Introduction to

General Relativity

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1 Special Relativity

From 1687 until the beginning of the 20th century, Newton's notion of absolute space and time were the de-facto standard for physical theories describing mechanical motion of point particles and extended objects. It is so pervasive of the realm of physics, and in accordance with our everyday experiences, that it is still the foundation of most courses in physics and engineering. The equations for building ships, shooting rockets, up to the movement of stars and planets, rested on the foundation of Newton's three laws of mechanics, as well was the law of gravity.

It was just in 1887, when it turned out that Newtonian notions of space and time, together with the well-known Galilei-transformation, had to be extended to describe velocities close to the speed of light. In thie first chapter, we will go over this generalisation.

First we recall Newton's notions of space and time, as well as that of inertial observers and Galilei-transformations.

In Newtonian physics, space has the form of a 3-dimensional affine space, while time has the structure of a 1-dimensional affine space, i.e. a line. An inertial observer is one on whom no physical forces act. Such an observer moves, as per Newton's first law of mechanics, either not at all, or along a straight line with constant velocity.

To each inertial observer O corresponds an associated inertial coordinate system, with coordinates x^i, where $i = 1, 2, 3$. In these coordinates, Newton's second law can be expressed as

$$\frac{d^2 x^i}{dt} = 0$$

for any curve $t \mapsto x^i(t)$ of an inertial observer. Of course, every inertial observer is at rest in their worn inertial coordinate system, namely at the origin $x^i = 0$.

All inertial observers are considered equivalent, so it is prudent to understand how the coordinates of one inertial observer can be transformed into those of another one.

Assume there are two inertial observers O and $\tilde{O}$, with respective coordinates x^i and $\tilde{x}^i$. From $\tilde{O}$'s perspective, O moves in positive $\tilde{x}^1$-direction with velocity v. If the axes of the two coordinate systems are parallel, then the two sets of coordinates are related by the Galilei transformation law

$$\begin{aligned} \tilde{t} &= t \\ \tilde{x}^1 &= x^1 + vt \\ \tilde{x}^2 &= x^2 \\ \tilde{x}^3 &= x^3 \end{aligned} \tag{1.1}$$

Here t and $\tilde{t}$ are the time measured by either observer, which agree, since time is independent of the observer. Assume that at $t = 0$, observer O throws a stone in positive x^1-direction with velocity $dx^1/dt = w$. Then $\tilde{O}$ would measure the velocity of that stone to be

$$\tilde{w} = \frac{d\tilde{x}^1}{d\tilde{t}} = \frac{d\tilde{x}^1}{dt} = \frac{dx^1}{dt} + v = w + v. \tag{1.2}$$

So velocities add up in Newtonian mechanics, which is what conforms to our everyday experience.

1.0.1 From Galilei to Lorentz

In 1887, Michelson and Morley attempted to measure the relative speed of the Earth with respect to the aether, which at that time was thought to be the medium in which electromagnetic waves propagate. The experiment, famously, turned out to have a negative result: No matter in which direction the measurement apparatus moved, it always measured a speed pf

$$c = 3 \cdot 10^8 \,\mathrm{ms}^{-1}.$$

This directly contradicts (1.2), which predicts that two differently moving observers should measure different speeds of light, differing by their relative difference in velocity. But they did not. As it turned out, every inertial observer measures the same value of the speed of light, directly contradicting the Galilei transformations. A way out was proposed by slightly changing the transformation laws (1.1).

As it turns out, the time t has to be transformed along side the other three coordinates x^i. To treat them all on the same footing, one introduces

$$x^0 := ct.$$

The collection of four coordinates are being denoted by x^μ, with $\mu = 0, 1, 2, 3$. The ansatz for the new transformation can be written as

$$\begin{aligned}
\tilde{x}^0 &= A\big(x^0 + Bx^1\big) \\
\tilde{x}^1 &= C\big(x^1 + vt\big) = C\big(x^1 + \beta x^0\big) \\
\tilde{x}^2 &= x^2 \\
\tilde{x}^3 &= x^3
\end{aligned}$$

with $\beta := \frac{v}{c}$.

Now assume that at time $t = 0$, O sends out a light ray in the x^1-direction. The coordinates on the light rays then satisfy $dx^1/dt = c$, so $x^1 = x^0$, or, more general,

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0. \tag{1.3}$$

To conform with the result from Michelsons and Morleys experiment, we have to demand that the curve on $\tilde{O}$'s system satisfies the same equation, just for the $\tilde{x}^\mu$. From this it follows that

$$\begin{aligned}
0 &= (\tilde{x}^0)^2 - (\tilde{x}^1)^2 - (\tilde{x}^2)^2 - (\tilde{x}^3)^2 \\
&= A^2\left(x^0 - Bx^1\right)^2 - C^2\left(x^1 - \frac{v}{c}x^0\right)^2 - (x^2)^2 - (x^3)^2 \\
&= \left(A^2 - C^2\frac{v^2}{c^2}\right)(x^0)^2 - \left(C^2 - A^2B^2\right)(x^1)^2 - (x^2)^2 - (x^3)^2 \\
&\qquad + \left(2C\frac{v}{c} - 2AB\right)x^0x^1.
\end{aligned}$$

From comparison with (1.3) we can read off that

$$A^2 - C^2\frac{v^2}{c^2} = 1, \tag{1.4}$$

$$C^2 - A^2B^2 = 1, \tag{1.5}$$

$$2C\frac{v}{c} - 2AB = 0. \tag{1.6}$$

This can be solved for A, B, and C, and we obtain[1]

$$B = \beta = \frac{v}{c}, \quad A = C =: \frac{1}{\sqrt{1-\beta^2}}\,\gamma.$$

which gives us the formula for the Lorentz transformation as

$$\begin{aligned}
\tilde{x}^0 &= \gamma\left(x^0 + \beta x^1\right), \\
\tilde{x}^1 &= \gamma\left(x^1 + \beta x^0\right), \\
\tilde{x}^2 &= x^2, \\
\tilde{x}^3 &= x^3.
\end{aligned} \tag{1.7}$$

This transformation law is an example of a Lorentz transformation, which are those affine coordinate transformations which preserve the speed of light for all inertial observers. To check how the law (1.2) changes in this case, consider the same situation as before, but now $\tilde{O}$'s coordinates are related to O's via (1.7) and not (1.1). One gets, for the

[1] Up to signs, which would correspond to a parity transformation, or time reflection in $\tilde{O}$'s system. These are allowed, but not the solutions we look at here.

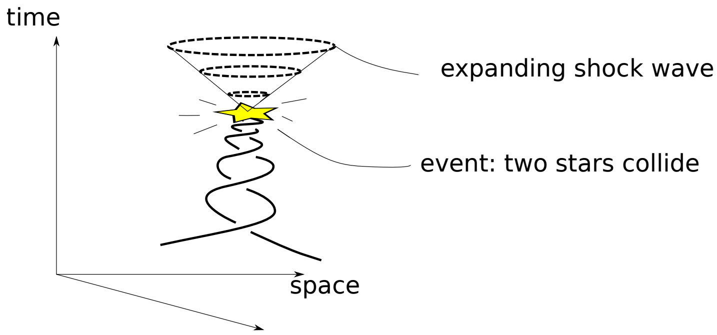

Figure 1.1: In a space-time diagram, times increases upwards, so a diagram of two stars encircling each other, colliding and exploding would roughly look like this.

speed measured in $\tilde{O}$'s system, that

$$\begin{aligned}\tilde{w} &= \frac{d\tilde{x}^1}{d\tilde{t}} = c\frac{d\tilde{x}^1}{d\tilde{x}^0} = c\left(\frac{d\tilde{x}^0}{dx^0}\right)^{-1}\frac{d\tilde{x}^1}{dx^0} \\ &= c\left(\frac{d}{dx^0}\Big(\gamma\big(x^0+\beta x^1\big)\Big)\right)^{-1}\frac{d}{dx^0}\Big(\gamma\big(x^1+\beta x^0\big)\Big) \\ &= \frac{c}{1+\beta\frac{w}{c}}\left(\frac{w}{c}+\beta\right) = \frac{v+w}{1+\frac{vw}{c^2}}\end{aligned}$$

For speeds small w.r.t. the speed of light, i.e.ü$v, w \ll c$; this tends towards (1.2), while for $w = c$ this leads to $\tilde{w} = c$, as demanded.

1.0.2 Minkowski space

A consequence of replacing Galilei- with Lorentz transformations is not just an adaptation of formulas, but a reworking of Newton's notion of space and time. In particular, one considers a four dimensional unification of space and time (the so called *Minkowski space* $\mathbb{R}^{1,3}$). Curves in Minkowski space describe motions, e.g. two stars circling each other and colliding. A point in Minkowski space is called an *event* (when and where something happens). In the coordinate system in Minkowski space, the point 0 is the *origin event*. The coordinate axes are $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$. We interpret this as follows: The parameter t is the elapsed time as measured by someone sitting at $x^1 = x^2 = x^3 = 0$ with the conversion factor c which works because c is the same in all frames of reference. The x^i, $i = 1, 2, 3$ are the distance from the x^0-axis. We write x^μ for a four vector. In our convention, Greek indices will always run from 0 to 3.
The curve that an object (observer, space ship...) traces out in Minkowski space is called a *world line*. A world line is a map

$$\phi \mapsto x^\mu(\phi) \tag{1.8}$$

with a more or less arbitrary curve parameter ϕ.

An *inertial system* is defined as a coordinate system that has the following properties:

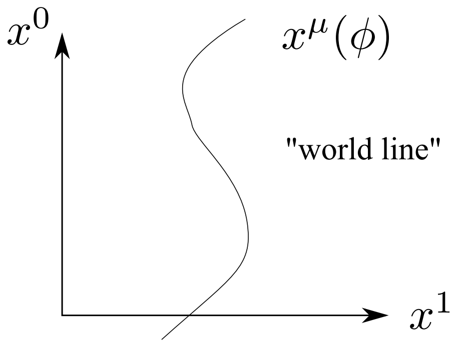

Figure 1.2: The world line of a (point-like) object, traced out in a coordinate system.

- Any freely moving object (i.e. one without force acting on it) is described by a straight line.

- Light rays always move on world lines with slope equal to 1.

The first condition is similar the old Newtonian physics: A coordinate system is called "inertial" if and only if $\vec{F} = m\ddot{\vec{x}}$ holds, i.e. $\ddot{\vec{x}} = 0$ for freely moving objects.
The second condition means that the speed of light is the same in all inertial systems.
To each inertial observer corresponds an inertial system. The world line of the observer in his own frame of reference is given by

$$x^\mu(t) = \begin{pmatrix} ct \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.9}$$

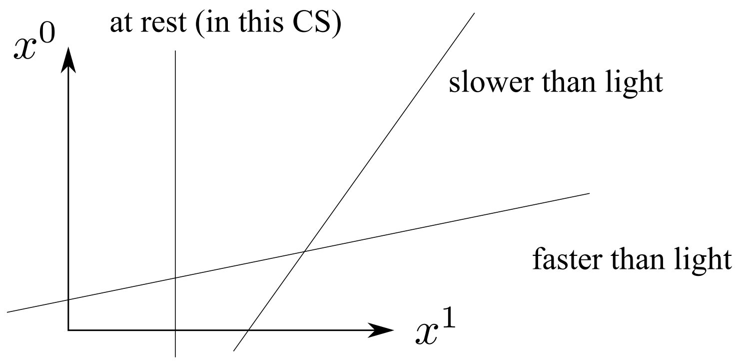

Figure 1.3: World lines of point particles moving freely. The slope of the line indicates how fast the particle moves.

1.1 Minkowski-distance squared

In Minkowski space, the distance between two points does not exist, but the distance squared does:

$$d(E_1, E_2)^2 := c^2(\Delta t)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2 = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \tag{1.10}$$

with the *Minkowski metric*

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.11}$$

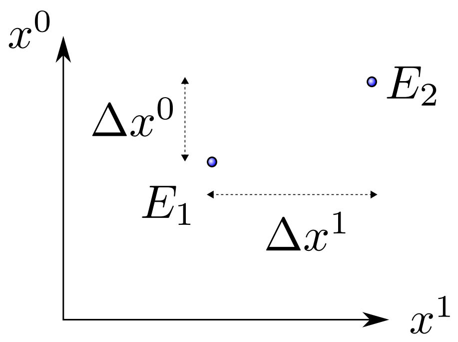

Figure 1.4: Two events E_1 and E_2 in Minkowski space.

From now on we want to save space and omit the sum over indices. Whenever there is an index appearing twice (once as upper, once as lower index), it is summed over. This is called the *Einstein convention*. Now, we can write

$$d(E_1, E_2)^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu. \tag{1.12}$$

Also, if an index appears only once (called a *free index*), it has to appear on both sides of the equation, and in the same position. An index is not allowed to appear more than twice. See table 1.1 for examples.

The Minkowski squared distance can have either sign, which is why $d(E_1, E_2)$ doesn't exist. We categorize the distances squared by their sign:

- $d(E_1, E_2)^2 > 0$: *time-like*
- $d(E_1, E_2)^2 < 0$: *space-like*
- $d(E_1, E_2)^2 = 0$: *light-like*

allowed	not allowed
$x^\mu = y^\mu$	$x^\mu = y_\mu$
$x^\mu = T^{\mu\nu}_{\nu\sigma} x^\sigma + y^\mu$	$x^\mu = y^\nu$
$\eta_{\mu\nu} x^\mu x^\nu = 3$	$x^\mu y^\nu = T^{\mu\nu} + \eta_{\mu\nu}$
$x^\mu y^\nu = T^{\mu\nu}$	$x^\mu = \eta_{\nu\sigma} y^\sigma z_\sigma T^{\sigma\nu\mu}$

Table 1.1: Allowed and disallowed index placements

Velocity vectors of curves have a Minkowski length squared:

$$\left|\frac{dx^\mu}{d\phi}\right|^2 = \eta_{\mu\nu}\frac{dx^\mu}{d\phi}\frac{dx^\nu}{d\phi}. \tag{1.13}$$

The velocity vector does not express the velocity directly, the velocity is the inverse slope of $dx^\mu/d\phi$. A curve is called time-like/space-like/light-like if $|dx^\mu/d\phi|^2 > 0/ < 0/ = 0 \forall \phi$. The Minkowski length of a time-like curve with $[0,1] \ni \phi \mapsto x^\mu(\phi)$ is given by

$$l(x) = \int_0^1 d\phi \sqrt{\left|\frac{dx^\mu}{d\phi}\right|^2}. \tag{1.14}$$

This length $l(x)$ does not depend on the parameterization of x. Here, x is the complete world line, while x^μ describes its components.

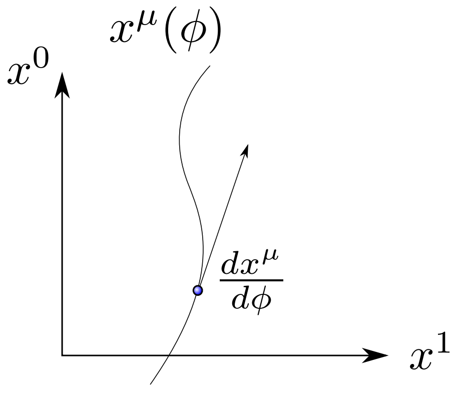

Figure 1.5: Velocity vector of a world line.

We assume that no physical signal can travel faster than light. So an event E can not have an influence on all other events E', not even all of those which, in an inertial system, have a larger x^0-coordinate (i.e. lie in the future of E). In fact, for two events E, E', which are space-like with respect to each other, different inertial observes can disagree on which of the two happens before the other!

Because of this, it is convenient to define "future" and "past" in a coordinate-independent way. First of all, we call a curve $\phi \to x^\mu(\phi)$ *causal*, if $|dx^\mu/d\phi|^2 \geq 0$. We call it *future pointing* if $dx^0/d\phi > 0$, and *past-pointing* if $dx^0/d\phi < 0$.

For an event E, the set $\mathcal{J}^+(E)$ of all events E' which are in the *causal future* of E are those which can be reached from E by a future-pointing, causal curve, i.e.:

$$\mathcal{J}^+(E) := \left\{ E' \,\middle|\, \text{there is a future pointing, causal curve from } E \text{ to } E' \right\} \tag{1.15}$$

Similarly, $\mathcal{J}^-(E)$, the *causal past* of E, is the set of all events with a causal, future pointing curve from E' to E, or alternatively, a causal past-pointing curve from E to E'. The *chronological future/past* $\mathcal{I}^\pm(E)$ consist those events E', which can be reached from E by a future-(past-pointing time-like curve. The *light-cone* of E, denoted V_E, is given by all events E' that can be reached by alight signal from E, i.e.

$$V := \left\{ E' \,\middle|\, d(E,E')^2 = 0 \right\}. \tag{1.16}$$

The *future / past light cone* $V^\pm$ is defined as one would expect (see figure 1.6).

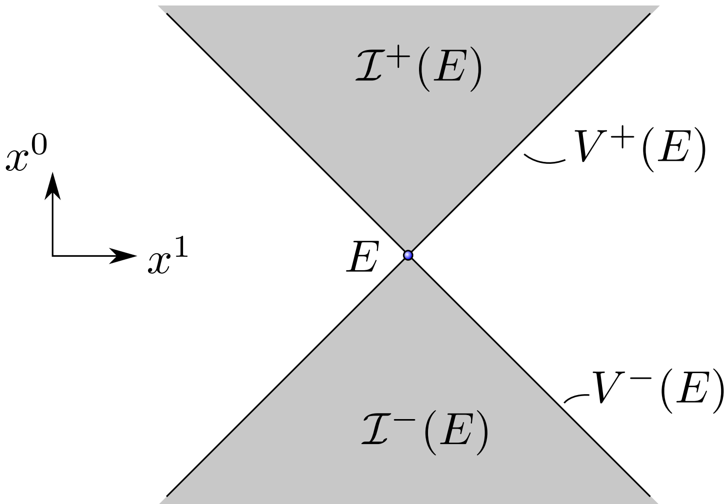

Figure 1.6: Chronological future and past of an event E. Here, $\mathcal{J}^+(E) = \mathcal{I}^+(E) \cup V^+(E)$, similarly for $x \leftrightarrow -$.

1.2 Change of inertial systems

Let the coordinates of an event in one inertial system be called x^μ and in another inertial system $\tilde{x}^\mu$. How are they related?
Remember: Free particles move along straight lines. That means the transformation law has to be affine-linear. An affine transformation conserves points, straight lines and planes. The transformation law should look something like this:

$$y^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{1.17}$$

Here, $\Lambda^\mu{}_\nu$ is a 4×4 matrix which does not depend on x. The upper index labels the rows, the lower index the columns. The vector a^μ is a translation independent of x. It can be arbitrary, the interesting question is what forms Λ can take.

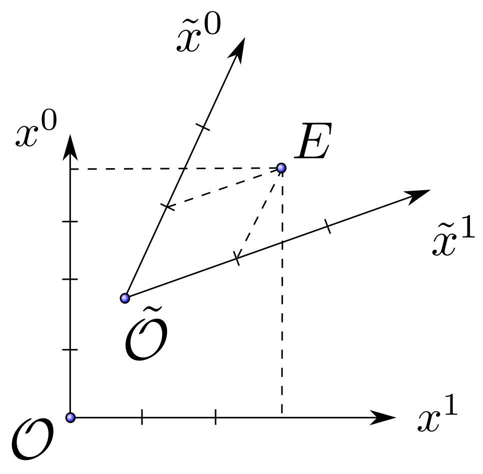

Figure 1.7: Two inertial systems, describing Minkowski space. The same event E has coordinates $x^0 = 4$, $x^1 = 3$, and $\tilde{x}^0 = 1$, $\tilde{x}^1 = 1$.

The speed of light is the same in all inertial systems. If $d(E_1, E_2)^2 = 0$ in one inertial system, it has to be 0 in all inertial systems. In fact, one can show that, if two observers use the same scale of measurement, and use only light signals to construct their coordinate systems, they will always agree on the value $d(E_1, E_2)^2$ between two events E_1, E_2. So we shall demand to only allow coordinate transformations that leave $d(E_1, E_2)^2$ unchanged. Label the coordinates of the events in one inertial system x^μ and y^μ, in the other inertial system $\tilde{x}^\mu$ and $\tilde{y}^\mu$. Then the length squared in the first system is

$$\begin{aligned} d_{IS_1}(E_1, E_2)^2 &= \eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu) \\ &\overset{!}{=} d_{IS_2}(E_1, E_2)^2 \\ &= \eta_{\mu\nu}(\tilde{x}^\mu - \tilde{y}^\mu)(\tilde{x}^\nu - \tilde{y}^\nu) \\ &= \eta_{\mu\nu}\Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho (x^\sigma - y^\sigma)(x^\rho - y^\rho) \\ &= \eta_{\sigma\rho}\Lambda^\sigma{}_\mu \Lambda^\rho{}_\nu (x^\mu - y^\mu)(x^\nu - y^\nu). \end{aligned} \tag{1.18}$$

Here, we used the same $\eta_{\mu\nu}$ and in the last step, simply renamed the indices that are summed over. This equation holds for all events E_1, E_2, i.e. all coordinates x^μ, y^μ. It follows that

$$\eta_{\mu\nu} \overset{!}{=} \eta_{\sigma\rho}\Lambda^\sigma{}_\mu \Lambda^\rho{}_\nu. \tag{1.19}$$

The Lorentz-transformations in matrix form are

$$\Lambda^T \eta \Lambda = \eta. \tag{1.20}$$

They form the *Lorentz group*, which is the group

$$\mathcal{L} = O(1,3) = \{\Lambda \in \mathbb{R}^{4\times 4} | \Lambda^T \eta \Lambda = \eta\}. \tag{1.21}$$

The group $O(n)$ is the group of orthogonal $n \times n$ matrices, meaning its elements are matrices that are real and whose inverse is equal to their transpose. The group is called the *orthogonal group*.
Important members of $O(1,3)$ are

- rotations in space:
$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix} \tag{1.22}$$
with a rotation matrix
$$R \in SO(3) = \{R \in \mathbb{R}^{3\times 3} | R^T R = \mathbb{1}, \det R = 1\}. \tag{1.23}$$
The group $SO(n)$ is called the *special orthogonal group* and is a subgroup of $O(n)$. It contains all elements of $O(n)$ with determinant equal to 1. These matrices describe rotations, which is why the group sometimes is called the *rotation group*.

- boosts that connect two inertial systems for observers with constant velocity with respect to each other. A boost in x^1-direction has the form
$$\Lambda = \begin{pmatrix} \cosh\psi & -\sinh\psi & & \\ -\sinh\psi & \cosh\psi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.24}$$
with $\psi \in \mathbb{R}$.

- further elements: Time reversal
$$T^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.25}$$
and parity (reversal of spatial orientation)
$$P^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{1.26}$$
Warning: Even though p and η look exactly alike, they are not identical! This is already apparent from their index positions (η has two indices downstairs, P one up- and one downstairs), but also from their respective role: η is used to construct the Minkowski distance between events, while P is a coordinate transformation, i.e. it is a way to compute one set of coordinates of an event, from its coordinates in another coordinate system.

Time dilation

In each inertial system for each observer, the coordinates are what the observer at rest would measure with rods and clocks. In particular, to each inertial system corresponds one inertial observer, and vice versa. In the future, we will use them interchangeably. [2] From the coordinate $x^0 = ct$, $t = x^0/c$ is the time that passed for the observer, the other three are the distance from the observer, along the three main axes. In Minkowski space, all points that have the same t-coordinate form a plane of simultaneous events. In equation **??**, we already calculated how the world lines of two observers look in the same system. Now assume that at 0, they meet and synchronize their clocks. I.e., in IS_1, O_1 is at rest and O_2 moves in negative x^1-direction with speed v. At the event where they meet, both clocks show $t = s = 0$. At $t = t_0$, O_1 looks at his clock and asks what O_2's clock shows at this moment. The event E is "the position of O_2 at $t = t_0$". In IS_1, E has the coordinates $(ct_0, -vt_0, 0, 0)^T$ with $v = c\tanh\psi$. Transform this to IS_2:

$$\begin{aligned}\begin{pmatrix}\cosh\psi & \sinh\psi & & \\ \sinh\psi & \cosh\psi & & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix}ct_0 \\ -ct_0\tanh\psi \\ 0 \\ 0\end{pmatrix} &= \begin{pmatrix}ct_0\left(\cosh\psi - \frac{\sinh^2\psi}{\cosh\psi}\right) \\ 0 \\ 0 \\ 0\end{pmatrix} \\ &= \begin{pmatrix}\frac{ct_0}{\cosh\psi} \\ 0 \\ 0 \\ 0\end{pmatrix}.\end{aligned} \tag{1.27}$$

The x^1 coordinate of E is 0 because O_2 hasn't moved in this system. In the second line, the identity $\cosh^2\psi - \sinh^2\psi = 1$ was used. This factor of $\cosh\psi$ depends on v the following way:

$$\begin{aligned}\cosh\psi &= \sqrt{\frac{\cosh^2\psi}{\cosh^2\psi - \sinh^2\psi}} \\ &= \frac{1}{\sqrt{1 - \tanh^2\psi}} \\ &= \frac{1}{\sqrt{1 - v^2/c^2}} \\ &=: \gamma.\end{aligned}$$

Since $v \leq c$, the expression $\frac{1}{\cosh\psi}$ is always smaller or equal to 1. So the event E has, in IS_2, the coordinates $(ct_0\sqrt{1 - v^2/c^2}, 0, 0, 0)^T$. That means that for O_1, it seems like less time has passed for O_2 by a factor of $\sqrt{1 - v^2/c^2}$.

[2] In fact, in the classical literature about relativity, much time is devoted to the technical aspects of how an inertial observer would construct their inertial coordinate system, with light signals being sent back and forth.

Length contraction

O_1 sits in a car of length L and passes O_2 with speed v. The front and the back of the car have the following coordinates:

$$x_b^\mu(t) = \begin{pmatrix} ct \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad x_f^\mu(t) = \begin{pmatrix} ct \\ L \\ 0 \\ 0 \end{pmatrix}. \tag{1.28}$$

World lines for front and back in IS_1

$$\tilde{x}_b^\mu(t) = \begin{pmatrix} ct\cosh\psi \\ ct\sinh\psi \\ 0 \\ 0 \end{pmatrix} \tag{1.29}$$

and

$$\tilde{x}_f^\mu(t) = \begin{pmatrix} \cosh\psi & \sinh\psi & & \\ \sinh\psi & \cosh\psi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} ct \\ L \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} ct\cosh\psi + L\sinh\psi \\ ct\sinh\psi + L\cosh\psi \\ 0 \\ 0 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ L' \\ 0 \\ 0 \end{pmatrix}.$$

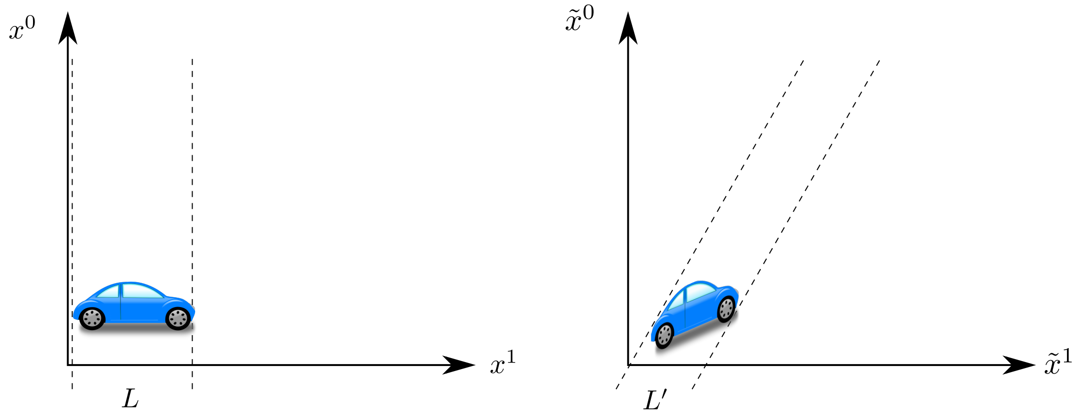

Figure 1.8: A car at rest in coordinates x^μ, and moving with constant speed v in coordinates $\tilde{x}^\mu$.

