# The cosmic microwave background (CMB)



## Lecture 1 Cosmological perturbation theory

Fluctuations in the CMB are small. Therefore they can be treated (to good accuracy) by first order perturbation theory.
$\Rightarrow$ We obtain <u>linear</u> eqns. which can be solved with high accuracy. $\Rightarrow$ comparison with data yields accurate "measurements," "estimates" of the parameters which describe the (background) universe and the initial conditions for the perturbations.

### 1.1 Definition of cosmological perturbations, gauges

$$ds^2 = a^2(t)\left[-(1+2A)dt^2 - 4B_i\, dt\, dx^i + (1+2H_L)\gamma_{ij}dx^i dx^j + 2H_{ij}\, dx^i dx^j\right] = (\bar{g}_{\mu\nu} + a^2 h_{\mu\nu})dx^\mu dx^\nu$$

with $h = h_i{}^i = h_{ij}\gamma^{ij} = 0$.



The split into a background $\bar{g}_{\mu\nu}$ and a perturbation $h_{\mu\nu}$ is not unique. The only measurable quantity is the true physical metric $g_{\mu\nu}$.
The same is true with the energy momentum tensor.
Let us the physical fields $g_{\mu\nu}$ and $T_{\mu\nu}$ which are related via Einstein's eqn,

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}.$$

If we now choose some foliation of spacetime and average both $g_{\mu\nu} \to \bar{g}_{\mu\nu}$, $T_{\mu\nu} \to \overline{T_{\mu\nu}}$

in general we will not have $G_{\mu\nu}(\bar{g}_{\mu\nu}) = 8\pi G\, \overline{T_{\mu\nu}}$ (non-linearity of Einstein's equation).
We call an averaging procedure admissible only if it yields Einstein's eqn. for the averaged metric & energy-momentum tensor.
For a geometry which is close to Friedmann, there will be many admissible averaging procedures which satisfy

$$\frac{|T_{\mu\nu} - \overline{T_{\mu\nu}}|}{|\max_{\{\mu\nu\}} T_{\mu\nu}|} \ll 1 \quad \text{and} \quad \frac{|g_{\mu\nu} - \bar{g}_{\mu\nu}|}{\max_{\{\mu\nu\}} |g_{\mu\nu}|} \ll 1.$$

Since $\dfrac{|T_{\mu\nu} - \overline{T_{\mu\nu}}|}{|T_{\mu\nu}|}$ is small, of order $\varepsilon$, say, ~~different~~ admissible backgrounds differ only by order $\varepsilon$.

Let us now fix an admissible background $(\bar{T}_{\mu\nu}, \bar{g}_{\mu\nu})$.
Since the theory is invariant under diffeomorphisms, the perturbations are not unique. For any diffeomorphism $\phi$ the metric $g$ and $\phi_*(g)$ describe the same geometry. Here $\phi_*$ is the "push-foreward".
We consider only diffeomorphisms which leave the background invariant $(\bar{T}_{\mu\nu}, \bar{g}_{\mu\nu})$. gauge transformations
Such diffeom. can deviate from the identity only by order $\varepsilon$:

$$\phi(x) = x + \varepsilon X,$$

where $X$ is a vector-field.
Hence the push-forward is given by

$$\phi_* = id + \varepsilon L_X$$

$$g = \bar{g} + \varepsilon a^2 h \longmapsto \phi_*(g) = \bar{g} + \varepsilon(a^2 h + L_X \bar{g}) + \mathcal{O}(\varepsilon^2)$$

$\Rightarrow$ Under such an 'infinitesimal coordinate transformation' the metric perturbations transform with the Lie-derivative of the background metric,

$$a^2 h \longmapsto a^2 h + L_X \bar{g}$$



But this is true for an arbitrary perturbed tensor field $S = \bar{S} + S^{(1)}$

$$S^{(1)} \longrightarrow S^{(1)} + L_X \bar{S}.$$

Since every vector field $X$ induces an infinitesimal coordinate transformation $\phi_\varepsilon^X$, where $\phi_s^X$ denotes the flux of $X$, this implies that a perturbation is invariant under infinitesimal coordinate transformations ($\equiv$ gauge transformations) if and only if

$$L_X \bar{S} = 0 \quad \forall X.$$

This is the Stewart + Walker Lemma (1974).

<u>Harmonic decomposition</u>

Since the $\{t = const.\}$ hypersurfaces are homogeneous and isotropic, it is reasonable to perform a harmonic analysis. A (spatial) tensor field on these hypersurfaces can be decomposed into components which transform irreducibly under translations and rotations
For functions in the case $K=0$ this is simply the Fourier decomposition

$$f(\underline{x}, t) = \int d^3k\, f(\underline{k}, t) e^{-i\underline{k}\underline{x}}$$

If $K \neq 0$ it is the decomposition into eigenfunctions of the Laplacian,

$$\Delta Q_k^{(S)} = -k^2 Q_k^{(S)}$$

$$k^2 = K(\ell+2)\ell\,, \quad \ell \in \mathbb{N} \quad \text{for } K>0$$

$$k^2 \geq -K \quad \text{for } K<0$$

For a spatial vector field, it is the decomposition into a gradient and a curl:

$$V_i = \nabla_i \varphi + B_i \quad , \quad B^i{}_{|i} \equiv \underline{\nabla}\cdot\underline{B} = 0$$

Here $_{|i}$ is the covariant derivative w.r.t. the 3-metric $\gamma_{ij}$. $\varphi$ is the spin zero component of $\underline{V}$ and $\underline{B}$ is its spin $\underline{1}$ component.

A symmetric spatial tensor field can be decomposed into two spin 0, two spin = 1 and two spin = 2 degrees of freedom:

$$H_{ij} = H_L \gamma_{ij} + (\nabla_i \nabla_j - \tfrac{1}{3}\gamma_{ij}\Delta) H_T + \tfrac{1}{2}(H^{(V)}_{i|j} + H^{(V)}_{j|i}) + H^{(T)}_{ij}$$

where $H^{(V)i}{}_{|i} = H^{(T)i}{}_{i} = H^{(T)ij}{}_{|j} = 0$

As a basis for vector and tensor modes we use vector and tensor type eigenfunctions of the Laplacian



$$\Delta Q_j^{(V)} = -k^2 Q_j^{(V)}$$

$$\Delta Q_{ij}^{(T)} = -k^2 Q_{ij}^{(T)}$$

where $Q_j^{(V)}$ is a transvers vector and $Q_{ij}^{(T)}$ is a traceless, symmetric, transvers tensor,

$$Q^{(V)i}{}_{|i} = 0\,, \quad Q^{(T)i}{}_{i} = Q^{(T)i}{}_{j|i} = 0\,.$$

Both, $Q_j^{(V)}$ and $Q_{ij}^{(T)}$ have two degrees of freedom.

If curvature vanishes, each mode is determined by a wave vector $\underline{k}$. Using an orthonormal basis $(\underline{e}^{(1)}, \underline{e}^{(2)})$ in the plane normal to $\underline{k}$, we can define helicity basis vectors

$$\underline{e}^{(\pm)} = \frac{1}{\sqrt{2}}(\underline{e}^{(1)} \pm i\,\underline{e}^{(2)})$$

We develop $Q^{(V,T)}$ in terms of helicity eigen states

$$Q_i^{(V)} = Q^{(+)} e_i^{(+)} + Q^{(-)} e_i^{(-)}$$

$$Q_{ij}^{(T)} = Q^{(+2)} e_{ij}^{(+2)} + Q^{(-2)} e_{ij}^{(-2)} \quad \text{where}$$

$$e_{ij}^{(\pm 2)} = e_i^{(\pm)} e_j^{(\pm)} \quad (= e_{ij}^{+} \pm i\, e_{ij}^{\times})$$

$$(e^{(d)}_{ij}) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad (e^{\times}_{ij}) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

Under rotations around $\underline{k}$ $\underline{e}^{(\pm)}$ and $e^{(\pm 2)}_{ij}$ simply transform with a phase $e^{\pm i\varphi}$ and $e^{\pm 2i\varphi}$. Hence they have magnetic quantum number (i.e. helicity) $\pm 1$ and $\pm 2$ respectively. These positive & negative helicity basis can also be defined for $K \neq 0$.

