The Quaternion-Kähler Manifold $\mathrm{SO}(4,20)/(\mathrm{SO}(4)\times\mathrm{SO}(20))$ from the c-map and as Moduli Space of K3 Surfaces

Die quaternion-Kählermannigfaltigkeit
$\mathrm{SO}(4,20)/(\mathrm{SO}(4)\times\mathrm{SO}(20))$ aus der c-Abbildung und als
Moduliraum von K3-Flächen

Diplomarbeit

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Zusammenfassung

In dieser Diplomarbeit betrachten wir quaternion-Kählermannigfaltigkeiten, insbesondere den symmetrischen Raum $\frac{SO(4,20)}{SO(4)\times SO(20)}$, und Koordinaten auf diesen. Zum einen erhalten wir diese aus der c-Abbildung, die jeder speziellen Kählermannigfaltigkeit eine quaternion-Kählermannigfaltigkeit zuordnet. Die c-Abbildung wird realisiert durch eine Konstruktion in der $N = 2$ Supergravitation, die kurz zusammengefasst wird. Als Beispiel wird die Klasse symmetrischer spezieller Kählermannigfaltigkeiten $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n-1)}{SO(2)\times SO(n-1)}$ untersucht, denen unter der c-Abbildung die Klasse der quaternion-Kählermannigfaltigkeiten $\frac{SO(4,n+1)}{SO(4)\times SO(n+1)}$ zugeordnet wird. Diese Mannigfaltigkeit tritt, für den Fall $n + 1 = 20$, auch als Moduliraum von K3-Mannigfaltigkeiten auf. Die Moduli erhält man bei der Kompaktifizierung der Typ IIA Supergravitation von 10 auf 6 Dimensionen auf einer solchen K3-Mannigfaltigkeit. Es wird versucht, eine Beziehung zwischen den Koordinaten auf $\frac{SO(4,20)}{SO(4)\times SO(20)}$ herzustellen, die zum einen aus der c-Abbildung erhalten werden und zum anderen von den Modulifeldern der K3-Mannigfaltigkeit.

Abstract

In this diploma thesis quaternion-Kähler manifolds, in particular the symmetric manifold $\frac{SO(4,20)}{SO(4)\times SO(20)}$, and coordinates on them, are considered. On the one hand these are obtained from the c-map which assigns to every special Kähler manifold a quaternion-Kähler manifold. The c-map is realised by a construction in $N = 2$ Supergravity which will be reviewed. As an example the class of symmetric special Kähler manifolds $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n-1)}{SO(2)\times SO(n-1)}$ is considered to which the c-map assigns the class of quaternion-Kähler manifolds $\frac{SO(4,n+1)}{SO(4)\times SO(n+1)}$. This manifold, for the case $n + 1 = 20$, is also related to the moduli space of a K3 surface. The moduli are obtained by compactification of type IIA Supergravity from 10 down to 6 dimensions on the K3 surface. The aim is to establish a relationship between the coordinates on the space $\frac{SO(4,20)}{SO(4)\times SO(20)}$ obtained from the c-map on the one hand and from the moduli fields of the K3 surface on the other.

Contents

Chapter 1

Introduction

The Standard Model of particle physics has been very successful in describing the fundamental particles and their interactions by Quantum Field Theories (QFT's), in particular quantised Yang-Mills theories for the electroweak and strong forces (see e.g. [23]). However, the Standard Model does not include the gravitational force, for which a successful description exists only classically by the General Theory of Relativity (GR, see e.g. [22]). It is possible that a theory that describes all known (and yet unknown) particles and their interactions including gravity may look very different from the familiar theories QFT and GR, although these must be included in some sense as a low-energy or classical limit, respectively. One of the most promising candidates in this regard is String Theory (see e.g. [18]).

In String Theory the fundamental physical objects are one-dimensional vibrating strings as opposed to point-like particles in the Standard Model. In the Standard Model there are several different types of particles, the quarks and leptons (and their antiparticles) which can be generated and observed by today's particle accelerators and detectors, as well as the gauge bosons that mediate the forces between them. In String Theory there is only one type of string and the different kinds of particles are explained as excitations of the different vibrational modes of the string. The string can either be open, i.e. with two lose ends, or closed.

String Theory took its starting point in the late 1960's where it was invented as a possible theory of strong interactions. It occured, however, that in these theories there is always a massless spin-2 particle present. This in particular has led to the insight that String Theory might play a role in the description of a fundamental theory of all interactions including gravity where one needs a spin-2 field to describe the graviton, the particle mediating the gravitational force.

To describe fermions within String Theory one imposes Supersymmetry on the action of the theory which relates the bosonic and fermionic degrees of freedom with one another, resulting in Superstring Theory. There are five known consistent Superstring Theories which all describe strings in a spacetime background of 10 dimensions. These are known as type I, type IIA/B, $SO(32)$-heterotic and $E_8 \times E_8$-heterotic Superstring Theories. The type I theory describes open strings as well as closed strings, whereas the type II and heterotic theories describe closed strings only. In this thesis we are concerned with the low-energy effective theory of type IIA/B String Theories which are $N = 2$ Supergravity theories.

Although the five Superstring Theories seem to be different there are connections between them known as dualities, denoted as S-, T- and U-duality, that suggest that they in fact describe the same physics and can thus be seen as different aspects of one theory only. In this thesis, only T-duality of the type IIA/B theories is considered.

For the concept of T-duality one assumes that one of the 10 dimensions is periodic, i.e. the spacetime manifold has the topology $M^9 \times S^1_R$ where M^9 is 9-dimensional Minkowsi space and S^1_R is a circle of radius R. In this case we say that one dimension of the original 10-dimensional spacetime is compactified. Two theories are said to be related by T-duality if one theory compactified on a circle with radius R is equivalent to the other theory compactified on a circle with radius $\frac{\alpha'}{R}$, where α' is the Regge slope.[1]

To explain our familiar 4-dimensional physical spacetime within String Theory one assumes that 6 of the 10 dimensions are compact and in particular so small that they are not noticeable at scales accessible by today's high energy experiments. This means one assumes that the 10 dimensional spacetime is of the form $M^4 \times Y^6$ where M^4 is Minkowski spacetime and Y^6 is a compact Ricci-flat 6-dimensional manifold. In case where Y^6 is a Calabi-Yau manifold the resulting low-energy effective theory of the compactified type IIA/B theory is an $N = 2$ Supergravity theory in $D = 4$ dimensions, coupled to a number of vector multiplets and hypermultiplets which are the basic particle multiplets in $N = 2$, $D = 4$ Supersymmetry.

The effect of a duality of two theories should also be encountered between their low-energy effective theories as well as in their dimensionally reduced versions. For the type IIA and type IIB theories, compactified on the same Calabi-Yau manifold Y^6, T-duality is implemented by a further dimensional reduction from 4 to 3 spacetime dimensions, performed on the low-energy effective $N = 2$ Supergravity Lagrangian. The result of this procedure is a mapping that relates the vector multiplet sector and the hypermultiplet

[1] α' is the fundamental quantity in String Theory and has dimensions of lenght-squared.

sector of $N = 2$ Supergravity in 4 dimensions with each other. This mapping is referred to as the c-map in the literature [6, 12].

Each vector multiplet contains a complex scalar field and each hypermultiplet four real scalar fields. These fields are interpreted as coordinates on a Riemannian manifold, the target manifold. Supersymmetry restricts the type of manifold which are allowed as target spaces. For vector multiplets and hypermultiplets coupled to $N = 2$ Supergravity these manifolds are restricted to be special Kähler manifolds and quaternion-Kähler manifolds, respectively. Since the c-map maps the vector multiplet sector to the hypermultiplet sector it can be seen as a mapping between these two classes of manifolds. The manifolds in the image of the c-map are called dual quaternion-Kähler manifolds. We will consider as an example of the c-map the symmetric manifolds $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n-1)}{\mathrm{SO}(2)\times\mathrm{SO}(n-1)} \mapsto \frac{\mathrm{SO}(4,n+1)}{\mathrm{SO}(4)\times\mathrm{SO}(n+1)}$.

As an intermediate step of compactification from 10 to 4 dimensions one also considers dimensional reduction on a compact, complex 2-dimensional Calabi-Yau manifold, leading to a theory in 6 spacetime dimensions. The only examples of compact, 2-dimensional Calabi-Yau manifolds are the 4-Tori, which are flat, and the K3 surfaces, which are hyper-Kähler manifolds.

By performing a compactification, one splits the set of coordinates into spacetime and internal coordinates. Also, the components of fields are split in this way. The degrees of freedom of fields ascribed to the internal manifold can be interpreted as additional fields arising in the dimensionally reduced spacetime. These moduli fields describe the variation of the internal manifold over the spacetime manifold.

In this thesis we consider compactification of type IIA Supergravity on a K3 surface, see e.g. [11]. The allowed variations of the K3 manifold are the ones which leave the metric Ricci-flat. The variations are parametrised by a set of 58 moduli fields for the metric. The B-field in the type IIA Supergravity action yields a set of another 22 moduli. The moduli fields together form a set of coordinates on the moduli space. It was shown e.g. in [20] that these 80 fields are invariant under an $\mathrm{SO}(4, 20)$ symmetry and the moduli space is locally of the form $\frac{\mathrm{SO}(4,20)}{\mathrm{SO}(4)\times\mathrm{SO}(20)}$, which is the same manifold as in our example for the c-map.

In this thesis the aim is to compare the description of the manifold $\frac{\mathrm{SO}(4,20)}{\mathrm{SO}(4)\times\mathrm{SO}(20)}$ by the moduli fields of the K3 surface with the description of this manifold obtained from the c-map. Therefore the coordinates obtained from the c-map in [12] have to be brought into a similar form as the one given in [11] that describes the moduli space of the K3 surface.

Outline of the thesis

Chapter 2 gives an overview of $N = 2$ extended Supersymmetry and Supergravity. After some general remarks, the Supersymmetry algebra is written down in section 2.1. The massless representations of the $N = 2$ Supersymmetry algebra are described in section 2.2 and the field contents of the basic $N = 2$ multiplets are given. In section 2.3 the couplings of an arbitrary number of vector multiplets and hypermultiplets to $N = 2$ Supergravity are discussed in turn. For the vector multiplet coupling we follow [7] and explain that the scalar fields lying in the vector multiplets take values on a target space which is restricted to be a special Kähler manifold. For hypermultiplets coupled to Supergravity the scalar fields take values on a quaternion-Kähler manifold, as shown in [3]. The definitions of Kähler, special Kähler and quaternion-Kähler manifolds are given in appendix A.

In *Chapter 3* the c-map, which establishes a relationship between the vector multiplet and the hypermultiplet sectors of $N = 2$ Supergravity, is discussed. The c-map is realised by a construction performed on the $N = 2$ Supergravity Lagrangian. In section 3.1 we review the result of this construction as it is given in [12]. Mathematically speaking, the c-map is a way to construct a quaternion-Kähler metric from a given special Kähler metric. Appendix B gives more details on the calculations that lead to this result. In section 3.2 the calculations are carried out for the specific example $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n-1)}{\mathrm{SO}(2)\times\mathrm{SO}(n-1)}$ of a special Kähler manifold to which is assigned the quaternion-Kähler manifold $\frac{\mathrm{SO}(4,n+1)}{\mathrm{SO}(4)\times\mathrm{SO}(n+1)}$ under the c-map. The metric of the quaternion-Kähler space is given in explicit coordinates by the couplings of the scalar fields of the Lagrangian obtained by the c-map. The aim in section 3.3 is to write the Lagrangian in a way that allows to compare the scalar fields described by it with the moduli fields of a K3 surface, discussed in the next chapter.

In *chapter 4* compactification of type IIA Supergravity on a K3 surface is considered. The compactification gives rise to moduli fields of the K3 surface. Section 4.1 gives a brief introduction on K3 surfaces and the moduli space of Ricci-flat metrics on a K3 surface. The compactification is outlined in 4.2 where we discuss the result of [20] that the full moduli space of the K3 surface is of the form $\frac{\mathrm{SO}(4,20)}{\mathrm{SO}(4)\times\mathrm{SO}(20)}$ which is the same manifold as in our example in section 3.2. In section 4.3 we make a comparison of the moduli fields of a K3 surface with the coordinates that we get in 3.2 from the c-map.

Finally, in *chapter 5* we conclude with a summary and discussion of the work done in this thesis.

Chapter 2

N=2 Supergravity

In this chapter the basic features of Supersymmetry and Supergravity and their extended versions, in particular $N = 2$ Supergravity, are explained for the context in which we need them. For a general introduction to Supersymmetry and Supergravity see for example [23] or [24]. For $N = 2$ Supergravity we refer to [1, 3, 7, 8, 9, 10].

The concept of Supersymmetry was introduced in the 1970's when it was realised that besides the symmetries of abelian and non-abelian gauge field theories which are internal symmetries of the fields, there can be symmetries of a physical theory that relate bosonic and fermionic degrees of freedom with one another. The generators of these symmetries are fermionic in the sense that they transform in a spinor representation of the Lorentz algebra and are composed with each other by an anticommutator rather than a commutator. It was shown in [16] that there is a very restricted way in physics of extending the Poincaré algebra by Supersymmetry generators. The generators of the Poincaré spacetime symmetry together with the Supersymmetry generators form the super-Poincaré algebra.

Supersymmetry also arises in a natural way in String Theory when one describes strings with fermionic degrees of freedom. The low-energy effective theories of the known Superstring Theories are Supergravity theories in 10 dimensions, or Supergravity theories in lower dimensions for compactified String Theories.

In this chapter we review $N = 2$ Supergravity in $D = 4$ dimensions. In section 2.1 the algebra of N-extended Supersymmetry is discussed. An overview of the field contents of massless representations of the $N = 2$ Supersymmetry algebra is given in section 2.2. Then, in section 2.3, the coupling of vector and hypermultiplets to the $N = 2$ Supergravity multiplet is described.

2.1 Supersymmetry Algebra

The spacetime symmetry algebra of a physical theory in $(3+1)$-dimensional spacetime is the Poincaré algebra which is spanned by the momentum operators P^μ and the angular momentum and Lorentz boost operators which are arranged to form a skew-symmetric tensor $M^{\mu\nu}$ with $\mu, \nu = 0, 1, 2, 3$, the commutation relations of which are given by:

$$\begin{array}{r}\text{Poincaré}\\ \text{algebra}\end{array}\left\{\begin{aligned}[][P^\mu, P^\nu] &= 0,\\ [M^{\mu\nu}, P^\rho] &= i(\eta^{\nu\rho}P^\mu - \eta^{\mu\rho}P^\nu),\\ [M^{\mu\nu}, M^{\rho\sigma}] &= i(\eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}),\end{aligned}\right. \tag{2.1}$$

with the metric tensor $\eta = \mathrm{diag}(-1, +1, +1, +1)$.

To extend the Poincaré algebra to include a symmetry between bosons and fermions one can introduce the fermionic symmetry generators Q_α and $\bar{Q}^{\dot\alpha} = (Q_\alpha)^\dagger$, where $\alpha, \dot\alpha = 1, 2$ are Weyl spinor indices, dotted and undotted indices corresponding to transformation under the two different chiral representations of the Lorentz algebra. In general one can have N such generators Q^I_α, $\bar{Q}^{I|\dot\alpha} = (Q^I_\alpha)^\dagger, I = 1, \dots, N$, in which case one speaks of N-extended Supersymmetry. The super-Poincaré algebra is the extension of the Poincaré algebra by these Supersymmetry generators. They obey anticommutation relations with each other and commutation relations with the elements of the Poincaré algebra as follows:

$$\begin{array}{r}\text{supersymmetric}\\ \text{extension}\end{array}\left\{\begin{aligned}[][P^\mu, Q^I_\alpha] &= 0, & [P^\mu, \bar{Q}^{I|\dot\alpha}] &= 0,\\ [Q^I_\alpha, M^{\mu\nu}] &= i(\sigma^{\mu\nu})_\alpha^\beta Q^I_\beta, & [\bar{Q}^{I|\dot\alpha}, M^{\mu\nu}] &= i(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\dot\beta}\bar{Q}^{I|\dot\beta},\\ \{Q^I_\alpha, Q^J_\beta\} &= 0, & \{\bar{Q}^{I|\dot\alpha}, \bar{Q}^{J|\dot\beta}\} &= 0,\\ \{Q^I_\alpha, \bar{Q}^{J|\dot\beta}\} &= 2\delta^{IJ}(\sigma^\mu)^{\dot\beta}_\alpha P_\mu,\end{aligned}\right. \tag{2.2}$$

where $\sigma^\mu = (\mathbf{1}_{2\times2}, \sigma^i)$, $\bar\sigma^\mu = (\mathbf{1}_{2\times2}, -\sigma^i)$, $\sigma^i, i = 1, 2, 3$ being the Pauli matrices and $\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)$ as well as $\bar\sigma^{\mu\nu} = \frac{1}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)$. In the third line of the equations 2.2 we have omitted any central charges that can appear on the right hand sides of both anticommutation relations. However, the central charges are not present in the case where there are no mass parameters present in the Lagrangian of the theory.