The way for O_2 to determine the length L' in their coordinate system is to ask "what is the $\tilde{x}^1$-component of the world line of the front of the car, at that moment when the back of the car passes my position?" To compute this, one needs to compute the intersection of $\tilde{x}_f^\mu(t)$ with the $\tilde{x}^1$-axis. Let us assume that happens at the curve parameter $t = t_0$, so

$$ct_0 = -L\tanh\psi = -\frac{Lv}{c} \tag{1.30}$$

and

$$\begin{aligned} L' &= ct_0 \sinh\psi + L\cosh\psi = -L\tanh\psi\sinh\psi + L\cosh\psi \\ &= L\left(\frac{-\sinh^2\psi}{\cosh\psi} + \frac{\cosh^2\psi}{\cosh\psi}\right) = \frac{L}{\cosh\psi} = \frac{L}{\gamma}. \end{aligned}$$

The car appears shorter for O_2 by a factor of $\sqrt{1 - v^2/c^2}$.

Elapsed time of a moving observer

How much time passes for an observer moving along an arbitrary (not necessarily straight) world line x between events $E_1 = x(0)$ and $E_2 = x(1)$? To calculate this, we subdivide the interval $[0, 1]$ into small bits:

$$0 = \phi_0 \le \phi_1 \le \phi_2 \le \ldots \le \phi_N = 1. \tag{1.31}$$

On the segment between $x^\mu(\phi_j)$ and $x^\mu(\phi_{j+1})$, the curve is nearly linear (if the curve is smooth). We transform to the system in which the observer is at rest (for this moment):

$$^{(j)}\tilde{x}_1^\mu = \Lambda_{(j)}{}^\mu{}_\nu x^\nu(\phi_j) + a_j^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.32}$$

$$^{(j)}\tilde{x}_2^\mu = \Lambda_{(j)}{}^\mu{}_\nu x^\nu(\phi_{j+1}) + a_j^\mu = \begin{pmatrix} c\Delta t_j \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.33}$$

The elapsed time is

$$\Delta t_j = \frac{\sqrt{d\left(^{(j)}\tilde{x}_1, {}^{(j)}\tilde{x}_2\right)^2}}{c}. \tag{1.34}$$

The Minkowski distance squared is invariant under Lorentz transformations, which means that

$$\Delta t_j = \sqrt{d(x^\mu(\phi_j), x^\mu(\phi_{j+1}))^2} = \frac{1}{c}\sqrt{\eta_{\mu\nu}(x^\mu(\phi_{j+1}) - x^\mu(\phi_j))(x^\nu(\phi_{j+1}) - x^\nu(\phi_j))}. \tag{1.35}$$

Use the approximation

$$x^\mu(\phi_{j+1}) \approx x^\mu(\phi_j) + \frac{dx^\mu}{d\phi_j}\Delta\phi_j + \mathcal{O}(\Delta\phi_j^2) \tag{1.36}$$

to obtain

$$\Delta t_j \approx \frac{1}{c}\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\phi}\frac{dx^\nu}{d\phi}}\Delta\phi_j + \mathcal{O}(\Delta\phi_j^2). \tag{1.37}$$

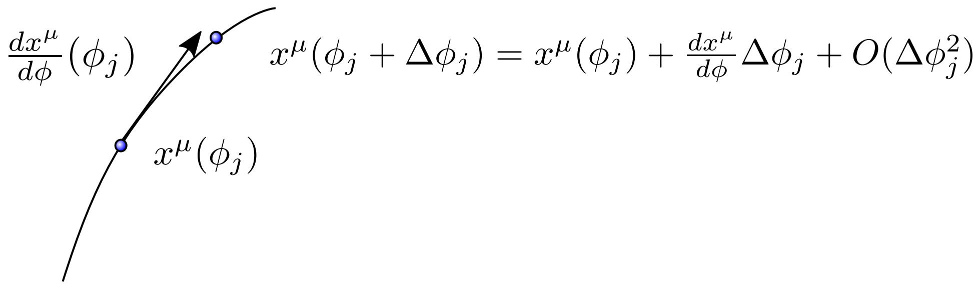

Figure 1.9: Approximation of a curve with short straight lines.

The total elapsed time is the sum over all intervals of time:

$$T = \lim_{N\to\infty} \sum_{i=0}^{N-1} \Delta t_j = \frac{1}{c}\int_0^1 d\phi \sqrt{\left|\frac{dx^\mu}{d\phi}\right|^2} = \frac{1}{c} l(x). \tag{1.38}$$

Timelike curves in Minkowski space can be parameterized by the *proper time* (Minkowski length):

$$s(\phi) = \int_{\phi_0}^{\phi} d\phi' \sqrt{\left|\frac{dx^\mu}{d\phi'}\right|^2}. \tag{1.39}$$

The curve is *regular*, when $dx^\mu/d\phi' \neq 0$ holds for all ϕ'. The relation can be inverted as $s \mapsto \phi(s)$, so that we can write

$$\hat{x}^\mu(s) := x^\mu(\phi(s)). \tag{1.40}$$

The length of the curve is given by

$$l(\hat{x}) = \int_{s_0}^{s_1} ds = s_1 - s_0 \tag{1.41}$$

where ds is called the *infinitesimal line element*. It is given by

$$ds^2 = c^2(dt)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \tag{1.42}$$

It contains all information about the spacetime geometry and tells how much time passes for an observer that moves through spacetime.

1.2.1 Relativistic kinematics

Observers in special relativity move along world lines, usually parameterised by proper time $s \mapsto x^\mu(s)$. The 4-velocity for a particle that is obtained by deriving with respect to proper time is given by

$$u^\mu := c\frac{dx^\mu}{ds}.$$

Note that the vector u has the dimension of a velocity. It satisfies $u^2 := \eta_{\mu\nu}u^\mu u^\nu = c^2$. The 3-velocity measured by the inertial observer who belongs to the coordinate system is $\vec{v}$, which has components v^i with $i = 1, 2, 3$, given by

$$v^i \;:=\; \frac{dx^i}{dt} \;=\; c\frac{dx^i}{dx^0}. \tag{1.43}$$

Note that the proper time s (time passing for the particle) and x^0 (time passing for the inertial observer) do not run at the same speed, there is a difference

$$ds \;=\; \frac{1}{\gamma}dx^0$$

coming from time-dilation. Here γ is the factor

$$\gamma \;=\; \frac{1}{\sqrt{1-\frac{\vec{v}^2}{c^2}}},$$

where $\vec{v}$ is the velocity (1.43). Note that, for a general, accelerated world line, this will depend on s. With this we get

$$u^\mu \;=\; c\frac{d}{ds}x^\mu \;=\; \gamma c\frac{d}{dx^0}x^\mu.$$

So u is the vector with components

$$u \;=\; \gamma\big(c, \vec{v}\big).$$

For relativistic kinematics, an important physical quantity is the so-called 4-momentum p of a particle, which is defined by

$$p^\mu \;:=\; m_0\, u^\mu.$$

Here m_0 is the so-called rest mass, which is the inertial mass measured in the rest frame of the particle. Note that m_0 is the same in every inertial system by definition, while the inertial mass (i.e. the "resistance" of a particle against being accelerated) might depend on the frame. One also writes

$$p \;=\; \big(\gamma m_0 c,\, \gamma m_0 \vec{v}\big) \;=:\; \big(m_{\mathrm{rel}} c,\, m_{\mathrm{rel}}\vec{v}\big).$$

$m_{\mathrm{rel}} := m_0\gamma$ is called the "relativistic mass". It captures the inertial properties of the particle. Note that the 0-component of p^μ, divided by c, has dimensions of an energy, and indeed one writes

$$p \;=\; \big(E/c,\, \vec{p}\big).$$

The one has

$$p^2 \;:=\; \eta_{\mu\nu}p^\mu p^\nu \;=\; \frac{E^2}{c^2} - \vec{p}^2$$

while on the other hand, one has $p^2 = m_0^2 u^2 = m_0^2 c^2$, so

$$E^2 = c^2 \vec{p}^2 + m_0^2 c^4.$$

Indeed, for small velocities $|\vec{v}| \ll c$, one gets

$$E = m_0 c^2 + \frac{1}{2} m_0 \vec{v}^2 + \ldots.$$

So E indeed measures the energy stored in a particle, which has both a rest contribution $m_0 c^2$, as well as a kinetic contribution, which, to lowest order in the velocity, coincides with the non-relativistic kinetic energy.

For interactions between particles, it is the sum of all p^μ which is conserved, both in classical relativistic mechanics, as well as in QFT. This incorporates both momentum- and energy conservation in physical processes. Mechanical laws are, as in Newtonian mechanics, written in the form

$$\frac{d}{ds} p^\mu = f^\mu,$$

where f^μ is the relativistic analogue of a force. The precise form of the force depends on the physical theory at hand.

1.2.2 Non-inertial observers

So far, we have only considered inertial observers, which means coordinate systems of observers moving freely. But in reality, observers are often non-inertial (accelerated) because they are subject to forces (electromagnetism, gravity, ...). The coordinate systems of those observers will not be inertial. Because of that, there will be *inertial forces.* Those are forces that arise in non-inertial reference frames due to their acceleration and seem to have no physical origin, for example centrifugal forces in rotating frames. Because of this, they are often also called *pseudo forces.*
Consider an observer O moving along the world line

$$x^\mu(s) = \begin{pmatrix} d \sinh \frac{s}{d} \\ d \cosh \frac{s}{d} \\ 0 \\ 0 \end{pmatrix} \tag{1.44}$$

that is parameterized by the proper time s. Construct a coordinate system with coordinates y^μ for O. This will not be an inertial system!

Between the proper time s and $s + \Delta s$, the observer nearly moves along a straight line (if $\Delta s \ll 1$):

$$x^\mu(s + \Delta s) = x^\mu(s) + \frac{dx^\mu}{ds} \Delta s + \mathcal{O}(\Delta s^2) \tag{1.45}$$

where we encounter the proper velocity dx^μ / ds. This means that for a short time interval, O is moving almost along the same world line as the inertial observer O_I with world line

$$x^\mu_{(I)}(\tau) = x^\mu(s) + \frac{dx^\mu}{ds} d\tau \tag{1.46}$$

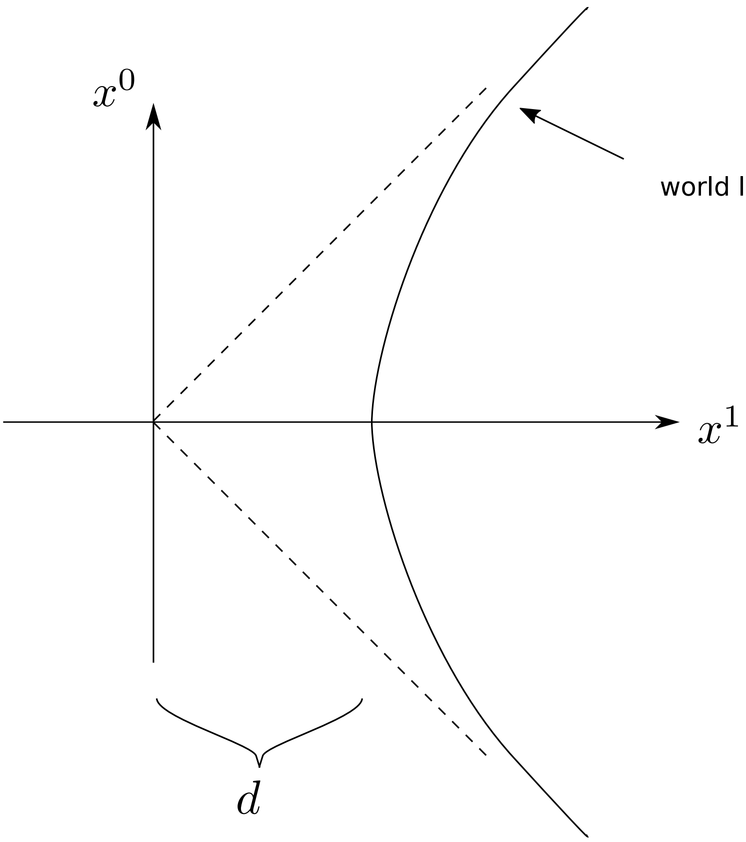

Figure 1.10: World line of the Rindler observer 1.44.

with τ being the proper time of O_I. So at least for this short moment, O and O_I should see the world (nearly) the same way, so they should have the same plane of simultaneous events (at least in their near vicinity) which is given by

$$\Sigma_s = \left\{ x^\mu(s) + X^\mu \,\middle|\, \eta_{\mu\nu} X^\mu \frac{dx^\nu}{ds}(s) = 0 \right\}. \tag{1.47}$$

It is orthogonal to $x^\mu_{(I)}$. Having the same plane of simultaneous events means having the same notion of things happening right now.

For this moment, the world line for O_I is a straight line tangential to the world line of O at the point $x^\mu(s) = x^\mu_{(I)}(0)$. The vector X^μ comes from the origin and crosses x^μ and $x^\mu(s)$.

Because O is accelerating, Σ_s and $\Sigma_{s'}$ will not be parallel for $s \neq s'$. We calculate the proper velocity as

$$\frac{dx^\mu}{ds}(s) = \begin{pmatrix} \cosh \frac{s}{d} \\ \sinh \frac{s}{d} \\ 0 \\ 0 \end{pmatrix} \tag{1.48}$$

(the prefactor d vanished because we get an additional factor $1/d$ from the derivative).

The directions orthogonal (in the Minkowski inner product) to that are

$$e_1^{(s)} = \begin{pmatrix} \sinh \frac{s}{d} \\ \cosh \frac{s}{d} \\ 0 \\ 0 \end{pmatrix}, \qquad e_2^{(s)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad e_3^{(s)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.49}$$

All of these (3d hyper-)planes Σ_s intersect in $\{x^\mu : x^0 = x^1 = 0, x^2, x^3 \text{ arbitrary}\}$ (sinh is only 0 if its argument is 0). At these points, the coordinates y^μ break down: The event $\{x^\mu = 0\}$ would be having several different y^μ-coordinates, but coordinates must uniquely define events.

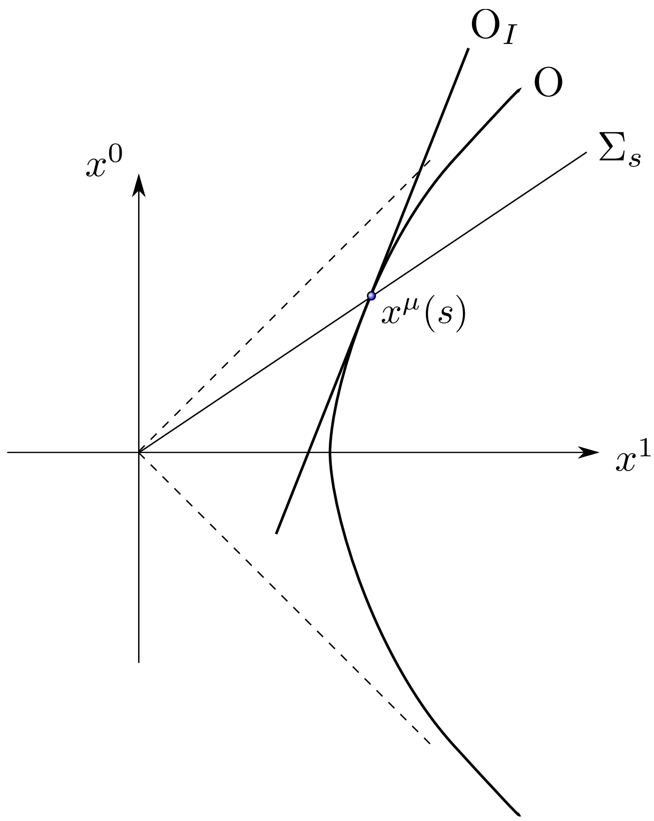

Figure 1.11: Plane of simultaneity Σ_s for the Rindler observer O at proper time s. It coincides with that of the inertial observer O_I.

Coordinate system O is y^μ (with $y^0 = s$) such that his world line in his coordinate system is given by

$$y^\mu(s) = \begin{pmatrix} s \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.50}$$

The transformations between the coordinate systems are

$$x^0 = (y^1 + d)\sinh\frac{y^0}{d} \tag{1.51}$$
$$x^1 = (y^1 + d)\cosh\frac{y^0}{d} \tag{1.52}$$
$$x^2 = y^2 \tag{1.53}$$
$$x^3 = y^3. \tag{1.54}$$

Here we already see that in O's system, $y^1 = -d$ is a problem. If $y^1 = -d$, $y^2 = y^3 = 0$, all points for all y^0 get mapped to $\{x^\mu = 0\}$. Since coordinates need to uniquely identify events, we can only allow $y^1 > -d$.

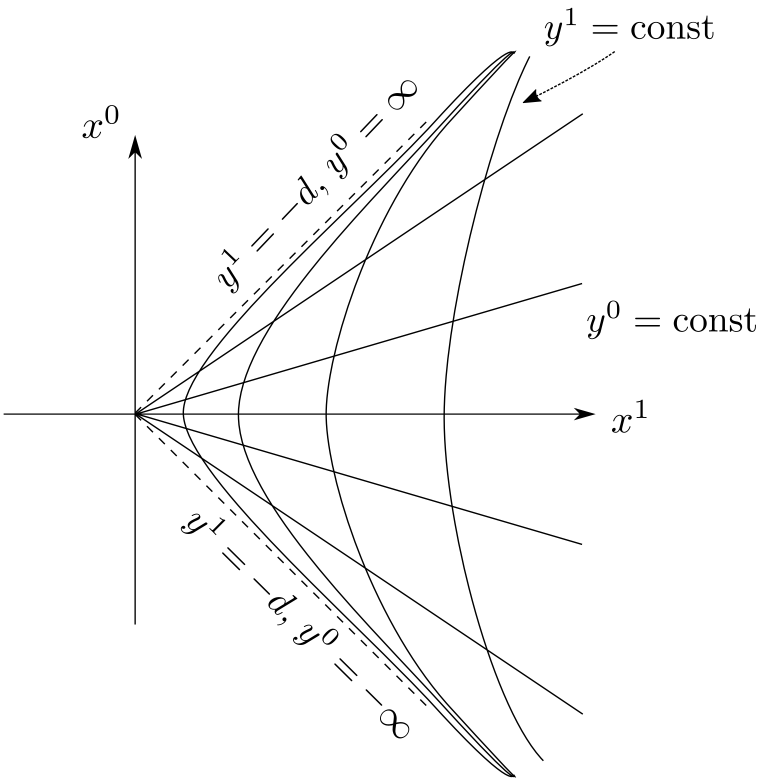

Figure 1.12: Rindler coordinates y^μ, covering the Rindler wedge. They break down at the boundary of that wedge.

For $y^1 > -d$, the coordinates $x^\mu(y^\mu)$ can be inverted:

$$y^0 = d \cdot \operatorname{arctanh}\frac{x^0}{x^1} \tag{1.55}$$
$$y^1 = \sqrt{(x^1)^2 - (x^0)^2} - d \tag{1.56}$$
$$y^2 = x^2 \tag{1.57}$$
$$y^3 = x^3. \tag{1.58}$$

This only works for $|x^1| > |x^0|$, because otherwise, y^1 would become imaginary. So we only take $x^1 > |x^0|$. This is called the *Rindler wedge*. In the inertial system, if we only look at the x^0, x^1-plane, this is only the right area outside of the light cone. In O's system, it corresponds to a *Rindler horizon* at $y^1 = -d$ where O's world ends, so to speak.

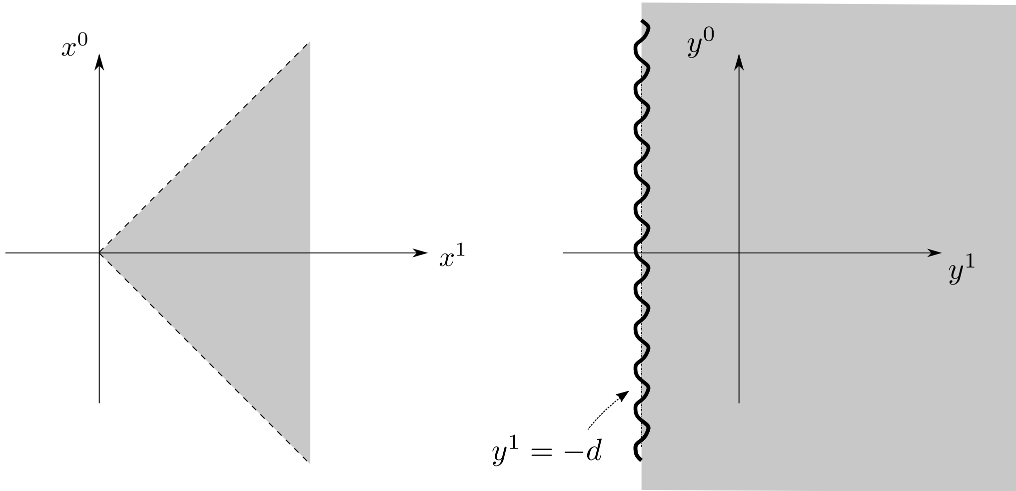

Figure 1.13: Coordinate patches in the Rindler wedge. In Rindler coordinates, the region $y^1 \leq -d$ is not accessible ("behind the horizon").

In O's system, a freely moving observer does not move along a straight line. Assume that O has, in his space ship, a rock. That rock is tossed out of the airlock at $s = 0$. From that point on, the rock moves freely. In the inertial system, the world line of rock is given by

$$x^\mu_{(r)}(\tau) = \begin{pmatrix} \tau \\ d \\ 0 \\ 0 \end{pmatrix}, \tag{1.59}$$

the rock moves on a straight line. In O's system however, the world line takes the form

$$y^\mu_{(r)}(\tau) \begin{pmatrix} d \cdot \operatorname{arctanh} \frac{\tau}{d} \\ \sqrt{d^2 - \tau^2} - d \\ 0 \\ 0 \end{pmatrix}. \tag{1.60}$$

The rock never crosses the horizon because for the same reasons as before, $|\tau| > |d|$ is never possible. However, τ approaches d. The rock comes infinitely close to the horizon, but never crosses it.

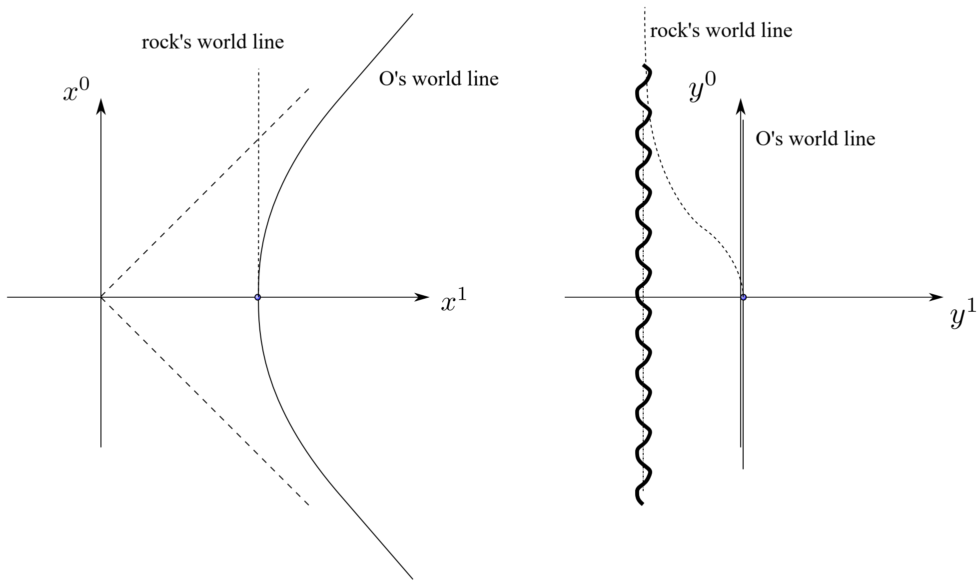

Figure 1.14: World line of the Rindler observer, and a rock that is being dropped, both in Minkowski coordinates and Rindler coordinates.

For small $|\tau| \ll 1$, we can expand $y^0_{(r)}$ and $y^1_{(r)}$ in a Taylor expansion:

$$y^0_{(r)}(\tau) \approx \tau \tag{1.61}$$

$$y^1_{(r)}(\tau) \approx -\frac{1}{2d}\tau^2. \tag{1.62}$$

We expanded $\sqrt{d^2 - x} - d \approx d - d - \frac{x}{2d} + \ldots$ with $x = \tau^2$. Equations 1.61 and 1.62 hold for a short time. Compare this to "free fall" in a constant gravitational field: The change in height is given by

$$\Delta h(t) = -\frac{1}{2}gt^2 \tag{1.63}$$

with the acceleration g.
For $\tau \to d$, $y^0(\tau) \to \infty$ and $y^1(\tau) \to -d$. This means that the rock can reach the horizon after an infinite amount of time. From O's point of view, the rock approaches the horizon, but never reaches it. Also, it seems to "freeze in time".
The Rindler observer feels a force in negative y^1-direction with an acceleration

$$a = -\frac{1}{y^1 - d} \tag{1.64}$$

which diverges at the horizon. The force that this Rindler observer experiences is "fictitious" (it has no physical cause, so to speak, like the centrifugal or the Coriolis force). They are a result of being in a non-inertial system.

1.3 The gravitational force and relativity

Special relativity posed a new framework for the motion of point particles, which was an extension of Newton's mechanics, in particular his three laws of motion. It also fit perfectly with Maxwell's electrodynamics, which describes the electric interaction of particles. It was clear pretty quickly, that also Newton's law of gravity should be adapted to fit into the relativistic framework proposed by Einstein. In particular, the Coulomb potential of a charge at rest and the gravitational potential of a mass at rest both looked fairly similar, so the two forces should be quite similar as well, right?

Unfortunately, writing a law of gravitational attraction between massive particles in a way that fit with Einstein's special relativity theory proved quite difficult. There were several reasons for that, but one of the major ones was that, although gravity and electromagnetism might look similar in some way, they were quite different in other important ways. For instance, the electric charge, which is the source for the electromagnetic field, does not change under Lorentz transformations (it is also called a *Lorentz-scalar*), while the mass of a particle, which is the source of the gravitational field, turned out to be dependent of the observer. This proved to be an insurmountable obstacle for writing down a law of gravity which works for relativistic particles in Minkowski space.

However, Einstein found a solution to this problem, in the so-called *equivalence principle*. Namely, he realised that an observer falling freely in the gravitational field (e.g. of the Earth), feels weightless during their fall. In Newtonian mechanics, this is a consequence of the fact that an observer, who is accelerated by the gravitational field, feels a fictitious force which exactly counteracts the gravitational force. Einstein, however, realised that the freely falling observer themself is an *inertial observer*, and therefore feels weightless, while the person standing on the surface of the Earth does not feel a physical force, but feels a fictitious force – just like the Rindler observer, who keeps a constant distance to the horizon.