We also introduce scalar type vectors & tensors and vector type tensors via

$$Q^{(S)}_{j} = -k^{-1} Q^{(S)}_{|j}, \quad Q^{(S)}_{ij} = k^{-2} Q^{(S)}_{|ij} + \frac{1}{3}\gamma_{ij} Q^{(S)}$$

$$Q^{(V)}_{ij} = \frac{-1}{2k}\left(Q^{(V)}_{i|j} + Q^{(V)}_{j|i}\right)$$

The decomposition of the $k$-mode of a vector or tensor field is then of the form

$$V_i = V Q^{(S)}_i + V^{(V)} Q^{(V)}_i$$

$$H_{ij} = H_L Q^{(S)} \gamma_{ij} + H_T Q^{(S)}_{ij} + H^{(V)} Q^{(V)}_{ij} + H^{(T)} Q^{(T)}_{ij}.$$

A generic metric perturbation is of the form

$$h_{\mu\nu} dx^\mu dx^\nu = -2A\,dt^2 - 2B_i\,dt\,dx^i + 2H_{ij}\,dx^i dx^j$$



We want to determine the behavior of $h_{\mu\nu}$ under gauge transformations

$$X = T\partial_t + L^i \partial_i$$

$$(L_X \bar{g})_{\mu\nu} = X^\alpha g_{\mu\nu;\alpha} + X^\alpha{}_{,\mu} g_{\alpha\nu} + X^\alpha{}_{;\nu} g_{\mu\alpha}$$

$$L_X \bar{g} = a^2\left[-2(\mathcal{H}T + \dot{T})dt^2 + 2(\dot{L}_i - T_{|i})dt\,dx^i + (2\mathcal{H}T\gamma_{ij} + L_{i|j} + L_{j|i})dx^i dx^j\right]$$

$(\mathcal{H} = \frac{\dot{a}}{a})$

$\Rightarrow$

$$A \to A - \frac{1}{2}a^{-2}(L_X\bar{g})_{00} = A + \mathcal{H}T + \dot{T}$$

$$B_i \to B_i - \dot{L}_i + T_{|i}$$

$$H_{ij} \to H_{ij} + \mathcal{H}T\gamma_{ij} + L_{i|j} + L_{j|i}$$

Splitting $L^i$ into its scalar and vector component we find for the harmonic components, $L^i = L Q^{(S)i} + L^{(V)} Q^{(V)i}$

$$H^{(T)} \to H^{(T)}$$

$$H^{(V)} \to H^{(V)} - kL^{(V)}$$

$$H_T \to H_T - kL$$

$$H_L \to H_L + \mathcal{H}T + \frac{k}{3}L$$

$$B^{(V)} \to B^{(V)} - \dot{L}^{(V)}$$

$$B \to B - \dot{L} - kT, \quad A \to A + \mathcal{H}T + T$$

⇒ For scalar perturbations e.g. we can choose

$L = k^{-1} H_T$ and $T = k^{-1} B - k^{-1} \dot{H}_T$ so

that after this gauge transformation

$H_T = B = 0$.

This is called <u>longitudinal</u> gauge.

In this gauge the scalar metric perturbations are given by

$$h^{(S)}_{\mu\nu} dx^\mu dx^\nu = -2\Psi dt^2 - 2\Phi \gamma_{ij} dx^i dx^j .$$

Since they are (most probably) not relevant, we shall not discuss vector perturbations any further.

Tensor perturbations are gauge invariant.

$\Psi$ and $\Phi$ are the so called Bardeen potentials.

In a generic gauge they are given by

$$\Psi = A - \mathcal{H} k^{-1} \sigma - k^{-1} \dot{\sigma}$$

$$\Phi = -H_L - \tfrac{1}{3} H_T + \mathcal{H} k^{-1} \sigma \,, \quad \sigma = k^{-1} \dot{H}_T - B$$



<u>Perturbations of the energy momentum tensor</u>

$$T_\mu{}^\nu = \bar{T}_\mu{}^\nu + \theta_\mu{}^\nu$$

We define the perturbed energy density $\rho$ and energy flux 4-vector $u$ as the timelike eigenvalue and eigen-vector of $T^\mu{}_\nu$:

$$T^\mu{}_\nu u^\nu = -\rho u^\mu \,, \quad u^2 = -1$$

$$\rho = \bar{\rho}(1+\delta) \,, \quad u = u^0 \partial_t + u^i \partial_i$$

$$u^0 = \frac{1}{a}(1-A) \,, \quad u^i = \frac{1}{a} v^i = \frac{1}{a} v Q^{(S)i}$$

Be $P^\alpha{}_\beta \equiv u^\alpha u_\beta + \delta^\alpha{}_\beta$ the projector onto the subspace of tangent space normal to $u$.

We define the stress tensor

$$\tau^{\mu\nu} = P^\mu{}_\alpha P^\nu{}_\beta T^{\alpha\beta}$$

The components $\tau^\mu{}_0$ and $\tau^0{}_\mu$ are determined by the previously defined perturbations.

$$\tau^i{}_j = \bar{P}[(1+\pi_L)\delta^i{}_j + \Pi^i{}_j] \,, \quad \Pi^i{}_i = 0$$

$\pi_L$: pressure perturbation; $\Pi^i{}_j$: anisotropic stress

Under gauge transformations we have for the modes

$$\delta \to \delta - 3(1+w)\mathcal{H}T$$

$$\pi_L \to \pi_L - 3\frac{c_s^2}{w}(1+w)\mathcal{H}T$$

$$v \to v - \dot{L}$$

$\Pi \to \Pi$ is gauge invariant

$\Gamma = \pi_L - \frac{c_s^2}{w}\delta$ is also gauge invariant.

It can be shown that $\Gamma$ is proportional to the divergence of the entropy flux $\Rightarrow$

$\Gamma = 0$ for <u>adiabatic</u> perturbations.

Gauge invariant velocity and density perturbations can be found by combining $v$ and $\delta$ with metric perturbations:

$$V := v - \frac{1}{k}\dot{H}_T = v^{(long)}$$

$$D_s := \delta + 3(1+w)\mathcal{H}(k^{-2}\dot{H}_T - k^{-1}B) = \delta^{(long)}$$

$$D := \delta^{(long)} + 3(1+w)\mathcal{H}k^{-1}V$$

$$D_g := \delta^{(long)} - 3(1+w)\Phi$$

since metric perturbations are suppressed by a factor $(kt)^2$ with respect to density perturbations*, all these density perturbations are equivalent on sub-Hubble scales i.e. for $k \sim \frac{1}{\lambda} \gg \mathcal{H} \sim \frac{1}{t}$.



To see (*) consider that the 00 Einstein eqn. is generically of the form

$$\frac{\delta\rho}{\rho}\cdot 8\pi G\rho a^2 = \mathcal{O}\left(\frac{1}{t^2}h + \frac{k}{t}h + k^2 h\right)$$

$$\mathcal{O}(\mathcal{H}^2) = \mathcal{O}\left(\frac{1}{t^2}\right)$$

$$\Rightarrow \delta \simeq \mathcal{O}(1 + kt + (kt)^2)h = \mathcal{O}((kt)^2 h)$$

for $kt \gg 1$.