2.2 Multiplets of $N = 2$ Supersymmetry

Equations 2.1 and 2.2 together make up the symmetry algebra of a supersymmetric physical theory. In the following representations of the $N = 2$ Supersymmetry algebra on physical states are discussed for the massless case. Since the Supersymmetry generators $Q^I, \bar{Q}^I$ do not commute with the helicity operator $h = \vec{J} \cdot \frac{\vec{p}}{|\vec{p}|}$, where $J^i = \frac{1}{2}\epsilon^{ijk}M_{jk}$, $i, j, k \in \{1, 2, 3\}$, the irreducible representations of the Supersymmetry algebra contain states of different helicities. Acting with a Supersymmetry generator Q^I (or $\bar{Q}^I$) on a state results in a state with helicity raised (lowered) by $\frac{1}{2}$. For the case $N = 2$, starting with a state of highest helicity λ in a multiplet, by acting with $\bar{Q}^1$ or $\bar{Q}^2$ on that state, one obaines two different states of helicity $\lambda - \frac{1}{2}$. Acting successively with $\bar{Q}^1$ and $\bar{Q}^2$ results, since $\bar{Q}^1\bar{Q}^2 = -\bar{Q}^2\bar{Q}^1$, in one state with helicity $\lambda - 1$. Since $\bar{Q}^1\bar{Q}^1 = \bar{Q}^2\bar{Q}^2 = 0$ there are no further states in the multiplet.

In this thesis we consider multiplets with helicities $(2, \frac{3}{2}, 1)$ (Supergravity multiplet), $(1, \frac{1}{2}, 0)$ (vector multiplet) and $(\frac{1}{2}, 0, -\frac{1}{2})$ (hypermultiplet). To make a Lorentz invariant theory, to every multiplet one has to include the CPT conjugate multiplet with opposite helicities. Note, however, that the hypermultiplet is its own CPT conjugate. The multiplets for the different values of λ together with their CPT conjugate multiplets are listed in table 2.1. The field contents of the on-shell representations of the Supersymmetry algebra on multiplets of classical fields for $N = 2$ Supergravity in $D = 4$ spacetime dimensions are, e.g. given in [8]:

- Supergravity multiplet:
$$\{e_\mu^a; \psi_\mu^i; A_\mu\}, \tag{2.3}$$
where e_μ^a is the vierbein of the metric representing the spin-2 graviton, $\psi_\mu^i, i = 1, 2$ is a doublet of spin-$\frac{3}{2}$ gravitini and A_μ is the spin-1 graviphoton.

- Vector multiplet:
$$\{F_{\mu\nu}; \Omega_i; X\}, \tag{2.4}$$
where $F_{\mu\nu}$ is the field strength of a spin-1 gauge boson, $\Omega_i, i = 1, 2$ a doublet of spin-$\frac{1}{2}$ fermions and X a complex scalar.

- Hypermultiplet:
$$\{\chi_i; \phi^a\}, \tag{2.5}$$
where $\phi^a, a = 1, 2, 3, 4$ are four real scalar fields and $\chi_i, i = 1, 2$ is a doublet of spinor fields.

These fields are also listet in table 2.1.

Multiplet	Helicity	CPT Conjugate	Field	Degrees of Freedom
Vector multiplet	1	-1	A_μ	2
	$\frac{1}{2}$	$-\frac{1}{2}$	Ω_i	4
	0	0	X	2
Hypermultiplet	$\frac{1}{2}$	$-$	χ_i	2
	0	$-$	ϕ^a	4
	$-\frac{1}{2}$	$-$	χ_i	2
Supergravity multiplet	2	-2	e_μ^a	2
	$\frac{3}{2}$	$-\frac{3}{2}$	ψ_μ^i	4
	1	-1	A_μ	2

Table 2.1: $N = 2$ supermultiplets

2.3 Coupling of Vector and Hypermultiplets to $N = 2$ Supergravity

In this section we want to discuss the on-shell Lagrangians one obtains by coupling to $N = 2$ Supergravity an arbitrary number of vector multiplets, as for example discussed in [7], and hypermultiplets, described in [3]. We focus on the bosonic parts of the Lagrangians. For the coupling it is important to note that while the physical theory obtained is symmetric under the super-Poincaré group, the underlying symmetry of the off-shell Lagrangian is the superconformal group. For $D = 4$, $N = 2$, the superconformal group is

$$\mathrm{SU}(2,2|N=2) \supset \mathrm{SU}(2,2) \otimes U(1) \otimes \mathrm{SU}(2), \tag{2.6}$$

where the $\mathrm{SU}(2,2)$ factor is identified as the conformal group which contains the Poincaré group enlarged by dilatations and conformal transformations.

Supergravity Multiplet

A representation of the superconformal group in terms of classical fields is given by the Weyl multiplet which contains $24 + 24$ bosonic + fermionic degrees of freedom:

$$\left\{e_\mu^a; \psi_\mu^i; b_\mu; A_\mu; \mathcal{V}_\mu^{ij}; T_{ab}^{ij}; \chi^i; D\right\}, \tag{2.7}$$

Here, e_μ^a is the vierbein of the metric representing the graviton and ψ_μ^i is the gavitino doublet ($i = 1, 2$). The fields b_μ and A_μ are the gauge fields of dilatational symmetry and $U(1)$ transformations, respectively, and the antihermitean traceless tensor $\mathcal{V}_\mu^{ij}, i, j = 1, 2$ contains the gauge fields for the SU(2) transformations. The real tensor T_{ab}^{ij}, antisymmetric in the SU(2)-indices i, j as well as in the Lorentz indices $a, b = 0, 1, 2, 3$, the spinor doublet χ^i and the real scalar D are auxilliary fields, which are eliminated by their equations of motion.

Upon fixing the dilatational, $U(1)$ and SU(2) symmetries, of the fields in the Weyl multiplet only the vierbein e_μ^a and gravitini ψ_μ^i remain as physical degrees of freedom. The physical graviphoton in 2.3 is coming, as we will see in the next section, from the spin-1 field of a vector multiplet.

Vector Multiplets Coupled to Supergravity

To couple n vector multiplets to Supergravity one introduces $n+1$ vector multiplets to start with, labelled by $I = 0, \ldots, n$. One of the $n+1$ vector multiplets ($I = 0$ by convention) is a compensating multiplet for the remaining gauge degrees of freedom of the superconformal group.

The scalars X^I span an $(n+1)$-dimensional complex space but as a result of the $U(1)$ and dilatational symmetry one of them can be eliminated by going to inhomogeneous coordinates $Z^A = X^A/X^0, A = 1, \ldots, n$. These fields Z^A form coordinates on an n-dimensional complex manifold which is restricted to be a projective (or local) special Kähler manifold (c.f. appendix A.3).

The result of coupling vector multiplets to $N = 2$ Supergravity is a Lagrangian that is encoded in a single holomorphic function $F(X^0, \ldots, X^n)$, called the prepotential, which has to be of homogeneous degree 2, that is $F(\lambda X^0, \ldots, \lambda X^n) = \lambda^2 \cdot F(X^0, \ldots, X^n)$. The Lagrangian is given by

$$e^{-1}\mathscr{L}_{vec} = \frac{1}{2}R - K_{A\overline{B}}\partial_\mu Z^A \partial^\mu \bar{Z}^{\overline{B}} + \frac{1}{4}(\mathfrak{Im}\mathscr{N})_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} - \frac{1}{4}(\mathfrak{Re}\mathscr{N})_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}, \tag{2.8}$$

where the field strength tensors $F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu$ are derived from vector potentials A^I_μ and $\tilde{F}^I_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{I|\rho\sigma}$ is the dual field strength tensor. Also, R is the Ricci scalar and e the determinant of the vierbein e^a_μ. The Kähler metric $K_{A\overline{B}} = \frac{\partial^2 K}{\partial Z^A \partial \bar{Z}^{\overline{B}}}$ of the projective special Kähler target manifold is computed from the Kähler potential given by (c.f. definition A.10)

$$K = -\ln i\left(\bar{X}^I F_I - X^I \bar{F}_I\right), \tag{2.9}$$

where F_I denotes the derivative of the prepotential with respect to X^I.

The matrix $\mathscr{N}$ that describes the couplings of the field strengths $F^I_{\mu\nu}$ of the vector bosons A^I_μ is given by

$$\mathscr{N}_{IJ} = \bar{F}_{IJ} + 2i\frac{(\mathfrak{Im}F_{IK})(\mathfrak{Im}F_{JL})X^K X^L}{(\mathfrak{Im}F_{LK})X^K X^L}, \tag{2.10}$$

where F_{IJ} are the second derivatives of the prepotential.

Hypermultiplets Coupled to Supergravity

The coupling of hypermultiplets to $N = 2$ Supergravity has been worked out in [3]. Every hypermultiplets contains four real scalar fields. For the case in which there are m hypermultiplets present we denote these scalar fields by $\phi^u, u = 1, \ldots, 4m$. These scalar fields form coordinates on a real $4m$-dimensional target manifold which, shown in [3], is restricted by Supergravity to be a quaternion-Kähler manifold (c.f. appendix A.2 for the definition). The result of the coupling is a Lagrangian the bosonic part of which is given by

$$e^{-1}\mathscr{L}_{hyp} = \frac{1}{2}R + h_{uv}\partial_\mu\phi^u\partial^\mu\phi^v, \tag{2.11}$$

where $h_{uv}, u, v = 1, \ldots, 4m$ is the quaternion-Kähler metric of the target manifold of the scalar fields.

Summary

From the two previous paragraphs, the Lagrangian of an arbitrary number of n vector multiplets and m hypermultiplets coupled to $N = 2$ Supergravity can be summarised as

$$\begin{aligned} e^{-1}\mathscr{L} =& \frac{1}{2}R + h_{uv}\partial_\mu\phi^u\partial^\mu\phi^v - K_{A\overline{B}}\partial_\mu Z^A\partial^\mu\bar{Z}^{\overline{B}} \\ &+ \frac{1}{4}(\mathfrak{Im}\mathscr{N})_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} - \frac{1}{4}(\mathfrak{Re}\mathscr{N})_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}. \end{aligned} \tag{2.12}$$

The Lagrangian is fully specified by the quaternion-Kähler metric h_{uv} and the holomorphic prepotential F. The Kähler potential K is obtained from equation 2.9 and the coupling matrix $\mathscr{N}$ of the fields $F^I_{\mu\nu}$ from equation 2.10.

Chapter 3

Quaternion-Kähler manifolds from the c-map

As mentioned in the introduction, the c-map of $N = 2$ Supergravity which we are going to discuss in the next section, is related to T-dualtity of type IIA/IIB Superstring Theory. The known Superstring Theories are defined in 10 dimensions. To explain our familiar 4-dimesional physical spacetime within String Theory one assumes that 6 of the 10 dimensions are compact, spacetime being of the form $M^4 \times Y^6$ where M^4 is Minkowski space and Y^6 is a 6-dimensional compact Ricci-flat manifold, which we take to be a Calabi-Yau 3-fold (3 complex dimensions = 6 real dimensions).[1] The low-energy effective theory of type IIA/B String Theories compactified in this way is an $N = 2$ Supergravity in 4 dimensions coupled to vector and hypermultiplets, the Lagrangian of which we have described in section 2.3.

T-duality is now implemented by a further dimensional reduction of the $N = 2$ Supergravity Lagrangian on a circle with radius R from 4 to 3 dimensions. This work was carried out in [12], which we will review in section 3.1. The effect of the construction is that the vector multiplet sector is mapped onto the hypermultiplet sector. Since the target manifolds of vector and hypermultiplets are special Kähler and quaternion-Kähler manifolds, respectively, the c-map can be viewed as a mapping between these two classes of manifolds

$$c : \{\text{special Kähler manifolds}\} \to \{\text{quaternion-Kähler manifolds}\}\,. \qquad (3.1)$$

In section 3.2 we consider as an example for the c-map the classes of symmetric manifolds $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n-1)}{\mathrm{SO}(2)\times\mathrm{SO}(n-1)} \mapsto \frac{\mathrm{SO}(4,n+1)}{\mathrm{SO}(4)\times\mathrm{SO}(n+1)}$, which was also discussed in [1]. We then analyse coordinates on these manifolds in section 3.3.

[1] For a definition of Calabi-Yau manifolds see appendix A.1.

3.1 The c-map of $N = 2$ Supergravity

As discussed in detail in [6], the c-map is a mapping between the target spaces of the vector multiplet and hypermultiplet sectors of $N = 2$ Supergravity, special Kähler and quaternion-Kähler manifolds, respectively. It thus gives an explicit way of constructing quaternion-Kähler metrics. It is named after a similar construction of E. Calabi in [5] where the first examples of hyper-Kähler metrics were constructed. The explicit computation of the metric on the quaternion-Kähler spaces is given in [12]. We now outline the steps of this construction and discuss the result. More details of the calculations can be found in appendix B. There we also compute explicitely the three fundamental 2-forms associated with the quaternionic structure of the quaternion-Kähler manifold.

The first step of the construction is a dimensional reduction from 4 to 3 dimensions of the $N = 2$ vector multiplet Lagrangian 2.8:

$$e^{-1}\mathscr{L}_{vec} = \frac{1}{2}R - K_{A\overline{B}}\partial_\mu Z^A \partial^\mu \bar{Z}^{\overline{B}} + \frac{1}{4}(\mathfrak{Im}\mathscr{N})_{IJ} F^I_{\mu\nu} F^{J|\mu\nu} - \frac{1}{4}(\mathfrak{Re}\mathscr{N})_{IJ} F^I_{\mu\nu} \tilde{F}^{J|\mu\nu}, \tag{3.2}$$

with $F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu$. By dimensional reduction, every 4-vector field A^I_μ splits into a 3-vector $\hat{A}^I_{\hat{\mu}}$ and a scalar $\zeta^I := A^I_3$, whereas the vierbein e^a_μ splits into a dreibein $e^{\hat{a}}_{\hat{\mu}}$, a 3-vector $B_{\hat{\mu}}$ and a scalar ϕ. Here, indices with a hat run only over $0, 1, 2$ whereas normal indices run over $0, 1, 2, 3$. The scalar fields $Z^A, \bar{Z}^A, A = 1, \ldots, n$ are reduced to scalar fields in 3 dimensions which we denote by $\hat{Z}^A, \hat{\bar{Z}}^A$.

In 3 dimensions, the 3-tensor field strengths $F^I_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}}\hat{A}^I_{\hat{\nu}} - \partial_{\hat{\nu}}\hat{A}^I_{\hat{\mu}}$ and $H_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}}B_{\hat{\nu}} - \partial_{\hat{\nu}}B_{\hat{\mu}}$ can be dualised to scalars fields. (In 3 dimensions, an antisymmetric tensor is Hodge dual to a vector field; then, these vector fields can be dualised to scalars by Lagrange multipliers, for details see appendix B). In this way the 3-vectors $\hat{A}^I_{\hat{\mu}}$ and $B_{\hat{\mu}}$ are replaced in the Lagrangian by scalar fields which we denote by $\tilde{\zeta}^I$ and $\tilde{\phi}$, respectively. The resulting Lagrangian now describes a theory of only scalars in three dimensions, namely the $4n + 4$ fields ϕ, $\tilde{\phi}$, ζ^I, $\tilde{\zeta}^I$, $I = 0, \ldots, n$ and $\hat{Z}^A$, $\hat{\bar{Z}}^A$, $A = 1, \ldots, n$.

This 3-dimensional theory is now reinterpreted again as a theory in 4 dimensions. The resulting Lagrangian is given by (c.f. equation B.20 of appendix B):

$$e^{-1}\tilde{\mathscr{L}} = \frac{1}{2}R - K_{A\bar{B}}\partial_\mu \hat{Z}^A \partial^\mu \hat{\bar{Z}}^{\bar{B}} + \frac{1}{4\phi^2}\left((\partial_\mu\phi)^2 + \left(\partial_\mu\tilde{\phi} - \tilde{\zeta}_I\partial_\mu\zeta^I + \zeta^I\partial_\mu\tilde{\zeta}_I\right)^2\right) + \frac{1}{2\phi}\left(\mathscr{N}_{IK}\partial_\mu\zeta^K + \partial_\mu\tilde{\zeta}_I\right)(\mathfrak{Im}\mathscr{N})^{-1|IJ}\left(\bar{\mathscr{N}}_{JL}\partial^\mu\zeta^L + \partial^\mu\tilde{\zeta}_J\right). \tag{3.3}$$

It is shown in [12] that the metric of the manifold described by this Lagrangian is quaternion-Kähler. It is thus a possible target manifold for hypermultiplets coupled to $N = 2$ supergravity. In fact, the Lagrangian 3.3 describes the bosonic part of a theory of $n + 1$ hypermultiplets coupled to $N = 2$ supergravity. The effect of the described procedure thus is that the vector multiplet sector of $N = 2$ supergravity is mapped onto the hypermultiplet sector (and vice versa, by the inverse c-map which we have not considered here). In this way, to each special Kähler target manifold of the vector multiplets is assigned a quaternion-Kähler manifold as possible target manifold of the scalars in the hypermultiplets. Note, however, that not every quaternion-Kähler manifold can be obtained in this way. The manifolds in the image of the c-map are called dual quaternion-Kähler manifolds.

