_{11} = 1 \otimes 1 \otimes \sigma^2 \ . \tag{B.13}$$

Now let ϵ be a ten-dimensional Majorana-Weyl spinor that decomposes under (B.10) as

$$\epsilon = \psi \otimes \chi \otimes \theta \ . \tag{B.14}$$

Here ψ is a spinor of $Spin(1,4)$, χ is a spinor of $Spin(5)$ and θ is a two-component spinor. Note that the smallest spinor representation of $Spin(1,4)$ and $Spin(5)$ is eight-dimensional [83]. The spinors ψ and χ each have four complex components, however we cannot impose the Majorana condition in five-dimensions [83]. For this, consider the following: With the decomposition (B.14), the ten-dimensional chirality condition

$$\Gamma_{11}\epsilon = -\epsilon \ , \tag{B.15}$$

implies via (B.13) that

$$\sigma^2\theta = -\theta \ . \tag{B.16}$$

We define the charge conjugate of the ten-dimensional spinor ϵ by [20]

$$\epsilon^c = C_{10}\epsilon^* \ , \tag{B.17}$$

where ϵ^* denotes the complex conjugate of the spinor ϵ and C_{10} decomposes into

$$C_{10} = C_{1,4} \otimes C_5 \otimes \sigma^1 \ . \tag{B.18}$$

Thus the charge conjugates of the lower-dimensional spinors are defined as

$$\psi^c = C_{1,4}\psi^* \ , \quad \chi^c = C_5\chi^* \ , \quad \theta^c = \sigma^1\theta^* \ . \tag{B.19}$$

The Majorana condition in ten dimensions, $\epsilon = \epsilon^c$, then implies

$$\theta = \sigma^1\theta^* \ . \tag{B.20}$$

However, in five-dimensions we have $\psi^{cc} = -\psi$ and $\chi^{cc} = -\chi$ and thus one cannot impose the Majorana condition on these spinors [83]. Instead, the spinors $\psi_1 = \psi$ and $\psi_2 = \psi^c$ have to satisfy the additional condition

$$\psi^{\mathcal{A}} = \varepsilon^{\mathcal{AB}}(\psi^B)^c \ . \tag{B.21}$$

More generally, a set of spinors ψ_α for $\alpha = 1, 2, ..., 2n$ is called symplectic Majorana spinors, if [83]

$$\psi^\alpha = \Omega^{\alpha\beta}(\psi^\alpha)^c \ . \tag{B.22}$$

Here $\Omega^{\alpha\beta}$ is a non-degenerate, antisymmetric matrix that satisfies

$$\Omega_{\alpha\beta}\Omega^{\beta\gamma} = \delta_\alpha^\gamma \ . \tag{B.23}$$

Later on, we will also need the charge conjugates of the gamma matrices γ^μ and ρ^m. It can be shown that these transform as [83]

$$(\gamma^\mu)^c = -\gamma^\mu \ , \tag{B.24}$$

while

$$(\rho^\mu)^c = \rho^\mu \ . \tag{B.25}$$

The sign difference between (B.24) and (B.25) is related to the signatures of the metrics on AdS_5 and SE_5.

B.3 Supersymmetry on $\mathbf{AdS}_5 \times \mathbf{SE}_5$

Let us now derive the amount of supersymmetry preserved in compactifications of type IIB supergravity on Sasaki-Einstein manifolds. The fermionic part of the type IIB spectrum contains the gravitino $\hat{\Psi}_M$ and the dilatino $\hat{\lambda}$, which are complex Weyl spinors of $Spin(1,9)$. Equivalently, we can split both spinors according to

$$\begin{aligned} \hat{\Psi}_M &= \hat{\Psi}^1_M + i\hat{\Psi}^2_M \ , \\ \hat{\lambda} &= \hat{\lambda}^1 + i\hat{\lambda}^2 \ , \end{aligned} \tag{B.26}$$