So the equivalence principle states that gravity is equivalent to a fictitious force one feels due to an accelerated motion.

Consider the following four observers observing the fall of an apple:

I The first observer in in a room on earth. The apple is accelerated downwards with $\vec{g}$.

II The observer is in a room that is accelerated upwards in space (far from gravity sources) by an angel that is carrying it with $\vec{g}$ in the opposite direction to the acceleration in the first example.

III The observer is in a freely falling room near earth.

IV The observer is in a room that is moving freely in space.

Let's look at how Newton views these systems:

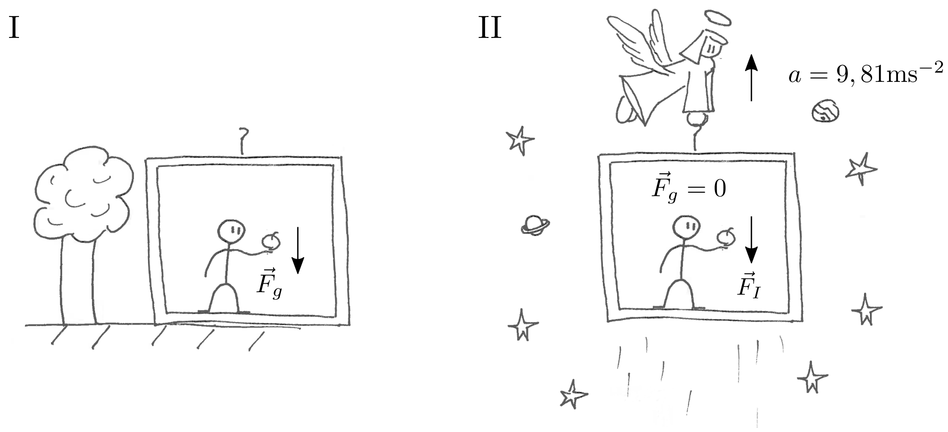

Figure 1.15: The equivalence principle states that an observer (in a small room with no windows) can not distinguish whether she is feeling a downward force because the is standing on the ground of the Earth, or is in space, but being accelerated upwards with constant acceleration (like a Rindler observer).

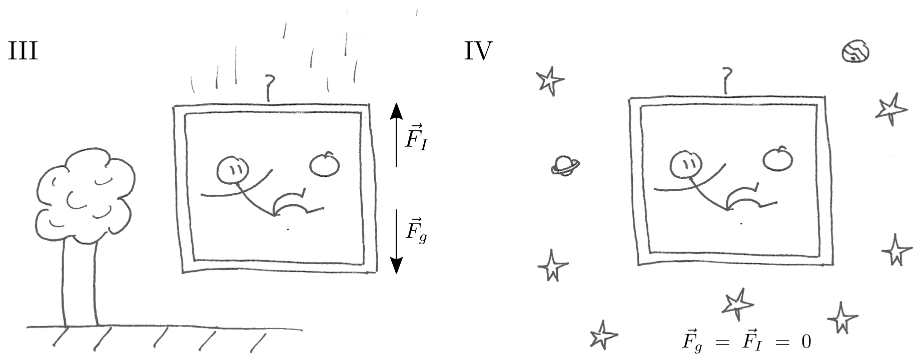

Figure 1.16: Equally, the equivalence principle states that an observer (in a small windowless room) can not distinguish whether she is falling freely in a gravitational field, or floating weightless in space.

I This is an inertial system: It is at rest. There is a physical force acting on the apple given by

$$\vec{F}_g = m_g \vec{g} \tag{1.65}$$

where m_G is the gravitational mass.

II This is not an inertial system: It is accelerated and the apple is at rest. To explain its movement, we need to introduce a fictitious (or *inertial*) force

$$\vec{F}_I = m_I \vec{g} \tag{1.66}$$

with the inertial mass m_I.

III This is not an inertial system: It is accelerated. There is a physical force $\vec{F}_G$ pulling him downwards, and there is a fictitious force since he is accelerated:

$$\vec{F}_g + \vec{F}_I = (m_g - m_I)\vec{g} = 0. \tag{1.67}$$

IV This is an inertial system: There are no forces whatsoever.

Now let's look at how Einstein sees things:

I This is not an inertial system: The observer feels a force.

II This is not an inertial system: The observer feels a force.

III This is an inertial system: There are no forces.

IV This is an inertial system: There are no forces.

So they only agree on 2 and 4.
The Einstein point of view rests on $m_I = m_g$. This is called the *equivalency principle*. As far as we know, it seems to hold (within experimental precision in 2017: $|m_I/m_g - 1| < 10^{-13}$).

1.3.1 Movement of an inertial observer in a gravitational field

Einstein's idea of how to include gravitational force into relativity was as follows: The gravitational force that people feel is no physical force, but a fictitious force. Therefore, every observer which is not accelerated does not feel any gravitational force (even if it is being moved around by the gravitational field). Therefore, it has to be treated the same way as an inertial observer in special relativity.

Well, not quite: an important point is that the equivalence principle only holds exactly for point-like observers. It only holds approximately for small observers, i.e. those who do not have a large spatial extension. If your body is several thousand kilometres large, you can feel the gravitational field of the earth pulling you in different directions, i.e. one can feel the *tidal forces* of the gravitational field.

In other words, an observe who is only influenced by the gravitational field is an inertial observer – but her inertial system might not encompass all of space-time, but only a small neighbourhood around her world line. But in that coordinate system, the laws of special relativity should hold! Assume such an inertial observer has an inertial system with coordinates ξ^μ. Any *other* observer, which moves only under the influence of the gravitational field, has a world line $\xi^\mu(s)$ satisfying

$$\frac{d^2\xi^\mu}{ds^2} = 0. \tag{1.68}$$

A general, non-inertial observer O (such as the Rindler observer) will have a different coordinate system, and might not perceive the inertially moving particles as moving along a straight line. What is the equation of motion in their coordinates x^μ?

We assume that locally the two coordinate system can be transformed into one another, and that $x^\mu(\xi)$ and $\xi^\mu(x)$ are smooth and inverse to one another. This means that all partial derivatives $\frac{\partial\xi^\mu}{\partial x^\nu}$ exist, are smooth, and satisfy

$$\frac{\partial\xi^\mu}{\partial x^\nu}\frac{\partial x^\nu}{\partial\xi^\rho} = \delta^\mu{}_\rho, \qquad \frac{\partial x^\mu}{\partial\xi^\nu}\frac{\partial\xi^\nu}{\partial x^\rho} = \delta^\mu{}_\rho. \tag{1.69}$$

We express the world line $s \mapsto \xi^\mu(s)$ in O's coordinates x^μ. ONe has

$$\begin{aligned} 0 &= \frac{d^2\xi^\mu}{ds^2} = \frac{d}{ds}\left(\frac{d}{ds}\xi^\mu(x(s))\right) = \frac{d}{ds}\left(\frac{\partial\xi^\mu}{\partial x^\nu}\frac{dx^\nu}{ds}\right) \\ &= \frac{d}{ds}\left(\frac{\partial\xi^\mu}{\partial x^\nu}\right)\frac{dx^\nu}{ds} + \frac{\partial\xi^\mu}{\partial x^\nu}\frac{d^2x^\nu}{ds^2} = \frac{\partial^2\xi^\mu}{\partial x^\rho\partial x^\nu}\frac{dx^\rho}{ds}\frac{dx^\nu}{ds} + \frac{\partial\xi^\mu}{\partial x^\nu}\frac{d^2x^\nu}{ds^2}. \end{aligned}$$

This can be expressed as:

$$0 = \frac{\partial x^\sigma}{\partial\xi^\mu}\frac{\partial\xi^\mu}{\partial x^\nu}\frac{d^2x^\nu}{ds^2} + \frac{\partial x^\sigma}{\partial\xi^\mu}\frac{\partial^2\xi^\mu}{\partial x^\rho\partial x^\nu}\frac{dx^\rho}{ds}\frac{dx^\nu}{ds}. \tag{1.70}$$

Using the fact that the matrices with entries $\frac{\partial x^\sigma}{\partial\xi^\mu}$ and $\frac{\partial\xi^\mu}{\partial x^\nu}$ are inverse to one another, i.e. (1.69), we get:

$$\frac{d^2x^\mu}{ds^2} + \frac{\partial x^\mu}{\partial\xi^\sigma}\frac{\partial^2\xi^\sigma}{\partial x^\rho\partial x^\nu}\frac{dx^\rho}{ds}\frac{dx^\nu}{ds} = 0. \tag{1.71}$$

This equation of motion still contains terms which depend on the inertial coordinates ξ^μ, to which our observer O might not have any access (or might not be interested in). But there is a better way to express this equation of motion, by introducing the so-called *space-time metric*. This is a local version of the Minkowski distance.

Consider two events close to one another, which, in the inertial observer's system, have coordinates ξ^μ and $\xi^\mu + \Delta\xi^\mu$. Since for the inertial observer the laws of special relativity should hold, the space-time distance between the two events are

$$\Delta s^2 = \eta_{\mu\nu}\Delta\xi^\mu\Delta\xi^\nu.$$

We express these in O's coodrdinates x^μ, and get

$$\Delta\xi^\mu = \frac{\partial\xi^\mu}{\partial x^\rho}\Delta x^\rho + \ldots, \tag{1.72}$$

up to terms of higher order in the Δx^ρ. The partial derivatives are to be taken at x^μ the coordinates of the first event. This gives us

$$\Delta s^2 = \eta_{\mu\nu}\frac{\partial\xi^\mu}{\partial x^\rho}\frac{\partial\xi^\nu}{\partial x^\sigma}\Delta x^\rho\Delta x^\sigma. \tag{1.73}$$

In the limit of the two events approaching one another, one gets

$$ds^2 =: g_{\rho\sigma}dx^\rho dx^\sigma. \tag{1.74}$$

where we have defined the *space-time metric*

$$g_{\mu\nu} := \eta_{\rho\sigma}\frac{\partial\xi^\rho}{\partial x^\mu}\frac{\partial\xi^\sigma}{\partial x^\nu}. \tag{1.75}$$

These are coefficients which measure the infinitesimal space-time distance between events.

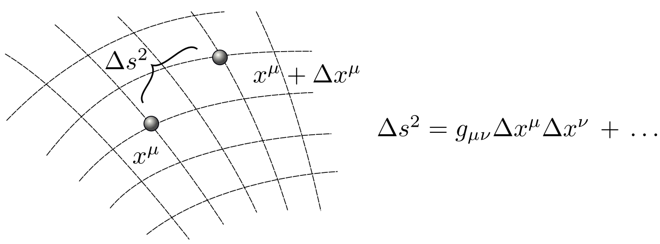

Figure 1.17: Two events which are close to one another have the space-time distance Δs^2, which can be expressed with the help of the metric coefficients $g_{\mu\nu}$. If the coordinates x^μ are actually inertial coordinates, we have $g_{\mu\nu} = \eta_{\mu\nu}$. In general, however, $g_{\mu\nu}$ will be different from $\eta_{\mu\nu}$, and can even change from point to point.

The equation of motion (1.71) can now be expressed in terms of the metric coefficients, and their partial derivatives. This is great, because we do not need to make a reference to an inertial observer, but can work entirely with physical quantities which can be measured entirely by O. For this we need the *inverse metric* $g^{\mu\nu}$, which is defined by the inverse matrix to $g_{\mu\nu}$, i.e.

$$g_{\mu\nu} g^{\nu\rho} = \delta^{\rho}_{\mu}, \quad g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}. \tag{1.76}$$

With these we define the so-called *Christoffel symbols* $\Gamma^{\mu}_{\nu\rho}$, by

$$\Gamma^{\mu}_{\nu\rho} := \frac{1}{2} g^{\mu\sigma} \left(\frac{\partial g_{\sigma\nu}}{\partial x^{\rho}} + \frac{\partial g_{\sigma\rho}}{\partial x^{\nu}} - \frac{\partial g_{\nu\rho}}{\partial x^{\sigma}} \right). \tag{1.77}$$

The Christoffel symbols are symmetric in the lower two indices.

$$\Gamma^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\rho\nu} \qquad \text{für alle } \mu, \nu, \rho = 0, \ldots, 3. \tag{1.78}$$

Nex we express the inerse metric in terms of the inverse Minkowski metric $\eta^{\mu\nu}$:

$$g^{\mu\nu} = \frac{\partial x^{\mu}}{\partial \xi^{\rho}} \frac{\partial x^{\nu}}{\partial \xi^{\sigma}} \eta^{\rho\sigma}. \tag{1.79}$$

Insert now (1.75) into the expression of the brackets in (1.77), and we get:

$$\begin{aligned} & \frac{\partial g_{\sigma\nu}}{\partial x^{\rho}} + \frac{\partial g_{\rho\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\rho\nu}}{\partial x^{\sigma}} \qquad (1.80) \\ &= \eta_{\lambda\tau} \left(\frac{\partial}{\partial x^{\rho}} \left(\frac{\partial \xi^{\lambda}}{\partial x^{\sigma}} \frac{\partial \xi^{\tau}}{\partial x^{\nu}} \right) + \frac{\partial}{\partial x^{\nu}} \left(\frac{\partial \xi^{\lambda}}{\partial x^{\rho}} \frac{\partial \xi^{\tau}}{\partial x^{\sigma}} \right) - \frac{\partial}{\partial x^{\sigma}} \left(\frac{\partial \xi^{\lambda}}{\partial x^{\rho}} \frac{\partial \xi^{\tau}}{\partial x^{\nu}} \right) \right) \\ &= \eta_{\lambda\tau} \left(\frac{\partial^2 \xi^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial \xi^{\tau}}{\partial x^{\nu}} + \frac{\partial \xi^{\lambda}}{\partial x^{\sigma}} \frac{\partial^2 \xi^{\tau}}{\partial x^{\nu} \partial x^{\rho}} + \frac{\partial^2 \xi^{\lambda}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial \xi^{\tau}}{\partial x^{\sigma}} + \frac{\partial \xi^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 \xi^{\tau}}{\partial x^{\nu} \partial x^{\sigma}} \right. \\ &\qquad \left. - \frac{\partial^2 \xi^{\lambda}}{\partial x^{\sigma} \partial x^{\rho}} \frac{\partial \xi^{\tau}}{\partial x^{\nu}} - \frac{\partial \xi^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 \xi^{\tau}}{\partial x^{\sigma} \partial x^{\nu}} \right) \\ &= 2 \eta_{\lambda\tau} \frac{\partial^2 \xi^{\tau}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial \xi^{\lambda}}{\partial x^{\sigma}}. \qquad (1.81) \end{aligned}$$

Here we have used the symmetry of the metric: $g_{\mu\nu} = g_{\nu\mu}$. Now we use (1.79), and get for the Christoffel symbol:

$$\begin{aligned} \Gamma^{\mu}_{\nu\rho} &= \frac{\partial x^{\mu}}{\partial \xi^{\alpha}} \frac{\partial x^{\sigma}}{\partial \xi^{\beta}} \eta^{\alpha\beta} \eta_{\lambda\tau} \frac{\partial^2 \xi^{\tau}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial \xi^{\lambda}}{\partial x^{\sigma}} \\ &= \frac{\partial x^{\mu}}{\partial \xi^{\tau}} \frac{\partial^2 \xi^{\tau}}{\partial x^{\nu} \partial x^{\rho}}. \end{aligned}$$

But this is exaclty the expression in (1.71), which is why we can write :

$$\frac{d^2 x^{\mu}}{ds^2} + \Gamma^{\mu}_{\nu\rho} \frac{dx^{\nu}}{ds} \frac{dx^{\rho}}{ds} = 0. \tag{1.82}$$

So in order to incorporate the gravitational force, Einstein assumed the following:

- Space-time is still four-dimensional, and events can be expressed in coordinates, but the space might not have the structure of an affine space anymore.
- Observers which are only being influenced by the gravitational force are the inertial observers. In their respective inertial coordinate systems ξ^μ. the laws of special relativity hold. But these coordinate systems do not necessarily cover all of space-time any more.
- The geometry of space-time is encoded in the space-time metric $g_{\mu\nu}$, which measures the space-time distance between nearby events. In an inertial coordinate system $g_{\mu\nu} = \eta_{\mu\nu}$, but in general (non-inertial) coordinates x^μ, the metric will be different, and depend on the point. It will be a *field*.
- In general coordinates, the equations of motion for a particle under the influence of gravity is (1.82).

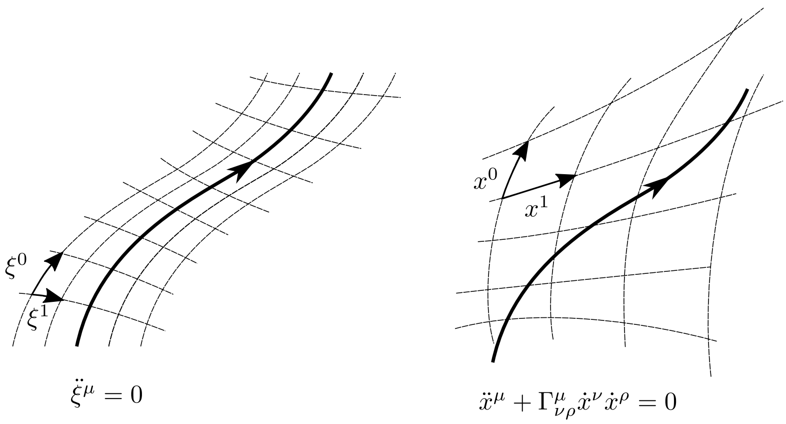

Figure 1.18: The same world line satisfies $\ddot{\xi}^\mu = 0$ in the inertial coordinate system, and (1.82) in an arbitrary CS.

2 Tensor fields on arbitrary manifolds

The treatment of tensor fields on arbitrary manifolds is a mathematical discipline called *differential geometry*. Here, we can only give a brief introduction into a very wide field. We will treat, in general, n-dimensional space with arbitrary coordinates. Even though mathematically, this fits into the realm of geometry, it is actually ubiquitous in physics: The concepts appear as coordinates in space(-time) $(x^0, ..., x^n,\ \varphi, \theta, \rho,\ ...)$ and as thermodynamic properties (temperature T, pressure p, entropy S) that are coordinates on some manifold.
The notation in this chapter will be as follows: In general n-dimensional spaces, we denote coordinates by $x^i, i = 1, ..., n$. We use Greek indices $\mu, \nu = 0, 1, ..., n-1$ almost exclusively when talking about the spacetime in general relativity.

2.1 Vectors, dual vectors and tensors

2.1.1 Vectors

In classical physical systems with finitely many degrees of freedom, one almost exclusively deals with real, finite-dimensional vector spaces V. A vector v from such a space will always be an abstract object, but in physics it is important to do computations with real numbers. To be able to do this, one needs to introduce a **basis**, which consists of a choice of vectors $e_1, e_2, \dots, e_n$, such that each abstract vector v can be decomposed into this basis, via

$$v = \sum_{i=1}^{n} v^i e_i, \tag{2.1}$$

with unique real numbers v^i, $i = 1, \dots, n$, which are called the **components** of V (with repsect to the basis $\{e_i\}_{i=1}^n$). As customary, we will also here use the Einstein convention and omit the sum, i.e. we will always write $v = v^i e_i$. The number n is called the dimension of V.

Instead of working with the abstract v, one can work with its components. In other words, a basis provides a linear isomorphism between V and $\mathbb{R}^n$. It is important to notice that this isomorphism depends on the basis: If a vector has certain components with respect to one basis, it might have completely different components with respect to another basis. Assume that we have given a second set of basis vector s$\tilde{e}_i$, then we also have

$$v = \tilde{v}^i \tilde{e}_i. \tag{2.2}$$

How do the components $\tilde{v}^i$ and v^i depend on each other? That depends on how the two sets of basis vectors relate to one another. Each of the $\tilde{e}_i$ can be decomposed with respect to the basis vectors e_i, so we have

$$\tilde{e}_i = M_i{}^j e_j \tag{2.3}$$

for some real numbers $M_i{}^j$. Since the decomposition also has to work in the other way, the coefficients $M_i{}^j$ form an invertible matrix M. With this, we get

$$\tilde{v}^i \tilde{e}_i = \tilde{v}^i M_i{}^j e_j \overset{!}{=} v^i e_i, \tag{2.4}$$

and since the coefficients are unique, this means that $\tilde{v}^i M_i{}^j = v^j$ (for all j, and with an implied sum over the i).

Now let $N^i{}_j$ denote the coefficients of the matrix N, which satisfies $N^{-1} = M^T$, or, in coefficients:

$$\begin{aligned} M_i{}^j N^k{}_j &= \delta^i_k, \\ N^i{}_j M^j{}_k &= \delta^k_i. \end{aligned}$$

The matrix N is also called **contragredient** to M. Multiplying with $N^k{}_j$ and summing over j, we get

$$N^k{}_j v^j = N^k{}_j \tilde{v}^i M_i{}^j = \tilde{v}^i \delta^k_i = \tilde{v}^k, \tag{2.5}$$

or in other words

$$\tilde{v}^i = N^i{}_j v^j. \tag{2.6}$$

2.1.2 Dual vectors

To every vector space V there is the so-called **dual space**, denoted by V^*, which consists of linear forms on $\mathbb{R}$, i.e.

$$V^* = \left\{ \alpha : V \to \mathbb{R} \,\middle|\, \alpha \text{ is linear} \right\} \tag{2.7}$$

Any dual basis vector α is completely determined by its values $\alpha(e_i)$ on the basis vectors. For any basis $\{e_i\}_{i=1}^n$ of V, there is a dual basis $\{\theta^i\}_{i=1}^n$ of V^* (in particular, the two have the same dimension). The dual basis is defined by

$$\theta^i(e_j) = \delta^i_j. \tag{2.8}$$

Every dual vector α can be decomposed into these, i.e.

$$\alpha = \alpha_i \theta^i. \tag{2.9}$$

The coefficients α_i are real numbers, and they are precisely the values of α on the basis vectors, i.e.

$$\alpha(e_i) = \left(\alpha_j \theta^j\right)(e_i) = \alpha_j \delta^j_i = \alpha_i. \tag{2.10}$$

Under a change of basis vectors $e_i \to \tilde{e}_i$, also the dual basis changes. The dual basis vectors $\tilde{\theta}^i$ can be decomposed into the θ^i, and we make the ansatz $\tilde{\theta}^i = A^i{}_j\theta_j$. Plugging this into the definition of the dual basis, and using the linearity of the θ^i, we obtain

$$\delta^i_j \;=\; \tilde{\theta}^i(\tilde{e}_j) \;=\; A^i{}_l\theta^l\big(M_j{}^k e_k\big) \;=\; A^i{}_l M_j{}^k \delta^l_k \;=\; A^i_k M_j{}^k. \tag{2.11}$$

We see that the matrix A satisfies the same equation as N, so they have to agree: $A = N$. Very similarly, one shows the the coefficients of a dual vector changes under a change of basis, such that

$$\tilde{\alpha}_i \;=\; M_i{}^j \alpha_j. \tag{2.12}$$

Again, it is important to note that, under a change of basis, neither a vector v nor a dual vector α change – it is just that their coefficients v^i and α_i change, because they are now computed with respect to a different basis (or a different dual basis).

2.1.3 Higher order tensors

Vectors and dual vectors are two examples of a more general construction in physics: tensors.[1] Given two vector spaces V and W, one can form the tensor product $V \otimes W$ between them. For our intents and purposes, the tensor product consists of all formal linear combinations of $v \otimes w$, with $v \in V$ and $w \in W$, such that

$$\begin{aligned} \alpha(v \otimes w) &= (\alpha v) \otimes w \;=\; v \otimes (\alpha w) \\ v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2 \\ (v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w. \end{aligned}$$

Because of this, if $\{e_i\}_{i=1}^n$ is a basis for V and $\{f_j\}_{j=1}^m$ is a basis for W, then $e_i \otimes f_j$ form a basis for $V \otimes W$. In particular, every element T in $V \otimes W$ can be decomposed as

$$T \;=\; T^{ij} e_i \otimes f_j. \tag{2.13}$$

Notably, $\dim(V \otimes W) = \dim(V)\dim(W)$. The T^{ij} are called the components of T, and as usual, they depend on the choice of bases in V and W. In general, one can form arbitrarily long tensor products $V \otimes W$, $V \otimes W \otimes U$ ... of arbitrary vector spaces, but we will in the following only be concerned with a specific sort of tensor products. For a vector space V, we will only consider tensor products of some copies of V and some copies of V^* (in any order). We will call the **type** of the tensor the numbers (r, s), where r is the number of V's and s the number of V^*'s in the tensor product.

Careful: the tensor product is not commutative, so one has

$$V \otimes W \neq W \otimes V. \tag{2.14}$$

[1] Although this is meant with respect to a specific vector space. Higher order tensors also form a larger vector space themselves.

Although the two vector spaces on either side of (2.14) are isomorphic, it is a good idea to be aware that the order of the tensor product, and therefore the position of indices, is in general important.

As with dual vectors, a change of basis of V also changes the components of more general tensors. For example, if we have a tensor T out of the tensor product space $V \otimes V^* \otimes V$, its components $T^i{}_j{}^k$ are defined by

$$T = T^i{}_j{}^k \, e_i \otimes \theta^j \otimes e_k. \tag{2.15}$$

Under a basis change $e_i \to \tilde{e}_i$, also the the θ^i change, and the tensor product basis in general. So we have

$$T = \tilde{T}^i{}_j{}^k \, \tilde{e}_i \otimes \tilde{\theta}^j \otimes \tilde{e}_k \stackrel{!}{=} T^i{}_j{}^k \, e_i \otimes \theta^j \otimes e_k. \tag{2.16}$$

Since the (dual) basis vectors change as $\tilde{e}_i = M_i{}^j e_j$ and $\tilde{\theta}^i = N^i{}_j \theta^j$, we get, with a very similar calculation than in the previous subsection, that

$$\tilde{T}^i{}_j{}^k = N^i{}_r \, M_j{}^s \, N^k{}_t \, T^r{}_s{}^t. \tag{2.17}$$

This is a general rule for the way in which the components of a tensor transform under change of basis:

each upper ("contravariant") index gets a N

each lower ("covariant") index gets a M.

The names contravariant and covariant are not used very much any more nowadays.

2.1.4 Some physical examples

So far, tensors have been rather abstract objects. But they do appear in many places in classical physics. Almost all physical quantities are tensors, where the underlying vector space $V = \mathbb{R}^3$ (in Newtonian mechanics), or $V = \mathbb{R}^4$ (in special relativity). The reason is that many of these quantities do somehow depend on directions in space(-time), and thus their components change when the basis in space(-time), i.e. the Cartesian coordinates, change. In the following we will look at a few examples:

1. Temperature T
   This is a tensor of type $(0,0)$, also called a scalar. The reason is that temperature is just a number, which does not depend on any direction.

2. Velocity v^i
   This is very clearly a vector (indeed, the underlying vector space is $\mathbb{R}^3$, which is the space of translations of Euclidean space in which Newtonian physics is happening. Since velocity is an infinitesimal translation, the components have one upper index.

3. Force F_i
   In Newtonian mechanics, one does not often see the difference between vectors and dual vectors. But if one pays close attention, one realizes that the physical concept of force is a dual vector. This is due to the fact that force essentially tells you how much (potential) energy W a particle gains, if it moves a bit Δx^i in a certain direction. Therefore, if this Δx is a vector, the energy gain is

$$W = -F(\Delta x) = -F_i \Delta x^i. \tag{2.18}$$

   The minus sign comes from the fact that a force field always points into the direction of where a particle would lose the most energy.