The Lagrangian 3.3 shows that the hypermultiplet sector with a dual quaternion-Kähler target manifold - like the vector multiplet Lagrangian - can be encoded in a single holomorphic function F. Note, however, that in the case of both vector and hypermultiplets coupled to Supergravity we need two holomorphic functions, one for the vector multiplet sector and one for the hypermultiplet sector. Also note that the fields $\hat{Z}^A, \hat{\bar{Z}}^A$ of the hypermultiplet sector are then indepentent of the fields $Z^A, \bar{Z}^A$ from the vector multiplet sector. The Lagrangian of a full theory in 2.12 with h_{uv} taken to be a dual quaternion-Kähler metric is thus given by

$$\begin{aligned} e^{-1}\mathscr{L} = &\frac{1}{2}R - K_{A\overline{B}}\partial_\mu Z^A \partial^\mu \bar{Z}^{\overline{B}} + \frac{1}{4}(\mathfrak{Im}\mathscr{N})_{IJ} F^I_{\mu\nu} F^{J|\mu\nu} - \frac{1}{4}(\mathfrak{Re}\mathscr{N})_{IJ} F^I_{\mu\nu} \tilde{F}^{J|\mu\nu} \\ &- \hat{K}_{A\bar{B}}\partial_\mu \hat{Z}^A \partial^\mu \hat{\bar{Z}}^{\bar{B}} + \frac{1}{4\phi^2}\left((\partial_\mu \phi)^2 + \left(\partial_\mu \tilde{\phi} - \tilde{\zeta}_I \partial_\mu \zeta^I + \zeta^I \partial_\mu \tilde{\zeta}_I \right)^2 \right) \\ &+ \frac{1}{2\phi}\left(\hat{\mathscr{N}}_{IK} \partial_\mu \zeta^K + \partial_\mu \tilde{\zeta}_I \right) (\mathfrak{Im}\hat{\mathscr{N}})^{-1|IJ} \left(\bar{\hat{\mathscr{N}}}_{JL} \partial^\mu \zeta^L + \partial^\mu \tilde{\zeta}_J \right), \end{aligned} \tag{3.4}$$

where the Kähler metric K and matrix $\mathscr{N}$ is derived from a holomorphic function $F(Z)$ and $\hat{K}$ and $\hat{\mathscr{N}}$ are derived from an in general different holomorphic function $\hat{F}(\hat{Z})$.

In the following sections we will only consider the part of the Lagrangian describing the dual quaternion-Kähler space and therefore will again omit the hats there.

3.2 An Example of Symmetric Manifolds

In this section we want to compute all data for the example of the c-map assigning to each other the symmetric spaces [1, 6]:

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n-1)}{\mathrm{SO}(2)\times\mathrm{SO}(n-1)} \leftrightarrow \frac{\mathrm{SO}(4,n+1)}{\mathrm{SO}(4)\times\mathrm{SO}(n+1)}, \quad n \geq 2. \tag{3.5}$$

The special Kähler manifold $\mathrm{SU}(1,1)/\mathrm{U}(1)\times\mathrm{SO}(2,n-1)/(\mathrm{SO}(2)\times\mathrm{SO}(n-1))$ can be described by a holomorphic prepotential (c.f. appendix A.3) which is given by [1]:

$$\begin{aligned} F(X^0,\ldots,X^n) &= \frac{X^1}{2X^0}\cdot\eta_{MN}X^M X^N, \\ \eta_{MN} &= \mathrm{diag}(-1,+1,\ldots,+1), \quad M,N = 2,\ldots,n. \end{aligned} \tag{3.6}$$

The Kähler potential K is then given by (see appendix A, definition A.10)

$$\exp(-K) = i(\bar{X}^I F_I - X^I \bar{F}_I) = \frac{-i}{2}(Z^1-\bar{Z}^1)\cdot\eta_{MN}(Z^M-\bar{Z}^M)(Z^N-\bar{Z}^N), \tag{3.7}$$

where $I = 0,\ldots,n$ and we have set $X^0 = 1$ by introducing the inhomogeneous coordinates $X^I := Z^I = (1, Z^A)$ with $Z^A = X^A/X^0$, $A = 1,\ldots,n$. From K, the Kähler metric $K_{A\bar{B}} = \frac{\partial^2 K}{\partial Z^A \partial \bar{Z}^B}$ is computed to be

$$K_{A\bar{B}} = \begin{pmatrix} -\frac{1}{(Z^1-\bar{Z}^1)^2} & \mathbf{0} \\ \mathbf{0} & \frac{2\eta_{A\bar{B}}}{(Z_M-\bar{Z}_M)(Z^M-\bar{Z}^M)} - \frac{4\eta_{AM}(Z^M-\bar{Z}^M)\eta_{\bar{B}N}(Z^N-\bar{Z}^N)}{((Z_M-\bar{Z}_M)(Z^M-\bar{Z}^M))^2} \end{pmatrix}, \tag{3.8}$$

where we have written $Z_M = \eta_{MN}Z^N$, but note that M,N only run over $2,\ldots,n$.

The matrix F_{IJ} (the second derivatives of the prepotential) is given by

$$F_{IJ} = \begin{pmatrix} Z^1\cdot(Z_M Z^M) & -\frac{1}{2}(Z_M Z^M) & -Z^1\cdot Z_J \\ -\frac{1}{2}(Z_M Z^M) & 0 & Z_J \\ -Z^1\cdot Z_I & Z_I & Z^1\cdot\eta_{IJ} \end{pmatrix}. \tag{3.9}$$

We can now compute the coupling matrix $\mathcal{N}$ of the gauge bosons. Therefore note that

$$\begin{aligned}(F_{IJ}-\bar{F}_{IJ})Z^J = \Big(&-\frac{1}{2}Z^1\cdot(Z_M Z^M - \bar{Z}_M\bar{Z}^M) + \bar{Z}^1\cdot\bar{Z}_M(Z^M-\bar{Z}^M), \\ &\frac{1}{2}(Z_M-\bar{Z}_M)(Z^M-\bar{Z}^M), (Z^1-\bar{Z}^1)\cdot(Z_I-\bar{Z}_I)\Big), \end{aligned} \tag{3.10}$$

as well as

$$Z^I(F_{IJ}-\bar{F}_{IJ})Z^J=(Z^1-\bar{Z}^1)\cdot(Z_M-\bar{Z}_M)(Z^M-\bar{Z}^M). \tag{3.11}$$

From equation 2.10 we then get

$$\begin{aligned}\mathscr{N}_{IJ}&=\bar{F}_{IJ}+\frac{(F_{IK}-\bar{F}_{IK})Z^K(F_{JL}-\bar{F}_{JL})Z^L}{(Z^1-\bar{Z}^1)(Z_M-\bar{Z}_M)(Z^M-\bar{Z}^M)}\\ &=\begin{pmatrix} P & Q & R_J\\ Q & \frac{(Z_M-\bar{Z}_M)(Z^M-\bar{Z}^M)}{4(Z^1-\bar{Z}^1)} & \frac{1}{2}(Z_J+\bar{Z}_J)\\ R_I & \frac{1}{2}(Z_I+\bar{Z}_I) & \bar{Z}^1\eta_{IJ}+\frac{(Z^1-\bar{Z}^1)(Z_I-\bar{Z}_I)(Z_J-\bar{Z}_J)}{(Z_M-\bar{Z}_M)(Z^M-\bar{Z}^M)}\end{pmatrix},\end{aligned} \tag{3.12}$$

where the abbreviations $P,Q,R_M,M=2,\ldots,n$ stand for

$$\begin{aligned}P=&\Big(\big(Z^1/2\cdot(Z_MZ^M+\bar{Z}_M\bar{Z}^M)-\bar{Z}^1\cdot Z_M\bar{Z}^M\big)^2-(Z^1-\bar{Z}^1)^2\cdot\\ &\cdot(Z_MZ^M)(\bar{Z}_N\bar{Z}^N)\Big)\Big/\Big((Z^1-\bar{Z}^1)\cdot(Z_M-\bar{Z}_M)(Z^M-\bar{Z}^M)\Big),\\ Q=&\frac{-Z^1/2\cdot(Z_MZ^M+\bar{Z}_M\bar{Z}^M)+\bar{Z}^1\cdot Z_M\bar{Z}^M}{2(Z^1-\bar{Z}^1)},\\ R_M=&\Big(-Z^1/2\cdot(Z_M-\bar{Z}_M)\cdot(Z_NZ^N-\bar{Z}_N\bar{Z}^N)+\\ &\bar{Z}^1\cdot(Z_M\bar{Z}_N-\bar{Z}_MZ_N)(Z^N-\bar{Z}^N)\Big)\Big/\Big((Z_N-\bar{Z}_N)(Z^N-\bar{Z}^N)\Big).\end{aligned} \tag{3.13}$$

To compute all the couplings in the Lagrangian 3.3 one also needs to know the real and imaginary parts of the matrix $\mathscr{N}$, as well as their inverses. The real part of $\mathscr{N}$ is given by

$$(\mathfrak{Re}\mathscr{N})_{IJ}=\begin{pmatrix}\frac{1}{8}(Z^1+\bar{Z}^1)\cdot(Z_M+\bar{Z}_M)^2 & -\frac{1}{8}(Z_M+\bar{Z}_M)^2 & -\frac{1}{4}(Z^1+\bar{Z}^1)(Z_J+\bar{Z}_J)\\ -\frac{1}{8}(Z_M+\bar{Z}_M)^2 & 0 & \frac{1}{2}(Z_J+\bar{Z}_J)\\ -\frac{1}{4}(Z^1+\bar{Z}^1)(Z_I+\bar{Z}_I) & \frac{1}{2}(Z_I+\bar{Z}_I) & \frac{1}{2}(Z^1+\bar{Z}^1)\eta_{IJ}\end{pmatrix}, \tag{3.14}$$

where we used the shorthand notation $(Z_M-\bar{Z}_M)^2=(Z_M-\bar{Z}_M)(Z^M-\bar{Z}^M)$. This matrix can be easily inverted:

$$(\mathfrak{Re}N)^{-1|IJ}=\begin{pmatrix}\frac{32}{(Z^1+\bar{Z}^1)(Z_M+\bar{Z}_M)^2} & \frac{8}{(Z_M+\bar{Z}_M)^2} & \frac{8(Z^J+\bar{Z}^J)}{(Z^1+\bar{Z}^1)(Z_M+\bar{Z}_M)^2}\\ \frac{8}{(Z_M+\bar{Z}_M)^2} & 0 & \frac{4(Z^J+\bar{Z}^J)}{(Z_M+\bar{Z}_M)^2}\\ \frac{8(Z^I+\bar{Z}^I)}{(Z^1+\bar{Z}^1)(Z_M+\bar{Z}_M)^2} & \frac{4(Z^I+\bar{Z}^I)}{(Z_M+\bar{Z}_M)^2} & \frac{2\eta^{IJ}}{Z^1+\bar{Z}^1}\end{pmatrix}. \tag{3.15}$$

The imaginary part of $\mathscr{N}$ is given by

$$\mathfrak{Im}\mathscr{N} = \begin{pmatrix} \mathfrak{Im}P & \mathfrak{Im}Q & \mathfrak{Im}R_J \\ \mathfrak{Im}Q & \frac{-i(Z_M-\bar{Z}_M)^2}{4(Z^1-\bar{Z}^1)} & 0 \\ \mathfrak{Im}R_I & 0 & i(Z^1-\bar{Z}^1)\left[\frac{1}{2}\eta_{IJ} - \frac{(Z_I-\bar{Z}_I)(Z_J-\bar{Z}_J)}{(Z_M-\bar{Z}_M)^2}\right] \end{pmatrix}, \tag{3.16}$$

with the abbreviations

$$\begin{aligned}
\mathfrak{Im}P =& \Big[Z^1\bar{Z}^1(Z_M-\bar{Z}_M)\left(Z^M(\bar{Z}_N\bar{Z}^N) - \bar{Z}^M(Z_NZ^N)\right) + \left((Z^1)^2+(\bar{Z}^1)^2\right)\cdot \\
& \cdot \left(\frac{1}{2}(Z_M\bar{Z}^M)^2 - (Z_MZ^M)(\bar{Z}_N\bar{Z}^N) + \frac{1}{8}(Z_MZ^M+\bar{Z}_M\bar{Z}^M)^2\right)\Big]\Big/ \\
& \left[i(Z^1-\bar{Z}^1)\cdot(Z_M-\bar{Z}_M)(Z^M-\bar{Z}^M)\right], \\
\mathfrak{Im}Q =& \frac{i(Z^1+\bar{Z}^1)(Z_M-\bar{Z}_M)^2}{8(Z^1-\bar{Z}^1)}, \\
\mathfrak{Im}R_M =& \frac{i(Z^1-\bar{Z}^1)}{2(Z_N-\bar{Z}_N)(Z^N-\bar{Z}^N)}\Big[\frac{1}{2}(Z_M-\bar{Z}_M)(Z_NZ^N-\bar{Z}_N\bar{Z}^N) \\
& + (Z_M\bar{Z}_N-\bar{Z}_MZ_N)(Z^N-\bar{Z}^N)\Big].
\end{aligned} \tag{3.17}$$

To invert this matrix one can use the identity

$$(\mathfrak{Im}\mathscr{N})^{-1|IJ} = 2i\left(-N^{-1|IJ} + \frac{\bar{Z}^IZ^J + Z^I\bar{Z}^J}{\bar{Z}NZ}\right), \tag{3.18}$$

with $N_{IJ} = F_{IJ} - \bar{F}_{IJ}$ and $\bar{Z}NZ = Z^IN_{IJ}Z^J$. This formula can be checked by multiplying with $\mathfrak{Im}\mathscr{N}$ obtained from equation 2.10. From equation 3.9 one can compute $F_{IJ} - \bar{F}_{IJ}$ and can check that

$$\begin{aligned}
N^{-1|IJ} =& \left[(Z^1-\bar{Z}^1)(Z_M-\bar{Z}_M)(Z^M-\bar{Z}^M)\right]^{-1}\cdot \\
& \begin{pmatrix} 4 & 2(Z^1+\bar{Z}^1) & 2(Z^J+\bar{Z}^J) \\ 2(Z^1+\bar{Z}^1) & 4Z^1\bar{Z}^1 & 2(Z^1Z^J+\bar{Z}^1\bar{Z}^J) \\ 2(Z^I+\bar{Z}^I) & 2(Z^1Z^I+\bar{Z}^1\bar{Z}^I) & 2(Z^I\bar{Z}^J+\bar{Z}^IZ^J)+(Z_M-\bar{Z}_M)^2\eta^{IJ} \end{pmatrix}.
\end{aligned} \tag{3.19}$$

Also,

$$\bar{Z}NZ = \bar{Z}^IN_{IJ}Z^J = -\frac{1}{2}(Z^1-\bar{Z}^1)(Z_M-\bar{Z}_M)^2. \tag{3.20}$$

Now insert 3.19 and 3.20 into 3.18. Although some of the entries of $\mathfrak{Im}\mathscr{N}$ look rather complicated, $(\mathfrak{Im}\mathscr{N})^{-1}$ has a somewhat simpler form:

$$(\mathfrak{Im}\mathscr{N})^{-1|IJ} = -2i\left[(Z^1 - \bar{Z}^1)(Z_M - \bar{Z}_M)(Z^M - \bar{Z}^M)\right]^{-1} \cdot$$
$$\begin{pmatrix} 8 & 4(Z^1 + \bar{Z}^1) & 4(Z^J + \bar{Z}^J) \\ 4(Z^1 + \bar{Z}^1) & 8Z^1\bar{Z}^1 & 4(Z^1 Z^J + \bar{Z}^1\bar{Z}^J) \\ 4(Z^I + \bar{Z}^I) & 4(Z^1 Z^J + \bar{Z}^1\bar{Z}^J) & 4(Z^I\bar{Z}^J + \bar{Z}^I\bar{Z}^J) + (Z_M - \bar{Z}_M)^2\eta^{IJ} \end{pmatrix}. \tag{3.21}$$

We have thus computed all the data needed to describe the couplings in the Lagrangian 3.3.

3.3 Coordinates on Coset Spaces

The couplings of the scalar fields in the Lagrangian 3.3, computed in section 3.2, describe a metric on the space $\mathrm{SO}(4,n)/(\mathrm{SO}(4)\times\mathrm{SO}(n))$. Our aim in this section is to arrange the fields into a matrix $\mathscr{M} \in \mathrm{SO}(4,n)$ in a way that the Lagrangian can be written in the form $\mathrm{Tr}(\partial_\mu \mathscr{M}^{-1}\partial^\mu \mathscr{M})$. The motivation for this is that in chapter 4 we want to compare these coordinates to coordinates on the moduli space of K3 surfaces, for which the Lagrangian is written in this form in [11]. We start with a little more general approach.

3.3.1 Coordinates on $\mathrm{SO}(m,n)/(\mathrm{SO}(m)\times\mathrm{SO}(n))$

We consider symmetric spaces of the form $M = \mathrm{SO}(m,n)/(\mathrm{SO}(m)\times\mathrm{SO}(n))$. For the discussion we refer to [14]. On the level of Lie algebras we can write a representative of the coset $\mathfrak{so}(m,n)/(\mathfrak{so}(m)\oplus\mathfrak{so}(n))$ as a matrix

$$\left(\begin{array}{c|c} \mathbf{0} & B \\ \hline B^T & \mathbf{0} \end{array}\right), \quad B \in \mathrm{Mat(m \times n, \mathbb{R})}. \tag{3.22}$$

We get a representative of an element in M by exponentiating this matrix:

$$\mathscr{M} := \exp\left(\begin{array}{c|c} \mathbf{0} & B \\ \hline B^T & \mathbf{0} \end{array}\right) = \left(\begin{array}{c|c} \sqrt{\mathbf{1}+qq^T} & q \\ \hline q^T & \sqrt{\mathbf{1}+q^Tq} \end{array}\right), \tag{3.23}$$

where $q \in \mathrm{Mat}(m\times n, \mathbb{R})$ is given by:

$$q = B\left(\frac{\sinh B^TB}{B^TB}\right)^{\frac{1}{2}}. \tag{3.24}$$

Here, the divison by B^TB has to be understood formally for the power series expansion of sinh in B^TB since the matrix B^TB need not be invertible. From 3.24 one obtains for the $m\times m$ and $n\times n$ submatrices of $\mathscr{M}$:

$$\sqrt{1+qq^T} = (\cosh BB^T)^{1/2}, \quad \sqrt{1+q^Tq} = (\cosh B^TB)^{1/2}. \tag{3.25}$$

We now want to discuss the two examples of $\mathrm{SO}(2,n)/(\mathrm{SO}(2)\times\mathrm{SO}(n))$ and $\mathrm{SO}(4,n)/(\mathrm{SO}(4)\times\mathrm{SO}(n))$. We refer to the appendix of [1] where coordinates for the first example are called Calabi-Vesentini coordinates. By a simple analogy we then make a similar approach for the second example.