where $\hat{\Psi}^{\mathcal{A}}_M$ and $\hat{\lambda}^{\mathcal{A}}$ are ten-dimensional Majorana-Weyl fermions. The supersymmetry variations of these fermions are given by [20]

$$\begin{aligned} \delta\hat{\Psi}_M &= \hat{\nabla}_M\epsilon + \frac{i}{16\cdot 5!}\hat{F}_{5NOPQR}\Gamma^{NOPQR}\Gamma_M - \frac{1}{96}\left(\Gamma_M^{NOP}\hat{G}_{3NOP} - 9\Gamma^{NO}\hat{G}_{3MNO}\right)\epsilon^c \ , \\ \delta\hat{\lambda} &= \frac{1}{2}\left(\partial_M\hat{\phi} - ie^{\hat{\phi}}\partial_M\hat{C}_0\right)\Gamma^M\epsilon^c + \frac{i}{24}\Gamma^{NOP}\hat{G}_{3NOP}\epsilon \ , \end{aligned} \tag{B.27}$$

where $\hat{\nabla}_M$ is the covariant derivative associated to the type IIB metric $\hat{g}$. We want to study the supersymmetry preserved by solutions of the form

$$\begin{aligned} \hat{g} &= g + g_{\mathrm{SE}_5} \ , \\ \hat{F}_5 &= f(\mathrm{vol}_{\mathrm{AdS}_5} + \mathrm{vol}_{\mathrm{SE}_5}) \ , \\ \hat{\phi} &= const \ , \end{aligned} \tag{B.28}$$

with all other type IIB fields set to zero. Then the supersymmetry variation of the dilatino is trivial while the gravitino variation reduces to

$$\delta\hat{\Psi}_M = \hat{\nabla}_M \epsilon + \frac{i}{16 \cdot 5!} \hat{F}_{5NOPQR} \Gamma^{NOPQR} \Gamma_M \ . \tag{B.29}$$

The covariant derivative $\hat{\nabla}_M$ associated to $\hat{g}$ decomposes as

$$\begin{aligned} \hat{\nabla}_\mu &= \nabla_\mu \otimes 1 \otimes 1 \ , \\ \hat{\nabla}_m &= 1 \otimes \nabla^{\text{SE}}_m \otimes 1 \ , \end{aligned} \tag{B.30}$$

where ∇_μ is the covariant derivative associated to the metric $g_{\mu\nu}$ on AdS_5 and ∇^{SE}_m is the covariant derivative associated to the Sasaki-Einstein metric g_{SE_5}. Similarly, we find for the contraction $\hat{F}_{5MNOPQ}\Gamma^{MNOPQ}$ of the five-form [84],

$$\hat{F}_{5MNOPQ}\Gamma^{MNOPQ} = -if \cdot 5!(1 \otimes 1 \otimes \sigma^3 + i\sigma^1) \ . \tag{B.31}$$

Thus for $\hat{F}_{5M} := \hat{F}_{5NOPQR}\Gamma^{NOPQR}\Gamma_M$, the second term in (B.29) decomposes as

$$\begin{aligned} \frac{i}{16 \cdot 5!}\hat{F}_{5\mu} &= \frac{1}{16} f \gamma_\mu \otimes 1 \otimes (1 - \sigma^2) \ , \\ \frac{i}{16 \cdot 5!}\hat{F}_{5m} &= 1 \otimes \frac{i}{16} f \rho_m \otimes (1 - \sigma^2) \ . \end{aligned} \tag{B.32}$$

In order to study the supersymmetry preserved by the solution (B.28), we have to introduce Killing spinors. For a complete Riemannian spin manifold (M, g), a Killing spinor χ is defined to be a smooth section of the spin bundle such that

$$\nabla^{\text{SE}}_m \chi = \frac{i\alpha}{2} \rho_m \chi \ , \quad m = 1, ..., \dim M \ . \tag{B.33}$$