## 1.2 perturbation equations $\delta G_{\mu\nu} = 8\pi G\delta T_{\mu\nu}$

<u>Einstein's eqn:</u>

$$-(k^2 - 3K)\Phi = 4\pi G\rho a^2 D \qquad (00)$$

$$k(\mathcal{H}\Psi + \dot{\Phi}) = 4\pi G a^2(\rho + P)V \qquad (0i)$$

$$k^2(\Phi - \Psi) = 8\pi G a^2 \Pi \qquad (i \neq j)$$

$$\ddot{H}^{(T)} + 2\mathcal{H}\dot{H}^{(T)} + (2K + k^2)H^{(T)} = 8\pi G a^2 P\Pi^{(T)}$$

<u>Energy momentum conservation</u>

$$\dot{D} - 3w\mathcal{H}D = -\left(1 - \frac{3K}{k^2}\right)\left[(1+w)kV + 2\mathcal{H}w\Pi\right]$$

$$\dot{V} + \mathcal{H}V = k\left[\Psi + \frac{c_s^2}{1+w}D + \frac{w}{1+w}\Gamma - \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\frac{w}{1+w}\Pi\right]$$

or, in terms of $D_g$ and $V$:

Insert on gravity waves

If $\Pi^{(T)} = 0$ and $K = 0$, and $a \propto t^q$

The tensor perturbation eqn becomes

$$\ddot H + \frac{2q}{t}\dot H + k^2 H = 0 .$$

This is solved by spherical Bessel functions

$$H = \frac{1}{a}\Big(\underbrace{A(kt)\, j_{q-1}(kt)}_{\text{regular for } kt \to 0} + \underbrace{Bkt\, y_{q-1}(kt)}_{\text{diverges for } kt \to 0 \Rightarrow B = 0}\Big)$$

$$H \longrightarrow \begin{cases} \text{const}, & kt \to 0 \\ \underbrace{\frac{A}{a}\sin\left(kt - \frac{q-1}{2}\pi\right)}_{\text{gravity wave in a Friedmann universe}}, & kt \to \infty \end{cases}$$

$$\rho_H(k) = \underset{\uparrow\ \text{two hel. modes}}{2}\,\frac{\langle|\dot H|^2\rangle}{16\pi G} = \frac{A^2 k^2}{a^4\, 16\pi G}$$

If the spectrum of $H$ is scale invariant, $\langle|H|^2\rangle k^3 = \text{const} = \frac{A^2(k)}{2}k^3$

the energy density spectrum is blue:

$$\frac{d\rho_H}{d\ln k} = 4\pi k^3 \rho_H(k) = \frac{A^2(k)\,k^5}{a^4 \cdot 4G}$$



$$\dot D_g + 3(c_s^2 - w)\mathcal{H} D_g + (1+w)kV + 3w\mathcal{H}\Gamma = 0$$

$$\dot V + \mathcal{H}(1 - 3c_s^2)V = k(\Psi + 3c_s^2\Phi) + \frac{c_s^2 k}{1+w} D_g + \frac{wk}{1+w}\left[\Gamma - \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\Pi\right]$$

$\Gamma$ and $\Pi$ have to be determined via matter eqn.

Eg. for adiabatic perturbations of an ideal fluid $\Gamma = \Pi = 0$.

One can use the (00) and (i0) Einstein eqns to derive a second order eqn for $\Phi$ and $\Psi$ from the conservation eqns. Eliminating finally $\Phi$ with the ($i \neq j$) eqn yields the <u>Bardeen eqn.</u> (Ex. 1)

$$\ddot\Psi + 3\mathcal{H}(1 + c_s^2)\dot\Psi + \left[3(c_s^2 - w)\mathcal{H}^2 - (2 + 3w + 3c_s^2)K + c_s^2 k^2\right]\Psi = \mathcal{S}(\Pi, \Gamma)$$

If $\mathcal{S} \equiv 0$, this is a damped wave eqn. with damping term $3\mathcal{H}(1 + c_s^2)$ and (time dependent) mass term

$$m^2(t) = c_s^2 k^2 + 3(c_s^2 - w)\mathcal{H}^2 - (2 + 3w + 3c_s^2)K .$$

We are interested mainly in two cases:

## 1.3 Perturbations of dust / radiation for $K = 0$

i) matter (dust): $w = c_s^2 = 0$, $a \propto t^2$

$\Rightarrow \mathcal{H} = \frac{2}{t}$

$$\ddot{\Psi} + \frac{6}{t}\dot{\Psi} = 0 \Rightarrow \Psi = A + \underbrace{\frac{B}{(kt)^5}}_{\text{decaying mode can soon be neglected}} = \Psi_0$$

$$\dot{V} + \frac{2}{t}V = k\Psi = kA$$

$$(Vt^2)^{\cdot} = kAt^2$$

$$(Vt^2) = \frac{kA}{3}t^3 + C$$

$$V = \frac{A}{3}kt + \underbrace{C/t^2}_{\text{decaying mode can be neglected}}$$

$$\dot{D}_g + kV = 0, \quad \dot{D}_g = -\frac{A}{3}k^2 t$$

$$D_g = -\frac{A}{6}(kt)^2 + A'$$

can not be neglected, dominates on large scales, $kt \ll 1$

$$D_g = D - 3\frac{\mathcal{H}}{k}V - 3\Phi = \underbrace{\frac{-k^2}{4\pi G a^2 \rho}}_{\frac{3}{2}\mathcal{H}^2 = \frac{6}{t^2}}\phi - 3\frac{\mathcal{H}}{k}V - 3\phi$$

$\uparrow \frac{4}{t^2}$

$$D_g = -\left[\frac{(kt)^2}{6} + 3\right]\Psi - \frac{6}{kt}V = -\frac{A}{6}(kt)^2 - 3A - 2A$$

$$\Rightarrow A' = -5A$$



$$\Psi = \Psi_0 = \text{const}$$

$$V = \frac{1}{3}\Psi_0(kt) \propto \sqrt{a}$$

$$D_g = -5\Psi_0 - \frac{1}{6}\Psi_0(kt)^2, \quad D = -\frac{1}{6}\Psi_0(kt)^2 \propto a$$

ii) radiation

$w = c_s^2 = 1/3$, $a \propto t$, $\mathcal{H} = \frac{1}{t}$

$$\ddot{\Psi} + \frac{4}{t}\dot{\Psi} + \frac{k^2}{3}\Psi = 0$$

$$\Psi = \frac{\sqrt{3}}{kt}\left(A j_1\left(\tfrac{1}{\sqrt{3}}kt\right) + \underbrace{B y_1\left(\tfrac{1}{\sqrt{3}}kt\right)}\right)$$

not regular, diverges for $kt \to 0$ $\Rightarrow$ need to require $B = 0$

$$x := \frac{kt}{\sqrt{3}} = c_s kt$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \xrightarrow{x \to 0} \frac{x}{3}, \quad y_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x} \xrightarrow{x \to 0} -\frac{1}{x^2}$$

$$\Psi(x) \xrightarrow{x \to 0} \frac{A}{3}, \quad \Psi(x) = A\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right)$$

$$D_g = 2A\left[\cos x - \frac{2}{x}\sin x\right], \quad V = -\frac{\sqrt{3}}{4}D_g'$$

$\xrightarrow{x \to 0} -2A.$

$\xrightarrow{x \to \infty} 2A\cos x$ (acoustic peaks!)

## 1.4 perturbed Light like geodesics

At $T = T_{rec} \cong 3700K \cong 0.32eV$, $z_{rec} \cong 1370$, there are no longer enough photons with energy $\omega_\gamma > 1Ry = 13.6eV$ around to keep protons and electrons ionized. They recombine to neutral hydrogen, leaving a small ionization fraction $x_R \approx 10^{-4}$.

At $z_{dec} \cong 1100$, $T_{dec} \cong 3000K \cong 0.26eV$, the free electron density has become so sparse that the interaction of photons and electrons can be neglected. From then on, photons virtually "free stream" into our antennas.

This decoupling process is rather rapid. In a first approximation we assume it to be abrupt.

In this approximation we miss two important physical effects
- Silk damping on small scales and
- Polarization

Therefore, we will have to come back and do a more sophisticated Boltzmann equation approach in a second go.

To simplify the formulas we neglect curvature, $K = 0$.

From the last scattering surface into our antenna, photons move along geodesics of the perturbed



space time. We want to calculate the resulting temperature fluctuations for scalar perturbations.

We use longitudinal gauge

The unperturbed photon trajectory is given by

$$(x^\mu(t)) = (t, \underline{n}(t_0 - t) + \underline{x}_0)$$

for a photon which arrives at $\underline{x}_0$ at time $t_0$ and comes in from direction $\underline{n}$ (the direction of photon propagation is $-\underline{n}$).

Our metric is of the form

$$d\tilde{s}^2 = a^2 ds^2 \quad \text{with} \quad ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$$

the metrics $d\tilde{s}^2$ and $ds^2$ have the same light-like geodesics, only the affine parameters $\tilde{\lambda}$ and $\lambda$ are different.

$$n = \frac{dx}{d\lambda}, \quad \tilde{n} = \frac{dx}{d\tilde{\lambda}}, \quad n^2 = \tilde{n}^2 = 0, \quad n^0 = 1, \underline{n}^2 = 1$$

In an expanding univere, the physical photon momentum is redshifted, $\propto \frac{1}{a}$. Therefore the components with respect to a comoving, non expanding basis scale as $\frac{1}{a^2}$, hence $\tilde{\lambda} = a^2 \lambda$.