Calabi-Vesentini coordinates on $\mathrm{SO}(2,n)/(\mathrm{SO}(2)\times\mathrm{SO}(n))$

Here, the submatrix q of the coset representative $\mathscr{M}$ in (3.23) is a $2\times n$ matrix and we assemble the first two rows $\mathscr{M}_0^\Lambda, \mathscr{M}_1^\Lambda$ of the matrix $\mathscr{M}$ into one row of complex numbers:

$$\Phi^\Lambda = \frac{1}{\sqrt{2}}(\mathscr{M}_0^\Lambda + i\mathscr{M}_1^\Lambda), \quad \Lambda = 0,\dots,n+1. \tag{3.26}$$

The fact that $\mathscr{M}\in\mathrm{SO}(2,n)$ gives the orthonormality conditions

$$\bar{\Phi}^\Sigma\Phi^\Lambda\eta_{\Sigma\Lambda} = 1, \quad \Phi^\Sigma\Phi^\Lambda\eta_{\Sigma\Lambda} = 0, \quad \eta = \mathrm{diag}(1,1,-1,\dots,-1). \tag{3.27}$$

A solution to these equations is given by

$$\Phi^\Lambda = \frac{X^\Lambda}{\sqrt{\bar{X}^\Sigma X^\Lambda\eta_{\Sigma\Lambda}}}, \tag{3.28}$$

by setting

$$X^\Lambda = \left(\frac{1}{2}(1+y^2), \frac{i}{2}(1-y^2), y^a\right), \quad a = 2,\dots,n+1, \tag{3.29}$$

for a set of n independent complex coordinates y^a.

In fact, from 3.26, 3.28 and 3.29 we get for the entries of the upper left 2×2 submatrix of $\mathscr{M}$ in 3.23:

$$\mathscr{M}_0^0 = \frac{1+\frac{1}{2}(y^2+\bar{y}^2)}{\sqrt{1-2(y\bar{y})+y^2\bar{y}^2}}, \quad \mathscr{M}_1^1 = \frac{1-\frac{1}{2}(y^2+\bar{y}^2)}{\sqrt{1-2(y\bar{y})+y^2\bar{y}^2}}, \tag{3.30}$$

$$\mathscr{M}_1^0 = \mathscr{M}_0^1 = \frac{-i}{2}\frac{y^2-\bar{y}^2}{\sqrt{1-2(y\bar{y})+y^2\bar{y}^2}}. \tag{3.31}$$

For the $2\times n$ matrix q one has

$$q = \frac{1}{\sqrt{1-2(y\bar{y})+y^2\bar{y}^2}}\begin{pmatrix} y^1+\bar{y}^1 & \cdots & y^n+\bar{y}^n \\ -i(y^1-\bar{y}^1) & \cdots & -i(y^1-\bar{y}^1)\end{pmatrix}. \tag{3.32}$$

One can now check that for the upper left 2×2 matrix one has indeed

$$\begin{aligned}\begin{pmatrix}\mathscr{M}_0^0 & \mathscr{M}_0^1 \\ \mathscr{M}_1^0 & \mathscr{M}_1^1\end{pmatrix}^2 &= \frac{1}{1-2(y\bar{y})+y^2\bar{y}^2}\begin{pmatrix}(1+y^2)(1+\bar{y}^2) & y^2-\bar{y}^2 \\ y^2-\bar{y}^2 & (1-y^2)(1-\bar{y}^2)\end{pmatrix} \\ &= \mathbf{1}+qq^T. \end{aligned}\tag{3.33}$$

The lower right $n\times n$ submatrix of $\mathscr{M}$ can be computetd from the 2×2 matrix $\sqrt{\mathbf{1}+qq^T}$ by the formula

$$\sqrt{\mathbf{1}+q^Tq} = \mathbf{1}+q^T(\sqrt{\mathbf{1}+qq^T}-\mathbf{1})(qq^T)^{-1}q, \tag{3.34}$$

which can be easily checked by squaring the expression on the right-hand side, but we do not write down the explicit result here.

Coordinates on $\mathrm{SO}(4,n)/(\mathrm{SO}(4)\times\mathrm{SO}(n))$

We want to proceed in a similar way as in the previous paragraph to obtain coordinates on $\mathrm{SO}(4,n)/(\mathrm{SO}(4)\times\mathrm{SO}(n))$. q is now a $4\times n$ Matrix, and we assemble the first 4 rows of the matrix $\mathscr{M}$ into two rows of complex numbers by writing:

$$\Phi^\Lambda := \frac{1}{\sqrt{2}}(\mathscr{M}_0^\Lambda + i\mathscr{M}_1^\Lambda), \quad \Psi^\Lambda := \frac{1}{\sqrt{2}}(\mathscr{M}_2^\Lambda + i\mathscr{M}_3^\Lambda), \quad \Lambda = 0,\dots,n+3. \tag{3.35}$$

If we set

$$\Phi^\Lambda = \frac{X^\Lambda}{\sqrt{\bar{X}^\Sigma X^\Lambda \eta_{\Sigma\Lambda}}}, \quad \Psi^\Lambda = \frac{Y^\Lambda}{\sqrt{\bar{Y}^\Sigma Y^\Lambda \eta_{\Sigma\Lambda}}}, \tag{3.36}$$

X^Λ, Y^Λ still have to fullfil the orthogonality conditions

$$X^\Sigma X^\Lambda \eta_{\Sigma\Lambda} = X^\Sigma Y^\Lambda \eta_{\Sigma\Lambda} = X^\Sigma \bar{Y}^\Lambda \eta_{\Sigma\Lambda} = Y^\Sigma Y^\Lambda \eta_{\Sigma\Lambda} = 0, \tag{3.37}$$

$$\eta = \mathrm{diag}(1,1,1,1,-1,\dots,-1). \tag{3.38}$$

A solution of these is given by

$$\begin{aligned} X^\Lambda =&\Big(\frac{1}{2\sqrt{2}}(1+A+B), \frac{i}{2\sqrt{2}}(1-A+B), \frac{-1}{2\sqrt{2}}(1+A-B), \\ &\frac{-i}{2\sqrt{2}}(1-A-B), x^a\Big), \quad a = 4,\dots,n+3 \\ Y^\Lambda =&\Big(\frac{1}{2\sqrt{2}}(1+D+C), \frac{-i}{2\sqrt{2}}(1-D-C), \frac{1}{2\sqrt{2}}(1+D-C), \\ &\frac{-i}{2\sqrt{2}}(1-D+C), y^a\Big), \quad a = 4,\dots,n+3, \end{aligned} \tag{3.39}$$

where x^a, y^a are $2n$ independent complex parameters and A, B, C, D are expressed by them as

$$\begin{aligned} A = x^2, \quad B &= 2\frac{|x^2|^2\cdot y^2\cdot(x\bar{y}) - x^2\cdot y^2\cdot(\bar{x}\bar{y}) + x^2\cdot(\bar{x}y) - (xy)}{|x^2|^2\cdot|y^2|^2 - 1}, \\ D = y^2, \quad C &= 2\frac{|y^2|^2\cdot\bar{x}^2\cdot(xy) - \bar{x}^2\cdot y^2\cdot(x\bar{y}) + y^2\cdot(\bar{x}\bar{y}) - (\bar{x}y)}{|x^2|^2\cdot|y^2|^2 - 1}. \end{aligned} \tag{3.40}$$

It is not obvious how the fields ϕ, $\tilde{\phi}$, ζ^I, $\tilde{\zeta}^I$, $I = 0,\dots,n$ and $\hat{Z}^A$, $\hat{\bar{Z}}^A$, $A = 1,\dots,n$ of the Lagrangian 3.3 can be associated with the parameters x^a, y^a of equation 3.39 to write the Lagrangian in the form $\mathrm{Tr}(\partial_\mu\mathscr{M}^{-1}\partial^\mu\mathscr{M})$. In the next section we will therefore discuss some simpler examples.

3.3.2 Coordinates for the Example

We can at least write part of the Lagrangian 3.3 in the form $\mathrm{Tr}(\partial_\mu \mathscr{M}^{-1}\partial^\mu \mathscr{M})$ with some matrix $\mathscr{M}$. As the simplest example we start with the $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$ part of the special Kähler manifold of our example that is described by only one complex coordinate $Z^1 = X + iY$. The metric belonging to this part is as given in equation 3.8:

$$\mathrm{d}s^2 = \frac{1}{(Z^1 - \bar{Z}^1)^2}\mathrm{d}Z^1\mathrm{d}\bar{Z}^1 = -\frac{1}{4Y^2}\left((\mathrm{d}X)^2 + (\mathrm{d}Y)^2\right) = -\frac{1}{4}\left(\mathrm{d}S^2 + e^{-2S}\mathrm{d}X^2\right), \tag{3.41}$$

where we have introduced $S = \ln Y$. This can be written in the form

$$\mathrm{d}s^2 = \frac{1}{8}\mathrm{Tr}\left(\mathrm{d}\mathscr{M}^{-1}\mathrm{d}\mathscr{M}\right), \tag{3.42}$$

with the Matrix $\mathscr{M}$ and its invers given by [19]:

$$\mathscr{M} = \begin{pmatrix} e^S & e^S X \\ e^S X & e^{-S} + e^S X^2 \end{pmatrix}, \quad \mathscr{M}^{-1} = \begin{pmatrix} e^{-S} + e^S X^2 & -e^S X \\ -e^S X & e^S \end{pmatrix}. \tag{3.43}$$

This result can be generalised for the $\frac{\mathrm{SO}(2,n)}{\mathrm{SO}(2)\times\mathrm{SO}(n)}$ part of the special Kähler manifold. The metric given in 3.8 can be written with $Z^M = X^M + iY^M$, $M = 2, \ldots, n$, as

$$\mathrm{d}s^2 = \left(-\frac{\eta_{MN}}{2Y_L Y^L} + \frac{Y_M Y_N}{(Y_L Y^L)^2}\right)(\mathrm{d}Y^M\mathrm{d}Y^N + \mathrm{d}X^M\mathrm{d}X^N). \tag{3.44}$$

For the moment we will concentrate only on the part of the Lagrangian which doesn't involve the coordinates X^M:

$$\mathrm{d}s_Y^2 = \left(\frac{\eta_{MN}}{2Y_L Y^L} - \frac{Y_M Y_N}{(Y_L Y^L)^2}\right)\mathrm{d}Y^M\mathrm{d}Y^N. \tag{3.45}$$

If one introduces

$$S = -\frac{1}{2}\ln\left(-2Y_L Y^L\right), \tag{3.46}$$

3.45 can be expressed as

$$\mathrm{d}s_Y^2 = (\mathrm{d}S)^2 + e^{2S}\eta_{MN}\mathrm{d}Y^M\mathrm{d}Y^N = -\frac{1}{2}\mathrm{Tr}\left(\mathrm{d}\mathscr{M}^{-1}\mathrm{d}\mathscr{M}\right) + \frac{n-3}{2}(\mathrm{d}S)^2, \tag{3.47}$$

with the matrix $\mathscr{M}$ and the invers matrix $\mathscr{M}^{-1}$ given by

$$\mathscr{M} = \begin{pmatrix} e^S & e^S Y^2 & \ldots & e^S Y^n \\ e^S Y^2 & & & \\ \vdots & & e^{-S}\eta^{MN} + e^S Y^M Y^N & \\ e^S Y^n & & & \end{pmatrix}, \tag{3.48}$$

$$\mathscr{M}^{-1} = \begin{pmatrix} \frac{1}{2}e^{-S} & -e^S Y_2 & \dots & -e^S Y_n \\ -e^S Y_2 & & & \\ \vdots & & e^S \eta^{MN} & \\ -e^S Y_n & & & \end{pmatrix}. \tag{3.49}$$

We can write $\mathscr{M}$ in the form $\mathscr{M} = \mathscr{V}^T \mathscr{V}$ with the upper triangular matrix

$$\mathscr{V} = \begin{pmatrix} e^{\frac{S}{2}} & e^{\frac{S}{2}} Y^2 & \dots & e^{\frac{S}{2}} Y^n \\ 0 & & & \\ \vdots & & e^{-\frac{S}{2}} \eta^{MN} & \\ 0 & & & \end{pmatrix}. \tag{3.50}$$

Here, it was only possible for us to bring part of the coordinates of the Lagrangian 3.3 into the desired form. In particular, this looks difficult for the part of the Lagrangian involving the coordinates $\zeta^I, \tilde{\zeta}^I$ because of the complicated form of the coupling matrix $\mathscr{N}$ given in equation 3.12.

Chapter 4

Moduli Space of K3 Surfaces

In order to obtain a physical theory in 4 spacetime dimensions, the spacetime background of 10-dimensional String Theory or its low-energy Supergravity theory is considered to be of product form $M^4 \times Y^6$ of 4-dimensional Minkowski space and an internal 6-dimensional manifold Y^6 which is compact and Ricci-flat, for example a Calabi-Yau 3-fold (c.f. appendix A.1).

As an intermediate step of compactification down to 4 dimensions one can compactify on a compact 4-dimensional (complex 2-dimensional) manifold leading to a theory in 6 dimensions. Although there are a variety of complex 3-dimensional Calabi-Yau manifolds, there only exist two different classes of compact, complex 2-dimensional Calabi-Yau manifolds which are the flat 4-Tori on the one hand and the so-called K3 surfaces on the other, see e.g. [4] for the latter.

In this chapter we consider compactification of type IIA Supergravity on a K3 surface, spacetime being of the form $M^6 \times \mathrm{K3}$, as for example discussed in [11, 15]. By carrying out the dimensional reduction, the degrees of freedom of fields in the 10-dimensional theory that can be ascribed to the internal manifold (the K3 surface), are reinterpreted as additional fields arising in the 6-dimensional theory. These moduli fields correspond to the possible variation of the configuration of the internal K3 surface over M^6. The set of moduli fields forms coordinates on the moduli space. At the end of section 4.1 we give a brief definition of the moduli space of Ricci-flat metrics on a K3 surface. This space will be enlarged in section 4.2 where we also consider moduli fields coming from the B-field in the Supergravity action. It turns out that the moduli fields of the metric and the B-field together can be described in a form invariant under $O(4, 20)$ and that in fact the full moduli space is locally of the form $\frac{\mathrm{SO}(4,20)}{\mathrm{SO}(4)\times\mathrm{SO}(20)}$, the same space we obtained in section 3.2 from the c-map. In 4.3 we try to compare the fields which are obtained from the c-map in section 3.2 with the moduli fields of the K3 surface.

4.1 K3 Surfaces

In this section K3 surfaces are defined following [2] and some of their properties given. We also explain the notion of moduli space of Ricci-flat metrics on a K3 surface. For mathematical notation for k-forms and basic definitions of cohomology we refer to appendix C.

Definition 4.1. A **K3 surface** is a complex 2-dimensional manifold X with the following properties

(i) X has vanishing first de Rham cohomology: $H^1(X,\mathbb{R}) = 0$,

(ii) X has vanishing first Chern class.[1]

It can be shown from these properties that all K3 surfaces are diffeomorphic to each other. In particular, K3 surfaces are compact, simply connected manifolds that allow Ricci-flat metrics. Given a Ricci-flat metric on a K3 surface, it is indeed a hyper-Kähler manifold (c.f. appendix A.2).

The dimension of the second de Rham cohomology $H^2(X,\mathbb{R})$ of a K3 surface X is 22. It therefore contains 22 harmonic 2-forms $\Omega_i, i = 1,\dots,22$. One can decompose $H^2(X,\mathbb{R})$ as a direct sum of the space self-dual forms and anti-self-dual forms, denoted by $\mathcal{H}^+$ and $\mathcal{H}^-$, the elements of which are defined to satisfy $*\Omega = +\Omega$ and $*\Omega = -\Omega$, respectively, where $*$ is the Hodge-$*$-operator:

$$H^2(X,\mathbb{R}) = \mathcal{H}^+ \oplus \mathcal{H}^-, \tag{4.1}$$

with $\dim \mathcal{H}^+ = 3$ and $\dim \mathcal{H}^- = 19$.