Here $\alpha \in \mathbb{R}$ is the Killing constant.[2] We call χ parallel if $\alpha = 0$ and real if $\alpha \neq 0$. We then have the following theorem [78, Theorem 1.14]:

Theorem 8 *A complete simply-connected Sasaki-Einstein manifold admits at least 2 linearly independent real Killing spinors with $\alpha = +1, -1$ for dim $M = 2n - 1$ and $\alpha = +1, +1$ for dim $M = 2n$, respectively. Conversely, a complete Riemannian spin manifold admitting such Killing spinors in the respective dimensions is Sasaki-Einstein with Hol$(\bar{g}) \subset SU(n)$.*

In particular, this implies that every simply-connected Sasaki-Einstein manifold is a spin manifold. Note however that the existence of Killing spinors is more involved in the case that M is not simply-connected. For example, it was explained in [78] that $S^5/\mathbb{Z}_r$ only admits Killing spinors for $r = 0, 3$. Thus in supergravity applications, one usually

[2] Due to the sign difference between (B.24) and (B.25), we define Killing spinors on AdS_5 without the i on the right hand side of (B.33), i.e. $\nabla_\mu \psi = \frac{1}{2}\alpha\gamma_\mu\psi$ for ψ a spinor on AdS.

defines Sasaki-Einstein manifolds to always admit Killing spinors. This is the point of view that we will take in this thesis.

For the solution (B.28) to preserve supersymmetry, the type IIB gravitino variation has to vanish for some ten-dimensional supersymmetry parameter. This parameter is given in terms of two Majorana-Weyl spinors $\epsilon_{\mathcal{A}}$, which can be combined into a single spinor $\tilde{\epsilon} = \epsilon_1 + i\epsilon_2$. Given a real Killing spinor χ on SE_5 provided by Theorem 8, its charge conjugate satisfies the Killing equation with Killing constant $-\alpha$. Using the fact that $(\nabla^{\mathrm{SE}}_m \chi)^c = \nabla^{\mathrm{SE}}_m \chi^c$, this can be seen by taking the charge conjugate of the Killing equation (B.33), i.e.

$$\left(\nabla^{\mathrm{SE}}_m \chi\right)^c = \left(\frac{i\alpha}{2}\rho_m \chi\right)^c = -\frac{i\alpha}{2}\rho_m \chi^c \ . \tag{B.34}$$

Here we used (B.25) in the second step. Thus we can expand the ten-dimensional supersymmetry parameter $\epsilon_{\mathcal{A}}$ as

$$\epsilon_{\mathcal{A}} = \psi_{\mathcal{A}} \otimes \chi \otimes \theta + \psi^c_{\mathcal{A}} \otimes \chi^c \otimes \theta \ . \tag{B.35}$$

Here $\psi_{\mathcal{A}}$, $\psi^c_{\mathcal{A}}$ are four symplectic Majorana spinors of $Spin(1,4)$. This ansatz for $\epsilon_{\mathcal{A}}$ automatically satisfies the ten-dimensional Majorana condition $\epsilon^c_{\mathcal{A}} = \epsilon_{\mathcal{A}}$ by using (B.19). Then the linear combination $\tilde{\epsilon}$ is given by

$$\tilde{\epsilon} = \epsilon_1 + i\epsilon_2 = \tilde{\psi}_1 \otimes \chi \otimes \theta + \tilde{\psi}^c_2 \otimes \chi^c \otimes \theta \ , \tag{B.36}$$

where $\tilde{\psi}_1 = \psi_1 + i\psi_2$ and $\tilde{\psi}^c_2 = \psi^c_1 + i\psi^c_2$. Inserting this ansatz combined with (B.30) and (B.32) into the Gravitino variation, we find

$$\begin{aligned} \delta\hat{\Psi}_\mu &= \left[\nabla_\mu + \frac{1}{16} f \gamma_\mu \otimes 1 \otimes (1-\sigma^2)\right]\tilde{\epsilon} \ , \\ \delta\hat{\Psi}_m &= \left[\nabla_m + 1 \otimes \frac{i}{16} f \rho_m \otimes (1-\sigma^2)\right]\tilde{\epsilon} \ . \end{aligned} \tag{B.37}$$