As always for lightlike geodesics, the affine parameter is only fixed up to a multiplicative constant which we fix now via the condition

$\underline{n}^2 = 1$ and $\tilde{\lambda} = a^2 \lambda$.

We set $n^0 = 1 + \delta n^0$ , $n^i = \bar{n}^i + \delta n^i$

The geodesic eqn yields to first order

$$\frac{d\delta n^\mu}{d\lambda} = -\delta\Gamma^\mu_{\alpha\beta} n^\alpha n^\beta$$

$$\delta\Gamma^0_{\alpha\beta} = -\frac{1}{2}\left(h_{\alpha 0|\beta} + h_{\beta 0,\alpha} - \dot{h}_{\alpha\beta}\right)$$

$$\frac{d}{d\lambda}\delta n^0 = \underbrace{h_{\alpha 0,\beta} n^\alpha n^\beta}_{n^\alpha \cdot \frac{d h_{\alpha 0}}{d\lambda}} - \frac{1}{2}\dot{h}_{\alpha\beta} n^\alpha n^\beta \qquad \left(\text{since } n^\beta = \frac{dx^\beta}{d\lambda}\right)$$

$$\delta n^0\Big|_i^f \underset{\text{longitudinal gauge}}{=} h_{00}\Big|_i^f - \frac{1}{2}\int \dot{h}_{\mu\nu} n^\mu n^\nu \, d\lambda$$

The ratio of the energy of a photon measured by some observer at $t_f$ to the energy emitted at $t_i$ is

$$\frac{E_f}{E_i} = \frac{(\tilde{n} \tilde{\cdot} \tilde{u})_f}{(\tilde{n} \tilde{\cdot} \tilde{u})_i} = \frac{a_i}{a_f}\,\frac{(n\cdot u)_f}{(n\cdot u)_i}$$

where we have used $\tilde{n} \propto \frac{1}{a^2} n$, $\tilde{u} \propto \frac{1}{a} u$ and "$\tilde{\cdot} = a^2 \cdot$"



~~Hence~~ $u = (1-\Psi)\partial_t + \underset{"V^i"}{v^i}\partial_i = a\tilde{u}$.

Now $\frac{a_i}{a_f}$ is the ratio of the usual, unperturbed redshift which relates $\lambda$ and $\tilde{\lambda}$. It is also given by $\frac{a_i}{a_f} = \frac{T_f}{T_i}$ , where these are the unperturbed temperatures. They can be related to the measured final and initial temperatures $T_0$ and $T_{dec}$ via

$$T_f + \delta T_f = T_0 , \qquad T_i + \delta T_i = T_{dec}$$

Hence $$\frac{T_f}{T_i} = \frac{T_0 - \delta T_f}{T_{dec} - \delta T_i} = \frac{T_0}{T_{dec}}\left(1 - \frac{\delta T_f}{T_f} + \frac{\delta T_i}{T_i}\right)$$

$$= \frac{T_0}{T_{dec}}\left(1 + \frac{\delta T}{T}\Big|_i^f\right) = \frac{T_0}{T_{dec}}\left(1 - \frac{1}{4}\delta^{(long)}\Big|_i^f\right)$$

$$\Rightarrow \frac{E_f}{E_i} = \frac{T_0}{T_{dec}}\left\{1 - \left[\frac{1}{4}D_{gr} + \Phi - \Psi + \underline{V}_{(b)}\cdot\underline{n} + 2\Psi\right]\Big|_i^f + \int_i^f (\dot{\Psi} + \dot{\Phi})\,d\lambda - \int_i^f \dot{H}^{(T)}_{ij} n^i n^j \, d\lambda\right\}$$

$$\frac{\Delta T}{T} = \frac{\Delta T(\underline{n}_1) - \Delta T(n_2)}{T} = \left(\frac{E_f}{E_i}(n_1) - \frac{E_f}{E_i}(n_2)\right)\cdot\frac{T_{dec}}{T_0}$$

$\Rightarrow$ up to a dipole contribution we can set

$$\frac{\Delta T}{T}(\underline{n}) = \frac{1}{4}D_{gr}(\underline{x}_i, t_i) + \underline{V}_b(\underline{x}_i, t_i)\cdot\underline{n}$$
$$+ (\Phi+\Psi)(\underline{x}_i, t) + \int_{t_i}^{t_0}(\dot\Psi+\dot\Phi)(x(t), t)\,dt.$$

(RD '90)

acoustic peak + part of SW

SW

Doppler term

ISW

$x_i = x_0 + \underline{n}(t_0 - t_i)$

<u>Adiabatic</u> initial conditions require $D_{gr} = \frac{4}{3}D_{gm}$. But in a matter dominated universe $D_{gm} = -5\Psi$ on super-Hubble scales, $\underline{kt \ll 1}$

$\Rightarrow \quad D_{gr} = -\frac{20}{3}\Psi$ for $kt \ll 1$. Neglecting $\Pi$, so that $\Phi = \Psi$ we find

$$\frac{\Delta T}{T} \simeq \underbrace{-\frac{1}{3}\Psi}_{\substack{SW\\(OSW)}} + \underbrace{2\int_{t_i}^{t_0}\dot\Psi\,dt}_{ISW}.$$

As we have seen, on sub-Hubble scales, $D_{gr}$ is oscillating like a cosine. The maxima and minima of this cosine determine the acoustic peak positions.

For tensor perturbations (gravity waves) we obtain in the same way

$$\left(\frac{\Delta T}{T}(n)\right)^{(T)} = -\int_i^f \dot H_{ij}^{(T)} n^i n^j\,dt$$



## 1.5 Power spectra

We assume that the initial conditions for perturbations are 'random variables' in the sense that $D_g$ is not fixed. Only correlators, i.e. expectation values of the form

$$\langle D_g(x) D_g(y)\rangle \quad \text{or} \quad \langle V(\underline{k})\Psi(\underline{k}')\rangle$$

are determined by theory.

We furthermore assume that the process which generates the initial perturbations is statistically homogeneous and isotropic, i.e.

$\langle X(\underline{x}) X(\underline{y})\rangle$ depends only on the distance

$r = |\underline{x} - \underline{y}|$

One can then show that $\underline{k}$ modes are independent, (Ex.2)

$$\langle X(\underline{k}) X^*(\underline{k}')\rangle = (2\pi)^3\delta^3(\underline{k}-\underline{k}')\underbrace{P_X(k)}_{\text{power spectrum of the variable } X.}$$

• <u>Matter</u>: $\langle D_{gm}(\underline{k}, t_0) D_{gm}^*(\underline{k}', t_0)\rangle = (2\pi)^3\delta^3(\underline{k}-\underline{k}')P_D(k)$

• <u>grav. potential</u>:

$$\langle\Psi(\underline{k})\Psi^*(\underline{k}')\rangle k^3 = (2\pi)^3\underbrace{k^3 P_\Psi(k)}_{A_S(kt_0)^{n_S-1}}\delta(\underline{k}-\underline{k}')$$

$n_s = 1$ is the scale invariant so called Harrison-Zel'dovich spectrum.

It is roughly also the spectrum predicted by inflation: $n_s \lesssim 1$.