Since X is real 4-dimensional the Hodge dual of a harmonic 2-form is again a harmonic 2-form and thus a linear combination of the Ω^i:

$$*\Omega_i = H_i^j \Omega_j, \tag{4.2}$$

where H is a 22×22 matrix. On a K3 surface, applying the Hodge-$*$-operator twice yields the identity: $** = \mathbf{1}$ (c.f. appendix C). This implies

$$H_j^i H_k^j = \delta_k^i, \tag{4.3}$$

i.e. $H^{-1} = H$, which shows that the matrix H has eigenvalues ± 1 of which 3 are $+1$ and 19 are -1 because of 4.1.
The intersection matrix d is defined by

$$d_{ij} = \int_{\mathrm{K3}} \Omega_i \wedge \Omega_j, \tag{4.4}$$

[1]The first Chern class is an element of $H^2(X,\mathbb{Z})$, i.e. an equivalence class of 2-forms. A representative of the first Chern class is given by the Ricci form $\rho = \frac{i}{2\pi} R_{a\bar{b}} \mathrm{d}z^a \wedge \mathrm{d}\bar{z}^b$.

which is symmetric and also has signature $(3, 19)$. In fact, one can choose the Ω_i such that $d = \mathrm{diag}(1, 1, 1, -1, \ldots, -1)$. We also have

$$d_{ij} H^j_k = \int_{\mathrm{K3}} \Omega_i \wedge (*\Omega_k) = \int_{\mathrm{K3}} (*\Omega_i) \wedge \Omega_k = H^j_i d_{jk}, \tag{4.5}$$

showing that the matrix $d \cdot H$ is symmetric. From equation 4.3 and 4.5 together one gets $H^j_i d_{jk} H^k_l = d_{il}$, showing that H^j_i is an element of $\mathrm{SO}(3, 19)$.

Moduli Space of Ricci-flat Metrics on a K3 surface

Given a manifold M we denote the Ricci tensor with respect to a metric g on M by R^g_{mn}, $m, n = 1, \ldots, \dim M$. The set of all Ricci-flat metrics on M, i.e. metrics g with $R^g_{mn} = 0$, we denote by $\mathcal{R}(M)$. Two metrics g and $\tilde{g}$ are said to be equivalent if there exists a transformation ϕ of M (i.e. a diffeomorphism of M onto itself) such that $\tilde{g} = \phi^* g$. If $\mathcal{D}$ denotes the group of all transformations of M then the quotient under this equivalence relation,

$$\mathscr{M}_M = \mathcal{R}(M)/\mathcal{D}, \tag{4.6}$$

is called the *moduli space* of Ricci-flat metrics on M.

Infinitesimally, we can describe the moduli space locally around a given metric g as deformations of the metric $g \to g + \delta g$ for which

$$R^{g+\delta g}_{mn} = 0 \tag{4.7}$$

to leading order in δg. It is shown in [2] that the moduli space of a $K3$ surface is given by

$$\mathscr{M}_{\mathrm{K3}} = \mathrm{SO}(\Gamma_{3,19}) \backslash \mathrm{SO}(3, 19)/(\mathrm{O}(3) \times \mathrm{SO}(19)) \times \mathbb{R}_+ \tag{4.8}$$

where $\mathrm{SO}(\Gamma_{3,19})$ denotes the discrete subgroup of $\mathrm{SO}(3, 19)$ matrices leaving a $(3, 19)$-dimensional integer lattice $\Gamma_{3,19}$ invariant, acting on the space $\mathrm{SO}(3, 19)$ from the left whereas the subgroup $\mathrm{SO}(3) \times \mathrm{SO}(19)$ is acting on the right. The factor $\mathbb{R}_+$ corresponds to the modulus describing the volume of the K3 surface. The moduli space thus has dimension $3 \times 19 + 1 = 58$. The variation δg of the metric for constant volume of the K3 surface can be expanded as

$$\delta g_{mn} = \sum_{a=1}^{57} m^a \delta g^a_{mn}, \tag{4.9}$$

where $\delta g^a_{mn}, a = 1, \ldots, 57$ are a basis of infinitesimal deformations and the set of parameters $m^a, a = 1, \ldots, 57$ are the moduli. The extra modulus associated to the volume of the K3 surface will be denoted by ω.

4.2 Type IIA Supergravity Compactified on a K3 Surface

In this section compactification of 10-dimensional type IIA Supergravity on a K3 surface is considered, see e.g. [11]. We thus assume that spacetime is of product form $M^6 \times \mathrm{K3}$, with M^6 being 6-dimensional Minkowski space.

The bosonic spectrum from the Neveu-Schwarz-Neveu-Schwarz sector of type IIA Supergravity consists of the 10-dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ representing the gravition, the scalar dilaton field $\hat{\phi}$, and the Kalb-Ramond 2-form field $\hat{B}$. The action for this part of the theory is given by

$$\hat{S} = \int e^{-\hat{\phi}} \left(\frac{1}{2} \hat{R} * 1 + \frac{1}{2} \mathrm{d}\hat{\phi} \wedge *\hat{\phi} + \frac{1}{4} \hat{H} \wedge *\hat{H} \right), \tag{4.10}$$

where $\hat{H} = \mathrm{d}\hat{B}$ is the 'field strength' of the $\hat{B}$-field. Throughout we denote fields and quantities of the 10-dimensional theory with a hat, to distinguish them from fields of the dimensionally reduced theory.

For the compactification we split the coordinates as

$$\hat{x}^{\hat{\mu}} = (x^\mu, z^m), \quad \hat{\mu} = 0, \dots, 9, \quad \mu = 0, \dots, 5, \quad m = 6, 7, 8, 9. \tag{4.11}$$

The metric on the K3 manifold can vary with x^μ over the 6-dimensional spacetime so one can locally decompose the metric on the 10-dimensional spacetime as

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g^0_{mn}(z) + \delta g_{m,n}(x, z) \end{pmatrix}, \tag{4.12}$$

where we can expand the deformation δg_{mn} around g^0_{mn} as

$$\delta g_{mn}(x, z) = \sum_{j=1}^{57} m^j(x) \delta g^j_{mn}(z), \tag{4.13}$$

where the moduli fields $m^j, j = 0, \dots, 57$ now only depend on the coordinates x^μ. The modulus ω associated to the overall volume V of the K3 surface is given by:

$$\omega(x) = \frac{1}{V} \int_{\mathrm{K3}} *1. \tag{4.14}$$

With this, the 10-dimensional dilaton field $\hat{\phi}$ decomposes as

$$\hat{\phi}(\hat{x}) = \phi(x) + \ln \omega(z). \tag{4.15}$$

One also obtains moduli fields from the $\hat{B}$-field which can be expanded in terms of the 22 harmonic 2-forms $\Omega_i(z)$ on the K3 surface:

$$\hat{B}(x,z) = B(x) + \sum_{i=1}^{22} b^i(x)\Omega_i(z). \tag{4.16}$$

It is known that the full moduli space of a K3 surface is given by ([2, 20]):

$$\mathrm{SO}(\Gamma_{4,20})\backslash \mathrm{SO}(4,20)/(\mathrm{SO}(4)\times \mathrm{SO}(20)), \tag{4.17}$$

where again $\mathrm{SO}(\Gamma_{4,20})$ denotes the discrete subgroup of $\mathrm{SO}(4,20)$ matrices leaving invariant a $(4,20)$-dimensional integer lattice $\Gamma_{4,20}$.

After dimensional reduction and integration over the internal K3 surface the 6-dimensional action is [11]:

$$S = \int e^{-\phi}\left(\frac{1}{2}R*1 + \frac{1}{2}\mathrm{d}\phi\wedge *\mathrm{d}\phi + \frac{1}{4}H\wedge *H + \frac{1}{8}\mathrm{Tr}\left(\mathrm{d}\mathscr{M}^{-1}\wedge *\mathrm{d}\mathscr{M}\right)\right), \tag{4.18}$$

where the action of the scalar fields coming from the expansion of the metric and of the B-field have been arranged into a Matrix $\mathscr{M}$ which is an element of $\mathrm{SO}(4,20)$ and depends on the 80 parameters $b^i, i=1,\ldots,22$, $g^j, j=1,\ldots,57$ and ω as follows:

$$\mathscr{M} = \begin{pmatrix} \omega^{-1} & -2\omega^{-1}b^T & -2\omega^{-1}(\bar{b}b) \\ -2\omega^{-1}b & 4\omega^{-1}bb^T + Hd^{-1} & 4\omega^{-1}(\bar{b}b)b + 2Hb \\ -2\omega^{-1}(\bar{b}b) & 4\omega^{-1}(\bar{b}b)b^T + 2b^TH^T & \omega + 4\omega^{-1}(\bar{b}b)^2 + 4\bar{b}Hb \end{pmatrix}, \tag{4.19}$$

where $\bar{b} = b^Td$ and the 22×22 matrix H, defined in 4.2, depends only on the metric moduli $m^a, a=1,\ldots,57$. One can decompose the matrix $\mathscr{M}$ as follows (see e.g. [15]):

$$\mathscr{M} = \mathscr{V}^T\mathscr{V}, \quad \mathscr{V} = \begin{pmatrix} \omega^{-\frac{1}{2}} & -2\omega^{-\frac{1}{2}}b^T & -2\omega^{-\frac{1}{2}}(\bar{b}b) \\ 0 & v & 2vdb \\ 0 & 0 & \omega^{\frac{1}{2}} \end{pmatrix}, \tag{4.20}$$

where v is an upper triangular 22×22 matrix such that $v^Tv = H\cdot d^{-1}$. The upper triangular matrix $\mathscr{V}$ can be easily inverted:

$$\mathscr{V}^{-1} = \begin{pmatrix} \omega^{\frac{1}{2}} & 2\bar{b}v^{-1} & -2\omega^{-\frac{1}{2}}(\bar{b}b) \\ 0 & v^{-1} & -2\omega^{-\frac{1}{2}}db \\ 0 & 0 & \omega^{-\frac{1}{2}} \end{pmatrix}, \tag{4.21}$$

from which one can calculate $\mathscr{M}^{-1} = \mathscr{V}^{-1}(\mathscr{V}^{-1})^T$:

$$\mathscr{M}^{-1} = \begin{pmatrix} \omega + 4\bar{b}H^{-1}b + 4\omega^{-1}(\bar{b}b)^2 & 2\bar{b}H + 4\omega^{-1}(\bar{b}b)\bar{b} & -2\omega^{-1}(\bar{b}b) \\ 2dHb + 4\omega^{-1}(\bar{b}b)db & 4\omega^{-1}db\bar{b} + dH & -2\omega^{-1}db \\ -2\omega^{-1}(\bar{b}b) & -2\omega^{-1}b^T d & \omega^{-1} \end{pmatrix}. \tag{4.22}$$

From 4.19 and 4.22 the last term in the Lagrangian 4.18 can be computed, done in the next section.

4.3 Comparison of the Moduli Space of K3 with the Coordinates from the c-map

In section 3.3 we tried to arrange the coordinates of the quaternion-Kähler manifold obtained from the c-map into a matrix $\mathscr{M}$ to bring the Lagrangian into the form $\mathrm{Tr}(\partial_\mu \mathscr{M}^{-1}\partial^\mu \mathscr{M})$. This was only achieved for part of the coordinates $Z^A, \bar{Z}^A$ of the special Kähler manifold $\frac{\mathrm{SU(1,1)}}{\mathrm{U(1)}} \times \frac{\mathrm{SO(2,18)}}{\mathrm{SO(2)\times SO(18)}}$ which is embedded in the quaternion-Kähler space $\frac{\mathrm{SO(4,20)}}{\mathrm{SO(4)\times SO(20)}}$.

One can now compute the part of the Lagrangian 4.18 involving the moduli fields to compare it directly to the Lagrangian 3.3 obtained from the c-map in section 3.1. With the matrix $\mathscr{M}$ given in 4.19 and $\mathscr{M}^{-1}$ given in 4.22 one gets the result:

$$\frac{1}{8}\mathrm{Tr}\left(\partial_\mu \mathscr{M}^{-1}\partial^\mu \mathscr{M}\right) = -\frac{1}{4\omega^2}(\partial_\mu\omega)^2 - \frac{2}{\omega}(\partial_\mu b_i)H^i_j(\partial^\mu b^j) + \partial_\mu H^i_j \partial^\mu H^j_i. \tag{4.23}$$

This result looks rather simple, however, the major part of this expression is still encoded in the matrix H.

The form of 4.23 suggests that the modulus ω of the volume of the K3 surface can be associated with the field ϕ of the Lagrangian 3.3. Since the matrix H depends only on the metric moduli fields $m^a, a = 1, \ldots, 57$, the coupling of the fields $b^i, i = 1, \ldots, 22$ does not involve the fields b^i themselves. We have seen in section 3.3.2 that the coupling of the real parts X^A of the complex fields Z^A, described by equations 3.41 and 3.44, does not involve the fields X^A themselves. It could therefore be suggested that the $X^A, A = 1, \ldots, 19$ can be associated with the fields b^i. There are, however, only 19 of the coordinates X^A and it is not clear what the remaining 3 of the 22 coordinates b^i can be associated with. It is also not clear how the fields $\zeta^I, \tilde{\zeta}^I, I = 0, \ldots, 19$ can be arranged into the matrix H.

Chapter 5

Summary and Conclusions

In this thesis we have analysed two different sets of coordinates on the symmetric quaternion-Kähler manifold $\frac{\mathrm{SO}(4,20)}{\mathrm{SO}(4)\times\mathrm{SO}(20)}$ which arises in two different situations. On the one hand this space is in the image of the c-map and on the other it is related to the moduli space of K3 surfaces.

After a brief introduction to $N = 2$, $D = 4$ Supergravity in chapter 2, we have discussed the c-map in chapter 3. Starting from the bosonic $N = 2$ vector multiplet Lagrangian with a projective special Kähler target manifold, by the c-map one obtaines a Lagrangian that describes a theory of only scalar fields which are the component fields of hypermultiplets and thus parametrise a quaternion-Kähler target manifold. As a specific example we have considered the class of special Kähler manifolds $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n-1)}{\mathrm{SO}(2)\times\mathrm{SO}(n-1)}$ to which by the c-map is assigned the class of manifolds $\frac{\mathrm{SO}(4,n+1)}{\mathrm{SO}(4)\times\mathrm{SO}(n+1)}$. The c-map yields explicit coordinates on the quaternion-Kähler space. In order to compare these coordinates with the ones obtained for the moduli space of K3 surfaces in chapter 4 we have tried to arrange them into a matrix $\mathscr{M}$ such that Lagrangian can be written in the form $\mathrm{Tr}(\partial_\mu \mathscr{M}^{-1}\partial^\mu \mathscr{M})$. This was achieved only for part of the coordinates involved.

In chapter 4 we have discussed compactification of type IIA Supergravity on a K3 surface. We have seen that the 58 moduli fields arising from the allowed deformations of the metric on the K3 surface together with the 22 moduli from the expansion of the B-field in terms of harmonic 2-forms together describe a theory invariant under $\mathrm{SO}(4,20)$. The moduli space is locally of the form $\frac{\mathrm{SO}(4,20)}{\mathrm{SO}(4)\times\mathrm{SO}(20)}$. It was shown in [11] that the part of the Lagrangian describing the moduli fields can be written in the form $\mathrm{Tr}(\partial_\mu \mathscr{M}^{-1}\partial^\mu \mathscr{M})$ with an $\mathrm{SO}(4,20)$ matrix $\mathscr{M}$. In section 4.3 we have analysed how to compare the coordinates of the moduli space of K3 surfaces with those obtained from the c-map of our specific example in section 3.2.

Appendix A

Kähler, Hyper-Kähler and Quaternion-Kähler Geometry

In this chapter we give the basic definitions of the manifolds that appear as target spaces of vector and hypermultiplets in $N = 2$ Supersymmetry and Supergravity (chapter 2). We refer to [17] as a general reference for complex and Kähler geometry and to [4, 21] for hyper-Kähler and quaternion-Kähler geometry in section A.2. For special Kähler geometry in section A.3 we refer to [7, 13].

A.1 Complex and Kähler Manifolds

Definition A.1. An **almost complex manifold** M is a $2n$-dimensional real differentiable manifold on which is defined an **almost complex structure** J, that is a globally (i.e. on the whole of M) defined smooth $(1,1)$-tensor field

$$J : M \to \mathrm{End}(TM), \quad p \mapsto J_p \in \mathrm{End}(T_pM), \tag{A.1}$$

such that

$$J_p^2 = -\mathbf{1}_{T_pM} \quad \forall p \in M. \tag{A.2}$$

In a local coordinate chart $(U; x^1, \ldots, x^{2n})$, J can be given in components with respect to the vector fields $\frac{\partial}{\partial x^a}$ as $J(\frac{\partial}{\partial x^a}) = J_a^b \frac{\partial}{\partial x^b}$.

Definition A.2. An almost complex structure is said to be **integrable** if the Nijenhuis tensor N, which in components is given by

$$N^i_{jk} = 2\sum_{l=1}^{2n} \left(J^l_j \partial_l J^i_k - J^l_k \partial_l J^i_j - J^i_l \partial_j J^l_k + J^i_l \partial_k J^l_j \right), \tag{A.3}$$

vanishes. An almost complex manifold (M, J) is a **complex manifold** if and only if the almost complex structure is integrable in which case it is referred to as a **complex structure**. On a complex manifold one can find local coordinates $(U; x^1, \ldots, x^n, y^1, \ldots, y^n)$ on an open neighbourhood U such that $\frac{\partial}{\partial y^k} = J(\frac{\partial}{\partial x^k})$ for $k = 1, \ldots, n$ on U. One then introduces complex coordinates $z^k = x^k + iy^k, \bar{z}+ = x^k - iy^k$ with

$$\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right), \quad \frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right). \tag{A.4}$$

On a complex manifold, the transition function from one set of complex coordinates to another are holomorphic.