Since θ is an eigenspinor of σ^2 with eigenvalue -1, the last term in both variations simplifies. $\delta\hat{\Psi}_\mu = 0$ then implies

$$\begin{aligned} \nabla_\mu \tilde{\psi}_1 + \frac{1}{8} f \gamma_\mu \tilde{\psi}_1 &= 0 \ , \\ \nabla_\mu \tilde{\psi}^c_2 + \frac{1}{8} f \gamma_\mu \tilde{\psi}^c_2 &= 0 \ , \end{aligned} \tag{B.38}$$

while $\delta\hat{\Psi}_m = 0$ gives

$$\begin{aligned} \left(\frac{i}{8} f + \frac{i\alpha}{2}\right)\tilde{\psi}_1 &= 0 \ , \\ \left(\frac{i}{8} f - \frac{i\alpha}{2}\right)\tilde{\psi}^c_2 &= 0 \ . \end{aligned} \tag{B.39}$$

In the second step we used the fact that χ and χ^c are Killing spinors on SE_5. However, (B.39) can only be satisfied if one of the two spinors, say $\tilde{\psi}_2$, is zero. Then $\tilde{\psi}^1$ is a Killing spinor on AdS_5 satisfying

$$\nabla_\mu \tilde{\psi}_1 = \frac{\alpha}{2} \gamma_\mu \tilde{\psi}_1 \ . \tag{B.40}$$

Note that then ψ_1^c satisfies the Killing equation with constant $-\alpha$. Thus the ansatz (B.35) gives rise to two symplectic Majorana spinors on AdS_5 and the solution (B.28) preserves eight real supercharges in five dimensions, i.e. $\mathcal{N} = 2$ supersymmetry on AdS_5.

Appendix C

Consistent truncations of type IIB supergravity

In this appendix we briefly review the consistent truncations of type IIB supergravity on Sasaki-Einstein manifolds discussed in [36–43]. To this end, we first explain the ansatz used for the type IIB fields and then present the results in the special case of truncations on $T^{1,1}$ [43, 48, 49] .

Truncations on general Sasaki-Einstein manifolds

We explained in the previous Appendix A that the cone $C(M)$ over a Sasaki-Einstein manifold is Calabi-Yau. Moreover, it was proven in [85] that a real Killing spinor on (M, g) lifts to a parallel spinor on $(C(M), \bar{g})$. Using the fact that a Sasaki-Einstein manifold admits two linearly-independent spinors, we can then construct a nonzero holomorphic $(n, 0)$-form $\Omega_{(C(M)}$ on $C(M)$ [78, 86]. Along the same lines as (A.8), this defines a form Ω on M via[1]

$$\Omega_{C(M)} = t^2(dt^2 + it\eta) \wedge \Omega \ , \tag{C.1}$$

with exterior derivative

$$d\Omega = i3\eta \wedge \Omega \ . \tag{C.2}$$

Equipped with the Sasaki-Einstein structure forms $(\eta, d\eta, \Omega)$, we can now discuss the consistent truncations of type IIB supergravity on Sasaki-Einstein manifolds. We will follow here mostly [37, 39] with notation adapted to [51].