<u>Measurements</u> indicate $n_s = 0.95 \pm 0.02$ in perfect agreement with simple models of inflation

- <u>CMB</u>

$\frac{\Delta T}{T}$ is a function on the sphere of photon directions. Its harmonic expansion is therefore not in Fourier modes but in spherical harmonics

$$\frac{\Delta T}{T}(x_0, t_0, \underline{n}) = \sum_{\ell m} a_{\ell m}(x_0, t_0) Y_{\ell m}(\underline{n})$$

If the fluctuations are statistically isotropic

$$\langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell$$

CMB power spectrum

<u>The SW contribution</u>

$$\frac{\Delta T}{T}(\underline{x}_0, t_0, \underline{n}) \cong \frac{1}{3}\Psi(x_{dec}, t_{dec}) \quad , \quad x_0 = x_{dec} - \underline{n}(t_0 - t_{dec})$$

$$\frac{\Delta T}{T}(\underline{k}, t_0, \underline{n}) \cong \frac{1}{3}\Psi(\underline{k}, t_{dec}) e^{i\underline{k}\cdot\underline{n}(t_0 - t_{dec})}$$



$$e^{i\underline{k}\underline{n}(t_0 - t_{dec})} = \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell j_\ell(k(t_0 - t_{dec})) P_\ell(\hat{\underline{k}}\cdot\underline{n})$$

$$\left\langle \frac{\Delta T}{T}(\underline{n}) \frac{\Delta T}{T}(\underline{n}') \right\rangle_{\underline{n}\cdot\underline{n}' = \mu} = \sum_{\ell\ell' m m'} \langle a_{\ell m} a^*_{\ell' m'} \rangle Y_{\ell m}(\underline{n}) Y^*_{\ell' m'}(\underline{n}')$$

$$= \sum_\ell C_\ell \underbrace{\sum_m Y_{\ell m}(\underline{n}) Y^*_{\ell m}(\underline{n}')}_{\frac{2\ell+1}{4\pi} P_\ell(\mu)} = \frac{1}{4\pi} \sum_\ell (2\ell+1) C_\ell P_\ell(\mu)$$

$$\left\langle \frac{\Delta T}{T}(\underline{x}_0, t_0, \underline{n}) \frac{\Delta T}{T}(\underline{x}_0, t_0, \underline{n}') \right\rangle = \frac{(2\pi)^3}{(2\pi)^6} \int d^3k\, d^3k' \frac{1}{9} P_\Psi(k, t_{dec}) \cdot e^{i\underline{k}(\underline{n} - \underline{n}')(t_0 - t_{dec})} \delta^3(\underline{k} - \underline{k}')$$

$$e^{ix\mu} = \sum_\ell i^\ell (2\ell+1) j_\ell(x) P_\ell(\mu)$$

$$= \frac{1}{(2\pi)^3} \frac{1}{9} \int d^3k\, P_\Psi(k, t_{dec}) \sum_{\ell, \ell'} (2\ell+1)(2\ell'+1) j_\ell(k(t_0 - t_{dec})) \cdot j_{\ell'}(k(t_0 - t_{dec})) \underbrace{P_\ell(\hat{\underline{k}}\cdot\underline{n})}_{\frac{4\pi}{2\ell+1}\sum_m Y_{\ell m}(\hat{k}) Y^*_{\ell m}(\underline{n})} P_{\ell'}(\hat{\underline{k}}\cdot\underline{n}') i^{(\ell - \ell')}$$

$$\frac{4\pi}{2\ell'+1} \sum_{m'} Y^*_{\ell' m'}(\hat{\underline{k}}) Y_{\ell' m'}(\underline{n}')$$

$$d\Omega_{\hat{k}} \Rightarrow \delta_{\ell\ell'} \delta_{mm'}$$

$$= \frac{2}{\pi} \frac{1}{9} \sum_\ell \frac{2\ell+1}{4\pi} P_\ell(\underline{n}\cdot\underline{n}') \int dk\, k^2 P_\Psi(k, t_{dec}) j_\ell^2(k(t_0 - t_{dec}))$$

$$\Rightarrow C_\ell^{(SW)} \cong \frac{2}{9\pi}\int_0^\infty \frac{dk}{k} \underbrace{P_\psi(k)\, k^3}_{A_S (k t_0)^{n_S}} j_\ell^2(k t_0)$$

$$= \frac{2A_S}{9\pi}\underbrace{\int_0^\infty dx\, x^{n_S} j_\ell^2(x)}_{\frac{\pi}{2}\frac{\Gamma(3-n_S)\,\Gamma(\ell-\frac{1}{2}+\frac{n_S}{2})}{2^{3-n_S}\,\Gamma^2(2-\frac{n_S}{2})\,\Gamma(\ell+\frac{5}{2}-\frac{n_S}{2})}}$$

$$-3<n_S<3$$

For $n_S = 1$ we find

$$\ell(\ell+1)\, C_\ell^{(SW)} = \frac{A_S}{9\pi} = \text{const.} \left(\underset{\text{(measured)}}{\cong 7\times 10^{-10}}\right)$$

This is the dominant term for $\ell \lesssim 40$. $\vartheta_\ell \cong \frac{\pi}{\ell} \gtrsim 4^\circ$

On smaller scales, the acoustic peaks become important

$$C_\ell^{(AC)} \cong \frac{2}{\pi}\int \frac{dk}{k}(2\pi)^{-3}\left\langle \left|\frac{1}{4}D_r(\underline{k}, t_{dec})\, j_\ell(k t_0) + V_m(\underline{k}, t_{dec})\cdot j_\ell'(k t_0)\right|^2\right\rangle$$

$D_r(k, t_{dec}) \sim \cos\left(\frac{k\, t_{dec}}{\sqrt{3}}\right)$ has its maxima and minima at $k_n = \frac{n\pi\cdot\sqrt{3}}{t_{dec}}$, $\lambda_n = \frac{2\, t_{dec}}{\sqrt{3}\, n}$.



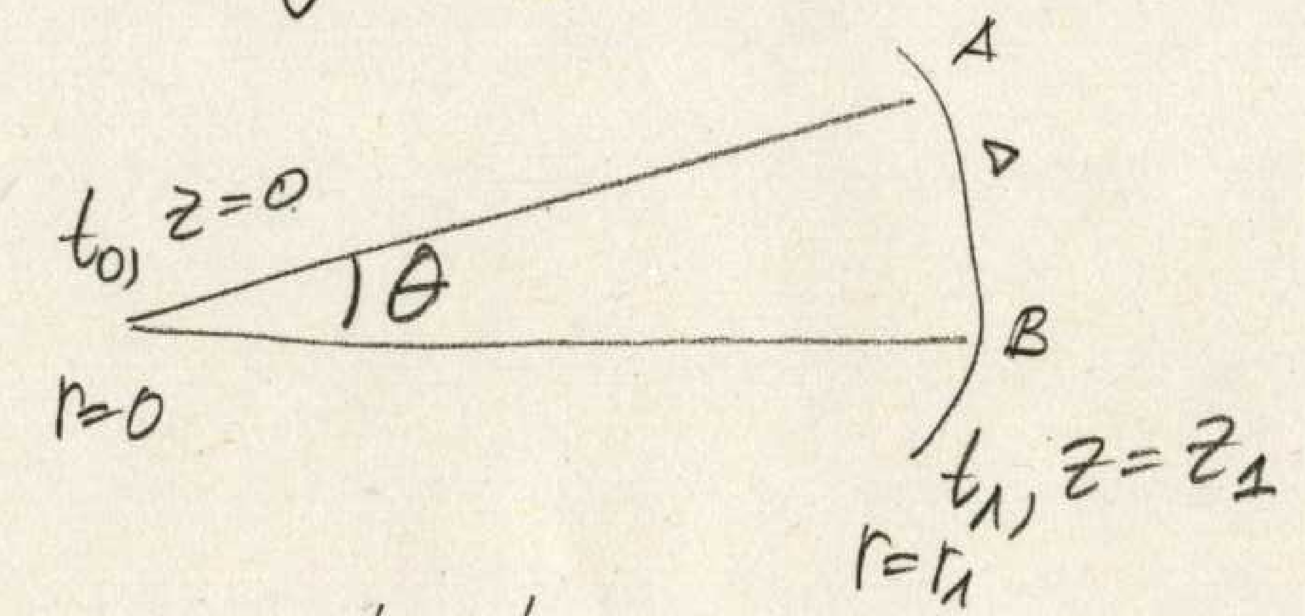
This wavelength is projected in the CMB sky in an angle of

$\theta_n = \frac{\lambda_n}{d_A(t_{dec})}$. Expanding in spherical harmonics, this corresponds to a harmonic $\ell_n \cong \frac{\pi}{\theta_n}$

$$\ell_n \cong \frac{\pi n\cdot\sqrt{3}\, d_A(t_{dec})}{2\, t_{dec}}.$$

Here, $d_A(t)$ is the angular diameter distance to the time $t$. It is defined as follows:

Let us consider two events at time $t_1$ which are an angle $\theta$ appart




The physical arc length between A and B is

$$\Delta = a(t_1)\,\chi(r_1)\cdot\vartheta = a(t_1)\,\chi(t_0-t_1)\cdot\vartheta$$

$$r_1 = t_0 - t_1$$

Hence $\vartheta = \frac{\Delta}{a(t_1)\,\chi(t_0-t_1)}$

We call $a(t_1)\,\chi(t_0-t_1) =: d_A(z_1)$ the angular diameter distance to $z_1$.