4. Stress σ_{ij}
   Stress is a tensor of type $(0,2)$. The reason being that it tells you which sort of force acts on which part of the surface of a small piece of matter. A bit of matter having normal vector A^i, where $A^i = An^i$, with the area A and the normal direction n^i, experiences, under some external deformation, a force

$$F_i = \sigma_{ij} A^j. \tag{2.19}$$

   For example, for a hydrostatic fluid, the stress tensor is

$$\sigma_{ij} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}, \tag{2.20}$$

   which is understandable, since the pressure acts perpendicular to the surface towards the center of the piece of matter. The non-diagonal entries of this tensor have to do with the shear a piece of matter is experiencing. For most physical cases, the stress tensor is symmetric, i.e. it satisfies $\sigma_{ij} = \sigma_{ji}$.

5. Magnetic field B_{ij}
   Contrary to the way in which the magnetic field os often depicted as a vector, it is actually a tensor of type $(0,2)$. It is different from the stress tensor, however, in that it is antisymmetric, i.e. it satisfies $B_{ij} = -B_{ji}$. The reason for this is that the magnetic field is actually a way to tell how much magnetic flux Φ is going through a small piece of are, which is being spanned by two vectors n^i and m^i. In particular, one has that

$$\Phi = B_{ij} n^i m^i. \tag{2.21}$$

   In the definition of the piece of surface, it is important to keep track of the orientation, otherwise the magnetic flux will change sign. This is why B is antisymmetric: it changes sign under exchange of n^i and m^i (which is the same as reversing the orientation of the surface).

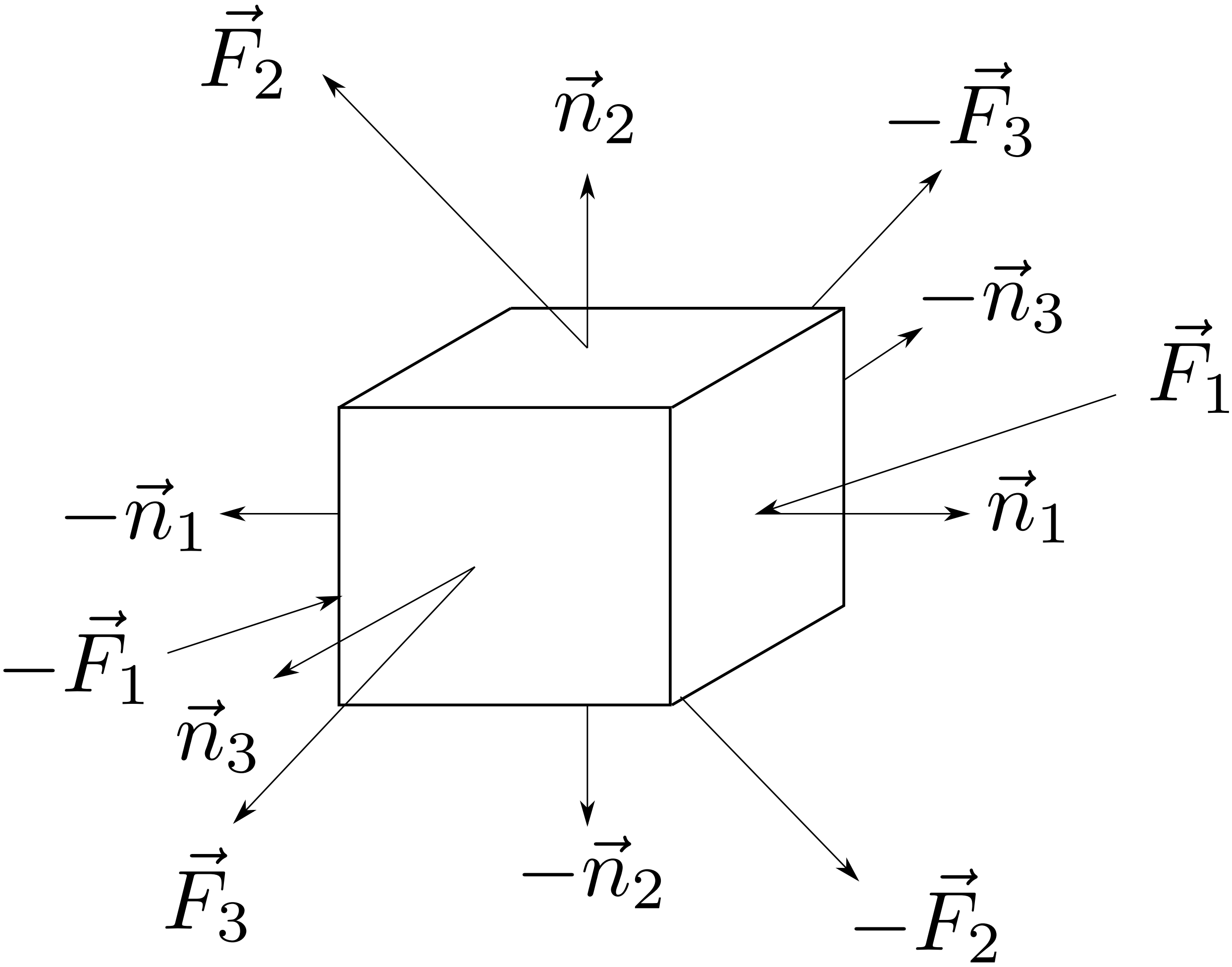

Figure 2.1: Different forces can act on different sides of a small piece of matter. This causes a deformation which can be described using the stress tensor σ_{ij}.

2.1.5 Operations on tensors

There are several operations on tensors which will turn out to be quite useful.

- Multiplication of tensors
  This operation takes e.g. a tensor T of type (p,q) and one S of type (r,s), and produces a tensor U of type $(p+r, q+s)$. For example: let T be from $V \otimes V^*$ (type $(1,1)$, components $T^i{}_j$), and S be from $V^* \otimes V^*$ (type $(0,2)$, components S_{ij}. The tensor U is an element in the tensor product $V \otimes V^* \otimes V^* \otimes V^*$. It has components $U^i{}_{jkl}$ are related to those of T and S via

$$U^i{}_{jkl} = T^i{}_j \, S_{kl}. \tag{2.22}$$

  Now, the ingeniousness about an equation like (2.22) is that it completely describes the relation between U, T and S – and it is true with respect to every basis!

Because of the way in which components of tensors like (...) transform under a change of basis, if equation (2.22) is true with respect to one basis, it is true with respect to *every* basis. Namely, one has

$$\tilde{U}^i{}_{jkl} = \tilde{T}^i{}_j \tilde{S}_{kl} = \left(N^i{}_r M_j{}^s T^r{}_s\right)\left(M_k{}^t M_l{}^u S_{tu}\right) = N^i{}_r M_j{}^s M_k{}^t M_l{}^u U^r{}_{stu} \tag{2.23}$$

In other words, the definition of (2.22) defines a set of numbers for every choice of basis. These numbers, for different choices of basis, are related to one another just as the components of a tensor of type $(1,3)$ would. In other words, this way one can *define* a tensor of that rank.

- Contraction of a tensor
  This operation takes a tensor of type (p,q) and produces a tensor of type $(p-1,q-1)$. For example, we take a tensor T with components $T^i{}_j{}^k$ (i.e. of type $(2,1)$), and choose one upper and one lower index. For this example, we choose the first and the second index. Then we define the components of a type $(1,0)$ tensor (i.e. a vector) S by

$$S^i = T^k{}_k{}^i. \tag{2.24}$$

  Again, if we do this with respect to another basis, then the numbers $\tilde{S}^i$ are related to the numbers S^i just as the components of a $(1,0)$ tensor would:

$$\tilde{S}^i = \tilde{T}^k{}_k{}^i = \underbrace{M^k{}_r N_k{}^s}_{=\delta^s_r} M^i{}_t T^r{}_s{}^t = M^i{}_t \delta^s_r T^r{}_s{}^t = M^i{}_t T^r{}_r{}^t = M^i{}_t S^t. \tag{2.25}$$

2.1.6 Symmetries of tensors

Very often, tensors in physics are not just elements in some tensor product space, but they often come with special symmetries regarding the exchange of their indices. As depicted in the examples earlier, for instance, the magnetic field B_{ij} is antisymmetric, i.e. satisfies $B_{ij} = -B_{ji}$.

Since we will deal with tensors with an arbitrary number k of indices, and its properties under exchange of these indices, it pays to remind ourselves of the notion of a **permutation**. A permutation σ in k elements is an invertible map of the numbers from 1 till k to itself, i.e.

$$\sigma:\ \{1,\ldots,k\} \to \{1,\ldots,k\}. \tag{2.26}$$

A special example of a permutation is an exchange of two elements $i \neq j$, e.g. $\sigma(i) = j$, $\sigma(j) = i$, and all other $\sigma(k) = k$. Every permutation can be written as a sequence of such a so-called transposition. The **sign** of a permutation σ is denoted by $(-1)^\sigma$ or $\operatorname{sgn}\sigma$, and it is ± 1, depending on whether σ consists of an even or an odd number of transpositions, i.e.

$$(-1)^\sigma = \begin{cases} 1 & \sigma \text{ is a sequence of an even number of transpositions} \\ -1 & \sigma \text{ is a sequence of an odd number of transpositions} \end{cases} \tag{2.27}$$

We denote the set of all permutations in k elements by $\mathfrak{S}_k$.

A tensor S of type $(k,0)$ is called **totally symmetric**, if its components do not change under permutation of indices, i.e.

$$S^{i_1 i_2 \cdots i_k} = S^{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(k)}} \qquad \text{for all } \sigma \tag{2.28}$$

Similarly, a tensor A of the same type is called **totally antisymmetric** if the components get get a minus sign whenever one exchanges two indices. In other words

$$A^{i_1 i_2 \cdots i_k} = (-1)^\sigma \, A^{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(k)}} \qquad \text{for all } \sigma \tag{2.29}$$

Examples for totally symmetric tensors are the stress tensor, which satisfies $\sigma_{ij} = \sigma_{ji}$, or the Minkowski metric, which satisfies $\eta_{\mu\nu} = \eta_{\nu\mu}$. Examples for totally antisymmetric tensors are the magnetic field, as already mentioned, or the epsilon tensor from the cross product: $\epsilon_{ijk} = \epsilon_{jki} = -\epsilon_{jik} = \cdots$.

Given an arbitrary type $(k,0)$ tensor T, one can project onto its symmetric part S, which is a totally symmetric type $(k,0)$ tensor, whose components are defined by

$$S^{i_1 i_2 \cdots i_k} = T^{(i_1 i_2 \cdots i_k)} := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} T^{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(k)}}. \tag{2.30}$$

Similarly, its antisymmetric part A is defined by

$$A^{i_1 i_2 \cdots i_k} = T^{[i_1 i_2 \cdots i_k]} := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma T^{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(k)}}. \tag{2.31}$$

In both cases, the sum ranges over all permutations in k elements. A the names suggest, the (anti-)symmetric part of T is a totally (anti-)symmetric tensor.

The above definitions are given with respect to a basis, but note that the equations satisfy our rules for well-defined tensor equations. So, if the components of a $(k,0)$-tensor are (anti-)symmetric with respect to one basis, they are so with respect to all bases.

All of these definitions have been made for type $(k,0)$-tensors, but one can similarly define totally (anti-) symmetric tensors of type $(0,k)$. What does *not* exist are totally (anti-)symmetric tensors of mixed type. For example, it does not make sense to demand that a type $(1,1)$ tensor is symmetric under its two indices: the condition $T^i{}_j = T^j{}_i$ does not satisfy the rules for well-defined tensor equations – so if it is true with respect to one basis, it is not necessarily true for all of them.

What one can define, however, are tensors which partially (anti-)symmetric. For instance, a tensor T with components $T^i{}_{jkl}$ could be only symmetric with respect to its first two lower indices, i.e. satisfy $T^i{}_{jkl} = T^i{}_{kjl}$. So

$$T^i{}_{jkl} = T^i{}_{(jk)l} := \frac{1}{2}\Big(T^i{}_{jkl} + T^i{}_{kjl}\Big). \tag{2.32}$$

If T would be symmetric with respect to only its first and third lower index, one would write that as $T^i{}_{jkl} = T^i{}_{(j|k|l)} = \frac{1}{2}(T^i{}_{jkl} + T^i{}_{lkj})$. One can quickly see where this is going, and one can define tensors of arbitrarily complex behaviour under permutation of their indices.

2.2 Manifolds

An n-dimensional (real) **manifold** is a "second countable Hausdorff space M with a maximal atlas". It's a space that locally looks like (a piece of) $\mathbb{R}^n$. "Second countable" means that M is not "too large", for example if the manifold is a straight line, it is not longer than the real line. A Hausdorff space is a space where two different points can always be separated by open sets. In practice, M can be described by local **coordinate charts**: (x, U) where U is some open subset of M and $x : U \to x(U) \subset \mathbb{R}^n$ a coordinate map.

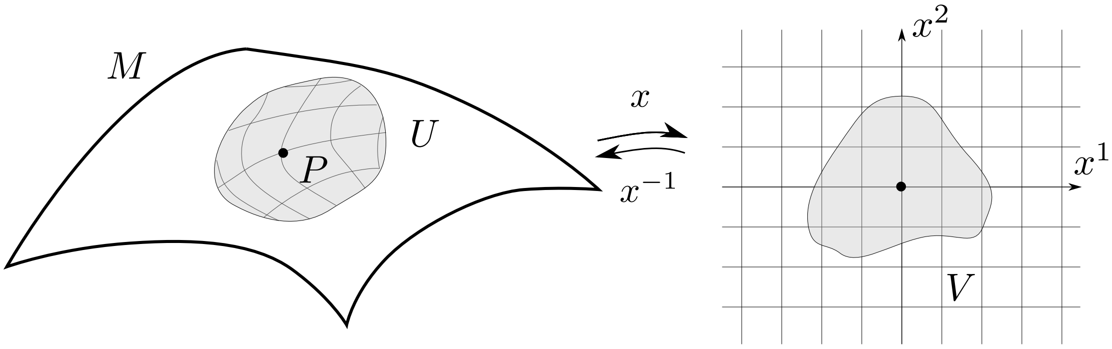

Figure 2.2: A manifold M is a space where the neighbourhood U of every point P loosk like a piece V of $\mathbb{R}^n$. Points in U get assigned coordinates $x^1, \ldots, x^n$.

Imagine as an example M being the two-dimensional surface of a three-dimensional torus. A part U of this surface is mapped by $x(U)$ to a two-dimensional, Cartesian coordinate system $\mathbb{R}^2$. Here, points in U can be described by coordinates (x^1, x^2). If two charts (x, U) and $(\tilde{x}, \tilde{U})$ overlap ($U \cap \tilde{U} \neq \emptyset$), the points in M which belong to both U and $\tilde{U}$ can be described by either coordinate system. We can map between the coordinate systems by $\tilde{x} \circ x^{-1}$ (from $x(U \cap \tilde{U})$ to $U \cap \tilde{U} \subset M$ to $\tilde{x}(U \cap \tilde{U})$) and by $x \circ \tilde{x}^{-1}$ in the other direction. Of course, we can only map the overlap between U and $\tilde{U}$ this way. These two maps are also denoted as $\tilde{x}^i(x)$ and $x^i(\tilde{x})$. They need to be smooth (infinitely often differentiable).

Let us consider some examples for manifolds:

1. The plane $M = \mathbb{R}^2$ is a manifold as obviously, all $\mathbb{R}^n$ look like $\mathbb{R}^n$ not only locally, but everywhere. On $\mathbb{R}^2$, there are several different coordinate charts. The most well-known are of course Cartesian coordinates (x^1, x^2) which cover all of M, but polar coordinates (r, φ) only cover part of M. They are not defined on the negative part of the x^1 axis and at 0.

2. The 2-sphere $S^2 = \{\vec{r} \in \mathbb{R}^3 | \|\vec{r}\|^2 = 1\}$ is the surface of a three-dimensional ball. Spherical coordinates (φ, θ) are only defined for $-\pi < \varphi < \pi$ and $0 < \theta < \pi$. The coordinates (φ, π), the north pole and the south pole are not defined. If they were included, the manifold would be non-continuous/-differentiable at those points.

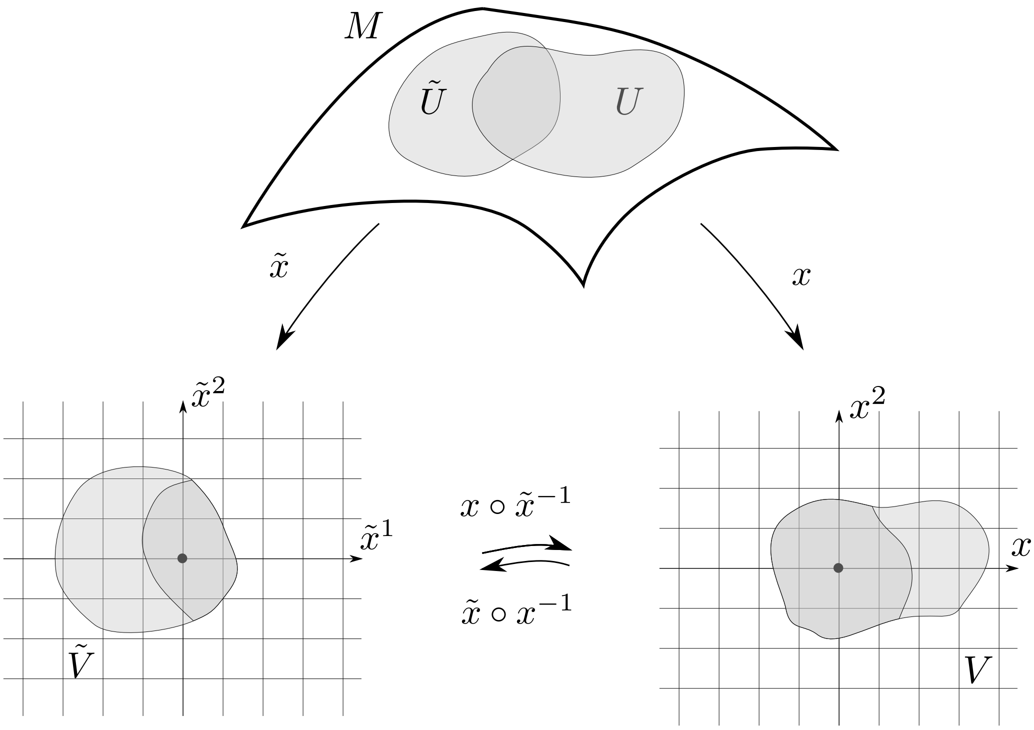

Figure 2.3: On the overlap of two coordinate charts one can change the coordinates. The change $x^i(\tilde{x})$ and its inverse $\tilde{x}^i(x)$ are infinitely often differentiable, so one can form partial derivatives, e.g. $\frac{\partial x^i}{\partial \tilde{x}^j}$.

3. Thermodynamics, for example the state variables of an ideal gas: p, V, T. Their equation of state ($pV = k_B T$) defines a two-dimensional manifold. Possible coordinates are either $(p, V), (T, V), (p, T)$ or more complicated coordinates which are functions of these three (the thermodynamic potentials $S, F, H, U, ...$). Thermodynamics is mostly a rewriting of differential geometry.

Note that in the examples, we can see that in practice, coordinates are not always called x^i, for example sometimes people use $\varphi, \theta, \psi, ...$ if they are angles.

2.2.1 Tangent vectors

Vectors can be regarded as "infinitesimal translations on M". Every tangent vector x at a point p in M arises as the velocity vector to a curve in M through p. In a coordinate chart (x, U), such a curve is given by $\phi \mapsto x^i(\phi)$. So the velocity vector of that curve is given by n numbers

$$X^i = \left.\frac{dx^i}{d\phi}\right|_{\phi=0}. \tag{2.33}$$

The coordinates of p are $x^i(0)$. The same curve in M in a different coordinate system $(\tilde{x}, \tilde{U})$ is given by

$$\tilde{x}^i(\phi) = \tilde{x}^i(x^j(\phi)). \tag{2.34}$$

So the same vector X in the other coordinate system is given by the coefficients

$$\tilde{X}^i = \frac{d\tilde{x}^i}{d\phi} = \frac{\partial \tilde{x}^i}{\partial x^j} X^j. \tag{2.35}$$

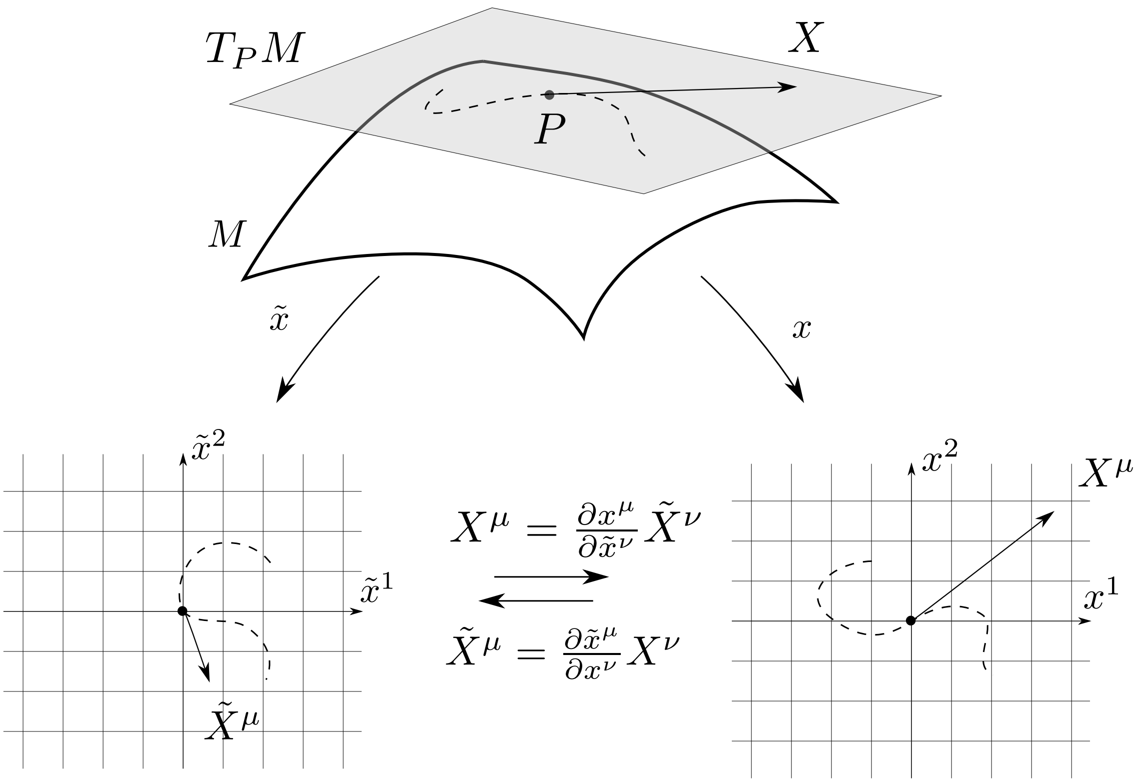

Figure 2.4: Tangent vectors are velocity vectors of curves. Their components change under a change of coordinates.

The vector is not a part of M. It is "stuck at point p", but as a tangential vector can lie outside of M. In the special case of Minkowski space, vectors point from one event to another, and they can be thought of as lying inside Minkowksi space itself.
All tangent vectors to a point p form a vector space called the **tangent space** T_pM. It is the space of all velocity vectors of curves through p. The dimension of T_pM is the same as that of M, which is n: $\dim T_pM = \dim M = n$. In drawings, people usually attach T_pM to p, but it is important to realize that T_pM is something completely different from M. Another mistake that is easily made is to think that the tangent spaces at two different points p and p' have something to do with each other. For example, the question could arise where two tangent spaces on a one-sphere intersect. That question is nonsensical as T_pM and $T_{p'}M$ are completely different things and a relation between them is only falsely suggested in drawings.

The collection of all T_pM for all p is called the **tangent bundle** TM.

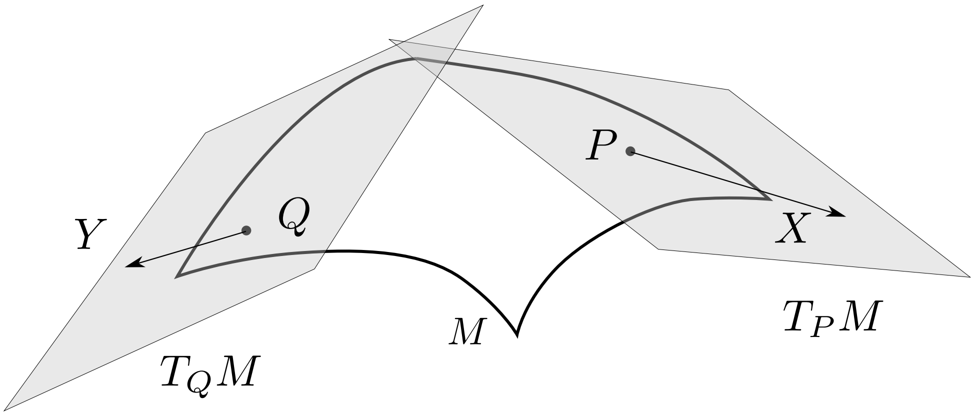

Figure 2.5: Careful: tanget spaces assigned to different points P and Q do not intersect, even if images might suggest so!

2.2.2 Basis for $T_p M$

Given a set of coordinates x^i around p, so that p has coordinates $(x^1(p), x^2(p), ..., x^n(p))$, there is a nice set of basis vectors

$$\left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, ..., \left.\frac{\partial}{\partial x^n}\right|_p. \tag{2.36}$$

The derivative $\partial/\partial x^i|_p$ is the tangent vector which belongs to the curve which in the coordinates $\{x^i\}$ is given by

$$\phi \mapsto \begin{pmatrix} x^1(p) \\ x^2(p) \\ \vdots \\ x^i(p) + \phi \\ \vdots \\ x^n(p) \end{pmatrix} \tag{2.37}$$

where all components are constant except for the ith component. An arbitrary curve $\phi \mapsto x^i(\phi)$ has a tangent vector at p, which can be decomposed as follows:

$$X_p = \frac{dx^i}{d\phi} \left.\frac{\partial}{\partial x^i}\right|_p = X^i \left.\frac{\partial}{\partial x^i}\right|_p. \tag{2.38}$$

Here, X_p is a vector in $T_p M$, X^i are numbers and the partial derivatives are vectors in $T_p M$. The construction of these basis vectors depend on coordinates. Their relation to the basis vectors with respect to a different coordinate chart $(\tilde{x}, \tilde{U})$ is:

$$\left.\frac{\partial}{\partial \tilde{x}^i}\right|_p = \frac{\partial x^j}{\partial \tilde{x}^i} \left.\frac{\partial}{\partial x^j}\right|_p. \tag{2.39}$$

2.2.3 Vector fields

A vector field X on M is a (smooth) assignment of tangent vectors X_p to every point p. In a coordinate system, (part of) X can be described by $\partial/\partial x^i|_p$:

$$X_p = X^i(x^j(p)) \left.\frac{\partial}{\partial x^i}\right|_p, \tag{2.40}$$

In short:

$$X = X^i \frac{\partial}{\partial x^i} \tag{2.41}$$

where X is a vector field, X^i are the n coefficient functions and the derivatives are the **basis vector fields** that are only defined on U. A vector field in two different coordinate systems is expressed as

$$X = X^j \frac{\partial}{\partial x^j} = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i}, \tag{2.42}$$

which means for the transformed components

$$\tilde{X}^i = \frac{\partial \tilde{x}^i}{\partial x^j} X^j. \tag{2.43}$$

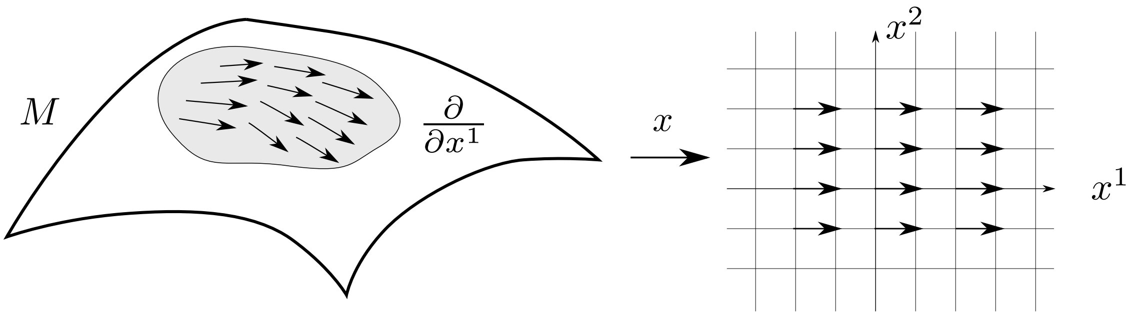

Figure 2.6: Basis vector fields $\frac{\partial}{\partial x^i}$ are those which, in local coordinates x are pointing in the i-th direction.