To obtain a consistent truncation to a five-dimensional theory, one expands the spectrum of type IIB supergravity in terms of the forms $(\eta, d\eta, \Omega)$ on the Sasaki-Einstein manifold. Since these forms are invariant under the isometry group of the Sasaki-Einstein manifold, the resulting truncation can be shown to be consistent [37, 39]. The expansion

[1] This is explained for the case $n = 4$ in [87].

of the metric is taken to be

$$\hat{g} = e^{2u_3 - 2u_1} g_5 + e^{2u_1} g_T + e^{-6u_3 - 2u_1} (\eta + A_0) \otimes (\eta + A_0) \ , \tag{C.3}$$

where $u_{1,3}$ are real scalar fields, A_0 is a one-form on the five-dimensional spacetime and g_T is the transverse metric of the Sasaki-Einstein metric.[2] While one assumes trivial dependence on the internal manifold for the Axion-Dilaton $\tau = \hat{\tau}$, the form fields also have to be expended in terms of $(\eta, d\eta, \Omega)$. As an example, we provide here the expansion of the two-forms $\hat{B}_2^a$ since this is where additional moduli will appear in the $T^{1,1}$ truncations. We find

$$\hat{B}_2^a = b_2^a + b_1^a \wedge (\eta + A_0) + \tfrac{1}{2} c^a d\eta + 2\mathrm{Re}\,[b^a \Omega] \ . \tag{C.4}$$

In total, the two-forms contribute $SL(2, \mathbb{R})$-doublets c^a of real scalars, complex scalars b^a, gauge fields b_1^a and two-forms b_2^a. The expansion for the three-forms $\hat{F}_3^a = d\hat{B}_2^a$ can then be obtained from (C.4) by computing the exterior derivative. Similarly, the expansion of the self-dual five-form provides three more scalars, two more gauge fields and two more two-forms. One can show [37, 39] that these fields can be arranged in a gravitational multiplet plus two vector multiplets of $\mathcal{N} = 4$ gauged supergravity in five dimensions[3]:

$$\begin{gathered} \{\text{graviton, 6 vectors, 1 real scalar}\} \\ 2\times\{\text{1 vector, 5 real scalars}\} \ . \end{gathered} \tag{C.5}$$

Here the scalar fields parametrize the manifold [37, 39]

$$SO(1,1) \times \frac{SO(5,2)}{SO(5) \times SO(2)} \ . \tag{C.6}$$

The fact that we obtain an $\mathcal{N} = 4$ supergravity theory in five-dimensions might be a bit surprising at first, since we have shown in Appendix B that type IIB solutions of the form $\mathrm{AdS}_5 \times \mathrm{SE}_5$ only preserve eight supercharges, i.e. $\mathcal{N} = 2$ in five dimensions. To this end, consider the two linearly independent Killing spinors χ, χ^c on SE_5 provided by Theorem 8. Similar to (B.35), we can use these to expand the type IIB gravitino $\hat{\Psi}^{\mathcal{A}}$ as

$$\hat{\Psi}_{\mathcal{A}} = \psi_{\mathcal{A}} \otimes \chi \otimes \theta + \psi_{\mathcal{A}}^c \otimes \chi^c \otimes \theta \ . \tag{C.7}$$

This gives rise to four symplectic Majorana gravitini in five dimensions, which is consistent with an $\mathcal{N} = 4$ supergravity theory. However, for a theory to preserve all supercharges, we must have $\langle \delta_\epsilon \hat{\Psi}^{\mathcal{A}} \rangle = 0$, where ϵ is the supersymmetry parameter. We have shown in Appendix B that for backgrounds of the form $\mathrm{AdS}_5 \times \mathrm{SE}_5$, ϵ has to be a complex spinor such that

$$\epsilon = \psi \otimes \eta \otimes \theta \ , \tag{C.8}$$

[2] The notation $u_{1,3}$ for the real scalars might seem a bit confusing at this point; however, in the case of truncations on $T^{1,1}$ another real scalar labelled u_2 will be present in the expansion of the metric. To keep notation consistent with [51], we therefore label the real scalars in the general case accordingly.

[3] The two-forms were dualized to vector fields in the process.

for ψ a Killing spinor on AdS$_5$. Since the smallest spinor representation of $Spin(1,4)$ is eight-dimensional [83], solutions of the truncated theory only preserve eight real supercharges, i.e. $\mathcal{N}=2$ supersymmetry. Indeed, this was found to be true in [37]. This concludes our review of consistent truncations on general Sasaki-Einstein manifolds. We will now turn to the special case of truncations on $T^{1,1}$.