Using $H = \frac{\dot a}{a}$, we can express it as

$$dt = \frac{da}{Ha}$$

$$d_A(z) = \frac{1}{H_0(1+z)|\Omega_K|^{1/2}} \chi_K\Big(\underbrace{\sqrt{|\Omega_K|}\, H_0}_{a_0} \int_0^z \frac{dz'}{H(z')}\Big).$$

$$H(z) = H_0\left(\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right)^{1/2}$$

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## 2. The Boltzmann equation for CMB anisotropies and polarization

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### 2.1 Basics

Photons (or any kind of "classical" particles) can be described by the distribution function defined on the 7-dimensional phasespace (the mass shell)

$$P_m = \{(x,p) \in T\mathcal{M} \,/\, g(x)(p,p) = -m^2\}$$

For photons then $m^2 = 0$.
The energy momentum tensor is given by integrals of the second moments over the fiber

$$P_m(x) = \{p \in T_x\mathcal{M} \,/\, g(x)(p,p) = -m^2\}.$$

$$f : P_m \to \mathbb{R} : (x,p) \mapsto f(x,p)$$



$$T^{\mu\nu}(x) = \int_{P_m(x)} \frac{\sqrt{|g(x)|}}{|p_0|} p^\mu p^\nu f(x,p)\, d^3p$$

(If we choose the coordinates $(x^\mu; p^i)$ on $P_m$ and consider $p^0$ as solution of $g_{\mu\nu}p^\mu p^\nu = -m^2$.)

With $p_\mu = \frac{\partial \mathcal{L}}{\partial \dot x^\mu} = m\dot x_\mu$ we have for free particles $\ddot x^\mu + \Gamma^\mu_{\alpha\beta}\dot x^\alpha \dot x^\beta = 0$

$$m\dot p^\mu = -\Gamma^\mu_{\alpha\beta} p^\alpha p^\beta$$

If there are no collisions, the distribution function remains constant in a comoving element of phase space volume

$$0 = \frac{df}{dt} = \dot x^\mu \partial_\mu f + \dot p^i \frac{\partial f}{\partial p^i} = \frac{1}{m}\left[p^\mu \partial_\mu f - \Gamma^i_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^i}\right]$$

This is the Liouville eqn.
If there are interactions, this zero is replaced by a collison integral

$$p^\mu \partial_\mu f - \Gamma^i_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^i} = \underbrace{C[f]}$$

This is the Boltzmann equation.

is well defined under certain assumption and then given like in special relativity, since collisons are local so curvature does not play a rôle.

In an unperturbed Friedmann universe, the Liouville eqn. just implies that physical momenta are redshifted. For the comoving momenta this implies $p^i \propto \frac{1}{a^2} \Leftrightarrow p := \sqrt{g_{ij}p^i p^j} \propto \frac{1}{a}$

setting $v = ap$ and interpreting $f = f(t, v)$ as a function of this redshift corrected momentum, the Liouville eqn. reduces to $\partial_t f = 0$. Hence $f = f(v)$ is a function of the redsift corrected momentum only.

We now define $f = \bar{f} + \delta f$ and

$$\mathcal{M}(\underline{n}) = \frac{1}{4}\frac{4\pi}{\rho_r a^4}\int v^3 \delta f \, dv = \frac{\Delta T}{T}(\underline{n})$$

The Liouville eqn. for scalar perturbation then becomes (lengthy derivation)

$$\partial_t \mathcal{M}^{(S)} + n^i \partial_i \mathcal{M}^{(S)} = -n^i \partial_i (\Psi + \Phi)$$

A formal solution of this eqn. is

$$\mathcal{M}^{(S)}(t, \underline{x}, \underline{n}) = \mathcal{M}^{(S)}(t_{in}, \underline{x} - \underline{n}(t - t_{in}), \underline{n}) - \int dt' \, n^i \partial_i (\Psi + \Phi)(t', \underline{x} - \underline{n}(t - t'))$$

$$= \mathcal{M}^{(S)}(t_{in}, \underline{x} - \underline{n}(t - t_{in}), \underline{n}) + (\Phi + \Psi)(t_{in}, \underline{x}_{in}) + \int_{t_{in}}^{t} dt' \, \partial_{t'} (\Psi + \Phi)(t', \underline{x} - \underline{n}(t - t'))$$



At early times, before decoupling, collisions strongly damp higher moments. Therefore if $t_{in} \ll t_{dec}$

$$\mathcal{M}^{(S)}(t_{in}, \underline{x} - \underline{n}(t - t_{in}), \underline{n}) = \left(\frac{1}{4} D_{gr} + \underline{n} \cdot \underline{V}^{(b)}\right)(t_{in}, \underline{x} - \underline{n}(t - t_{in}))$$

and $\mathcal{M}^{(S)} = \frac{\delta T}{T}$.

In Fourier space, the solution to the Liouville eqn is

$$\mathcal{M}^{(S)}(t_0, \underline{k}, \underline{n}) = e^{-ik\mu(t_0 - t_{in})} \mathcal{M}^{(S)}(t_{in}, \underline{k}, \mu) + \int_{t_{in}}^{t_0} dt \, e^{ik\mu(t - t_0)} \, ik\mu (\Phi + \Psi)(\underline{k}, t) \quad , \mu = \underline{n} \cdot \hat{\underline{k}}$$

## 2.2 Polarization

A photon with momentum $\omega \underline{n}$ is an electromagnetic wave propagating in direction $\underline{n}$. We define a polarization basis $\underline{\varepsilon}^{(1)}, \underline{\varepsilon}^{(2)}$ such that $(\underline{\varepsilon}^{(1)}, \underline{\varepsilon}^{(2)}, \underline{n})$ form a right handed orthonormal system. The electric field of the wave is of the form $\underline{E} = E_1 \underline{\varepsilon}^{(1)} + E_2 \underline{\varepsilon}^{(2)}$

The polarization tensor is defined by $P_{ab} = E_a E_b^*$

$P_{ab}$ is a hermitean $2 \times 2$ matrix and can therefore be written in the form

$$P_{ab} = \frac{1}{2}\left[I\sigma_{ab}^{(0)} + U\sigma_{ab}^{(1)} + V\sigma_{ab}^{(2)} + Q\sigma_{ab}^{(3)}\right]$$

where $\sigma^\mu$ are the Pauli matrices,

$$\sigma^{(0)} = \begin{pmatrix}1&0\\0&1\end{pmatrix},\ \sigma^{(1)} = \begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma^{(2)} = \begin{pmatrix}0&-i\\i&0\end{pmatrix},$$

$$\sigma^{(3)} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$I = |E_1|^2 + |E_2|^2$ is the intensity

$Q = |E_1|^2 - |E_2|^2$ is linear polarization in $\underline{\varepsilon}^{(1)}$ direction

$U = 2\,\mathrm{Re}(E_1^* E_2)$ is linear pol. in the direction at $45^\circ$ between $\underline{\varepsilon}^{(1)}$ and $\underline{\varepsilon}^{(2)}$

$V = 2\,\mathrm{Im}(E_1^* E_2)$ is circular polarisation.

Defining $\varepsilon^{(\pm)} = \frac{1}{\sqrt{2}}(\varepsilon^{(1)} \pm i\varepsilon^{(2)})$ and

$E = E^+\varepsilon^{(+)} + E^-\varepsilon^{(-)}$ one finds

$V = |E^+|^2 - |E^-|^2$.

The relevant collison process before recombination is non-relativistic Thomson scattering which does not generate circular polarization. We therefore set

$V \equiv 0$.