Definition A.3. A **Hermitean metric** on an almost complex manifold M is a Riemannian metric g, in components $g_{ab} = g(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b})$, which is invariant by the almost complex structure J, i.e.

$$(g_p)_{ab}(J_p)^a_c(J_p)^b_d = (g_p)_{cd} \quad \forall p \in M. \tag{A.5}$$

The fundamental 2-form with respect to g is defined by

$$\Phi = g_{ac}J^c_b \mathrm{d}x^a \wedge \mathrm{d}x^b. \tag{A.6}$$

If M is a complex manifold, introducing complex coordinates $z^1, \ldots, z^n$, $\bar{z}^1, \ldots, \bar{z}^n$, we denote the components of a metric g on M by

$$g_{ab} = g\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^b}\right), g_{a\bar{b}} = g\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right), g_{\bar{a}\bar{b}} = g\left(\frac{\partial}{\partial \bar{z}^a}, \frac{\partial}{\partial \bar{z}^b}\right). \tag{A.7}$$

For a Hermitean metric one has $g_{ab} = g_{\bar{a}\bar{b}} = 0$, and $g_{a\bar{b}}$ is a Hermitean matrix and one can write

$$\mathrm{d}s^2 = 2g_{a\bar{b}}\mathrm{d}z^a\mathrm{d}\bar{z}^b, \tag{A.8}$$

with $\mathrm{d}z^a = \mathrm{d}x^a + i\mathrm{d}y^a$ and $\mathrm{d}\bar{z}^b = \mathrm{d}x^b - i\mathrm{d}\bar{z}^b$ and the fundamental 2-form is

$$\Phi = -2ig_{a\bar{b}}\mathrm{d}z^a \wedge \mathrm{d}\bar{z}^b. \tag{A.9}$$

Definition A.4. A **Kähler manifold** M is a complex manifold M for which the complex structure J is parallel with respect to the Levi-Civita connection ∇^g induced by a Hermitean metric g, or, equivalently, for which the fundamental 2-form is closed:

$$(M, g)\,\text{Kähler} \Longleftrightarrow \nabla^g J = 0 \Longleftrightarrow \mathrm{d}\Phi = 0. \tag{A.10}$$

On a Kähler manifold the metric can be expressed locally on a coordinate chart $(U, z = (z_1, \ldots, z_n))$ in terms of a real-valued function K by

$$g_{a\bar{b}} = \frac{\partial^2 K}{\partial z^a \partial \bar{z}^b}. \tag{A.11}$$

The function K is referred to as the **Kähler potential**. The Kähler potential is not uniquely defined on U since

$$K(z, \bar{z}) \to K(z, \bar{z}) + f(z) + \bar{f}(\bar{z}), \tag{A.12}$$

where $f(z)$ is an arbitrary holomorphic function on U, yields a different Kähler potential that results in the same metric. This is called a Kähler transformation. In particular, on the overlap of two coordinate charts U_i, U_j, Kähler potentials K_i, K_j are related to each other by such a Kähler transformation.

Finally, an equivalent characterisation of Kähler manifolds is to say their holonomy group is contained in $\mathrm{U}(n) \subset \mathrm{SO}(2n)$.[1]

Definition A.5. A **Calabi-Yau** manifold is a Kähler manifold which in addition has vanishing Ricci curvature. These manifolds are characterised by their holonomy group being contained in $\mathrm{SU}(n) \subset \mathrm{U}(n)$.

A.2 Hyper-Kähler and Quaternion-Kähler Manifolds

Definition A.6. A **hyper-Kähler manifold** is a $4n$-dimensional Riemannian manifold M on which there are two globally defined complex structures I and J and a metric g such that

(i) (M, g) is a Kähler manifold with respect to both I and J,

(ii) $IJ = -JI$.

Note that on a hyper-Kähler manifold $K = IJ$ is another parallel complex structure and more generally, for any triplet of real numbers (x, y, z) with $x^2 + y^2 + z^2 = 1$, $xI + yJ + zK$ yields a parallel complex structure so that there is a whole manifold (isometric to S^2) of complex structures on M.

[1] The holonomy group of a Riemannian manifold M is obtained by all linear transformations on the tangent space T_pM at a given point p which are induced by parallel transporting a tangent vector at p around a closed, piecewise differentiable loop at p with respect to the Levi-Civita connection of M.

An equivalent characterisation of hyper-Kähler manifolds is to say their holonomy group is contained in $\mathrm{Sp}(n) \subset \mathrm{SO}(4n)$. Note that since $\mathrm{Sp}(n) \subset \mathrm{SU}(2n)$, hyper-Kähler manifolds are Calabi-Yau manifolds and thus automatically Ricci-flat.

Definition A.7. An **almost quaternionic manifold** is a $4n$-dimensional real differentiable manifold M for which there exists a covering of M by open sets $\{U_i\}$ such that:

(i) on each U_i there are two locally defined almost complex structures J_1 and J_2,

(ii) $J^1J^2 = -J^2J^1$, and we set $J^3 = J^1J^2$ which is another almost complex structure on U_i,

(iii) for all points $p \in U_i \cap U_j$ in an intersection, the vector space of endomorphisms spanned by J^1, J^2 and J^3 is the same for i and j.

Definition A.8. A **quaternion-Kähler manifold** is an almost quaternionic manifold M together with a metric g such that

(i) on each open set U_i the metric g is Hermitean with respect to J^1, J^2 and thus also J^3,

(ii) the Levi-Civita derivative of J^1, J^2 or J^3 lies again in the vector space spanned by J^1, J^2 and J^3.

An equivalent characterization of quaternion-Kähler manifolds is to say their holonomy group is contained in $\mathrm{Sp}(n).\mathrm{Sp}(1) := \mathrm{Sp}(n) \times_{\mathbb{Z}_2} \mathrm{Sp}(1) \subset \mathrm{SO}(4n)$, where the elements $\{-1, +1\}$ of both groups $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$ are identified. Quaternion-Kähler manifolds are Einstein manifolds, i.e. the Ricci tensor is proportional to the metric: $R^g_{ab} = \lambda g_{ab}$, for some $\lambda \in \mathbb{R}$. For $\lambda = 0$ a quaternion-Kähler manifold becomes hyper-Kähler but note that for $\lambda \neq 0$ on a quaternion-Kähler manifold in general there need not even exist a globally defined almost complex structure.

From the locally defined almost complex structures J^1, J^2, J^3 on an open set $U_i \subset M$ one again defines 2-forms defined on the same set U_i:

$$\Phi^i = g_{ac}(J^i)^c_b \mathrm{d}x^a \wedge \mathrm{d}x^b, \quad i = 1, 2, 3. \tag{A.13}$$

The 2-forms $\Phi^i, i = 1, 2, 3$ will in general not be closed unless $\lambda = 0$.

A.3 Special Kähler Manifolds

The Kähler manifolds that appear as target spaces of the scalar fields $X^I, I = 0, \ldots, n$ in the vector multiplets of $N = 2$ Supersymmetry and Supergravity are of some restricted type called special Kähler manifolds. One distinguishes between two types of special Kähler manifolds, *rigid or affine* special Kähler manifolds and *local or projective* special Kähler manifolds which appear in global (rigid) $N = 2$ Supersymmetry and local $N = 2$ Supergravity, respectively. The names affine / projective refer to the terminology used in the mathematics literature, whereas the terms rigid / local are used in the physics literature. Following [7], we give the definitions of the two types of manifolds which are adapted most for our use in $N = 2$ Supersymmetry / Supergravity. They both rely on the existence of a holomorphic function $F(X^I)$, called the prepotential, from which the Kähler potential of these manifolds is computed.

Definition A.9. An **affine (or rigid) special Kähler manifold** of complex dimension n is a Kähler manifold M satisfying the following conditions:

(i) On every coordinate chart $(U, z = (z_1, \ldots, z_n))$ of M there are n holomorphic functions $X = (X^1(z), \ldots X^n(z))$ and a holomorphic function $F(X)$ such that a Kähler potential for this chart is given by

$$K(z, \bar{z}) = i \left(X^I \frac{\partial \bar{F}(\bar{X})}{\partial \bar{X}^I} - \bar{X}^I \frac{\partial F(X)}{\partial X^I} \right). \tag{A.14}$$

(ii) The transition functions on the overlap of two coordinate charts U_i and U_j are given by

$$\begin{pmatrix} X \\ \frac{\partial F}{\partial X} \end{pmatrix}_{(i)} = e^{ic_{(ij)}} \cdot M_{(ij)} \begin{pmatrix} X \\ \frac{\partial F}{\partial X} \end{pmatrix}_{(j)} + \begin{pmatrix} U \\ V \end{pmatrix}_{(ij)}, \tag{A.15}$$

with $c_{(ij)} \in \mathbb{R}$, $M_{(ij)} \in \mathrm{Sp}(2n, \mathbb{R})$ and $(U, V)_{(ij)} \in \mathbb{C}^{2n}$.

(iii) The transition functions satisfy the cocycle condition on overlap regions of three charts.

Definition A.10. A **projective (or local) special Kähler manifold** of complex dimension n is a Kähler manifold M which satisfies the conditions:

(i) The cohomology class of the fundamental 2-form Φ on M is of even integer type, meaning that the integral of Φ over an arbitrary 2-cycle $\mathcal{C}$ is

$$\int_{\mathcal{C}} \Phi = 2\pi i n \quad \text{with } n \in \mathbb{Z}. \tag{A.16}$$

(ii) On every coordinate chart $(U, z = (z_1, \ldots, z_n))$ of M there are $n+1$ projective coordinate functions $X = (X^0(z), \ldots X^n(z))$ and a holomorphic function $F(X)$ which in addition is homogeneous of second degree such that a Kähler potential on this chart is given by

$$K(z,\bar{z}) = -\ln\left(i\bar{X}^I \frac{\partial F(X)}{\partial X^I} - iX^I \frac{\partial \bar{F}(\bar{X})}{\partial \bar{X}^I}\right). \tag{A.17}$$

(iii) The transition function on the overlap of two coordinate charts U_i and U_j is given by

$$\left(\begin{array}{c} X \\ \frac{\partial F}{\partial X} \end{array}\right)_{(i)} = \exp(f_{(ij)}) \cdot M_{(ij)} \left(\begin{array}{c} X \\ \frac{\partial F}{\partial X} \end{array}\right)_{(j)} \tag{A.18}$$

where $M_{(ij)} \in \mathrm{Sp}(2(n+1), \mathbb{R})$ and $f_{(ij)}$ is a holomorphic function on $U_i \cap U_j$. This amounts to a Kähler transformation of the potential $K_{(i)}(z,\bar{z}) = K_{(j)}(z,\bar{z}) + f_{(ij)}(z) + \bar{f}_{(ij)}(\bar{z})$.

(iv) The transition functions satisfy the cocycle condition on overlap regions of three charts.

On projective special Kähler manifolds we introduce inhomogeneous complex coordinates $Z^I = (1, Z^A)$ with $Z^A = \frac{X^A}{X^0}, A = 1, \ldots, n$ on a region with $X^0 \neq 0$.

Appendix B

Construction of the c-map

The bosonic part of the N=2 supergravity Lagrangian coupled to an arbitrary number of n vector multiplets is given by (c.f. equation 2.8):

$$e^{-1}\mathscr{L}_{vec} = \frac{1}{2}R - K_{A\overline{B}}\partial_\mu Z^A \partial^\mu \bar{Z}^{\overline{B}} + \frac{1}{4}(\mathfrak{Im}\mathscr{N})_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} - \frac{1}{4}(\mathfrak{Re}\mathscr{N})_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}. \tag{B.1}$$

The complex scalars $Z^A, \bar{Z}^A$ are coordinates of a projective special Kähler manifold. The construction of the c-map is performed by several manipulations of this Lagrangian that in the end yield a Lagrangian describing a manifold of only scalars which which are coordinates on a quaternion-Kähler manifold.

The first step is a dimensional reduction on a circle with radius R from 4 to 3 spacetime dimensions. Therefore one chooses the vierbein of the spacetime metric to be of the form

$$e^\mu_a = \begin{pmatrix} e^{\hat{\mu}}_{\hat{a}} & 0 \\ \phi B_{\hat{\mu}} & \phi \end{pmatrix}, \quad \mu, a = 0,1,2,3; \quad \hat{\mu}, \hat{a} = 0,1,2. \tag{B.2}$$

By this choice the 4-metric and its inverse take on the form

$$g_{\mu\nu} = \begin{pmatrix} \hat{g}_{\hat{\mu}\hat{\nu}} + \phi B_{\hat{\mu}} B_{\hat{\nu}} & \phi B_{\hat{\mu}} \\ \phi B_{\hat{\nu}} & \phi \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} \hat{g}^{\hat{\mu}\hat{\nu}} & -B^{\hat{\mu}} \\ -B_{\hat{\nu}} & B^2 + \frac{1}{\phi} \end{pmatrix}, \tag{B.3}$$

where now $\hat{g}_{\hat{\mu}\hat{\nu}} = \eta_{\hat{a}\hat{b}} e^{\hat{a}}_{\hat{\mu}} e^{\hat{b}}_{\hat{\nu}}$ is the 3-metric and $\hat{g}^{\hat{\mu}\hat{\nu}}$ its inverse.
The 4-dimensional Ricci scalar is

$$R^{(4)} = g^{\mu\rho} R^{(4)\ \nu}_{\mu\nu\rho} = g^{\mu\rho}(\partial_\nu \Gamma^\nu_{\mu\rho} - \partial_\mu \Gamma^\nu_{\nu\rho} + \Gamma^\sigma_{\mu\rho}\Gamma^\nu_{\sigma\nu} - \Gamma^\sigma_{\nu\rho}\Gamma^\nu_{\sigma\mu}), \tag{B.4}$$

the Christoffel symbols being defined by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}). \tag{B.5}$$

If one writes out the fourth components explicitely, the 4-dimensional Ricci scalar becomes

$$\begin{aligned} R^{(4)} &= R^{(3)} - \frac{1}{4}\phi\hat{g}^{\hat{\mu}\hat{\rho}}\hat{g}^{\hat{\nu}\hat{\sigma}}(\partial_{\hat{\mu}}B_{\hat{\nu}} - \partial_{\hat{\nu}}B_{\hat{\mu}})(\partial_{\hat{\rho}}B_{\hat{\sigma}} - \partial_{\hat{\sigma}}B_{\hat{\rho}}) + \frac{1}{2\phi^2}\hat{g}^{\hat{\mu}\hat{\rho}}(\partial_{\hat{\mu}}\hat{\phi})(\partial_{\hat{\rho}}\hat{\phi}) \\ &= R^{(3)} - \frac{1}{4}\phi H_{\hat{\mu}\hat{\nu}}H^{\hat{\mu}\hat{\nu}} + \frac{1}{2\phi^2}(\partial_{\hat{\mu}}\phi)(\partial^{\hat{\mu}}\phi), \end{aligned} \tag{B.6}$$

where we have introduced the field strength $H_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}}B_{\hat{\nu}} - \partial_{\hat{\nu}}B_{\hat{\mu}}$ of the field $B_{\hat{\mu}}$ and the 3-indices are now lowered and raised by the 3-metric and its inverse, respectively.
The 4-vectors A^I_μ of the vector multiplets are split in the following way into 3-vectors $\hat{A}^I_{\hat{\mu}}$ and scalars ζ^I:

$$A^I_\mu = \left(\hat{A}^I_{\hat{\mu}} + B_{\hat{\mu}}\zeta^I, \zeta^I\right), \tag{B.7}$$

where $B_{\hat{\mu}}$ is the same field as in the metric. Writing out the fourth components explicitely, the field strength tensors of the vector fields A^I_μ are

$$F^I_{\mu\nu} = \begin{pmatrix} \partial_{\hat{\mu}}(\hat{A}^I_{\hat{\nu}} + B_{\hat{\nu}}\zeta^I) - \partial_{\hat{\nu}}(\hat{A}^I_{\hat{\mu}} + B_{\hat{\mu}}\zeta^I) & \partial_{\hat{\mu}}\zeta^I \\ -\partial_{\hat{\nu}}\zeta^I & 0 \end{pmatrix}. \tag{B.8}$$

By this the remaining terms in the Lagrangian B.1 become:

$$\begin{aligned} \frac{1}{4}(\mathfrak{Im}\mathcal{N})_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} =& \frac{1}{4}(\mathfrak{Im}\mathcal{N})_{IJ}\Big[(\hat{F}^I_{\hat{\mu}\hat{\nu}} + H_{\hat{\mu}\hat{\nu}}\zeta^I)(\hat{F}^{J|\hat{\mu}\hat{\nu}} + H^{\hat{\mu}\hat{\nu}}\zeta^J) \\ &+ \frac{2}{\phi}(\partial_{\hat{\mu}}\zeta^I)(\partial^{\hat{\mu}}\zeta^J)\Big], \end{aligned} \tag{B.9}$$

$$\frac{1}{4}(\mathfrak{Re}\mathcal{N})_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu} = (\mathfrak{Re}\mathcal{N})_{IJ}\epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}}\left(\hat{F}^I_{\hat{\mu}\hat{\nu}} + H_{\hat{\mu}\hat{\nu}}\zeta^I\right)\left(\partial_{\hat{\rho}}\zeta^J\right). \tag{B.10}$$