$\mathcal{N}=4$ truncations of type IIB on $T^{1,1}$

The consistent truncations we studied in the previous section can be generalized for the particular example of $T^{1,1}$ [43, 48, 49]. For this, note that $T^{1,1} \cong S^2 \times S^3$ and thus the second and third de-Rham cohomolgy classes are non-trivial. Hence, we can extend the truncations on general Sasaki-Einstein manifolds by adding a closed, left-invariant two-form $\mathcal{Y} \in H^2(T^{1,1}, \mathbb{R})$ into the expansion, i.e. the set of expansion forms is now $(\eta, d\eta, \Omega, \mathcal{Y})$. Since $T^{1,1}$ can be constructed as an $U(1)$-bundle over $S^2 \times S^2$, there exist two sets of left-invariant $SU(2)$ one-forms σ_i and Σ_i [88]. Define the combinations

$$\begin{aligned} E_1 &= \tfrac{1}{6}(\sigma_1 + i\sigma_2)\ , \\ E_2 &= \tfrac{1}{6}(\Sigma_1 + i\Sigma_2)\ , \\ E_2' &= E_2 + v\bar{E}_1\ , \\ E_5 &= \eta + A_0\ , \\ \eta &= \tfrac{1}{3}(\sigma_3 + \Sigma_3) \end{aligned} \tag{C.9}$$

where v is a complex scalar. Then we can write the expansion of the ten-dimensional metric as [43, 48, 49]

$$\hat{g} = e^{2u_3 - 2u_1} ds_5^2 + e^{2u_1 + 2u_2} E_1 \bar{E}_1 + e^{2u_1 - 2u_2} E_2' \bar{E}_2' + e^{-6u_3 - 2u_1} E_5 E_5\ , \tag{C.10}$$

where $u_{1,2,3}$ are three real scalars. Similarly, we can write

$$\begin{aligned} d\eta &= \tfrac{i}{4}(E_1 \wedge \bar{E}_1 + E_2 \wedge \bar{E}_2)\ , \\ \mathcal{Y} &= \tfrac{i}{2}(E_1 \wedge \bar{E}_1 - E_2 \wedge \bar{E}_2)\ , \\ \Omega &= E_1 \wedge E_2\ . \end{aligned} \tag{C.11}$$

Then we find that the expansion of the two-forms $\hat{B}_2^a$ receives an additional term coming from $\mathcal{Y}$ [43, 48, 49],

$$\hat{B}_2^a = b_2^a + b_1^a \wedge (\eta + A_0) + \tfrac{1}{2} c^a d\eta + e^a \mathcal{Y} + 2\mathrm{Re}\,[b^a \Omega], \tag{C.12}$$

where e^a is a doublet of real scalars. These scalars appear as moduli of AdS backgrounds in the Betti-hyper truncation in Chapter 4. Moreover, the expansion of the three-forms also contains an additional term proportional to $\mathcal{Y} \wedge \eta$, i.e.

$$\hat{F}_3^a = d\hat{B}_2^a + j^a \mathcal{Y} \wedge \eta\ , \tag{C.13}$$

where the j^a are constant topological charges. Similarly, one adds additional terms in the expansion of the five-form $\hat{F}_5$. In total, one finds the following field content in the truncation: the five-dimensional spacetime metric $g_{\mu\nu}$, the graviphoton A_0, four real vectors which we label $\{A^1_\mu, A^2_\mu, b^a_{1\mu}\}$, four real tensor fields $\{b^a_{2\mu\nu}, L_{\mu\nu}, \bar{L}_{\mu\nu}\}$ and 16 real scalars

$$\{u_1, u_2, u_3, c^a, e^a, k, \tau, \bar{\tau}, v, \bar{v}, b^a, \bar{b}^a\} \ . \tag{C.14}$$