Since $\mathcal{M} = \frac{1}{4}\frac{\delta\rho}{\rho} = \frac{1}{4}\frac{\delta I}{I}$, we also define

$$q = \frac{1}{4}\frac{Q}{I} \quad \text{and} \quad \mathcal{U} = \frac{1}{4}\frac{U}{I}$$



$$\mathcal{P} := q\,\sigma^{(3)} + \mathcal{U}\,\sigma^{(1)}$$

is a real symmetric traceless matrix.

We define also

$$\mathcal{P}^{(+2)} \equiv \mathcal{P}_{++} = 2\,\mathcal{P}_{ab}\,\varepsilon_a^{(+)}\varepsilon_b^{(+)} = q + i\mathcal{U}$$

$$\mathcal{P}^{(-2)} \equiv \mathcal{P}_{--} = 2\,\mathcal{P}_{ab}\,\varepsilon_a^{(-)}\varepsilon_b^{(-)} = q - i\mathcal{U}$$

Under a rotation by an angle $\varphi$ around $\underline{n}$ we have

$$\varepsilon^{(1)} \to \cos\varphi\,\varepsilon^{(1)} + \sin\varphi\,\varepsilon^{(2)};\quad \varepsilon^{(2)} \to \cos\varphi\,\varepsilon^{(2)} - \sin\varphi\,\varepsilon^{(1)}$$

$$\varepsilon^{(\pm)} \to e^{\mp i\varphi}\varepsilon^{(\pm)}$$

$$q \pm i\mathcal{U} \to e^{\pm 2i\varphi}(q \pm i\mathcal{U})$$

$\mathcal{P}_{ab}$ is a spin two tensor field on the sphere.

$\mathcal{P}_{\pm\pm}$ are its helicity $+2$ and $-2$ components.

Such tensor fields can be expanded in terms of spin weighted spherical harmonics, ${}_sY_{\ell m}$.

For $s=0$ these are the usual spherical harmonics while for $s \neq 0$ they depend not only on $\underline{n}$ but also on the basis on the sphere, which is traditionally $(\underline{e}_\vartheta, \underline{e}_\varphi)$, $\underline{e}_\vartheta = \partial_\vartheta$, $\underline{e}_\varphi = \frac{1}{\sin\vartheta}\partial_\varphi$.

${}_s Y_{\ell m}$ transforms with helicity $s$ under a rotation around $\underline{n}$. For $\ell < s$, ${}_s Y_{\ell m} = 0$.

With $\underline{e}^{\pm} = \frac{1}{\sqrt{2}}(\underline{e}_\vartheta \pm i \underline{e}_\varphi)$ we can now expand

${}_s Y_{\ell m}(\vartheta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}} D^{(\ell)}_{m-s}(\varphi, \vartheta, 0)$ around z, around y

$$(q \pm i U)(\underline{n}) = \sum_{\ell \geq 2, m} a_{\ell m}^{(\pm 2)} \, {}_{\pm 2}Y_{\ell m}(\underline{n})$$

$$= \sum_{\ell, m} (e_{\ell m} \pm i b_{\ell m}) \, {}_{\pm 2} Y_{\ell m}(\underline{n})$$

$$e_{\ell m} = \frac{1}{2}\left(a_{\ell m}^{(2)} + a_{\ell m}^{(-2)}\right), \quad b_{\ell m} = \frac{-i}{2}\left(a_{\ell m}^{(2)} - a_{\ell m}^{(-2)}\right)$$

Similar to the quantum mechanical angular momentum operators $L_\pm$, helicity raising $\eth$ and lowering $\eth^*$ operators can be defined so that

$$\eth \, {}_s Y_{\ell m} \propto {}_{s+1}Y_{\ell m}, \quad \eth^* \, {}_s Y_{\ell m} \propto {}_{s-1}Y_{\ell m}$$

actually $\eth = -\sqrt{2}\,\nabla_+$, $\eth^* = -\sqrt{2}\,\nabla_-$

One finds

$$\eth^2({}_{-2}Y_{\ell m}) = \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \, Y_{\ell m}$$

$$(\eth^*)^2({}_2 Y_{\ell m}) = \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \, Y_{\ell m} \quad \text{so that}$$

$$(\eth^{(*)})^2 (q \pm i U)(n) = \sum_{\ell, m} \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \, a_{\ell m}^{(\pm 2)} Y_{\ell m}.$$

$$\eth^2(q - iU) + (\eth^*)^2(q + iU) = 2(\nabla_- \nabla_- P_{++} + \nabla_+ \nabla_+ P_{--})$$



We can now define

$$\mathcal{E} = \sum_{\ell, m} e_{\ell m} Y_{\ell m}, \quad \mathcal{B} = \sum_{\ell, m} b_{\ell m} Y_{\ell m}$$

Note that there is no local operator which transforms $\mathcal{P}$ into $\mathcal{E}$ and $\mathcal{B}$

However

$$\tilde{\mathcal{E}} = \frac{1}{2}\left[(\eth^*)^2 (q + iU) + \eth^2 (q - iU)\right]$$

$$= \sum_{\ell m} \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \, e_{\ell m} Y_{\ell m} \quad \text{and}$$

$$\tilde{\mathcal{B}} = \sum_{\ell m} \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \, b_{\ell m} Y_{\ell m} = -\frac{i}{2}\left[ \cdots - \cdots \right]$$

can be obtained by local measurements.

Under parity operation, $\underline{n} \to -\underline{n}$ $\underline{e}_\varphi$ changes sign, but $\underline{e}_\vartheta$ does not. Therefore

$$\underline{e}^{(\pm)} \longrightarrow \underline{e}^{(\mp)} \quad \text{so that} \quad {}_{(\pm 2)}Y_{\ell m} \to (-1)^\ell \, {}_{(\mp 2)}Y_{\ell m}$$

$$\Rightarrow \quad a_{\ell m}^{(\pm 2)} \longrightarrow (-1)^\ell a_{\ell m}^{(\mp 2)}$$

$$e_{\ell m} \longrightarrow (-1)^\ell e_{\ell m}, \quad b_{\ell m} \longrightarrow -(-1)^\ell b_{\ell m}.$$

The temperature expansion coefficients transform of course also like $a_{\ell m} \longrightarrow (-1)^\ell a_{\ell m}$.

Therefore, if the primordial process which has generated perturbation (inflation) was parity invariant we must have $\langle a_{\ell m} b_{\ell' m'} \rangle = \langle e_{\ell m} b_{\ell' m'} \rangle = 0$

$C_\ell^{B} := \langle |b_{\ell m}|^2 \rangle$ , $C_\ell^{E} := \langle |e_{\ell m}|^2 \rangle$ ,

$C_\ell := \langle |a_{\ell m}|^2 \rangle$ , $C_\ell^{\mathcal{M}E} := \langle a_{\ell m} e^*_{\ell m} \rangle$ .

(*) Actually

$(\eth^*)^2 (Q + i\mathcal{U}) = 2 \nabla_- \nabla_- \mathcal{P}_{++}$ , $\eth^2 (Q - i\mathcal{U}) = 2 \nabla_+ \nabla_+ \mathcal{P}_{--}$

So that

$$\tilde{\mathcal{E}} = \nabla_- \nabla_- \mathcal{P}_{++} + \nabla_+ \nabla_+ \mathcal{P}_{--} = 2 \nabla_i \nabla_j P_{ij} = 2\,\mathrm{div\,div}\,P$$