Vector fields can act on scalar fields ("functions"): Let $f : M \to \mathbb{R}$ be a function and X a vector field on M. Then $X(f)$ is also a function:

$$X(f)(p) := X^i \left.\frac{\partial f}{\partial x^i}\right|_{x^i(p)}, \tag{2.44}$$

where we have used the symbol f for both the function $f : M \to \mathbb{R}$, as well as the function in a local coordinate chart $f \circ x^{-1} : x(U) \to \mathbb{R}$. The action of a vector field on a function is therefore defined as

$$X(f) = X^i \frac{\partial f}{\partial x^i}. \tag{2.45}$$

The **commutator** of vector fields X and Y is again a vector field, and its action on a function is

$$[X, Y](f) := X(Y(f)) - Y(X(f)). \tag{2.46}$$

In local coordinates, this reads

$$[X,Y](f) = X^i \frac{\partial}{\partial x^i}\left(Y^j \frac{\partial f}{\partial x^j}\right) - Y^i \frac{\partial}{\partial x^i}\left(X^j \frac{\partial f}{\partial x^j}\right)$$
$$= X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} - Y^i \frac{\partial X^j}{\partial x^i}\frac{\partial f}{\partial x^j} - X^j Y^i \frac{\partial^2 f}{\partial x^i \partial x^j}. \tag{2.47}$$

The terms with the double derivative are the same because we sum over all i and j, so the commutator can be written as

$$[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}. \tag{2.48}$$

2.2.4 1-forms

The dual object to vector fields are **1-forms**. Where a vector field assigns a vector to each point, a 1-form ω assigns to each point a dual vector $\omega_p \in (T_pM)^*$, so ω_p is a linear map from T_pM to $\mathbb{R}$. In local coordinates, a basis of T_pM is given by $\partial/\partial x^i|_p$. The dual basis of T_p^*M is given by $dx^i|_p$ with the property

$$dx^i|_p \left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \delta^i{}_j. \tag{2.49}$$

It is a local basis only defined on U. Every 1-form ω can in local coordinates be written as $\omega = \omega_i dx^i$ with ω_i being n coefficient functions. A change of coordinates can be done by

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j \tag{2.50}$$

so that $\omega = \omega_i dx^i = \tilde{\omega}_i d\tilde{x}^i$, which means

$$\tilde{\omega}_i = \frac{\partial x^j}{\partial \tilde{x}^i}\omega_j. \tag{2.51}$$

2.2.5 Curve integrals of 1-forms

Let ω be a 1-form on M and γ a curve in M. In local coordinates, γ is given by $\phi \mapsto x^i(\phi)$ where $a \le \phi \le b$. From $\omega = \omega_i dx^i$ follows

$$\int_\gamma \omega := \int_a^b d\phi\, \omega\left(\frac{d\gamma}{d\phi}\right) := \int_a^b d\phi\, \omega_i \frac{dx^i}{d\phi} \tag{2.52}$$

where $d\gamma/d\phi$ is the tangent vector. This holds assuming that γ fits into one coordinate chart and is called the **integral of ω over γ**. Examples are

1. $M = \mathbb{R}^3$ with ω a force field and γ the path of a particle in ω. Then $\int_\gamma \omega$ is the change of energy of that particle as it moves along γ.

2. $M = \{(T,p,V) : k_B T = pV\}$, e.g. with $\omega = dU$ the change of energy or $\omega = dS$ the change of entropy. The curve γ is the thermodynamic process. Then $\int \omega$ is the total change of energy/entropy after the process.

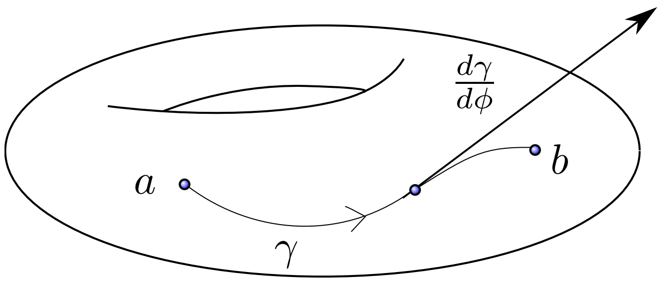

Figure 2.7: A 1-form can be integrated along a curve γ. The value is the integral over the velocity vector $\frac{d\gamma}{\phi}$, evaluated at $\omega_{\gamma(\phi)}$.

2.2.6 General tensor fields on M

A tensor field T of type (r, s) is locally described by its coefficients, e.g. $T_i{}^j{}_k$ (r indices upstairs, s indices downstairs). The $T_i{}^j{}_k$ are functions of the coordinates $\{x^i\}$. Locally, we write

$$T = T_i{}^j{}_k dx^i \otimes \frac{\partial}{\partial x^j} \otimes dx^k. \tag{2.53}$$

Physicists usually don't write the basis vectors. Under a change of coordinates, the same tensor T has different coefficients

$$\tilde{T}_i{}^j{}_k = \frac{\partial x^l}{\partial \tilde{x}^i} \frac{\partial \tilde{x}^j}{\partial x^m} \frac{\partial x^n}{\partial \tilde{x}^k} T_l{}^m{}_n. \tag{2.54}$$

Careful: contravariant (upstairs) indices get the matrix $\partial\tilde{x}/\partial x$ while covariant (downstairs) indices get the inverse matrix $\partial x/\partial\tilde{x}$. So the transformation behaviour is precisely the same as in the last chapter, just that the (constant) matrix $M_i{}^j$ has been replaced with $\frac{\partial x^j}{\partial \tilde{x}^i}$ (which now depends on the coordinates x^i), and $N^i{}_j$ has been replaced with $\frac{\partial \tilde{x}^i}{\partial x^j}$.

So the only things that have changed compared to the previous chapter, where we talked about tensors, is: tensor fields on manifolds have a tensor at each point p in the manifold, and each is a tensor with respect to a different vector space (namely T_pM). A basis for the tangent spaces is given by the coordinate basis vectors $e_i = \frac{\partial}{\partial x^i}$, and its dual basis $\theta^i = dx^i$. So, each time we change our local coordinates, we also automatically change the basis and dual basis for each (co-)tangent vector space in the region where the coordinates are defined. The matrices M and N are then given by the Jacobian of the change of coordinates (and its inverse).

2.2.7 The Cartan derivative

A totally anti-symmetric tensor field of type $(0, k)$ is called a k-form $\omega_{i_1 \dots i_k} = \omega_{[i_1 \dots i_k]}$. The **Cartan derivative** for a 0-form (function) is defined as

$$df = \frac{\partial f}{\partial x^i} dx^i. \tag{2.55}$$

For higher k, $d\omega$ is a $(k+1)$-form with

$$(d\omega)_{i_1\dots i_{k+1}} = \partial_{[i_1}\omega_{i_2\dots i_{k+1}]} = \frac{1}{(k+1)!}\sum_p (-1)^p \partial_{i_{p(1)}}\omega_{i_{p(2)}\dots i_{p(k+1)}}. \tag{2.56}$$

Of course, $dd\omega = 0$ for all ω. If $d\omega = 0$, then we call ω **closed**. If we can write $\omega = d\eta$, then we call ω **exact**. Every exact k-form is closed, but not necessarily the other way round. Curve integrals for exact 1-forms $\omega = df$ over a curve $\gamma : \phi \mapsto x^i(\phi)$ are given by

$$\begin{aligned}\int_\gamma \omega &= \int_\gamma df \\ &= \int_a^b d\phi \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial \phi} \\ &= \int_a^b d\phi \frac{d}{d\phi}(f(x(\phi))) = f(x(b)) - f(x(a)).\end{aligned} \tag{2.57}$$

2.3 Metrics, connections and curvature

So far, we have no a priori notion of "length of a tangent vector" or "angle between vectors". To get those, we need some additional structure, called **metric**. A metric g is a symmetric tensor field of type $(0,2)$. In coordinates, it is given by

$$g = g_{ij}dx^i dx^j \tag{2.58}$$

(here, we omitted the $\otimes$) with $g_{ij} = g_{ji}$. (Note that it is not a 2-form since it is symmetric, not anti-symmetric. For example, the Cartan derivative of a metric makes no sense.) It provides at each point p an inner product on T_pM between two tangent vectors X_p and Y_p:

$$\langle X_p, Y_p\rangle = g_p(X_p, Y_p) := g_{ij}X_p^i Y_p^j \tag{2.59}$$

which again is symmetric: $\langle X_p, Y_p\rangle = \langle Y_p, X_p\rangle$. The **signature** of a metric is (p,q,r) with the constraint $p+q+r = n$. Here, p is the number of positive eigenvalues (with multiplicity) of g_{ij}. Analogously, q is the number of negative and r the number of 0 eigenvalues. If $r \neq 0$, we call the metric **degenerate**. In this case, it is not invertible. Accordingly, a metric with $r = 0$ is called **non-degenerate** and its inverse metric exists. The inverse metric g^{-1} is a symmetric tensor field of type $(2,0)$ and satisfies

$$g^{ij}g_{jk} = \delta^i{}_k; \quad g_{ij}g^{jk} = \delta_i{}^k. \tag{2.60}$$

If $p = n$, the metric is **positive definite**, the inner product $\langle X_p, X_p\rangle$ is positive for all nonzero X_p. In this case, $q = r = 0$ and the metric is also called a **Riemannian metric**. A metric with $p = 1$, $q = n-1$ and $r = 0$ is called a **Lorentzian metric**. Examples are

1. $\mathbb{R}^3$ with the standard inner product:

$$g_{ij} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \delta_{ij} \tag{2.61}$$

in Cartesian coordinates.

2. $\mathbb{R}^{1,3}$ with Minkowski metric η:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.62}$$

   Here, $p = 1$, $q = 3$, $r = 0$.

3. $M = S^2$ with coordinates (φ, θ). Then the components are given by

$$g_{\varphi\varphi} = \sin^2\theta; \quad g_{\varphi\theta} = g_{\theta\varphi} = 0; \quad g_{\theta\theta} = 1. \tag{2.63}$$

   The metric is given by

$$g = d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes \varphi = d\theta^2 + \sin^2\theta d\varphi^2 =: ds^2. \tag{2.64}$$

One can use a non-degenerate metric to raise and lower indices. For example:

$$T_{ijk} = g_{jj'} T_i{}^{j'}{}_k \tag{2.65}$$

$$T^{ij}{}_k = g^{ii'} T_{i'}{}^j{}_k. \tag{2.66}$$

By the way, special relativity is a special case of the analysis on manifolds, just chose the metric $(1, -1, -1, -1)$ and restrict to only coordinate systems which are inertial systems. That means that, in that case, the change of coordinates $\tilde{x}^\mu = \Lambda^\mu{}_\nu + a^\mu$ leads to $\frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu$, which is constant (which makes many thing easier in special relativity).

2.3.1 The covariant derivative

In Minkowski space, one can form derivatives of tensors ∂T. If T has e.g. components $T_i{}^j{}_k$, then ∂T has coefficients $\partial_l T_i{}^j{}_k$. This is quite important, e.g. for formulating physical laws like the conservation law in electrodynamics $\partial_\mu j^\mu = 0$. On general manifolds, this doesn't work so easily any more.

Assume we have a vector field X which in one coordinate system has the components X^i, and in another $\{\tilde{x}^i\}$ the components $\tilde{X}^i$. If we form the derivatives in both coordinate systems, i.e. $\partial X^i / \partial x^j$ and $\partial \tilde{X}^i / \partial \tilde{x}^j$, these do not form the coefficients of a tensor. To see this, we make the transformation from one coordinate system to the other:

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^j} \tilde{X}^i &= \frac{\partial}{\partial \tilde{x}^j} \left(\frac{\partial \tilde{x}^i}{\partial x^k} X^k \right) \\ &= \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial X^k}{\partial \tilde{x}^j} + \frac{\partial x^m}{\partial \tilde{x}^j} \frac{\partial^2 \tilde{x}^i}{\partial x^m \partial x^k} X^k \\ &= \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial x^m}{\partial \tilde{x}^j} \left(\frac{\partial}{\partial x^m} X^k \right) + \frac{\partial x^m}{\partial \tilde{x}^j} \frac{\partial^2 \tilde{x}^i}{\partial x^m \partial x^k} X^k. \end{aligned} \tag{2.67}$$

Here, we first plugged in the definition of the coordinate transformation matrix. In the next step, we acted with the derivative operator $\partial / \partial \tilde{x}^j$ according to the chain rule and inserted $\partial x^m / \partial x^m = 1$ in the second term in order to avoid $\partial \tilde{x}^i / \partial \tilde{x}^j$. In the last step, we did the same in the first term so that X^k is derived with respect to the coordinates of the system it is defined in.
The first term of equation 2.67 is exactly how a tensor T^k_m of rank $(1,1)$ would transform, but the second term spoils the transformation behavior. Obviously, the numbers $\partial X^i / \partial x^j$ do not form the coefficients of tensors. To interpret this geometrically, we ask the question: What would the directional derivative of a vector field along a curve be? To answer this question, consider a path γ through a vector field X. In local coordinates, γ has the form

$$x^i(\phi) = x^i(0) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \phi \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.68}$$

where the j^{th} component is non-zero. We define the point p as $p = \gamma(0)$, i.e. p has coordinates $x^i(0)$. The point q has coordinates $x^i(0) + \phi \delta^i{}_j$. We can then try to use the usual definition of the derivative:

$$\partial_i X^i = \lim_{\phi \to 0} \frac{X^i(x^k(0) + \phi \delta^k{}_j) - X^i(x^k(0))}{\phi}. \tag{2.69}$$

But $X^i(x^k(0) + \phi \delta^k{}_j)$ and $X^i(x^k(0))$ are (coefficients of) vectors from different tangent spaces T_pM and T_qM. It makes no sense to subtract them. To define a useful notion of a directional derivative, we need a notion of **parallel transport**: First, we transport the vector $X_q \in T_qM$ to T_pM "without changing its direction". Then, we can subtract X_p and form the limit. The mathematical structure needed to define this is called **connection**. A general connection is defined by its Christoffel symbols. Having a non-degenerate metric (no matter what the signature is) allows for a special one, called **Levi-Civita connection**, which we will construct now.

We first try to get some geometrical intuition by considering a simple example: A 2d surface in $\mathbb{R}^3$, which is given by a function $f(x, y) = z$. This surface is a manifold M. The standard coordinates on M are $x^1 = x, x^2 = y$.

Now, from a point q with tangent space T_qM, we move the vector $X_q \in T_qM$ to p and denote it by $\hat{X}_p$. In general, $\hat{X}_p$ will not be an element of T_pM. Because of that, we project $\hat{X}_p$ to T_pM and denote it by $\tilde{X}_p$. Note that we use the ambient $\mathbb{R}^3$ for that construction.

The embedding of M in $\mathbb{R}^3$ provides a metric g on M, simply because every vector tangent to M can be viewed as a vector in $\mathbb{R}^3$, and that has a standard notion of length. Because of this, g is also called the induced metric.

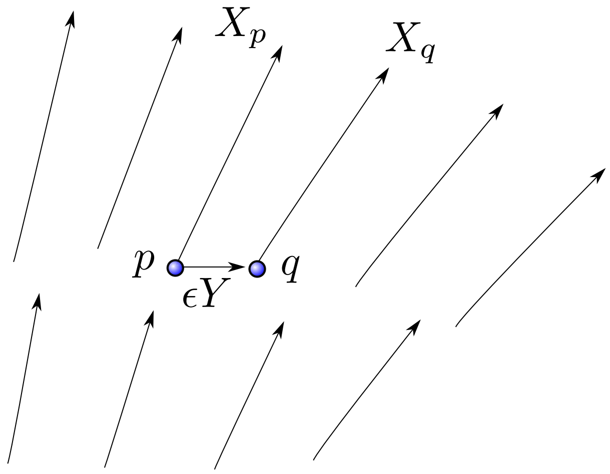

Figure 2.8: The naive way of defining the derivative of a vector field X in the direction of Y would be to form a difference quotient between $X^i(x)$ and $X^i(x+\epsilon Y)$, dividing by ϵ and taking the limit of $\epsilon \to 0$. But thie is problematic, since the two vectors at different points are living in different vector spaces, so they cannot be added or subtracted from one another.

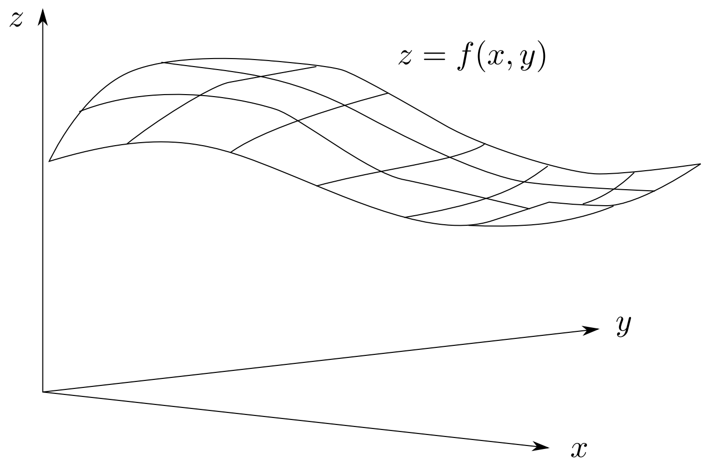

Figure 2.9: We first consider a $2d$ manifold embedded in $\mathbb{R}3$, which is given by the equation $z = f(x,y)$. This manifold automatically inherits an induced metric from the ambient $\mathbb{R}^3$.

Let's construct the basis vector fields. The point p has coordinates x^1, x^2. Then

$$\frac{\partial}{\partial x^1}\bigg|_p = \frac{d}{d\phi}\bigg|_{\phi=0} \begin{pmatrix} x^1+\phi \\ x^2 \\ f(x^1+\phi, x^2) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x^1}(x^1,x^2) \end{pmatrix} \tag{2.70}$$

$$\frac{\partial}{\partial x^2}\bigg|_p = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial x^2}(x^1,x^2) \end{pmatrix}. \tag{2.71}$$

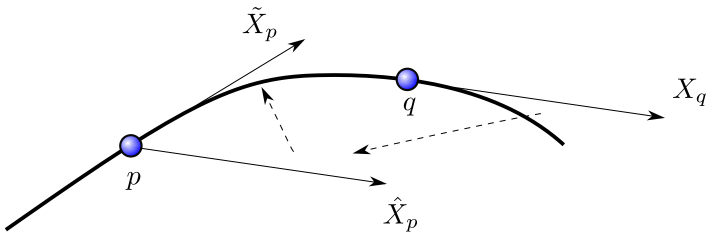

Figure 2.10: To define parallel transport of a vector X_q to a vector $\tilde{X}_p$, we first translate X_q to $\hat{X}_p$ at p, which might not be in T_pM any more. We then project it to T_pM, calling the result $\tilde{X}_p$.

For derivatives, we introduce the notation

$$f_{,i} := \frac{\partial f}{\partial x^i}, \quad f_{,ij} := \frac{\partial^2 f}{\partial x^i \partial x^j}. \tag{2.72}$$

The induced metric on M on coordinates x^1, x^2 is

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \delta_{ij} + f_{,i} f_{,j} = \begin{pmatrix} 1 + \left(\frac{\partial f}{\partial x^1}\right)^2 & \frac{\partial f}{\partial x^1}\frac{\partial f}{\partial x^2} \\ \frac{\partial f}{\partial x^1}\frac{\partial f}{\partial x^2} & 1 + \left(\frac{\partial f}{\partial x^2}\right)^2 \end{pmatrix}. \tag{2.73}$$

The derivatives 2.70 and 2.71 are the basis of the tangent space T_pM. The basis of the tangent space T_qM is constructed analogously, so that the vector in q and its parallel transported version on p can be written as

$$X_q = X^i_q \left.\frac{\partial}{\partial x^i}\right|_q \tag{2.74}$$

$$\tilde{X}_p = \tilde{X}^i_p \left.\frac{\partial}{\partial x^i}\right|_p . \tag{2.75}$$

If we have these, how do we get $\tilde{X}^i_p$? We use equations 2.70 and 2.71 to write

$$X_q = X^i_q \begin{pmatrix} e_i \\ f_{,i}(x^k + \phi \delta^k{}_j) \end{pmatrix} \tag{2.76}$$

where $x^k = (x^1, x^2)$ are the coordinates of p, $e_1 = (1,0)^t$ and $e_2 = (0,1)^t$. The coordinates of q are $x^k + \phi\delta^k{}_j$. We also write

$$\tilde{X}_p = \tilde{X}^i_p \begin{pmatrix} e_i \\ f_{,i}(x^k) \end{pmatrix}. \tag{2.77}$$

The projected vector is

$$\hat{X}_p = X^i_q \begin{pmatrix} e_i \\ f_{,i}(x^k + \phi \delta^k{}_j) \end{pmatrix}. \tag{2.78}$$

This is just a translation in $\mathbb{R}^3$, it doesn't change the components of 3-vectors. The scalar product of $\hat{X}_p$ with the basis vectors is

$$\begin{aligned}
g_{ij}\tilde{X}^j_p &= \left\langle \hat{X}_p, \left.\frac{\partial}{\partial x^i}\right|_p \right\rangle \\
&= X^k_q \begin{pmatrix} e_k \\ f_{,k}(x^l + \phi\delta^l{}_j) \end{pmatrix} \begin{pmatrix} e_i \\ f_{,i}(x^l) \end{pmatrix} \\
&= X^k_q(\delta_{ik} + f_{,i}(x^i) f_{,k}(x^l + \phi\delta^l{}_j)) \\
&= X^k_q(\underbrace{\delta_{ik} + f_{,i}(f_{,k}}_{g_{ik}} + \phi f_{,jk} + \mathcal{O}(\phi^2))) \\
&= g_{ik}X^k_q + \phi f_{,i} f_{,jk} X^k_q + \mathcal{O}(\phi^2).
\end{aligned} \tag{2.79}$$

In the fourth line, we expand $f_{,k}(x^l + \phi\delta^l{}_j)$ into a Taylor expansion in ϕ to first order. We then calculate

$$g_{ij,k} = \frac{\partial}{\partial x^k}(\delta_{ij} + f_{,i}f_{,j}) = f_{,i}f_{,jk} + f_{,ik}f_{,j} \tag{2.80}$$

which leads us to

$$\begin{aligned}
g_{ij,k} + g_{ik,j} - g_{jk,i} &= f_{,i}f_{,jk} + f_{,j}f_{,ik} + f_{,k}f_{,ij} + f_{,i}f_{,jk} - f_{,k}f_{,ij} - f_{,j}f_{,ik} \\
&= 2f_{,i}f_{,jk}.
\end{aligned} \tag{2.81}$$

Plugging this into equation 2.79 (multiplied with the inverse metric g^{il}) yields

$$\tilde{X}^i_p = X^i_q + \phi X^k_q \underbrace{\frac{1}{2}g^{il}(g_{il,k} + g_{kl,j} - g_{ik,l})}_{=:\Gamma^i_{jk}} + \mathcal{O}(\phi^2). \tag{2.82}$$

With this, we define the directional derivative in j-direction as

$$\begin{aligned}
(\nabla_j X)^i &= \lim_{\phi\to 0} \frac{\tilde{X}^i_p - X^i_p}{\phi} \\
&= \lim_{\phi\to 0} \left(\frac{X^i_q - X^i_p}{\phi} + \Gamma^i_{jk}X^k_q + \mathcal{O}(\phi) \right).
\end{aligned} \tag{2.83}$$

If we then take the limit of this with $q = p$, we get the **covariant derivative**

$$(\nabla_j X)^i = \partial_j X^i + \Gamma^i_{jk}X^k. \tag{2.84}$$

It depends on the metric via Christoffel symbols Γ^i_{jk}. While the derivation was specific to 2d surfaces on $\mathbb{R}^3$, equation 2.84 depends only on the metric. We take this as a guess for a derivative on arbitrary manifolds. Let M be a general manifold with a non-degenerate metric g, and X a vector field. In x^i coordinates, we write the Christoffel symbols as

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m}). \tag{2.85}$$

We change to other coordinates $\tilde{x}^i$:

$$\tilde{\Gamma}^i_{jk} = \frac{1}{2}\tilde{g}^{im}(\tilde{g}_{mj,k} + \tilde{g}_{mk,j} - \tilde{g}_{jk,m}). \tag{2.86}$$

Look at how $g_{ij,k}$ transforms:

$$\begin{aligned}
\tilde{g}_{mj,k} &= \frac{\partial \tilde{g}_{mj}}{\partial \tilde{x}^k} \\
&= \frac{\partial x^{k'}}{\partial \tilde{x}^k}\frac{\partial}{\partial x^{k'}}\left(\frac{\partial x^{m'}}{\partial \tilde{x}^m}\frac{\partial x^{j'}}{\partial \tilde{x}^j}g_{m'j'}\right) \\
&= \frac{\partial x^{k'}}{\partial \tilde{x}^k}\frac{\partial x^{m'}}{\partial \tilde{x}^m}\frac{\partial x^{j'}}{\partial \tilde{x}^j}g_{m'j',k'} + g_{m'j'}\frac{\partial^2 x^{m'}}{\partial \tilde{x}^m \partial \tilde{x}^k}\frac{\partial x^{j'}}{\partial \tilde{x}^j} + g_{m'j'}\frac{\partial^2 x^{j'}}{\partial \tilde{x}^j \partial \tilde{x}^k}\frac{\partial x^{m'}}{\partial \tilde{x}^m}.
\end{aligned} \tag{2.87}$$

In the first line, we just write out $\tilde{g}_{mj,k}$. In the second line, we transformed both the derivative and the metric according to tensor transformation laws. In the next line, we carried out the derivative with respect to $x^{k'}$ using the chain rule. Using this, we get

$$\tilde{\Gamma}^i_{jm} = \frac{\partial \tilde{x}^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial \tilde{x}^j}\frac{\partial x^{m'}}{\partial \tilde{x}^m}\Gamma^{i'}_{j'm'} + \frac{\partial \tilde{x}^i}{\partial x^{i'}}\frac{\partial^2 x^{i'}}{\partial \tilde{x}^j \partial \tilde{x}^m}. \tag{2.88}$$

This is not a tensor transformation as is obvious from the additional term. Transforming equation 2.84 however yields

$$\begin{aligned}
\tilde{\nabla}_i \tilde{X}^j &= \partial_i \tilde{X}^j + \tilde{\Gamma}^j_{ik}\tilde{X}^k \\
&= \frac{\partial \tilde{X}^j}{\partial \tilde{x}^i} + \tilde{\Gamma}^j_{ik}\tilde{X}^k \\
&= \frac{\partial \tilde{x}^j}{\partial x^{j'}}\frac{\partial x^{i'}}{\partial \tilde{x}^i}(\partial_{i'}X^{j'} + \Gamma^{j'}_{i'k'}X^{k'}) + \left(\frac{\partial \tilde{x}^j}{\partial x^{j'}}\frac{\partial^2 x^{j'}}{\partial \tilde{x}^i \partial \tilde{x}^k}X^k + \frac{\partial}{\partial x^{k'}}\left(\frac{\partial \tilde{x}^j}{\partial x^{l'}}\right)X^{l'}\frac{\partial x^{k'}}{\partial \tilde{x}^i}\right) \\
&= \frac{\partial \tilde{x}^j}{\partial x^{j'}}\frac{\partial x^{i'}}{\partial \tilde{x}^i}(\nabla_{i'}X^{j'}) + \underbrace{X^{k'}\frac{\partial}{\partial x^{k'}}\underbrace{\left(\frac{\partial \tilde{x}^j}{\partial x^{j'}}\frac{\partial x^{j'}}{\partial \tilde{x}^i}\right)}_{=\delta^j_i}}_{=0} \\
&= \frac{\partial \tilde{x}^j}{\partial x^{j'}}\frac{\partial x^{i'}}{\partial \tilde{x}^i}(\nabla_{i'}X^{j'}).
\end{aligned} \tag{2.89}$$

Here, we again just put in the definition in the first line and spelled out the derivative in the second line. In the third line, we transformed $\tilde{X}^j$, the partial derivative and $\tilde{\Gamma}^j_{ik}$ according to equation 2.88 and the normal tensor transformation rules. The respective transformations of index k cancel in the first term. The end result shows that the covariant derivative that we defined transforms like a tensor. We see that the covariant derivative transforms like a $(1,1)$-tensor.