These scalars parametrize the coset space [43, 48, 49]

$$SO(1,1) \times \frac{SO(5,3)}{SO(5) \times SO(3)} \ . \tag{C.15}$$

One can then show [43, 48, 49] that these fields can be reorganized into the gravitational multiplet coupled to three vector multiplets[4] in $\mathcal{N} = 4$ supergravity. Thus the addition of the non-trivial cohomology form $\mathcal{Y}$ gives rise to an additional vector multiplet in the truncated theory. Due to its origin, this multiplet is usually referred to as the $\mathcal{N} = 4$ Betti-vector multiplet. This multiplet will be particularly interesting in the study of AdS vacua in $\mathcal{N} = 2$ truncations on $T^{1,1}$ studied in Chapter 4.

[4] The four tensor fields obtained from the expansion of the type IIB fields are dual to four vector fields in five dimensions.

Appendix D

Non-existence of AdS vacua in the NS-sector truncation

Let us briefly discuss the NS-sector truncation (4.5) of type IIB supergravity compactified on $T^{1,1}$ and prove that it does not admit supersymmetric AdS vacua. To begin with, we review the scalar field geometries present in the NS-sector truncation and provide the gauged isometries and associated moment maps [51]. The NS-sector truncation leads to an $\mathcal{N} = 2$ theory that contains the gravity multiplet coupled to two vector multiplets and two hypermultiplets. Even though, this truncation is different from the Betti-vector truncation. The 10 scalar fields[1]

$$\{\phi + 4u_1, u_3, \phi - 4u_1, u_2, c, e, b, \bar{b}, v, \bar{v}\} \tag{D.1}$$

are kept after truncating the RR-sector from type IIB supergravity and then reducing to five-dimensions along the lines of Appendix C. The scalars $\phi + u_1$ and u_3 in the vector multiplets parametrize the projective special real manifold [51]

$$M_V^{NS} = SO(1,1) \times SO(1,1) \ , \tag{D.2}$$

with special geometric data given by

$$h^0 = e^{4u_3} \ , \quad h^1 = e^{-2u_1 - 2u_3 - \phi/2} \ , \quad h^2 = e^{2u_1 - 2u_3 + \phi/2} \ , \tag{D.3}$$

where $C_{012} = \frac{1}{6}$ and all others zero. Lowering the index I according to (2.11) we find

$$h_0 = \tfrac{1}{3} e^{-4u_3} \ , \quad h_1 = \tfrac{1}{3} e^{2u_1 + 2u_3 + \phi/2} \ , \quad h_2 = \tfrac{1}{3} e^{-2u_1 + 2u_3 - \phi/2} \ . \tag{D.4}$$

In the hypermultiplets, the scalars $\{\phi - 4u_1, u_2, c, e, b, \bar{b}, v, \bar{b}\}$ parametrize the quaternionic Kähler manifold [51]

$$M_H^{NS} = \frac{SO(4,2)}{SO(4) \times SO(2)} \ . \tag{D.5}$$

[1] Here we dropped the index $a = 2$ on the fields c^2, e^2, b^2 and $\bar{b}^2$.

The metric on M_H^{NS} can be read off from the kinetic terms of the hypermultiplet Lagrangian

$$\begin{aligned}\mathcal{L}^{NS}_{Hyper} =& -\tfrac{1}{2}e^{-4(u_1+u_2)+\phi}\tilde{g}\wedge *\tilde{g} - \tfrac{1}{2}e^{-4(u_1-u_2)+\phi}\tilde{G}\wedge *\tilde{G} - 4e^{-4u_1+\phi}\tilde{f}\wedge *\bar{\tilde{f}} \\ & -\tfrac{1}{4}d(4u_1-\phi)\wedge *d(4u_1-\phi) - 4du_2\wedge *du_2 - e^{-4u_2}Dv\wedge *D\bar{v}\ ,\end{aligned} \tag{D.6}$$

where

$$\begin{aligned}\tilde{g} &= (1-|v|^2)Dc + (1+|v|^2)De - 4\mathrm{Im}\,(vDb)\ , \\ \tilde{G} &= Dc - De\ , \\ \tilde{f} &= Db + \tfrac{i}{2}\bar{v}(Dc-De)\ .\end{aligned} \tag{D.7}$$