In 3 dimensions the field strengths $\hat{F}_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}}\hat{A}_{\hat{\nu}} - \partial_{\hat{\nu}}\hat{A}_{\hat{\mu}}$ and $H_{\hat{\mu}\hat{\nu}}$ can be converted to vector fields by Hodge dualisation, i.e. we define $\hat{F}_{\hat{\mu}} = -\frac{1}{2}\epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{F}^{\hat{\nu}\hat{\rho}}$ and $H_{\hat{\mu}} = -\frac{1}{2}\epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}}H^{\hat{\nu}\hat{\rho}}$. The scalar fields $Z^A, \bar{Z}^A$ simply reduce to scalar fields in three dimensions which we denote by $\hat{Z}^A, \hat{\bar{Z}}^A$. The dimensionally reduced Lagrangian then reads

$$\begin{aligned} e^{-1}_{(3)}\mathscr{L}^{(3)}_{vec} =& \frac{1}{2}R^{(3)} - \frac{1}{4}\phi^2 H_{\hat{\mu}}H^{\hat{\mu}} + \frac{1}{4\phi^2}(\partial_{\hat{\mu}}\phi)(\partial^{\hat{\mu}}\phi) - K_{A\bar{B}}\partial_{\hat{\mu}}\hat{Z}^A\partial^{\hat{\mu}}\hat{\bar{Z}}^{\bar{B}} \\ &+ (\mathfrak{Re}\mathcal{N})_{IJ}(\hat{F}^I_{\hat{\mu}} + H_{\hat{\mu}}\zeta^I)(\partial^{\hat{\mu}}\zeta^J) \\ &+ \frac{1}{2}\phi(\mathfrak{Im}\mathcal{N})_{IJ}(\hat{F}^I_{\hat{\mu}} + H_{\hat{\mu}}\zeta^I)(\hat{F}^{J|\hat{\mu}} + H^{\hat{\mu}}\zeta^J) - \frac{1}{2\phi}(\mathfrak{Im}\mathcal{N})_{IJ}(\partial_{\hat{\mu}}\zeta^I)(\partial^{\hat{\mu}}\zeta^J). \end{aligned} \tag{B.11}$$

Here $e_{(3)}$ denotes the determinant of the dreibein $e^{\hat{a}}_{\hat{\mu}}$, i.e. we have $\phi \cdot e_{(3)} = e$. The next step in the construction of the c-map is to convert the 3-dimensional vector fields $\hat{F}^I_{\hat{\mu}}$ and $H_{\hat{\mu}}$ in the Lagrangian B.11 into scalars. Therefore note that since the field strenghts $\hat{F}^I_{\hat{\mu}\hat{\nu}}$, $H_{\hat{\mu}\hat{\nu}}$ are derived from vector potentials $\hat{A}^I_{\hat{\mu}}$, $B_{\hat{\mu}}$, they obey Bianchi identities: $\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}}\partial_{\hat{\mu}}\hat{F}^I_{\hat{\nu}\hat{\rho}} = 0$, $\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}}\partial_{\hat{\mu}}H_{\hat{\nu}\hat{\rho}} = 0$. We modify the Lagrangian B.11 by adding Lagrange multipliers to it so that the Bianchi identities become field equations of the modified Lagrangian $\mathscr{L}^{(3)}_{mod}$:

$$e^{-1}_{(3)}\mathscr{L}^{(3)}_{mod} = e^{-1}_{(3)}\mathscr{L}^{(3)}_{vec} - \hat{F}^{I|\hat{\mu}}\partial_{\hat{\mu}}\tilde{\zeta}_I + \frac{1}{2}H^{\hat{\mu}}\partial_{\hat{\mu}}(\tilde{\phi} - \zeta^I\tilde{\zeta}_I). \tag{B.12}$$

The field equations of the fields $\tilde{\phi}$ and $\tilde{\zeta}_I$ are

$$0 = \partial_{\hat{\mu}}\left(\frac{\partial\mathscr{L}^{(3)}_{mod}}{\partial(\partial_{\hat{\mu}}\tilde{\zeta}_I)}\right) - \frac{\partial\mathscr{L}^{(3)}_{mod}}{\partial\tilde{\zeta}_I} = -\partial_{\hat{\mu}}\hat{F}^{I|\hat{\mu}} - \frac{1}{2}(\partial_{\hat{\mu}}H^{\hat{\mu}})\zeta^I, \tag{B.13}$$

$$0 = \partial_{\hat{\mu}}\left(\frac{\partial\mathscr{L}^{(3)}_{mod}}{\partial(\partial_{\hat{\mu}}\tilde{\phi})}\right) - \frac{\partial\mathscr{L}^{(3)}_{mod}}{\partial\tilde{\phi}} = \frac{1}{2}\partial_{\hat{\mu}}H^{\hat{\mu}}. \tag{B.14}$$

By inserting these back into B.12 we can restore, up to a divergence term, the original Lagrangian $\mathscr{L}^{(3)}_{vec}$. Instead, however, we can solve for the equations of motion for the fields $\hat{F}^I_{\hat{\mu}}$ and $H_{\hat{\mu}}$ which are

$$0 = (\mathfrak{Re}\mathscr{N})_{IJ}\partial^{\hat{\nu}}\zeta^J - \phi(\mathfrak{Im}\mathscr{N})_{IJ}(\hat{F}^{J|\hat{\nu}} + H^{\hat{\nu}}\zeta^J) + \partial^{\hat{\nu}}\tilde{\zeta}_I, \tag{B.15}$$

and

$$0 = -\frac{1}{2}\phi^2 H^{\hat{\nu}} + (\mathfrak{Re}\mathscr{N})_{IJ}\zeta^I\partial^{\hat{\nu}}\zeta^J - \phi(\mathfrak{Im}\mathscr{N})_{IJ}\zeta^I(\hat{F}^{J|\hat{\nu}} + H^{\hat{\nu}}\zeta^J) - \frac{1}{2}\partial^{\hat{\nu}}(\tilde{\phi} - \zeta^I\tilde{\zeta}_I), \tag{B.16}$$

and solve these to get

$$\hat{F}^{I|\hat{\nu}} + H^{\hat{\nu}}\zeta^I = \frac{1}{\phi}(\mathfrak{Im}\mathscr{N})^{-1|IJ}\left((\mathfrak{Re}\mathscr{N})_{JK}\partial^{\hat{\nu}}\zeta^K + \partial^{\hat{\nu}}\tilde{\zeta}_J\right), \tag{B.17}$$

$$H^{\hat{\nu}} = -\frac{1}{\phi^2}\left(\partial^{\hat{\nu}}\tilde{\phi} - (\partial^{\hat{\nu}}\zeta^I)\tilde{\zeta}_I + \zeta^I\partial^{\hat{\nu}}\tilde{\zeta}_I\right). \tag{B.18}$$

Here, as always in the following, the inverse of the matrix $\mathfrak{Im}\mathscr{N}$ is written with upper indices.

By inserting B.17 and B.18 into B.12 we get the result:

$$\begin{aligned} e^{-1}_{(3)}\tilde{\mathscr{L}}_{sca} = &\frac{1}{2}R^{(3)} - K_{A\bar{B}}\partial_{\hat{\mu}}\hat{Z}^A\partial^{\hat{\mu}}\hat{\bar{Z}}^{\bar{B}} + \frac{1}{2\phi}\Big[(\mathfrak{Im}\mathscr{N})_{IJ}(\partial_{\hat{\mu}}\zeta^I)(\partial^{\hat{\mu}}\zeta^J) \\ &- \left((\mathfrak{Re}\mathscr{N})_{IK}\partial_{\hat{\mu}}\zeta^K + \partial_{\hat{\mu}}\tilde{\zeta}_I\right)(\mathfrak{Im}\mathscr{N})^{-1|IJ}\left((\mathfrak{Re}\mathscr{N})_{JL}\partial^{\hat{\mu}}\zeta^L + \partial^{\hat{\mu}}\tilde{\zeta}^L\right)\Big] \\ &+ \frac{1}{4\phi^2}\left[(\partial_{\hat{\mu}}\phi)^2 + \left(\partial_{\hat{\mu}}\tilde{\phi} - (\partial_{\hat{\mu}}\zeta^I)\tilde{\zeta}_I + \zeta^I\partial_{\hat{\mu}}\tilde{\zeta}_I\right)^2\right]. \end{aligned} \tag{B.19}$$

The Lagrangian $\tilde{\mathscr{L}}_{sca}$ now describes, apart from the curvature term, a theory of only scalar fields, ϕ, $\tilde{\phi}$, ζ^I, $\tilde{\zeta}_I$ $(I = 0, \dots, n)$, Z^A, $\bar{Z}^{\bar{A}}$ $(A = 1, \dots, n)$, in 3 dimensions. The Lagrangian is now reinterpreted again as describing a theory in 4 dimensions:

$$\begin{aligned} e^{-1}\tilde{\mathscr{L}} = & \frac{1}{2}R^{(4)} - K_{A\bar{B}}\partial_\mu Z^A \partial^\mu \bar{Z}^{\bar{B}} + \frac{1}{2\phi}(\mathfrak{Im}\mathscr{N})_{IJ}(\partial_\mu\zeta^I)(\partial^\mu\zeta^J) \\ & - \frac{1}{2\phi}\left((\mathfrak{Re}\mathscr{N})_{IK}\partial_\mu\zeta^K + \partial_\mu\tilde{\zeta}_I\right)(\mathfrak{Im}\mathscr{N})^{-1|IJ}\left((\mathfrak{Re}\mathscr{N})_{JL}\partial^\mu\zeta^L + \partial^\mu\tilde{\zeta}^L\right) \\ & + \frac{1}{4\phi^2}\left[(\partial_\mu\phi)^2 + \left(\partial_\mu\tilde{\phi} - (\partial_\mu\zeta^I)\tilde{\zeta}_I + \zeta^I\partial_\mu\tilde{\zeta}_I\right)^2\right]. \end{aligned} \tag{B.20}$$

One can write B.20 in a more compact form by introducing the complex fields

$$\sigma_\mu := \partial_\mu\phi + i\left(\partial_\mu\tilde{\phi} - (\partial_\mu\zeta^I)\tilde{\zeta}_I + \zeta^I\partial_\mu\tilde{\zeta}_I\right), \tag{B.21}$$

$$W_{I|\mu} := \mathscr{N}_{IJ}\partial_\mu\zeta^J + \partial_\mu\tilde{\zeta}_I. \tag{B.22}$$

The Lagrangian then reads

$$e^{-1}\tilde{\mathscr{L}} = \frac{1}{2}R - K_{a\bar{b}}\partial_\mu Z^a\partial^\mu\bar{Z}^b + \frac{1}{4\phi^2}\sigma_\mu\bar{\sigma}^\mu - \frac{1}{2\phi}(\mathfrak{Im}\mathscr{N})^{-1|IJ}W_{I|\mu}\bar{W}^\mu_J. \tag{B.23}$$

To see that this Lagrangian describes a quaternion-Kähler manifold, one can specify a vielbein of the Lagrangian B.20, find the connection 1-forms ω, and compute from these the curvature. One then has to show that the curvature 2-forms $\Omega = \mathrm{d}\omega + \omega\wedge\omega$ take their values in $\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)$. Then the holonomy group of the manifold is contained in $\mathrm{Sp}(1).\mathrm{Sp}(n)$, showing that it is a quaternion-Kähler manifold (c.f. section A.2).[1]
A vielbein of the Lagrangian B.23 is given by

$$\begin{aligned} u &= \frac{1}{\sqrt{\phi}}\frac{1}{\sqrt{i\bar{Z}NZ}}(F_I\mathrm{d}\zeta^I + Z^I\mathrm{d}\tilde{\zeta}_I) = \frac{1}{\sqrt{\phi}}\frac{Z^I}{\sqrt{i\bar{Z}NZ}}\left(\mathscr{N}_{IJ}\partial_\mu\zeta^J + \partial_\mu\tilde{\zeta}_I\right), \\ e^A &= e^A_I\mathrm{d}Z^I, \quad A = 1, \dots, n, \\ v &= \frac{1}{2\phi}(\mathrm{d}\phi + i(\mathrm{d}\tilde{\phi} + \tilde{\zeta}_I\mathrm{d}\zeta^I - \zeta^I\mathrm{d}\tilde{\zeta}_I)), \\ E^A &= \frac{\sqrt{i\bar{Z}NZ}}{\sqrt{\phi}}e^A_I N^{-1|IK}(\mathscr{N}_{KL}\mathrm{d}\zeta^L + \mathrm{d}\tilde{\zeta}_K), \quad A = 1, \dots, n, \end{aligned} \tag{B.24}$$

together with their complex conjugate 1-forms. Here e^A_I denotes a vielbein of the original special Kähler manifold, i.e. we have $e^A_I(e^A_J)^* = K_{I\bar{J}}$. Also, $\bar{Z}NZ = \bar{Z}^I N_{IJ}Z^J$ and $N_{IJ} = F_{IJ} - \bar{F}_{IJ}$.

[1]This is a consequence of the Ambrose-Singer theorem, see e.g. [17].

To prove that B.24 indeed is a vielbein one shows that the Lagrangian can be written as

$$e^{-1}\tilde{\mathscr{L}} = e^A \otimes \bar{e}^A + E^A \otimes \bar{E}^A + u \otimes \bar{u} + v \otimes \bar{v}, \tag{B.25}$$

where the notation $\otimes$ from [12] denotes the composition of two 1-forms $\psi = \psi_\nu(x)\mathrm{d}x^\nu$, $\omega = \omega_\nu(x)\mathrm{d}x^\nu$, defined by $\psi \otimes \omega = \eta^{\mu\nu}\psi(\partial_\mu)\cdot\omega(\partial_\nu)$. We see that $e^A \otimes \bar{e}^A = \eta^{\mu\nu}(e^A_I \partial_\rho Z^I \mathrm{d}x^\rho)(\partial_\mu)\cdot(\bar{e}^A_{\bar{J}}\partial_\sigma \bar{Z}^{\bar{J}}\mathrm{d}x^\sigma)(\partial_\nu) = K_{I\bar{J}}\partial_\mu Z^I \partial^\mu \bar{Z}^{\bar{J}}$ and $v \otimes \bar{v} = \frac{1}{4\phi^2}\sigma_\mu\bar{\sigma}^\mu$. For the remaining term in B.23 one computes:

$$\begin{aligned}
E^A \otimes \bar{E}^A &= \frac{i\bar{Z}NZ}{\phi}K_{I\bar{J}}N^{-1|IK}N^{-1|JL}(\mathscr{N}_{KM}\partial_\mu\zeta^M + \partial_\mu\tilde{\zeta}_K)(\bar{\mathscr{N}_{LN}}\partial^\mu\zeta^N + \partial^\mu\tilde{\zeta}_L)\\
&= \frac{-i}{\phi}\left(-N^{-1|IJ} + \frac{\bar{Z}^I Z^J}{\bar{Z}NZ}\right)(\mathscr{N}_{IM}\partial_\mu\zeta^M + \partial_\mu\tilde{\zeta}_I)(\bar{\mathscr{N}_{JN}}\partial^\mu\zeta^N + \partial^\mu\tilde{\zeta}_J),\\
u \otimes \bar{u} &= \frac{1}{\phi}\frac{Z^I\bar{Z}^J}{i\bar{Z}NZ}(\mathscr{N}_{IM}\partial_\mu\zeta^M + \partial_\mu\tilde{\zeta}_I)(\bar{\mathscr{N}_{JN}}\partial^\mu\zeta^N + \partial^\mu\tilde{\zeta}_J).
\end{aligned} \tag{B.26}$$

By the identity (c.f. equation 3.19)

$$-N^{-1|IJ} + \frac{\bar{Z}^I Z^J + Z^I\bar{Z}^J}{\bar{Z}NZ} = \frac{1}{2i}(\mathfrak{Im}\mathscr{N})^{-1|IJ}, \tag{B.27}$$

which is checked by multiplying with $\mathfrak{Im}\mathscr{N}$ from equation 2.10, one gets

$$\begin{aligned}
E^A \otimes \bar{E}^A + u \otimes \bar{u} &= -\frac{1}{2\phi}(\mathscr{N}_{IM}\partial_\mu\zeta^M + \partial_\mu\tilde{\zeta}_I)(\mathfrak{Im}\mathscr{N})^{-1|IJ}(\bar{\mathscr{N}_{JN}}\partial^\mu\zeta^N + \partial^\mu\tilde{\zeta}_J)\\
&= -\frac{1}{2\phi}(\mathfrak{Im}N)^{-1|IJ}W_{I|\mu}\bar{W}^\mu_J,
\end{aligned} \tag{B.28}$$

which prooves B.25.
The vielbein 1-forms B.24 are now arranged into a $2 \times (2n+2)$ matrix $V^{\alpha\Gamma}$ where $\alpha = 1, 2$ and $\Gamma = 0, \ldots, 2n-1$:

$$V^{\alpha\Gamma} = \begin{pmatrix} u & v \\ e^A & E^A \\ \bar{v} & -\bar{u} \\ \bar{E}^A & -\bar{e}^A \end{pmatrix}. \tag{B.29}$$