We can also define the derivative of a vector field X in the direction of another vector field Y, which gives us yet another vector field $\nabla_Y X$ with coefficients

$$(\nabla_Y X)^i = Y^j \nabla_j X^i, \tag{2.90}$$

so

$$\nabla_i \equiv \nabla_{\frac{\partial}{\partial x^i}}. \tag{2.91}$$

One can not only form the covariant derivative of vector fields, but of arbitrary tensor fields. For this, one wants the product rule to hold, i.e.

$$\nabla_X (T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S) \tag{2.92}$$

for two tensor fields T, S. Also, the covariant derivative of a function is identical to the partial derivative:

$$\nabla_k f = \partial_k f. \tag{2.93}$$

One can show that the only way to do this is to define the covariant derivative of a (r, s)-tensors to be a $(r, s+1)$-tensor with components

$$\nabla_k T_{i_1 \ldots i_s}{}^{j_1 \ldots j_r} = \partial_k T_{i_1 \ldots i_s}{}^{j_1 \ldots j_r} + \Gamma^{j_1}_{km} T_{i_1 \ldots i_s}{}^{m j_2 \ldots j_r} + \ldots + \Gamma^{j_r}_{km} T_{i_1 \ldots i_s}{}^{j_1 \ldots j_{r-1} m} \tag{2.94}$$

$$- \Gamma^m_{k i_1} T_{m i_2 \ldots i_s}{}^{j_1 \ldots j_r} - \ldots - \Gamma^m_{k i_s} T_{i_1 \ldots i_{s-1} m}{}^{j_1 \ldots j_r}. \tag{2.95}$$

With this definition, one has that for functions f and vector fields X, Y, and tensor fields T and S of the same type:

$$\nabla_{fX} T = f(\nabla_X T) \tag{2.96}$$

$$\nabla_X (T + S) = \nabla_X T + \nabla_X S \tag{2.97}$$

$$\nabla_{X+Y} T = \nabla_X T + \nabla_Y T. \tag{2.98}$$

A vector field X is called **covariantly constant along a curve** γ, if $\nabla_{\frac{d\gamma}{d\phi}} X = 0$ for all ϕ. In coordinates, γ is given by $\phi \mapsto x^i(\phi)$, so we have

$$\begin{aligned} 0 &= \nabla_{\frac{dx^i}{d\phi} \frac{\partial}{\partial x^i}} X^j \\ &= \frac{\partial x^i}{\partial \phi} \nabla_i X^j \\ &= \frac{dx^i}{d\phi} \frac{\partial X^j}{\partial x^i} + \Gamma^j_{ik} \frac{dx^i}{d\phi} X^k \end{aligned}$$

which leads us to

$$\frac{dX^j}{d\phi} + \Gamma^j_{ik} \frac{dx^i}{d\phi} X^k = 0. \tag{2.99}$$

This is an ordinary differential equation for the coefficients $X^i(x^j(\phi))$. Note that it is a differential equation that one can write down not only for vector fields which are defined everywhere, but also for vector fields which are only defined along the curve. For instance, given a curve $\phi \mapsto \gamma(\phi)$, in local coordinates $\phi \mapsto x^i(\phi)$, and a function $\phi \mapsto X(\phi)$, where each $X(\phi)$ is a tangent vector in $T_{\gamma(\phi)} M$. So we have a vector for each ϕ. The covariant derivative of X along the curve γ is then denoted by

$$\frac{D}{d\phi} X^i = \frac{dX^i}{d\phi} + \Gamma^i_{jk} \frac{dx^j}{d\phi} X^k. \tag{2.100}$$

Sometimes, people also write $\frac{\nabla}{d\phi}$ instead of $\frac{D}{d\phi}$. If $\frac{D}{d\phi}X = 0$, then the vector X is said to be **parallely transported** along the curve γ. Note that this is a first order differential equation for the $X^i(\phi)$. So, if an initial condition $X^i(0)$ is given, as well as a curve, then a unique solution exists – which is the result of parallely transporting the initial vector along the curve.

2.3.2 Geodesics

On a manifold M with non-degenerate metric g, one can measure the "length" of curves γ. We have to distinguish different metrics:

- Riemannian metrics: For all γ with $\phi \mapsto x^i(\phi)$, the length is given by

$$l(\gamma) = \int d\phi \sqrt{g_{ij}\dot{x}^i\dot{x}^j} \tag{2.101}$$

  where $\dot{x}^i := dx^i/d\phi$.

- Lorentzian metric, time-like curves: Then $g_{ij}\dot{x}^i\dot{x}^j > 0$ and

$$l(\gamma) = \int d\phi \sqrt{g_{ij}\dot{x}^i\dot{x}^j}. \tag{2.102}$$

- Lorentzian metric, space-like curves: Then $g_{ij}\dot{x}^i\dot{x}^j < 0$ and

$$l(\gamma) = -\int d\phi \sqrt{-g_{ij}\dot{x}^i\dot{x}^j}. \tag{2.103}$$

- Lorentzian metric, light-like curves: $g_{ij}\dot{x}^i\dot{x}^j = 0$ and

$$l(\gamma) = 0. \tag{2.104}$$

Consider two points p and q in M, with a Riemannian metric. The "distance" between them is $d(p,q) = \inf_{\gamma:p\to q} l(\gamma)$ [2] For Lorentzian metrics and timelike curves, we define $d(p,q) = \sup_{\gamma:p\to q} l(\gamma)$. Unfortunately, it becomes a bit tricky if one wants to define for space-like separated points, which is why one usually does not do that.

A curve between p and q such that $l(\gamma)$ is stationary (i.e. does not change under small changes of γ) is called a **geodesic**. In particular, if there is a curve such that $l(\gamma) = d(p,q)$, then that γ is a geodesic. Consider a curve in local coordinates $\phi \mapsto x^i(\phi)$ and let this curve be parameterized by the arc length (Riemannian: curve length; Lorentzian, time-like: proper time). This means that

$$g_{ij}\dot{x}^i\dot{x}^j = 1 \tag{2.105}$$

[2] Here we have to take the infimum instead of the minimum because, the minimum may not exist, for example consider the case of $M = \mathbb{R}^2\backslash(0,0)$, the plane without the origin. Two points on the real axis on either side of the origin have no direct line between them, without leaving M. That way, all curves between them must go through the plane, and can only approach the minimal length.

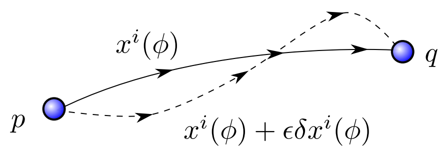

Figure 2.11: We consider a small variation δx^i of a path $\phi \to x^i(\phi)$.

Consider a small deviation from a curve: $\phi \mapsto x^i(\phi) + \epsilon\delta x^i(\phi)$. This other curve is not necessarily parameterized by arc length. We demand δx^i to vanish at the end points. The condition for $x^i(\phi)$ to be a geodesic is

$$\frac{d}{d\epsilon} l(x^i + \epsilon\delta x^i)|_{\epsilon=0} = 0 \text{ for all variations } \delta x^i. \tag{2.106}$$

This means

$$\begin{aligned} 0 &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int d\phi \sqrt{g_{ij}(x+\epsilon\delta x)\frac{d}{d\phi}(x^i+\epsilon\delta x^i)\frac{d}{d\phi}(x^j+\epsilon\delta x^j)} \\ &= \int d\phi \frac{1}{2\sqrt{g_{ij}(x)\frac{d}{d\phi}x^i\frac{d}{d\phi}x^j}}(g_{ij,k}\,\delta x^k\,\dot{x}^i\,\dot{x}^j + 2g_{ij}\delta\dot{x}^i\dot{x}^j) \\ \Leftrightarrow 0 &= \int d\phi(g_{ij,k}\delta x^k\dot{x}^i\dot{x}^j + 2g_{ij}\delta\dot{x}^i\dot{x}^j). \end{aligned} \tag{2.107}$$

The square root in the fraction is +1, due to parameterization. Use partial integration and $\delta x^i = 0$ at the end points:

$$\begin{aligned} 0 &= \int d\phi(g_{ij,k}\delta x^k\dot{x}^i\dot{x}^j - 2\delta x^i\frac{d}{d\phi}(g_{ij}\dot{x}^j)) \\ &= \int d\phi\delta x^k(g_{ij,k}\dot{x}^i\dot{x}^j - 2g_{kj,i}\dot{x}^i\dot{x}^j - 2g_{ik,j}\ddot{x}^j). \end{aligned} \tag{2.108}$$

The expression in brackets has to vanish:

$$0 = g_{ij,k}\dot{x}^i\dot{x}^j - 2g_{kj,i}\dot{x}^i\dot{x}^j - 2g_{ik,j}\ddot{x}^j \tag{2.109}$$

$$\Leftrightarrow 0 = \ddot{x}^i + \frac{1}{2}g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m})\dot{x}^j\dot{x}^k. \tag{2.110}$$

This gives us

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0. \tag{2.111}$$

This is the defining equation of motion for a geodesic. For Lorentzian metrics and time-like curves, these are the equations of motion for a freely falling observer. Note that, in the case of Lorentzian metrics, we have only defined this so far for time-like curves. However, the differential equation can be used for either of the cases of time-like, space-like or light-like. The derivation for the other two cases are completely analogous, and lead to the same differential equation (2.111). Note that (2.111) is a ordinary differential equation for the $x^i(\phi)$ of second order. So, for there to be a unique solution, one needs an initial position $x^i(0)$ and initial velocity $\dot{x}^i(0)$.

In Minkowski space, we have already seen that freely falling observers move along geodesics (which are just straight lines in that case). Here we have also seen that geodesics in general are those curves in which the length of the curve is stationary. Indeed, the "dawdling principle" (in German: "Trödelprinzip") states that an observer always moves from one event to the other so that they take the maximal time.

2.3.3 Parallel transport

On a general manifold M with non-degenerate metric g, we can use the Christoffel symbols to define a notion of parallel transport. (Note that our previous procedure relied on surfaces being embedded in $\mathbb{R}^3$). Parallel transport of a vector X_q (from the tangent space T_qM of a point q) along a curve is the solution to the ordinary differential equation 2.99 with the initial condition $X^i(\phi = 0) = X^i_q$. Having an ordinary differential equation and coefficients that are all smooth means that there is a unique solution.

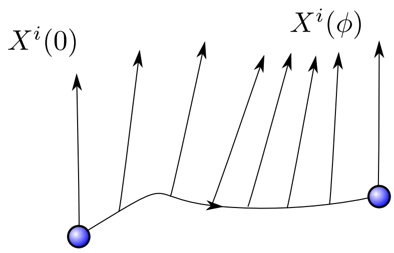

Figure 2.12: The covariant derivative allows the notion of parallel transport fo vectors along curves. A vector $X^i(\phi)$ is parallely transported along a curve, if $DX^i/d\phi = 0$.

The result of the parallel transport at p depends on which path one takes from q to p. For example, imagine we want to parallel transport a vector from the tangent space at the north pole of a sphere to the point on the equator "in the direction the vector points". If we take the shortest path, the vector will point directly "downwards". If we start with a path perpendicular to the first one, transport the vector to the equator and then move it along the equator to p, it will be perpendicular to the vector we got from the first path.

The Christoffel symbols define a notion of being **covariantly constant** and of parallel transport. We have already looked at the Levi-Civita connection with coefficients $\Gamma^k_{lm} = 1/2 g^{kj}(g_{jl,m} + g_{jm,l} - g_{lm,j})$. But any other choice of Christoffel symbols defines a connection, they just have to transform like

$$\tilde{\Gamma}^k_{lm} = \frac{\partial \tilde{x}^k}{\partial x^{k'}} \frac{\partial x^{l'}}{\partial \tilde{x}^l} \frac{\partial x^{m'}}{\partial \tilde{x}^m} \Gamma^{k'}_{l'm'} + \frac{\partial \tilde{x}^k}{\partial x^{m'}} \frac{\partial^2 x^{m'}}{\partial \tilde{x}^l \partial \tilde{x}^m}. \tag{2.112}$$

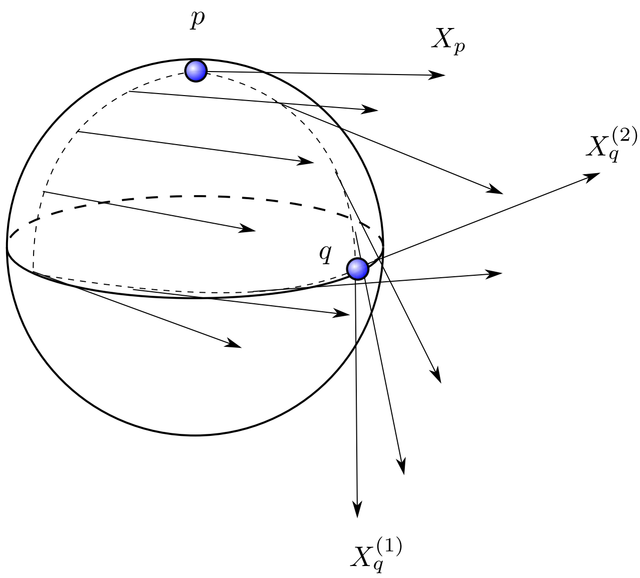

Figure 2.13: Parallely transporting a vector from a point p to a point q can yield different results, depending on which way one has chosen.

To define some other connection, simply chose some Γ^k_{lm} in one coordinate system. The $\tilde{\Gamma}^k_{lm}$ in other coordinate systems are then uniquely determined by equation 2.112. Still, the Levi-Civita connection is special: It is, for a given metric, the only one with the following two properties:

$$\nabla_k g_{ij} = 0 \tag{2.113}$$

and a connection which satisfies (2.113) is called **metric compatible**. This ensures that for a connection and a metric geodesics stationarize the length of a curve. The other property is

$$\Gamma^k_{lm} = \Gamma^k_{ml}, \tag{2.114}$$

and a connection which satisfies (2.114) is called **torsion-free**. In the following, we show this. Let g be a non-degenerate metric and Γ^k_{lm} the Christoffel symbols of any connection satisfying equations 2.113 and 2.114. Then, because of the way in which the covariant derivative acts on tensors fields (2.94), we have that

$$0 = \nabla_k g_{lm} = \partial_k g_{lm} - \Gamma^p_{kl} g_{pm} - \Gamma^p_{km} g_{lp}. \tag{2.115}$$

Similarly, by renaming the indices, we get

$$0 = \nabla_l g_{km} = \partial_l g_{km} - \Gamma^p_{lk} g_{pm} - \Gamma^p_{lm} g_{kp} \tag{2.116}$$

$$0 = \nabla_m g_{kl} = \partial_m g_{kl} - \Gamma^p_{mk} g_{pl} - \Gamma^p_{ml} g_{kp}. \tag{2.117}$$

Now, we use equation 2.114 to calculate 2.115 − 2.116 + 2.117:

$$-2\Gamma^p_{mk} g_{pl} + \partial_k g_{lm} - \partial_l g_{mk} + \partial_m g_{kl} = 0. \tag{2.118}$$

Rearranging and multiplying with g^{pl} from the left, we get

$$\Gamma^p_{mk} = \frac{1}{2} g^{pl} (g_{lm,k} + g_{lk,m} - g_{mk,l}), \tag{2.119}$$

which is the familiar expression of the Levi-Civita connection, which means that it is the only connection we get when assuming equations 2.113 (metric-compatible) and 2.114 (torsion-free).

2.3.4 Geometric interpretation of torsion

A connection corresponds to a "geometry" on a manifold, it relates different parts of the manifold and defines what they look like with respect to each other. A connection with torsion ($\Gamma^k_{lm} \neq \Gamma^k_{ml}$) describes a geometry in which parallelograms do not close. Imagine three points P, Q, R with coordinates $P : x^i$, $Q : x^i + \epsilon^i$, $R : x^i + \delta^i$ (ϵ^i and δ^i are translations). Accordingly, we call the vector from P to R δ and the one from P to Q ϵ. Now, we parallel transport δ to Q along ϵ and call the resulting vector Y, and ϵ to R along δ where we call the resulting vector X. The vectors X and Y don't point to the same point as they do for a torsion-free connection.

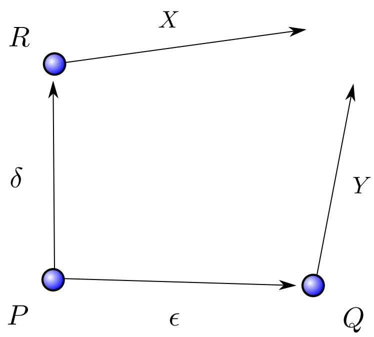

Figure 2.14: For a connection with torsion, small parallelograms, whose sides are given by the vectors ϵ, δ, X, and Y, do not close.

We denote their difference by Z. If $Z = \epsilon^i \delta^j \ldots + \mathcal{O}(\epsilon^3, \epsilon^2\delta, \delta^2\epsilon, \delta^3)$, i.e.

$$\lim_{\substack{|\epsilon|\to 0 \\ |\delta|\to 0}} \frac{Z}{|\epsilon|\,|\delta|} \neq 0, \tag{2.120}$$

then the connection is said to have torsion. The curve from P to R is given by $x^i(\phi) = x^i + \phi\delta^i$ with $0 \leq \phi \leq 1$. Let $X^k(\phi)$ be the solution to the geodesic equation

$$\dot{X}^k(\phi) = -\Gamma^k_{ml} \delta^m X^l(\phi) \tag{2.121}$$

with the initial condition $X^k(0) = \epsilon^k$. Now we assume that the components of δ are small: $|\delta^m| \ll 1$ (i.e. the Γ^k_{ml} don't vary much between P and R). We can then Taylor expand the Christoffel symbols to obtain

$$\begin{aligned}\dot{X}^k(\phi) &= -\Gamma^k_{lm}(\phi = 0)X^m\delta^l - \phi\Gamma^k_{lm}(\phi = 0)\delta^n X^m\delta^l + ...\\ &= -\Gamma^k_{lm}(\phi = 0)X^m\delta^l + \mathcal{O}(\delta^2).\end{aligned} \tag{2.122}$$

From the initial condition, when integrating we obtain

$$X^k(\phi) = -\Gamma^k_{lm}X^m(0)\delta^l\phi + \epsilon^k + \mathcal{O}(\delta^2). \tag{2.123}$$

We then get at the end of the curve

$$\begin{aligned}X^k &= X^k(\phi = 1)\\ &= -\Gamma^k_{lm}\epsilon^m\delta^l + \epsilon^k + \mathcal{O}(\delta^2)\\ &= (\delta^k_m - \delta^l\Gamma^k_{lm})e^m + \mathcal{O}(\delta^2).\end{aligned} \tag{2.124}$$

Similarly, we obtain

$$Y^k = Y^k(\phi = 1) = \delta^k - \epsilon^l\delta^m\Gamma^k_{lm} + \mathcal{O}(\epsilon^2) \tag{2.125}$$

and

$$\begin{aligned}Z^k &= -Y^k - \epsilon^k + \delta^k + X^k\\ &= \delta^l\epsilon^m(\Gamma^k_{lm} - \Gamma^k_{ml}) + \mathcal{O}(\epsilon^2, \delta^2)\\ &=: \delta^l\epsilon^m T^k_{lm}\end{aligned} \tag{2.126}$$

where we introduce the torsion tensor T^k_{lm}.
We get from a specific parallelogram to more general ones by looking at vector fields X, Y. Then

$$Z = T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] \tag{2.127}$$

and in coordinates

$$Z^k = T^k_{lm}X^l Y^m. \tag{2.128}$$

The torsion tensor is the anti-symmetric part of the connection: $T^k_{lm} = -T^k_{ml}$. The components of the torsion tensor are given by

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}. \tag{2.129}$$

An example of a connection with torsion is the “loxodromic connection” on $S^2\backslash\{N, S\}$, N and S being the north and south pole, respectively. The connection is defined by its parallel transport: A vector is parallel transported when the norm of the vector (as measured in the standard metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$) is constant and when the angle between the vector and the meridians is constant.

So we know the solution to parallel transport which means we can compute Γ^k_{lm} with $k, l, m = \phi, \theta$. Consider parallel transport of vectors along meridians and circles of latitude. First, we look at circles of latitude. Then

$$\dot{X}^\theta = -\Gamma^\theta_{\phi\theta}X^\theta - \Gamma^\theta_{\phi\phi}X^\phi \tag{2.130}$$

$$\dot{X}^\phi = -\Gamma^\phi_{\phi\theta}X^\theta - \Gamma^\phi_{\phi\phi}X^\phi. \tag{2.131}$$

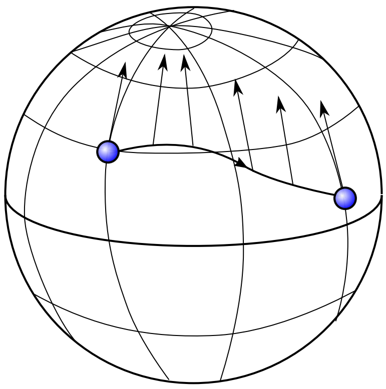

Figure 2.15: Parallel transport on S^2 with the loxodromic connection: if a vector points towards north, the parallely transported one also does.

In the following, we look at different solutions to the equations of parallel transport:
If we consider $X^\phi(\lambda) = 1$, $X^\theta(\lambda) = 0$, then obviously $\Gamma^\theta_{\phi\phi} = 0$, $\Gamma^\phi_{\phi\phi} = 0$.
In the case of $X^\phi(\lambda) = 0$, $X^\theta(\lambda) = 1$ (the vector points to the north pole), then $\Gamma^\theta_{\phi\theta} = 0$, $\Gamma^\phi_{\phi\theta} = 0$.
Along a meridian, the equations of motion take the form

$$\dot{X}^\theta = -\Gamma^\theta_{\theta\theta} X^\theta - \Gamma^\theta_{\theta\phi} X^\phi \tag{2.132}$$

$$\dot{X}^\phi = -\Gamma^\phi_{\theta\theta} X^\theta - \Gamma^\phi_{\theta\phi} X^\phi. \tag{2.133}$$

The solutions to parallel transport in this case are as follows:
For $X^\theta(\lambda) = 1$, $X^\phi(\lambda) = 0$ (the vector points to the south pole), we get $\Gamma^\phi_{\theta\theta} = 0$, $\Gamma^\theta_{\theta\theta} = 0$.
In the case of $X^\theta(\lambda) = 0$, $X^\phi(\lambda) = 1/\sin\theta(\lambda)$, with a curve $\theta(\lambda) = \lambda$, $\lambda \in (\lambda_0, \lambda_1)$, we have

$$0 = \Gamma^\phi_{\theta\theta} X^\theta - \underbrace{\Gamma^\theta_{\theta\phi}}_{=0} X^\phi \tag{2.134}$$

and

$$\dot{X}^\phi = \frac{d}{d\lambda}\frac{1}{\sin\lambda} = -\frac{\cos\lambda}{\sin^2\lambda} = -\Gamma^\phi_{\theta\phi}\frac{1}{\sin\lambda}, \tag{2.135}$$

from which we get that $\Gamma^\phi_{\theta\phi}(\lambda) = -\cot\lambda$.
So we can say that the only non-zero Christoffel symbol is $\Gamma^\phi_{\theta\phi}(\lambda) = -\cot\lambda \neq \Gamma^\phi_{\phi\theta} = 0$, so we have torsion!

In particular, different circles of latitude are parallel to each other with respect to the loxodromic connection (They aren't with respect to the Levi-Civita connection). Geodesics are called "loxodromes".

2.3.5 Curvature

In general, curvature (just like torsion) is a property of the connection. One can compute the curvature of any connection, but we will mostly have the Levi-Civita connection in

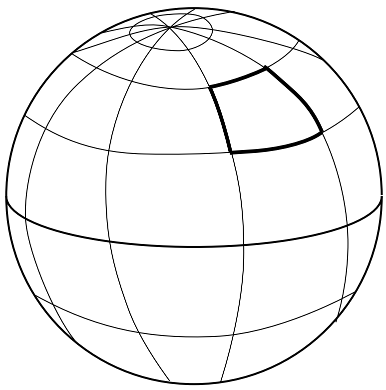

Figure 2.16: In the loxodromic connection, all meridians are parallel to one another. So closing parallelograms can have different lengths on opposing sides.

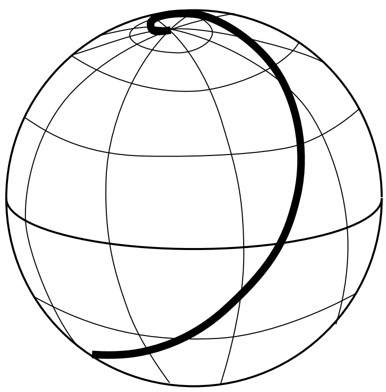

Figure 2.17: Geodesics on S^2 with the loxodromic connection are the loxodromes, which are no geodesics with respect to the Levi-Civita connection.

mind. Since that is determined by a non-degenerate metric g, its curvature can be regarded as a property of the metric.
Curvature means that covariant derivatives in different directions do not commute. This information is stored in the **Riemannian curvature tensor**, which is defined by

$$(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)^k =: R^k{}_{lmn} Z^l X^m Y^n. \tag{2.136}$$

For a geometric interpretation of $R^k{}_{lmn}$, consider two short paths between two points P and Q on a manifold. We chose one path to be along a small vector ϵ from P to a point R, then along another small vector δ to the point Q and the other path to be along δ to a point S first and then along ϵ to Q so that the two paths span a parallelogram. Now, we parallel transport a vector X_P from P to Q along both of these paths. We denote the vector obtained by parallel transport via R by $X_{(P\to R\to Q)}$ and analogously via S by $X_{(P\to S\to Q)}$.