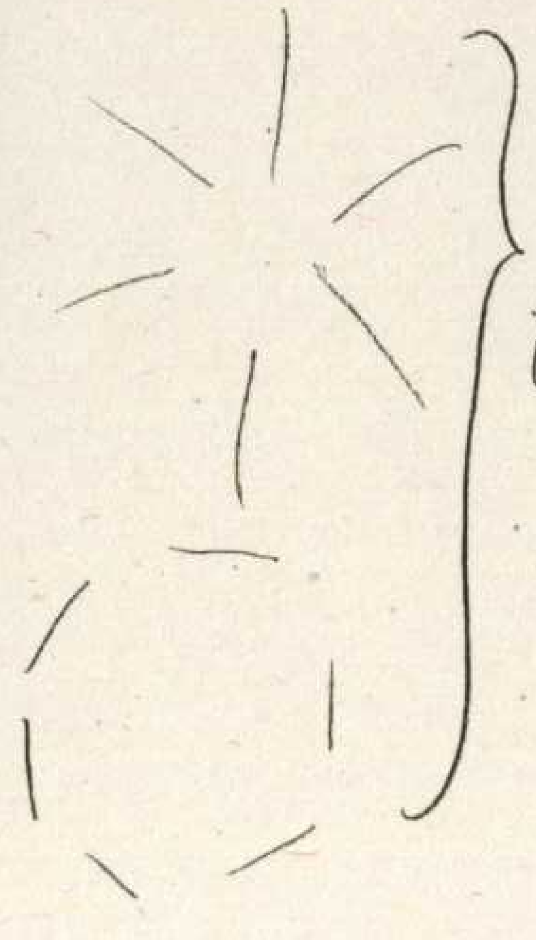
$$\tilde{\mathcal{B}} = \nabla_- \nabla_- \mathcal{P}_{++} - \nabla_+ \nabla_+ \mathcal{P}_{--} = 2 \varepsilon_{\ell m} \varepsilon_{ij} \nabla_\ell \nabla_i P_{mj} = 2\,\mathrm{curl\,curl}\,P$$

$\Rightarrow$ $\mathcal{E}$ measures gradient type polarization, while

$\mathcal{B}$ measures curl type "

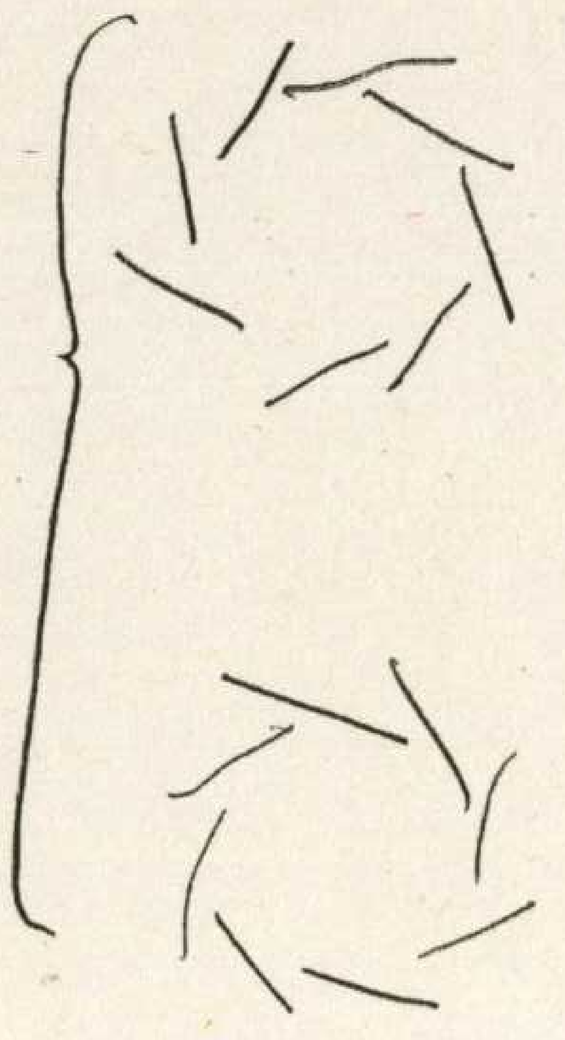
If $E_i = \nabla_i f$ (on the sphere) $\Rightarrow \mathcal{B} \equiv 0$ while

if $E_i = \varepsilon_{ij} \nabla_j g$ " $\Rightarrow \mathcal{E} = 0$



$\mathcal{E}$ type

$\mathcal{B}$ type





## 2.3 The collision term

We now consider incoming radiation from direction $\underline{n}'$ which then is scattered into direction $\underline{n}$ with

$\underline{n} \cdot \underline{n}' = \cos\beta$ .

For photons polarized in the scattering plane, the scattered field amplitude is suppressed by a factor $\cos\beta$, while normal to the plane it is not.

The scattering field generated per unit time in a plasma with electron density $n_e$ is proportional to $\sqrt{n_e \sigma_T}\, E$. In the rest frame of the electron we have

$$|E_\parallel^c|^2 = \frac{3}{8\pi} n_e \sigma_T \cos^2\beta\, |E_\parallel|^2 , \quad |E_\perp^c|^2 = \frac{3}{8\pi} n_e \sigma_T\, |E_\perp|^2$$

and the phase is not affected.

Choosing the polarisation $\underline{\varepsilon}^{(1)}(\underline{n})$ in the scattering plane

$\Rightarrow$ $\underline{\varepsilon}^{(2)}(n)$ normal to it, we have

$$I = |E_\parallel|^2 + |E_\perp|^2 , \quad Q = |E_\parallel|^2 - |E_\perp|^2 , \quad U = 2 E_\parallel E_\perp^*$$

$\Rightarrow$ For

$$V = \begin{pmatrix} I \\ Q + iU \\ Q - iU \end{pmatrix} \qquad V_c = \frac{n_e \sigma_T}{4\pi} S \cdot V$$

$$S = \frac{3}{4} \begin{pmatrix} \cos^2\beta + 1 & -\frac{1}{2}\sin^2\beta & -\frac{1}{2}\sin^2\beta \\ -\sin^2\beta & \frac{1}{2}(\cos\beta + 1)^2 & \frac{1}{2}(\cos\beta - 1)^2 \\ -\sin^2\beta & \frac{1}{2}(\cos\beta - 1)^2 & \frac{1}{2}(\cos\beta + 1)^2 \end{pmatrix}$$

To obtain $S$ with respect to the bases $(e_\vartheta(\underline{n}'), e_\varphi(n'))$ and $(\underline{e}_\vartheta(\underline{n}), \underline{e}_\varphi(\underline{n}))$ we first rotate $q \pm iU$ by an angle $\gamma'$ around $\underline{n}'$ to turn into $\varepsilon^{(1)}(\underline{n}'), \varepsilon^{(2)}(n')$ then apply $S'$ and rotate $(\varepsilon^{(1)}(\underline{n}), \varepsilon^{(2)}(n))$ back into $(\underline{e}_\vartheta(\underline{n}), \underline{e}_\varphi(n'))$ with angle $-\gamma$ around $\underline{n}$.

$\Rightarrow \quad S \longrightarrow R(-\gamma) S R(\gamma')$ where

$$R(\alpha) = \mathrm{diag}(1, e^{2i\alpha}, e^{-2i\alpha})$$

$$R(-\gamma) S R(\gamma) = \frac{1}{2}\sqrt{\frac{4\pi}{5}} \begin{pmatrix} Y_{20}(\beta,\gamma') + 2\sqrt{5}\, Y_{00}, & -\sqrt{\frac{3}{2}}\, Y_{2,-2}, & -\sqrt{6}\, Y_{22} \\ -\sqrt{6}\, Y_{22}\, e^{-2i\gamma} & 3\, {}_2Y_{2-2}\, e^{-2i\gamma} & 3\, {}_2Y_{22}\, e^{2i\gamma} \\ -\sqrt{\frac{3}{2}}\, {}_{-2}Y_{20} & 3\, {}_{-2}Y_{2-2}\, e^{2i\gamma} & 3\, {}_{-2}Y_{22}\, e^{2i\gamma} \end{pmatrix}$$

$\Rightarrow$

$$C[\nu]_{rest} = a\sigma_T n_e \left[ \frac{1}{10} \int d\Omega_{\underline{n}'} \sum_{m=-2}^{2} P_m(\underline{n}, \underline{n}') \nu(\underline{n}') - \nu(\underline{n}) + \frac{1}{4\pi} \int d\Omega_{n'} \mathcal{M}(n') \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]$$

$$P_m(n, n') = \begin{pmatrix} Y_{2m}(n) Y^*_{2m}(n') & -\sqrt{\frac{3}{2}}\, Y_{2m}(n)\, {}_2Y^*_{2m}(n') & \cdots \\ -\sqrt{6}\, {}_2Y_{2m}(n)\, Y^*_{2m}(n') & & \\ -\sqrt{6}\, {}_{-2}Y_{2m}(n)\, Y^*_{2m}(n') & & \end{pmatrix}$$

addi. thm of (spin weighted) spherical harmonics