The gauge group G^{NS} acts on M_H^{NS} via Killing vectors [51]

$$\begin{aligned}k_0 &= -(3ib\partial_b - 3i\bar{b}\partial_{\bar{b}}) + (3iv\partial_v - 3i\bar{v}\partial_{\bar{v}}) - j\partial_e\ , \\ k_1 &= 2\partial_c\ , \\ k_2 &= 0\ ,\end{aligned} \tag{D.8}$$

where we denote $j \equiv j^{a=2}$. Moreover, the associated moment maps are given by

$$\begin{aligned}\vec{\mu}_0 =& -[(3-\tfrac{1}{2}e^{\phi/2-2u_1}(e^{-2u_2}((1+|v|^2)j^2 + 2ivf - 2i\bar{v}\bar{f}) - e^{2u_2}j))\vec{e}_3 \\ & - (3\bar{v} + 2ie^{\phi/2-2u_1}(f-\tfrac{i}{2}\bar{v}j))\vec{e}_1 - (3v - 2ie^{\phi/2-2u_1}(\bar{f}+\tfrac{i}{2}vj))\vec{e}_2]\ , \\ \vec{\mu}_1 =& -[e^{\phi/2-2u_1}(e^{-2u_2}(1-|v|^2) + e^{2u_2})\vec{e}_3 - 2\bar{v}e^{\phi/2-2u_1}\vec{e}_1 - 2ve^{\phi/2-2u_1}\vec{e}_2\ , \\ \vec{\mu}_2 =&\ 0\ ,\end{aligned} \tag{D.9}$$

where we denote $f = f^{a=2}$.

With this data at hand, we can easily check that the NS-sector truncation does not admit supersymmetric vacua. For this, we use the conditions (2.79) computed in Section 2.2. We begin by computing the conditions from the third equation $\langle h^I k_I\rangle = 0$ in (2.79). Using the data described above, we find

$$\begin{aligned}0 = \langle h^I k_I\rangle =& -3ie^{4\langle u_3\rangle}\langle b\rangle\partial_b + 3ie^{4\langle u_3\rangle}\langle\bar{b}\rangle\partial_{\bar{b}} + 3ie^{4\langle u_3\rangle}\langle v\rangle\partial_v - 3ie^{4\langle u_3\rangle}\langle\bar{v}\rangle\partial_{\bar{v}} \\ & - e^{4\langle u_3\rangle}j\partial_e + 2e^{-2\langle u_1\rangle-2\langle u_3\rangle-\langle\phi\rangle/2}\partial_c\ .\end{aligned} \tag{D.10}$$

Since all coordinate vector fields appearing in this equation are linearly independent and the function $e^{-2\langle u_1\rangle-2\langle u_3\rangle-\langle\phi\rangle/2}$ cannot vanish for any value of the scalar fields u_1, u_3 and ϕ, this equation has no solution. Thus, without checking the moment map conditions (2.82), we immediately find that the NS-sector truncation does not admit supersymmetric AdS vacua. In particular, we anticipated this result since the Axion-Dilaton is not present in this truncation but it was shown in [25] that always is a modulus for type IIB solutions of the form $\mathrm{AdS}_5\times\mathrm{SE}_5$.

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Eidesstattliche Erklärung

Hiermit versichere ich an Eides statt, die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen benutzt zu haben.

Die eingereichte Fassung entspricht der auf dem elektronischen Speichermedium.

Die Dissertation wurde in der vorgelegten oder einer ähnlichen Form nicht schon einmal in einem früheren Promotionsverfahren angenommen oder als ungenügend beurteilt.

Hamburg, den 8. Mai 2017 Constantin Muranaka