Let P have the coordinates x^i. Then R has coordinates $x^i + \epsilon^i$ and the curve is

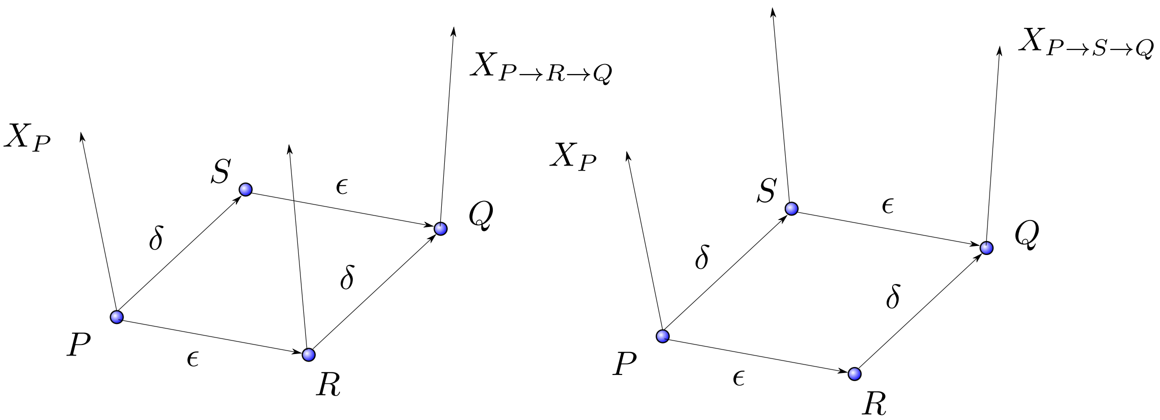

Figure 2.18: Parallely transporting a vector X_P to a point Q via two different intermediate points, R or S, can lead to different results.

$x^i(\phi) = x^i + \phi\epsilon^i$ with $0 \le \phi \le 1$. We write the equation for parallel transport:

$$\begin{aligned}
0 &= \dot{X}^k + \epsilon^l \Gamma^k_{lm} X^m \\
\Rightarrow \dot{X}^k(\phi) &= -\epsilon^l \Gamma^k_{lm}(\phi) X^m(\phi) \\
&= -\epsilon^l \left(\Gamma^k_{lm}(0) + \epsilon^p \partial_p \Gamma^k_{lm}(0)\phi\right)\left(X^m(0) + \epsilon^p \partial_p X^m(0)\phi\right) + \mathcal{O}(\epsilon^3) \\
&= -\epsilon^l \left(\Gamma^k_{lm}(0) + \epsilon^p \partial_p \Gamma^k_{lm}(0)\phi\right)\left(X^m(0) - \epsilon^p \Gamma^m_{pf} X^f(0)\phi\right) + \mathcal{O}(\epsilon^3) \\
&= -\epsilon^l \Gamma^k_{lm} X^m(0) + \phi \epsilon^l \epsilon^p \left(\Gamma^k_{lf}(0)\Gamma^f_{pm}(0) - \partial_p \Gamma^k_{lm}(0)\right) X^m(0) + \mathcal{O}(\epsilon^3)
\end{aligned} \tag{2.137}$$

where we carried out Taylor expansions in ϵ. The result is a very easy ordinary differential equation with initial condition $X^k(\phi = 0) = X^k_P$. The solution at $\phi = 1$ is

$$\begin{aligned}
X^k_R &= X^k(\phi = 1) \\
&= \left(\delta^k_m - \epsilon^l \Gamma^k_{lm} + \frac{1}{2}\epsilon^l \epsilon^p \left(\Gamma^k_{lf}\Gamma^f_{pm} - \partial_p \Gamma^k_{lm}\right)\right) X^m_P + \mathcal{O}(\epsilon^3).
\end{aligned} \tag{2.138}$$

Then we get the expression

$$X^k_{(P\to R\to Q)} = \left(\delta^k_m - \delta^l \Gamma^k_{lm}(R) + \frac{1}{2}\delta^l \delta^p \left(\Gamma^k_{lf}(R)\Gamma^f_{pm}(R) - \partial_p \Gamma^k_{lm}(R)\right)\right) X^m_R. \tag{2.139}$$

Here, all Christoffel symbols are the ones at R since R is the initial point of our curve. They are given by

$$\Gamma^k_{lm}(R) = \Gamma^k_{lm}(P) + \epsilon^p \partial_p \Gamma^k_{lm} + \mathcal{O}(\epsilon^2). \tag{2.140}$$

Then we can write

$$\begin{aligned}
X^k_{(P\to R\to Q)} = \Big(&\delta^k_m - \epsilon^l \Gamma^k_{lm} + \frac{1}{2}\left(\Gamma^k_{lf}\Gamma^f_{pm} - \partial_p \Gamma^k_{lm}\right)\epsilon^l \epsilon^p - \delta^l \Gamma^k_{lm} + \delta^l \epsilon^p \Gamma^k_{lf}\Gamma^f_{pm} \\
&- \delta^l \epsilon^p \partial_p \Gamma^k_{fm} + \frac{1}{2}\left(\Gamma^k_{lf}\Gamma^f_{pm} - \partial_p \Gamma^k_{lm}\right)\delta^l \delta^p\Big) X^m_P + \mathcal{O}(\epsilon^3, \epsilon^2\delta, \epsilon\delta^2, \delta^3).
\end{aligned} \tag{2.141}$$

For $X^k_{(P\to S\to Q)}$, we obtain the same expression with ϵ and δ exchanged. The difference between the parallel transported vectors is

$$\begin{aligned} X^k_{(P\to R\to Q)} - X^k_{(P\to S\to Q)} &= \left(\partial_p\Gamma^k_{lm} - \partial_l\Gamma^k_{pm} + \Gamma^k_{pf}\Gamma^f_{lm} - \Gamma^k_{lf}\Gamma^f_{pm}\right)\delta^p\epsilon^l X^m_P + \mathcal{O}(\epsilon^3,\epsilon^2\delta,\epsilon\delta^2,\delta^3) \\ &=: R^k{}_{mpl}\delta^p\epsilon^l X^m_P + \mathcal{O}(\epsilon^3,\epsilon^2\delta,\epsilon\delta^2,\delta^3) \end{aligned} \tag{2.142}$$

where we find the coefficients $R^k{}_{mpl}$ of the so called Riemann curvature tensor. It describes the (local) path-dependence of parallel transport, so it is a map $T_PM \to T_PM$. Two further "curvatures" are the **Ricci-tensor** which is obtained by contracting two indices of the Riemann curvature tensor:

$$R_{kl} := R^m{}_{kml} \tag{2.143}$$

And the **Ricci-scalar** which can be written as

$$R := g^{mn}R_{mn}. \tag{2.144}$$

A connection such that $R^k{}_{lmn} = 0$ is called **flat**. Parallel transport with respect to a flat connection does not change under a continuous change of the curve, so the resulting vector is independent of the path taken most of the times. Actually, this is only true for paths that can be continuously transformed into one another, e.g. if there is a hole in the manifold that lies between two paths in such a way that one cannot be transformed into the other continuously. In mathematics, these paths are called "not homotopy equivalent".
Also note that $\Gamma^k_{lm} = 0$ implies $R^k{}_{lmn} = 0$, but not the other way round (see e.g. Minkowski space in Rindler coordinates). Since $R^k{}_{lmn}$ is a tensor, it is zero in all coordinate systems if it is zero in one. The Christoffel symbols are not tensors, so they can be zero in some, but not all coordinate systems.
There are different conventions for the Riemann curvature tensor from different authors, related to ours as follows: $R^m{}_{nkl} = R^{(\text{Wald})}{}_{kln}{}^m = R^{(\text{Wikipedia})m}{}_{nkl} = R^{(\text{Fließbach})m}{}_{nlk} = -R^{(\text{Fließbach})m}{}_{nkl}$.

2.3.6 Symmetries of the Riemann curvature tensor

Here, we will talk about the Riemann curvature tensor of a Levi-Civita connection. We define

$$R_{klmn} := g_{ks}R^s{}_{lmn}, \tag{2.145}$$

a $(0,4)$ tensor that contains the same information as the Riemann curvature tensor, just that we lowered the first index. We write down

$$R_{klmn} = g_{ks}(\partial_m\Gamma^s_{nl} - \partial_n\Gamma^s_{ml} + \Gamma^s_{mt}\Gamma^t_{nl} - \Gamma^s_{nt}\Gamma^t_{ml}). \tag{2.146}$$

Using

$$g_{ks}(\partial_m\Gamma^s_{nl}) = \partial_m(g_{ks}\Gamma^s_{nl}) - (\partial_m g_{ks})\Gamma^s_{nl} \tag{2.147}$$

(simply employing the chain rule), inserting the defining relation for the Christoffel symbols

$$g_{ks}\Gamma^s_{\ nl} = \frac{1}{2}(g_{kn,l} + g_{kl,n} - g_{nl,k}) \tag{2.148}$$

and the identity

$$g_{ks,m} = g_{st}\Gamma^t_{\ mk} + g_{kt}\Gamma^t_{\ ms}, \tag{2.149}$$

we first obtain

$$g_{ks}(\partial_m \Gamma^s_{\ nl}) = \frac{1}{2}(g_{kn,lm} + g_{kl,nm} - g_{nl,km}) - g_{st}\Gamma^s_{\ nl}\Gamma^t_{\ mk} - g_{kt}\Gamma^t_{\ ms}\Gamma^s_{\ nl}. \tag{2.150}$$

Putting this together yields

$$R_{klmn} = \frac{1}{2}(g_{kn,ml} - g_{km,nl} + g_{ml,nk} - g_{nl,mk}) + g_{st}(\Gamma^t_{\ nk}\Gamma^s_{\ ml} - \Gamma^t_{\ mk}\Gamma^s_{\ nl}). \tag{2.151}$$

From this, it is now easy to read off the symmetries:

1. $R_{klmn} = -R_{klnm}$ (true for all connections)

2. $R_{klmn} = -R_{lkmn}$ (corresponds to parallel transport of vectors being norm-conserving)

3. $R_{klmn} = R_{mnkl}$ (only true for the Levi-Civita connection)

4. $R_{klmn} + R_{knlm} + R_{kmnl} = 0 \Rightarrow R_{k[lmn]} = 0$ (only true for the Levi-Civita connection, also known as the **first Bianchi identity**).

This list of symmetries of the Riemann curvature tensor is exhaustive. That means that if there is some tensor T_{klmn} satisfying 1 - 4, then there is a metric such that at some point P in some coordinate system, $R_{klmn}(P) = T_{klmn}$. This is no statement about R_{klmn} at other points.

2.3.7 Independent components of R_{klmn}

Let the dimension of the manifold be N and the indices k, l, m, n run from 1 to N. We can write conditions 1 to 3 as

$$R_{klmn} = R_{[kl][mn]} \tag{2.152}$$

where there is also the symmetry of exchanging the two pairs of indices. A pair $[mn]$ has $1, ..., N(N-1)/2 =: M$ independent components, so R can be seen as a symmetric $M \times M$ matrix. For a tensor satisfying conditions 1 to 3, there are $M(M+1)/2 = N(N-1)/2(N(N-1)/2+1)/2 = N(N-1)(N^2-N+2)/8$ independent components. Considering condition 4, because of 1 to 3 $R_{k[lmn]} = R_{[klmn]} = 0$ is only a condition if all four indices k, l, m, n are different. But if they are different, then the equation $R_{[klmn]} = 0$ is independent of conditions 1 to 3. That means the number of independent components for a tensor satisfying conditions 1 to 4 is $N(N-1)(N^2-N+2)/8-\binom{N}{4} = N^2(N^2-1)/12$ where $\binom{N}{4}$ is the number of possibilities to chose four different elements out of a set of N elements and is of course zero for $N < 4$. For $N = 0$, there are no components of R_{klmn} since that is a point which can't be curved. For $N = 1$ there are no components because a straight line might be curved (for example as a circle), but that is so-called extrinsic curvature while the Riemann curvature tensor only measures intrinsic curvature which a line does not have. For $N = 2$, there is one independent component since $R_{1212} = R_{2121} = -R_{1221} = -R_{2112}$ and all other components are zero. Then

$$R_{klmn} \propto g_{km}g_{ln} - g_{kn}g_{lm} \tag{2.153}$$

(both have the same symmetries 1 to 3). Choose a non.vanishing component of R_{klmn}, e.g. $k = 1, l = 2, m = 1, n = 2$, and then we have $R_{1212} \propto \det g$, where $\det g = g_{11}g_{22} - g_{12}g_{21}$. Hence

$$R_{klmn} = \frac{1}{\det g}(g_{km}g_{ln} - g_{kn}g_{lm})R_{1212}. \tag{2.154}$$

In two dimension, we can construct the Ricci tensor as

$$R_{kl} = g^{mn}R_{mknl} = g_{kl}R_{1212}/\det g \tag{2.155}$$

and the Ricci scalar as

$$R = g^{mn}R_{mn} = 2R_{1212}/\det g =: 2k \tag{2.156}$$

where k is the **Gauß curvature**. For $N = 3$, there are already six independent components, for $N = 4$ there are 20 and so on.

3 General Relativity

After 1905, relativistic physics had gained lots of credibility. It explained the negative results of the Michelson-Morley experiment and fit together perfectly with electromagnetism where Newtonian mechanics did not. On a mathematical level, the Cartesian coordinate systems, as well as the Galilei transformation which transformed between them, had to be replaced by inertial systems in SR, which were related by Lorentz transformations. This worked perfectly well, and the old, well-known classical mechanics emerged in the limit of $c \to \infty$.

While special relativity was perfectly able to explain all electromagnetic phenomena, the other force, gravity, was not described by it.[1] The initial attempts to also formulate the gravitational force in the language of special relativity, were not successful. One of the reasons for this was the fact that, while the source for the electric field (the electric charge e) was a Lorentz scalar, and therefore the same in every inertial coordinate system, the source for the gravitational field, the inertial/gravitational mass m_I, seemed to depend on the coordinate system, as $m_I = m_0/\sqrt{1 - v^2/c^2}$.

Albert Einstein overcame this problem by postulating two fundamental principles, on which he rested the development of a relativistic theory of gravity:

1. *Equivalence principle*: All gravitational forces are fictitious. An observer does not feel the difference between falling freely in a gravitational field and there being no gravity.

2. *General covariance*: There are no preferred coordinates. Physical laws should be form-invariant in every coordinate system. This means that they need to be formulated as laws between tensor fields on manifolds.

Geometrically, special relativity can also be formulated in the following way: space-time is a 4-dimensional manifold $M \simeq \mathbb{R}^4$, with a Lorentzian metric η, which in an inertial system is given by $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Freely falling observes move along geodesics of the Levi-Civita connection of that metric. The proper way to generalize this, Einstein realized, was to simply allow for general Lorentzian metrics g, and treating the case $g = \eta$ as the special case in which the gravitational field is very weak.

In the following we will consider the implications of this: space-time will be given by a 4-dimensional manifold with a Lorentzian metric g. Freely falling observers move along timelike geodesics of the Levi-Civity connection of g. Note that we assume that, in order

[1] At the time, only two forces were known to physicists. Strong and weak interaction would be discovered much later, with the advent of quantum field theory. Note that, while some quantum effects in and of itself were known experimentally at this point, it would only be 1925, 20 years after special relativity, that Heisenberg and Schrödinger would formulate the fundamental laws of quantum mechanics as we know them today.

to move along geodesics, observers need to be point-like and do not weigh anything: they move in the gravitational field, but do not influence it. Later we will consider how a distribution of mass results in a specific metric.

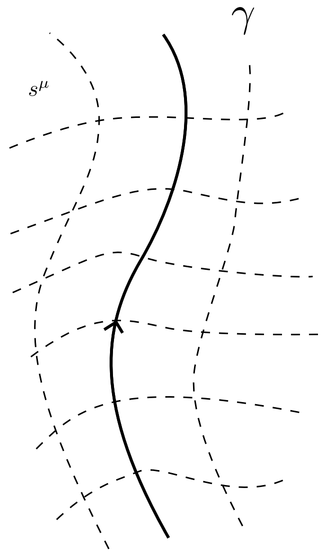

Figure 3.1: For a geodesic γ, the FNC coordinates are such that the geodesic itself, as well as the space-like coordinate lines are geodesics.

3.1 Fermi normal coordinates

We have seen that in Minkowski space, an inertial observer moves along a geodesic

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho = 0. \tag{3.1}$$

This is true in all coordinate systems. In inertial systems, these simplify to $\ddot{x}^\mu = 0$, because the Christoffel symbols vanish. For an inertial observer, there is one specific inertial system in which the observer themself is at rest, and their world line (parametrized in proper time) is given by $s \mapsto (s, 0, 0, 0)$. In fact, such a coordinate system exists for any geodesic

Now consider a general time-like geodesic γ in some spacetime with metric g. We now construct a local coordinate system, adapted to γ (only works in a neighbourhood around γ). Let γ be parameterized by proper time s, starting at P. At P, chose three space-like vectors $E_1, E_2, E_3 \in T_P M$, which are all orthonormal:

$$g(e_i, e_j) = -\delta_{ij} \tag{3.2}$$

where the minus sign appears because the vectors are space-like. They shall also be orthogonal to the velocity vector of γ at P:

$$E_0 := \left.\frac{d\gamma}{ds}\right|_{s=0} \in T_P M \tag{3.3}$$

with $g(E_0, E_i) = 0$ and $g(e_0, e_0) = 1$. Then parallel transport the vectors E_μ along the geodesic. They become a so-called **frame** $E_\mu(s)$ on γ. Since γ is a geodesic, $E_0(s)$ will always be equal to the velocity vector of γ at that point, and since the Levi-Civita connection is metric compatible, the $E_\mu(s)$ will always be orthonormal, i.e. $\langle E_\mu(s), E_\nu(s)\rangle = \eta_{\mu\nu}$ for all s. If, for a fixed μ, the four numbers E_μ^ρ denote the components of E_μ in some coordinates x^μ, and the metric components w.r.t. these coordinates are $g_{\mu\nu}$, then one has

$$g_{\rho\sigma} E_\mu^\rho(s) E_\nu^\sigma(s) = \eta_{\mu\nu}. \tag{3.4}$$

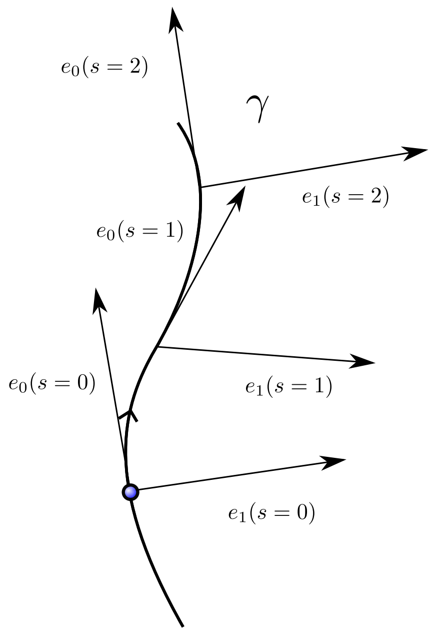

Figure 3.2: The basis vectors $E_\mu(s = 0)$ are parallelly transported along the whole geodesic $E_\mu(s)$. Because of the geodesic property, nad γ is parameterized by proper time, $E_0(s)$ is always the velocity vector of the curve.

On a point with proper time $s = s^0$, Q_{s^0}, define, for three numbers s^1, s^2, s^3, the geodesic $\gamma_{s^0}(\tau)$ which has initial position Q_{s^0} and initial velocity vector $\frac{d\gamma_{s^0}}{d\lambda}\Big|_{\lambda=0} = s^1 E_1 + s^2 E_2 + s^3 E_3$. Follow that geodesic until $\lambda = 1$. The point of the manifold that we ended up at then receives the coordinates s^0, s^1, s^2, s^3.

Because geodesics might overlap at some point, or not exist for a time $\lambda = 1$, it can happen that not every s^μ gets mapped to a point in the manifold, or that points in the manifold get assigned several different sets of coordinates s^μ. These points all have to be removed from the coordinate chart. But, one can show that at least in a small neighbourhood around the geodesic γ, the s^μ define a proper coordinate system. The s^μ are called **Fermi normal coordinates**.

In the FNC, the metric takes a very easy form on the geodesic: Let x^μ be some other,

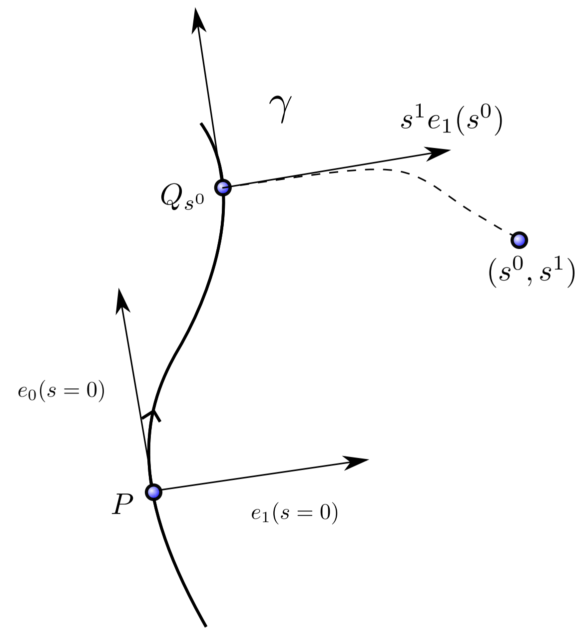

Figure 3.3: The event with Fermi normal coordinates s^μ is the one which can be reached from Q_{s^0} by a space-like geodesic with initial velocity vector $s^iE_i(s^0)$.

arbitrary coordinates. Then a geodesic starting at Q_{s^0} looks like this:

$$x^\mu(\lambda) = x^\mu(s^0,0,0,0) + \lambda s^iE_i - \frac{1}{2}\lambda^2\Gamma^\mu_{\nu\rho}E^\nu_iE^\rho_js^is^j + ... \tag{3.5}$$

where $x^\mu(s^0,0,0,0)$ is Q_{s^0} in x^μ coordinates. The vectors have components

$$E^\mu_i\frac{\partial}{\partial x^\mu} = E_i. \tag{3.6}$$

At $\lambda = 1$, one has $x^\mu(s^0,s^1,s^2,s^3) = x^\mu(s^0,0,0,0) + E^\mu_is^i + \mathcal{O}((s^i)^2)$. We can write

$$\left.\frac{\partial x^\mu}{\partial s^0}\right|_{s^i=0} = E^\mu_0(s^0) + \left.\frac{\partial E^\mu_i(s^0)}{\partial s^0}s^i\right|_{s^i=0} + ... = E^\mu_0 \tag{3.7}$$

and

$$\frac{\partial x^\mu}{\partial s^i} = 0 + E^\mu_i + \mathcal{O}(s^i)|_{s^i=0} = E^\mu_i. \tag{3.8}$$

So in the s^μ coordinate system, the metric takes coefficients $\tilde{g}_{\mu\nu}$, and on the geodesic (s^μ with $s^i = 0$)

$$\tilde{g}_{\mu\nu}|_{s^i=0} = \left.\frac{\partial x^\rho}{\partial x^\mu}\right|_{s^i=0}\left.\frac{\partial x^\sigma}{\partial x^\nu}\right|_{s^i=0}g_{\rho\sigma}|_{s^i=0} \tag{3.9}$$

with $g_{\rho\sigma}$ the metric in x^μ coordinates. Then on the geodesic, we have

$$\tilde{g}_{\mu\nu}(s^0,0,0,0) = g_{\rho\sigma}e^\rho_\mu E^\sigma_\nu = g(E_\mu,E_\nu) = \eta_{\mu\nu}. \tag{3.10}$$

In these coordinates, the metric on the geodesic takes the form $\eta_{\mu\nu}$.

Next consider the Christoffel symbols: The coordinate lines for spatial coordinates are geodesics, i.e. $y^0(\lambda) := s^0 = const$, $y^i(\lambda) := s^i(\lambda) = \lambda Y^i$ for some Y^i are geodesics. We can then write

$$0 = \ddot{y}^\mu + \Gamma^\mu_{\nu\rho}\dot{y}^\nu\dot{y}^\rho = 0 + \Gamma^\mu_{ij}(\lambda)Y^iY^j. \tag{3.11}$$

For $\lambda = 0$ we get

$$\Gamma^\mu_{ij}(s^0, 0, 0, 0)Y^iY^j = 0 \tag{3.12}$$

for all Y^i which means that

$$\Gamma^\mu_{ij}(s^0, 0, 0, 0) = 0. \tag{3.13}$$

Furthermore: Remember that the E_μ are parallel transported along the geodesic: On the curve $s^\mu(\lambda) = (\lambda, 0, 0, 0)^t$, the vector $X(\lambda) = X^\mu E_\mu(\lambda)$ with some fixed X^μ is parallel transported for all X^μ. That means it satisfies the equation of parallel transport

$$\dot{X}^\mu + \Gamma^\mu_{0\rho}(\lambda)X^\rho = 0 \tag{3.14}$$

where the index 0 appears because we move along the 0-direction. This implies that

$$\Gamma^\mu_{0\rho}(s^0, 0, 0, 0) = 0 \tag{3.15}$$

for all μ, ρ. Then $\Gamma^\mu_{\nu\rho} = 0$ on the geodesic (in FNC). Since $g_{\mu\nu,\rho} = g_{\mu\lambda}\Gamma^\lambda_{\rho\nu} + g_{\nu\lambda}\Gamma^\lambda_{\mu\rho}$, all first derivatives of $\tilde{g}_{\mu\nu}$ vanish on the geodesic. Directly on γ, $g_{\mu\nu} = \eta_{\mu\nu}$, $g_{\mu\nu,\rho} = 0$ in our coordinate system. The FNC are a reflection of the equivalence principle: An observer moving on a geodesic views the world locally as if he were in Minkowski space.

Note that this is true only in in the immediate surroundings of the world line of the observer γ. This can be rather limited if the second derivatives of $g_{\mu\nu}$ are large (they usually do not vanish, even in Fermi normal coordinates).

In second order, we get

$$g_{\mu\nu}(s^0, s^1, s^2, s^3) = \eta_{\mu\nu} + g_{\mu\nu,ij}s^is^j + \mathcal{O}((s^i)^3) = \eta_{\mu\nu} + ZR_{\mu i\nu j}s^is^j + ... \tag{3.16}$$

with

$$Z = \begin{cases} 1 & \mu = \nu = 0 \\ \frac{2}{3} & \mu = i, \nu = 0 \text{ or } \mu = 0, \nu = i \\ \frac{1}{3} & \mu, \nu \neq 0. \end{cases} \tag{3.17}$$

For each time-like geodesic γ, one can construct coordinates s^μ in a neighborhood of γ, so that in these coordinates the metric and Christoffel symbols on γ look as if one were in Minkowski space.

For an observer moving along γ, the Fermi normal coordinates are the most "natural" ones:

1. s^0 corresponds to the proper time along γ.