To find the connection 1-forms with respect to this vielbein, i.e. 1-forms $\omega^{\alpha I}_{\ \ \beta J}$ such that

$$\mathrm{d}V^{\alpha\Gamma} = \sum_{\beta,\Delta}\omega^{\alpha\Gamma}_{\ \ \beta\Delta}\wedge V^{\beta\Delta}, \tag{B.30}$$

one needs the exterior derivatives of the vielbein 1-forms given in [12] by:

$$\begin{aligned}
\mathrm{d}u &= \left[-\frac{1}{2}(v+\bar{v}) + \frac{\bar{Z}N\mathrm{d}Z - ZN\mathrm{d}\bar{Z}}{2\bar{Z}NZ}\right] \wedge u - \bar{E}^A \wedge e^A, \\
\mathrm{d}v &= v \wedge \bar{v} + u \wedge \bar{u} + E^A \wedge \bar{E}^A, \\
\mathrm{d}e^A &= \eta_B^A \wedge e^B, \\
\mathrm{d}E^A &= \left[-\eta - \frac{1}{2}(v+\bar{v}) - \frac{\bar{Z}N\mathrm{d}Z - ZN\mathrm{d}\bar{Z}}{2\bar{Z}NZ}\right] \wedge E^A - \bar{u} \wedge e^A \\
&\quad - \frac{1}{2}(\bar{Z}NZ)PN^{-1}(\mathrm{d}F)N^{-1}P^T \wedge \bar{E}^A,
\end{aligned} \tag{B.31}$$

where η_B^A is the connection of the original special Kähler manifold. P is an $n \times (n+1)$ matrix given by $P_I^A = e_I^A$, $P_0^I = -e_I^A Z^I$ and F is the matrix of the second derivatives F_{IJ} of the prepotential of the special Kähler manifold. By comparing B.31 with B.30 one can read off the connection 1-forms $\omega^{\alpha\Gamma}{}_{\beta\Delta}$. In fact, they decompose as

$$\omega = p \times \mathbf{1}_{(2n+2)\times(2n+2)} + \mathbf{1}_{2\times 2} \times q, \tag{B.32}$$

where the $(2n+2)\times(2n+2)$ matrix q is an element of $\mathfrak{sp}(n+1)$ and p is in $\mathfrak{sp}(1) = \mathfrak{su}(2)$. We only need p here:

$$p = \begin{pmatrix} \frac{1}{4}(v-\bar{v}) - \frac{\bar{Z}N\mathrm{d}Z - ZN\mathrm{d}\bar{Z}}{4\bar{Z}NZ} & -u \\ \bar{u} & -\frac{1}{4}(v-\bar{v}) + \frac{\bar{Z}N\mathrm{d}Z - ZN\mathrm{d}\bar{Z}}{4\bar{Z}NZ} \end{pmatrix}. \tag{B.33}$$

The $\mathfrak{sp}(1)$-curvature 2-form P computed from B.33 is given by:

$$\begin{aligned}
&P = \mathrm{d}p + p \wedge p = \\
&\begin{bmatrix} v\wedge\bar{v} - u\wedge\bar{u} + E^A\wedge\bar{E}^A - e^A\wedge\bar{e}^A & -2\left(u\wedge\bar{v} + e^A\wedge\bar{E}^A\right) \\ -2\left(v\wedge\bar{u} + E^A\wedge\bar{e}^A\right) & u\wedge\bar{u} - v\wedge\bar{v} - E^A\wedge\bar{E}^A + e^A\wedge\bar{e}^A \end{bmatrix}.
\end{aligned} \tag{B.34}$$

On a quaternion-Kähler manifold, the $\mathfrak{sp}(1) = \mathfrak{su}(2)$ part of the curvature is proportional to the fundamental 2-forms associated to the quaternionic structure of the manifold, also arraged into a 2×2 matrix:

$$P = \begin{pmatrix} P_1^1 & P_1^2 \\ P_2^1 & P_2^2 \end{pmatrix} = \mathrm{d}p + p \wedge p = i\lambda \begin{pmatrix} \Phi^3 & \Phi^1 - i\Phi^2 \\ \Phi^1 + i\Phi^2 & -\Phi^3 \end{pmatrix}. \tag{B.35}$$

In Supergravity the constant λ is restricted to be -1.

The components of P, expressed in the coordinates $\phi, \tilde{\phi}, \zeta^I, \tilde{\zeta}_I, Z^A, \bar{Z}^A$, are given by:

$$
\begin{aligned}
P_1^1 = -P_2^2 = &-\frac{i}{4\phi^2}\left(\mathrm{d}\phi \wedge \mathrm{d}\tilde{\phi} + \tilde{\zeta}_I \mathrm{d}\phi \wedge \mathrm{d}\zeta^I - \zeta^I \mathrm{d}\phi \wedge \mathrm{d}\tilde{\zeta}_I\right) + \frac{i}{2\phi}\mathrm{d}\zeta^I \wedge \mathrm{d}\tilde{\zeta}_I \\
&+ \frac{i}{\phi \bar{Z}NZ}\left(F_I \bar{F}_J \mathrm{d}\zeta^I \wedge \mathrm{d}\zeta^J + (F_I \bar{Z}^J - \bar{F}_I Z^J)\mathrm{d}\zeta^I \wedge \mathrm{d}\tilde{\zeta}_J + Z^I \bar{Z}^J \mathrm{d}\tilde{\zeta}_I \wedge \mathrm{d}\tilde{\zeta}_J\right) \\
&+ \frac{1}{2}\left(\frac{N_{IJ}}{\bar{Z}NZ} - \frac{(N\bar{Z})_I (NZ)_J}{(\bar{Z}NZ)^2}\right)\mathrm{d}Z^I \wedge \mathrm{d}\bar{Z}^J, \\
P_1^2 = -\bar{P}_2^1 = &\left(2\sqrt{\phi}\sqrt{i\bar{Z}NZ}\right)^{-1}\Bigg[\frac{1}{\phi}\Big(F_I \mathrm{d}(\phi - i\tilde{\phi}) \wedge \mathrm{d}\zeta^I + Z^I \mathrm{d}(\phi - i\tilde{\phi}) \wedge \mathrm{d}\tilde{\zeta}_I \\
&+ iF_I \tilde{\zeta}_J \mathrm{d}\zeta^I \wedge \mathrm{d}\zeta^J + i(Z^I \tilde{\zeta}_J + \zeta^I F_J)\mathrm{d}\tilde{\zeta}_I \wedge \mathrm{d}\zeta^J - iZ^I \zeta^J \mathrm{d}\tilde{\zeta}_I \wedge \mathrm{d}\tilde{\zeta}_J\Big) \\
&- 2F_{IJ}\mathrm{d}Z^J \wedge \mathrm{d}\zeta^I - 2\mathrm{d}Z^I \wedge \mathrm{d}\tilde{\zeta}_I\Bigg].
\end{aligned}
\tag{B.36}
$$

From equations B.34 and B.35 we have

$$
\begin{aligned}
\Phi^1 &= -\,\mathfrak{Im} P_1^2 = -i\left(u \wedge \bar{v} + v \wedge \bar{u} + e^I \wedge \bar{E}^I + E^I \wedge \bar{e}^I\right), \\
\Phi^2 &= -\,\mathfrak{Re} P_1^2 = -\left(u \wedge \bar{v} - v \wedge \bar{u} + e^I \wedge \bar{E}^I - E^I \wedge \bar{e}^I\right), \\
\Phi^3 &= -\,\mathfrak{Im} P_1^1 = i\left(v \wedge \bar{v} - u \wedge \bar{u} + E^I \wedge \bar{E}^I - e^I \wedge \bar{e}^I\right).
\end{aligned}
\tag{B.37}
$$

We now want to compute the three fundamental 2-forms for the example of the quaternion-Kähler manifold studied in section 3.2. Note that in the expression for P_1^1 one has, since the Kähler potential of the special Kähler manifold $K = -\ln i(\bar{Z}^I F_I - Z^I \bar{F}_I) = -\ln i\bar{Z}NZ$,

$$
\frac{N_{IJ}}{\bar{Z}NZ} - \frac{(N\bar{Z})_I (NZ)_J}{(\bar{Z}NZ)^2} = -\frac{\partial^2 K}{\partial Z^I \partial \bar{Z}^J} = -K_{I\bar{J}},
\tag{B.38}
$$

which is the Kähler metric, computed in equation 3.8. The three fundamental 2-forms are given by:

$$
\begin{aligned}
\Phi^1 = &\left(2\sqrt{\phi}\sqrt{i\bar{Z}NZ}\right)^{-1}\Bigg[\frac{1}{2\phi}\Big(i(F_I - \bar{F}_I)\mathrm{d}\phi \wedge \mathrm{d}\zeta^I + (F_I + \bar{F}_I)\mathrm{d}\tilde{\phi} \wedge \mathrm{d}\zeta^I \\
&+ i(Z^I - \bar{Z}^I)\mathrm{d}\phi \wedge \mathrm{d}\tilde{\zeta}_I + (Z^I + \bar{Z}^I)\mathrm{d}\tilde{\phi} \wedge \mathrm{d}\tilde{\zeta}_I - (F_I + \bar{F}_I)\tilde{\zeta}_J \mathrm{d}\zeta^I \wedge \mathrm{d}\zeta^J \\
&- ((Z^I + \bar{Z}^I)\tilde{\zeta}_J + \zeta^I(F_J + \bar{F}_J))\mathrm{d}\tilde{\zeta}_I \wedge \zeta^J + (Z^I + \bar{Z}^I)\zeta^J \mathrm{d}\tilde{\zeta}_I \wedge \mathrm{d}\tilde{\zeta}_J\Big) \\
&- iF_{IJ}\mathrm{d}Z^I \wedge \mathrm{d}\zeta^J + i\bar{F}_{IJ}\mathrm{d}\bar{Z}^I \wedge \mathrm{d}\zeta^J - i\mathrm{d}(Z^I - \bar{Z}^I) \wedge \mathrm{d}\tilde{\zeta}_I\Bigg],
\end{aligned}
\tag{B.39}
$$

$$
\begin{aligned}
\Phi^2 = &-\left(2\sqrt{\phi}\sqrt{i\bar{Z}NZ}\right)^{-1}\Bigg[\frac{1}{2\phi}\Big((F_I+\bar{F}_I)\mathrm{d}\phi\wedge\mathrm{d}\zeta^I - i(F_I-\bar{F}_I)\mathrm{d}\tilde{\phi}\wedge\mathrm{d}\zeta^I\\
&+(Z^I+\bar{Z}^I)\mathrm{d}\phi\wedge\mathrm{d}\tilde{\zeta}_I - i(Z^I-\bar{Z}^I)\mathrm{d}\tilde{\phi}\wedge\mathrm{d}\tilde{\zeta}_I + i(F_I-\bar{F}_I)\tilde{\zeta}_J\mathrm{d}\zeta^I\wedge\mathrm{d}\zeta^J\\
&+i((Z^I-\bar{Z}^I)\tilde{\zeta}_J+\zeta^I(F_J-\bar{F}_J))\mathrm{d}\tilde{\zeta}_I\wedge\mathrm{d}\zeta^J - i(Z^I-\bar{Z}^I)\zeta^J\mathrm{d}\tilde{\zeta}_I\wedge\mathrm{d}\tilde{\zeta}_J\Big)\\
&-F_{IJ}\mathrm{d}Z^I\wedge\mathrm{d}\zeta^J - \bar{F}_{IJ}\mathrm{d}\bar{Z}^I\wedge\mathrm{d}\zeta^J - \mathrm{d}(Z^I+\bar{Z}^I)\wedge\mathrm{d}\tilde{\zeta}_I\Bigg],
\end{aligned}
\tag{B.40}
$$

$$
\begin{aligned}
\Phi^3 = &\frac{1}{4\phi^2}\left(\mathrm{d}\phi\wedge\mathrm{d}\tilde{\phi}+\tilde{\zeta}_I\mathrm{d}\phi\wedge\mathrm{d}\zeta^I-\zeta^I\mathrm{d}\phi\wedge\mathrm{d}\tilde{\zeta}_I\right)-\frac{1}{2\phi}\mathrm{d}\zeta^I\wedge\mathrm{d}\tilde{\zeta}_I\\
&-\frac{1}{\phi\bar{Z}NZ}\left(F_I\bar{F}_J\mathrm{d}\zeta^I\wedge\mathrm{d}\zeta^J+(F_I\bar{Z}^J-\bar{F}_IZ^J)\mathrm{d}\zeta^I\wedge\mathrm{d}\tilde{\zeta}_J+Z^I\bar{Z}^J\mathrm{d}\tilde{\zeta}_I\wedge\mathrm{d}\tilde{\zeta}_J\right)\\
&+\left(\frac{2(Z_M-\bar{Z}_M)(Z_N-\bar{Z}_N)}{((Z_K-\bar{Z}_K)^2)^2}-\frac{\eta_{MN}}{(Z_K-\bar{Z}_K)^2}\right)\mathrm{d}Z^M\wedge\mathrm{d}\bar{Z}^N\\
&+\frac{1}{2(Z^1-\bar{Z}^1)^2}\mathrm{d}Z^1\wedge\mathrm{d}\bar{Z}^1.
\end{aligned}
\tag{B.41}
$$

Here one has, computed from the prepotential F in section 3.2,

$$
F_I = (F_0, F_1, F_I) = \left(-\frac{1}{2}Z^1Z_MZ^M, \frac{1}{2}Z_MZ^M, Z^1Z_I\right),
\tag{B.42}
$$

$$
F_{IJ} = \begin{pmatrix} Z^1\cdot(Z_MZ^M) & -\frac{1}{2}(Z_MZ^M) & -Z^1\cdot Z_J \\ -\frac{1}{2}(Z_MZ^M) & 0 & Z_J \\ -Z^1\cdot Z_I & Z_I & Z^1\cdot\eta_{IJ}\end{pmatrix}.
\tag{B.43}
$$

We have thus computed all the data needed to describe the quaternionic structure of the dual quaternion-Kähler manifold obtained from the c-map.

Appendix C

Mathematical notation

C.1 Notation for k-forms

Let M be a differentiable manifold of dimension m and $x^1, \ldots, x^n$ local coordinates on a neighbourhood U of some point $p \in M$. For a differential k-form on M, $0 \leq k \leq m$, we use the notation

$$\omega_k = \frac{1}{k!}\omega_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}, \tag{C.1}$$

where $\omega_{i_1 \ldots i_k}$ is totally antisymmetric in its indices.

The wedge (or exterior) product of a k-form ω_k and l-form η_l is given by

$$\omega_k \wedge \eta_l = \frac{1}{k!l!}\omega_{i_1 \ldots i_p}\eta_{j_1 \ldots j_l} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \wedge \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_l}, \tag{C.2}$$

defining a $(k+l)$-form.

The exterior differential d assigns to a k-form a $(k+1)$-form by

$$\mathrm{d}\omega_k = \frac{1}{k!}\partial_j \omega_{i_1 \ldots i_k} \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}. \tag{C.3}$$

The exterior differential is nilpotent: $\mathrm{d}^2 = 0$.

Given a Riemannian metric g_{ij} on M, the Hodge dual of a k-form is defined as

$$*\omega_k = \frac{1}{k!(m-k)!}\omega_{i_1 \ldots i_k}\varepsilon^{i_1 \ldots i_k}{}_{i_{k+1} \ldots i_n} \mathrm{d}x^{i_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{i_n}, \tag{C.4}$$

where $\varepsilon_{i_1, \ldots, i_n}$ is the totally antisymmetric tensor with $\varepsilon_{1 \cdots n} = 1$ and the indices are raised by the invers metric g^{ij}. The Hodge-$*$-operator applied twice satisfies

$$* * \omega_k = (-1)^{k(n-k)}\omega_k, \tag{C.5}$$

for any k-form on M.

C.2 Cohomology

A k-form ω_k is *closed* if $\mathrm{d}\omega_k = 0$. We denote the space of all closed k-forms on a differentiable manifold M, also called k-cocycles, by $Z^k(M)$. A k-form is *exact* if there is a $(k-1)$-form η_{k-1} such that $\mathrm{d}\eta_{k-1} = \omega_k$. We denote the space of all exact k-forms on M by $B^k(M)$. Since $\mathrm{d}^2 = 0$ we have $B^k(M) \subset Z^k(M)$. The obstruction for a closed k-form to be exact is called its cohomology class and the quotient space

$$H^k(M) = Z^k(M)/B^k(M) \tag{C.6}$$

is called the k-th de Rham cohomology of M. Also,

$$b_k = \dim H^k(M) \tag{C.7}$$

is called the k-th Betti number of M. If M is a compact Riemannian manifold, each equivalence class of $H^k(M)$ contains exactly one harmonic k-form. A harmonic form ω_k is defined by the condition $\Delta\omega_k = 0$, where Δ is the Laplacian on M being defined by $\Delta = \mathrm{d} * \mathrm{d} * + * \mathrm{d} * \mathrm{d}$. Thus, every closed k-form ω_k can be expanded as a linear combination of the b_k harmonic k-forms on M, modulo an exact k-form.

On a complex manifold M the space of k-forms decomposes as

$$\Lambda^k M = \bigoplus_{p+q=k} \Lambda^{p,q} M \tag{C.8}$$

where a (p,q)-form $\omega_{p,q} \in \Lambda^{p,q}$ is written as

$$\omega_{p,q} = \frac{1}{p!q!}\omega_{i_1\ldots i_p\bar{\imath}_1\ldots\bar{\imath}_q}\mathrm{d}z^{i_1} \wedge \cdots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\bar{z}^{\bar{\imath}_1} \wedge \cdots \wedge \mathrm{d}\bar{z}^{\bar{\imath}_q}. \tag{C.9}$$

Also, the exterior differential operator decomposes into $\mathrm{d} = \partial + \bar{\partial}$ with

$$\partial = \mathrm{d}z^i\partial_i, \quad \bar{\partial} = \mathrm{d}\bar{z}^{\bar{\imath}}\partial_{\bar{\imath}}, \tag{C.10}$$

where both operators are nilpotent: $\partial^2 = 0 = \bar{\partial}^2$. Further,

$$H^{p,q}_{\bar{\partial}}(M) = Z^{p,q}_{\bar{\partial}}(M)/B^{p,q}_{\bar{\partial}}(M) \tag{C.11}$$

is called the (p,q)-th Dolbeault cohomology and the Hodge numbers of M are defined to be

$$h^{p,q} = \dim H^{p,q}_{\bar{\partial}}(M). \tag{C.12}$$

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Hamburg, den 1. Juni 2009 Thomas Kecker