2. $-((s^1)^2 + (s^2)^2 + (s^3)^2)^{1/2}$ corresponds to the proper distance to the observer.

3. $E_\mu = \partial/\partial s^\mu|_\gamma$ are the coordinate axes and stay "constant" in time (they are parallel transported along γ).

Since in these coordinates $g_{\mu\nu} \approx \eta_{\mu\nu}$ and $\Gamma^\mu_{\nu\rho} \approx 0$, an observer moving along a time-like geodesic is the ideal generalization of an "inertial observer" to the case of curved metrics. A time-like geodesic is equivalent to a world line of a freely falling observer (who, by himself, is so light that he will not influence the gravitational field).

3.1.1 Curvature and tidal forces

A point-like observer (a *test mass*) follows a geodesic and does not feel the gravitational field. But two observers close to each other will realize that they are accelerated relative to one another. This is the *gravitational tidal force*: gravity acts differently on different regions of an extended body.

Consider a geodesic $\lambda \mapsto x_1^\mu(s)$ going through point P_1 at $\lambda = 0$, and another geodesic $\lambda \mapsto x_2^\mu(\lambda)$, passing through point P_2 at $\lambda = 0$ nearby (both parametrized by proper time λ). The connecting vector having components $\Delta x^\mu = x_2^\mu - x_1^\mu$. We assume that the two points are close to one another, i.e. the components of Δx^μ are assumed to be small.

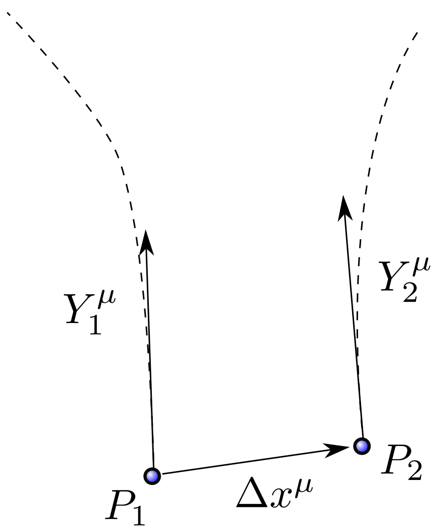

Figure 3.4: To compute tidal forces of the gravitational field, we consider two geodesics which are parallel at P_1 and P_2, only separated by a small distance vector Δx^μ, and compute the relative acceleration of the two geodesics.

We assume that, at P_1 and P_2 they are parallel, i.e. they have the "same" velocity vector. In other words, the velocity vectors

$$Y_1^\mu := \frac{dx_1^\mu}{d\lambda}_{|\lambda=0}, \qquad Y_2^\mu := \frac{dx_2^\mu}{d\lambda}_{|\lambda=0} \tag{3.18}$$

are the result of being parallel transported along $\Delta x^\mu(0) = \epsilon X^\mu$ to one another. The curve from P_1 to P_2 can be written as

$$y^\mu(\tau) = x_1^\mu(\lambda = 0) + \tau\,\Delta x^\mu. \tag{3.19}$$

This means that, up to quadratic order in the components Δx^μ, one can write

$$Y_2^\mu = Y_1^\mu - \Gamma^\mu_{\nu\rho}\Delta x^\nu Y_1^\rho + O((\Delta x)^2), \tag{3.20}$$

where the Christoffel symbol is evaluated at P_1. In the following, we will work in Fermi normal coordinates of geodesic 1, to make life easier for us. We note that, in these coordinates, because the Christoffel symbols vanish on the geodesic, and we get that $Y_2^\mu = Y_1^\mu + O((\Delta x)^2)$. So the velocity vectors being parallel transports of one another is equivalent to being at rest relative to one another (up to $(\Delta x)^2$-terms, in Fermi normal coordinates).

All of this only holds for the moment $\lambda = 0$, when both particles are at rest w.r.t one another. From then on, the two particles will drift apart, and this is captured in the difference of second derivatives of the respective geodesics. We define the relative velocity vector at $\lambda = 0$ to be:

$$\frac{dY_2^\mu}{d\lambda}_{|\lambda=0} - \frac{dY_1^\mu}{d\lambda}_{|\lambda=0} = \ddot{x}_2^\mu(\lambda = 0) - \ddot{x}_1^\mu(\lambda = 0). \tag{3.21}$$

In FNC, these equations become rather simple. Firstly, $\ddot{x}_1^\mu = 0$, since geodesic 1 is simply $\lambda \mapsto (\lambda, 0, 0, 0)$. For geodesic 2, we get that, because of the geodesic equation, at $\lambda = 0$ one has

$$\ddot{x}_2^\mu = -\Gamma^\mu_{\nu\rho}(P_2)Y_2^\nu Y_2^\rho = -\Gamma^\mu_{00}(P_2) = \Gamma^\mu_{00}(P_1) - \partial_\tau\Gamma^\mu_{00}(P_1)\Delta x^\tau + O((\Delta x)^2). \tag{3.22}$$

Now remember that in

$$R^\mu{}_{00\tau} = \partial_0\Gamma^\mu_{\tau 0} - \partial_\tau\Gamma^\mu_{00} + \Gamma^\mu_{0\nu}\Gamma^\nu_{\tau 0} - \Gamma^\mu_{\tau\nu}\Gamma^\nu_{00} \tag{3.23}$$

the Christoffel symbols vanish on the geodesic, so the only term remaining is $R^\mu{}_{00\tau} = -\partial_\tau\Gamma^\mu_{00}$. Then

$$\ddot{x}_2^\mu(\lambda = 0) = R^\mu{}_{00\tau}\Delta x^\tau + \mathcal{O}((\Delta s^\mu)^2). \tag{3.24}$$

Let us write $\Delta x^\mu = \epsilon X^\mu$. To linear order in ϵ, the relative acceleration vector at $\lambda = 0$ is given by

$$a^\mu := \lim_{\epsilon\to 0}\frac{\ddot{x}_2^\mu(\lambda = 0) - \ddot{x}_1^\mu(\lambda = 0)}{\epsilon} \tag{3.25}$$

In FNC, we have seen that this vector is given by

$$a^\mu = R^\mu{}_{00\tau}X^\tau = R^\mu{}_{\nu\sigma\tau}Y_1^\nu Y_1^\sigma X^\tau. \tag{3.26}$$

But this is a tensor equation! That means that if it is true in Fermi normal coordinates, it is true in every coordinate system. So for any geodesic γ_1 going through a point P_1

with velocity vector Y_1, a parallel geodesic γ_2 passing through a point P_2 which lies in direction ϵX, is turning towards a slightly different direction than γ_1. The relative acceleration, to lowest order in ϵ, is

$$\ddot{x}_2^\mu - \ddot{x}_1^\mu = \epsilon R(Y, X)Y + \mathcal{O}(\epsilon^2) \tag{3.27}$$

$$\ddot{x}_2^\mu(P_2) - \ddot{x}_1^\mu(P_1) = \epsilon a^\mu + \mathcal{O}(\epsilon^2). \tag{3.28}$$

Note that neither observer 1 nor 2 feels an acceleration: They both move along a geodesic. But they'll realize that they'll drift apart (or together).

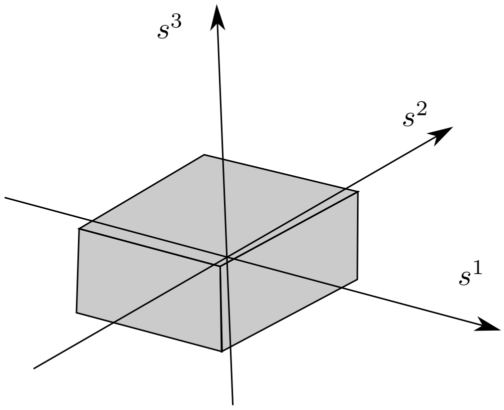

Figure 3.5: We consider many freely moving observers (e.g. dust particles), in a small box-like region.

Next, we will consider many idealized observers close to one another, e.g. light dust particles at rest in a cube. Each of them moves along its own geodesic. Again, we use Fermi normal coordinates. The cube gets compressed/expanded along s^0 due to tidal forces. How does the (3d-)volume of this box change over time (s^0), depending on the curvature tensor? We write

$$x^i(s^0) = x^i(0) + \frac{1}{2}(s^0)^2 R^i{}_{00j} x^j(0) + \mathcal{O}((x^i)^2, (s^0)^3) = M^i{}_j(s^0) x^j(0) \tag{3.29}$$

where we define the linear map

$$M^i{}_j(s^0) = \delta^i_j + \frac{1}{2} R^i{}_{00j}(s^0)^2 + ... \tag{3.30}$$

We denote the volume of a cube of length 1 by "Vol" and calculate it as

$$\mathrm{Vol}(s^0) = \det M^i{}_j(s^0) = \det\left(\delta^i_j + \frac{(s^0)^2}{2} R^i{}_{00j} + ...\right). \tag{3.31}$$

Here, the determinants are determinants of 3×3 matrices. We imposed the condition

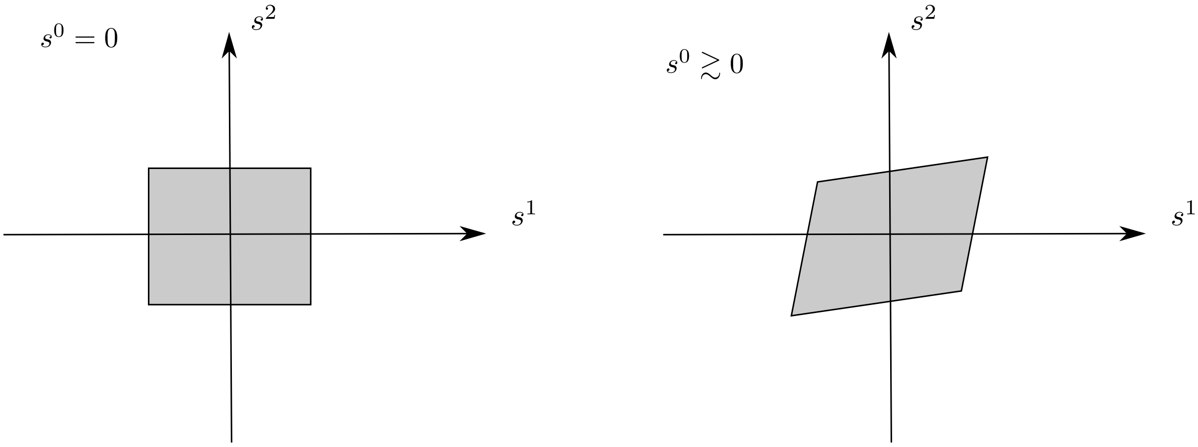

Figure 3.6: We consider the deformation of the box containing the dust, since all particles move on geodesics. For small time steps, this will be a linear deformation of the box, i.e. a combination of shear and compression/expansion. Note: the particles only move under the influence of an external gravitational field, but do not attract each other here, since they are so light.

$\mathrm{Vol}(0) = 1$. The derivatives are

$$\frac{d}{ds^0}\mathrm{Vol}(0) = 0 \tag{3.32}$$

$$\begin{aligned}\frac{d^2}{ds^{0\,2}}\mathrm{Vol}(0)&\\ &= \mathrm{Tr}(-R^i{}_{0j0})\\ &= -\sum_{i=1}^{3} R^i{}_{0i0}\\ &= -R^\mu{}_{0\mu0}\\ &= -R_{00}\\ &= -R_{\mu\nu}Y^\mu Y^\nu.\end{aligned} \tag{3.33}$$

where we used that $R^0{}_{000} = 0$. We get the 00-component of the Ricci tensor. The Y^μ are the initial velocities of the dust particles. Apparently, the Ricci tensor contains information about the rate of contraction/expansion of a congruence of geodesics, with initial velocity vector Y. The development of a volume is given by

$$\mathrm{Vol}(\lambda) = \mathrm{Vol}(0) - \frac{1}{2}\lambda^2 R_{\mu\nu}Y^\mu Y^\nu + \mathcal{O}(\lambda^3). \tag{3.34}$$

If the *Raychaudhuri scalar* $R_{\mu\nu}Y^\mu Y^\nu$ is bigger than zero, parallel geodesics get contracted.

3.2 Dynamics of general realtivity

3.2.1 The energy-momentum tensor

So far we have considered observers as test masses, i.e. as point-like particles of zero mass, which are influenced by the gravitational field, but do themselves not generate one. In fact, it is quite complicated in general relativity to consider the gravitational field generated by a point-like source. What one can do, however, is consider the gravitational field generated by a continuous matter distribution.

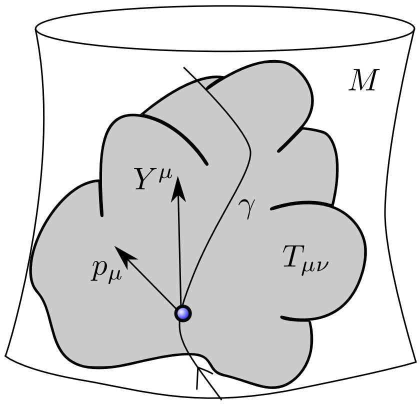

Figure 3.7: The energy-momentum tensor $T_{\mu\nu}$ of a matter field in space-time describes the relation between an observer with velocity vector Y^μ, and the measured energy / momentum flow density p_μ.

In Newtonian gravity, mass density ρ is the source of the gravitational potential Φ:

$$\Delta\Phi = 4\pi G_N \rho \tag{3.35}$$

The gravitational force is then given by

$$\vec{F} = -m\vec{\nabla}\Phi. \tag{3.36}$$

Note the similarity to electromagnetism:

$$4\pi j^\mu = \Box A^\mu \tag{3.37}$$

and

$$F = dA. \tag{3.38}$$

We would expect something similar to happen in general relativity. Newtonian gravity should be the non-relativistic limit of general relativity and the mass density ρ should lead to curvature of (the Levi-Civita-connection of) $g_{\mu\nu}$. But from special relativity, we know that mass corresponds to energy, so indeed any form of energy stored in matter (momentum, pressure, shear stress...) should lead to a curved metric. All of these are combined in the **energy-momentum tensor** $T_{\mu\nu}$. This is a tensor field, which contains information about the various forms of energy contained in a continuous distribution of matter. That can be a fluid, but also the electromagnetic field itself (which also counts as matter in this context). Quantum fields also have an energy-momentum tensor, which plays a prominent role in quantum field theory.

The content of this tensor field is as follows: An observer with the four-velocity Y^μ measures the four-momentum vector p^μ of the fluid

$$p_\mu = T_{\mu\nu} Y^\nu \tag{3.39}$$

One also often uses the version of the tensor where two indices have been raised, i.e. $T^{\mu\nu} := g^{\mu\rho} g^{\nu\sigma} T_{\rho\sigma}$, and also uses the name energy-momentum tensor for that. The meaning of the individual components is:

$$T^{\mu\nu} = \begin{pmatrix} T^{00} & & T^{0i} & \\ & T^{11} & T^{12} & T^{13} \\ T^{i0} & T^{21} & T^{22} & T^{23} \\ & T^{31} & T^{32} & T^{33} \end{pmatrix} = T^{\nu\mu} \tag{3.40}$$

where

$$T^{00} = \rho c^2 \tag{3.41}$$

is the energy density of the matter, and

$$T^{0i} = T^{i0} = c v^i \rho \tag{3.42}$$

the momentum density. The T^{ij}, for $i, j = 1, 2, 3$, are the components of the stress tensor of the fluid. This menas that the diagonal terms have an interpretation as the pressure (hydrostatic pressure in case of a fluid, or radiation pressure in the case of the electromagnetic field). The off-diagonal terms T^{ij} with $i \neq j$ contain information about shear deformation of the matter.
Examples:

- Perfect fluid in thermomdynamic equilibrium: Such a field can be described by the velocity vector field u^μ, which, at each point in space-time, describes the velocity vector of a particle co-moving with the fluid (with $g_{\mu\nu} u^\mu u^\nu = 1$). An observer at such a point, which is at rest with respect to the fluid, measures no relative flow, but only the energy density. The tensor has the form

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - g^{\mu\nu} p \tag{3.43}$$

  with energy density ρ and pressure p. In the comoving observer"s frame (i.e. in Fermi normal coordinates) the observer would measure

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{3.44}$$

- Electromagnetic field: In Minkowski space, where the field content can e.g. be described by the fields $\vec{E}$ and $\vec{B}$, the energy-momentum tensor is

$$T^{\mu\nu} = \begin{pmatrix} \frac{1}{2}(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}) & c\varepsilon_0(\vec{E}\times\vec{B})^T \\ c\varepsilon_0\vec{E}\times\vec{B} & \frac{1}{2}(\varepsilon_0 E^2 + \frac{B^2}{\mu_0})\delta_{ik} - \varepsilon_0 E_i E_k - \frac{1}{\mu_0}B_i B_k \end{pmatrix}. \tag{3.45}$$

  In a general space-time, the electromagnetic field is given in terms of the field strength tensor $F_{\mu\nu}$, and the energy-momentum tensor can be written as

$$T^{\mu\nu} = F^{\mu\gamma}F_{\gamma}{}^{\nu} + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{3.46}$$

  with $F^{\mu\nu} := g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}$.

- Klein-Gordon field: For a KG field $\phi(x)$ in Minkowski space, the energy-momentum tensor is given by

$$T_{\mu\nu} = (\nabla_\mu\phi)(\nabla_\nu\phi) - \frac{1}{2}g_{\mu\nu}\big(g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + m^2\phi^2\big). \tag{3.47}$$

In Fermi normal coordinates, where the velocity vector of an observer is Y^μ, the continuity equation for a fluid relates the change of energy density to the gradient of the momentum flow, i.e.

$$\partial_i T^{0i} = -\partial_0 T^{00} \tag{3.48}$$

where on the left hand side is the change of momentum and on the right hand side the change of energy density. Again, we compare this to electromagnetism:

$$\partial_0 j^0 = -\partial_i j^i. \tag{3.49}$$

Because of $\partial_i T^{0i} + \partial_0 T^{00} = \partial_\mu T^{\mu\nu}Y_\nu = \nabla_\mu T^{\mu\nu}Y_\nu$, this is equivalent to

$$\nabla_\mu T^{\mu\nu}Y_\nu = 0 \tag{3.50}$$

for all Y. So each observer measures the divergence

$$\nabla_\mu T^{\mu\nu} = 0. \tag{3.51}$$

This **continuity equation** plays an important role in terms of energy conservation in matter, and is satisfied by all sensible physical matter fields.

3.2.2 Connection between $T^{\mu\nu}$ and $g_{\mu\nu}$

The energy-momentum tensor should serve as a source for the curvature of $g_{\mu\nu}$. In analogy to electromagnetism, we would want this to be some second.order derivative operator acting on $g_{\mu\nu}$ being equal to $T^{\mu\nu}$. There are in principle several ways to achieve this, and one can not derive the correct way from first principles. But there are some good ad-hoc assumptions which lead to a useful set of equations.

Firstly, we note that, whichever equation we get, should contain Newtonian gravity in the non-relativistic limit. Now, it is surprisingly hard to actually define the term "non-relativistic" the proper way. For our purposes, it is enough to consider the limit of our formulas for small velocities $v^i \ll c$, and slowly-varying gravitational fields, i.e. $g_{\mu\nu,0} = 0$. We also assume that the deviations of the metric from $g_{\mu\nu}$, as well as spatial derivatives $g_{\mu\nu,i} \ll 1$ of the metric are small.

The proper velocity of an observer is

$$v^\mu = \frac{dx^\mu}{dx^0} \approx \begin{pmatrix} 1 \\ v^i/c. \end{pmatrix} \tag{3.52}$$

Obviously, $v^i/c \ll 1$ and $x^0 = ct$ with t being the time measured by the observer. We can expand this as

$$\frac{dx^\mu}{dx^0} = \frac{1}{c}\dot{x}^\mu = \begin{pmatrix} c \\ v^i \end{pmatrix}\left(1 + \mathcal{O}\left(\frac{v^i}{c}\right)\right) \tag{3.53}$$

and the second derivative as

$$\ddot{x}^\mu = \frac{1}{c^2}\frac{d^2x^\mu}{dt^2} \tag{3.54}$$

with

$$\begin{aligned}
\frac{d^2x^\mu}{dt^2} &= a^i \\
&= -\Gamma^i_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} \\
&= -\Gamma^i_{00}c^2\left(1 + \mathcal{O}\left(\frac{v^i}{c}\right)\right) \\
&= -\frac{c^2}{2}g^{i\lambda}(g_{\lambda 0,0} + g_{0\lambda,0} - g_{00,\lambda})\left(1 + \mathcal{O}\left(\frac{v^i}{c}\right)\right) \\
&\approx \frac{c^2}{2}g^{i\lambda}g_{00,\lambda} \\
&= \frac{c^2}{2}g^{ij}g_{00,j} \\
&= -\frac{c^2}{2}g_{00,i}.
\end{aligned} \tag{3.55}$$

Since g is not time-dependent, the derivative in x^0-direction $g_{,0}$ vanishes. Compare this to Newtonian physics:

$$\vec{a} = -\vec{\nabla}\Phi \tag{3.56}$$

$$\Rightarrow a^i = -\Phi_{,i} \tag{3.57}$$

$$\Rightarrow \partial_i\Phi = \frac{c^2}{2}g_{00,i}. \tag{3.58}$$

If $\Phi = 0$, then $g_{00} = 1$, so for slowly varying, weak gravitational potentials Φ, the geodesics in the metric

$$g_{\mu\nu} = \begin{pmatrix} 1+\frac{2\Phi}{c^2} & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \tag{3.59}$$

nearly behave as if they were to move in a gravitational potential $\Phi(x^1, x^2, x^3)$. In our limit, $2\partial_i\Phi/c^2 \ll 1$. The non-zero Christoffel symbols are

$$\Gamma^i_{00} = \frac{\partial_i\Phi}{c^2}; \quad \Gamma^0_{i0} = \Gamma^0_{0i} = \frac{\partial_i\Phi}{c^2}. \tag{3.60}$$

The Riemann tensor is

$$\begin{aligned} R^i{}_{0j0} &= \partial_j\Gamma^i_{00} - \underbrace{\partial_0\Gamma^i_{j\lambda}}_{=0} + \Gamma^i_{j\lambda}\Gamma^\lambda_{00} - \underbrace{\Gamma^i_{0\lambda}\Gamma^\lambda_{j0}}_{\propto(\partial_i\Phi)^2/c^4} \\ &= \frac{\partial_i\partial_j\Phi}{c^2}(1+\mathcal{O}(\Phi^2, (\partial_i\Phi)^2)). \end{aligned} \tag{3.61}$$

Similarly

$$R^0{}_{i0j} = -\frac{1}{c^2}\partial_i\partial_j\Phi(1+\mathcal{O}(\Phi^2, (\partial_i\Phi)^2)). \tag{3.62}$$

Then

$$R^0{}_{000} = 0, \tag{3.63}$$

so the 00 component of the Ricci-tensor is

$$R_{00} \approx \frac{1}{c^2}\Delta\Phi \tag{3.64}$$

and the space-like components

$$R_{ij} \approx -\frac{1}{c^2}\partial_i\partial_j\Phi. \tag{3.65}$$

The Ricci-scalar is

$$R = g^{00}R_{00} - R_{11} - R_{22} - R_{33} = \frac{2}{c^2}\Delta\Phi(1+\mathcal{O}(\Phi^2, (\partial_i\Phi)^2)). \tag{3.66}$$

How does one emulate $\Delta\Phi = 4\pi G_N\rho$? Consider a non-relativistic fluid with $T^{\mu\nu}$ approximately as in equation 3.44. We define

$$T := g_{\mu\nu}T^{\mu\nu} \approx \rho c^2 - 3p \approx \rho c^2 \tag{3.67}$$

as the trace. The equation

$$R = \frac{8\pi G_N}{c^4}T \tag{3.68}$$

then becomes, in the non-relativistic limit, equal to $\Delta\Phi = 4\pi G_N\rho$, which is good. But this equation cannot be the whole story, since not just the trace of the energy-momentum tensor T, but all components $T^{\mu\nu}$ should serve as a source for the curvature. That means we should have an equation

$$G^{\mu\nu} = \frac{8\pi G_N}{c^4} T^{\mu\nu} \tag{3.69}$$

for some tensor $G^{\mu\nu}$ with $G^{\mu\nu} g_{\mu\nu} = R$. So $G^{\mu\nu} = R^{\mu\nu} = g^{\mu\mu'} g^{\nu\nu'} R_{\mu'\nu'}$ would do the job, but so would $G^{\mu\nu} = \alpha R^{\mu\nu} + \beta g^{\mu\nu} R$ with $\alpha + 4\beta = 1$. Which one should we take?
We use the continuity equation: demand

$$\nabla_\mu G^{\mu\nu} = 0 \tag{3.70}$$

as a condition on α and β. To see what this implies for α and β, consider the identity

$$\nabla_\mu R^\lambda{}_{\nu\sigma\tau} + \nabla_\sigma R^\lambda{}_{\nu\tau\mu} + \nabla_\tau R^\lambda{}_{\nu\mu\sigma} = 0. \tag{3.71}$$

This is the *second Bianchi identity*[2]. Multiplying with δ^σ_λ yields the *contracted Bianchi identities*

$$\nabla_\mu R_{\nu\tau} + \nabla_\lambda R^\lambda{}_{\nu\tau\mu} - \nabla_\tau R_{\nu\mu} = 0. \tag{3.72}$$

Multiplying further with $g^{\nu\tau} g^{\mu\sigma}$ gives

$$g^{\mu\sigma} \nabla_\mu R - \nabla_\lambda R^{\sigma\lambda} - \nabla_\tau R^{\tau\sigma} = 0. \tag{3.73}$$

Using symmetry and renaming dummy indices, we obtain

$$g^{\mu\sigma} \nabla_\mu R = 2 \nabla_\mu R^{\mu\sigma} \tag{3.74}$$

which is true for all Levi-Civita connections. Now we use this to compute

$$\begin{aligned} 0 &\overset{!}{=} \nabla_\mu G^{\mu\nu} \\ &= \nabla_\mu (\alpha R^{\mu\nu} + \beta g^{\mu\nu} R) \\ &= \alpha \nabla_\mu R^{\mu\nu} + \underbrace{\beta (\nabla_\mu g^{\mu\nu}) R}_{=0} + \beta g^{\mu\nu} \nabla_\mu R \\ &\overset{3.74}{=} (\alpha + 2\beta) \nabla_\mu R^{\mu\nu}. \end{aligned} \tag{3.75}$$

In general $\nabla_\mu R^{\mu\nu} \neq 0$, so that means $\alpha + 2\beta \overset{!}{=} 0$. Together with $\alpha + 4\beta = 1$, we obtain

$$\alpha = -1; \quad \beta = \frac{1}{2}. \tag{3.76}$$

So we conclude that

$$G^{\mu\nu} = -R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R. \tag{3.77}$$

Usually, this is written as

[2]This equation is true for every connection, even ones different from the Levi-Civita connection. Is is presented here without proof.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G_N}{c^4} T_{\mu\nu}. \tag{3.78}$$

These are the famous **Einstein equations** where the left hand side equals (up to a sign) to $G_{\mu\nu}$, the **Einstein tensor**. We define the constant factor on the right hand side (Newton's constant up to a factor) as $\kappa := 8\pi G_N/c^4$.

Notice the similarity to electromagnetism in the equation

$$\Box A^\mu = 4\pi j^\mu \tag{3.79}$$

where on the left is a second order partial differential operator and on the right hand side a source term. In equation 3.78, $T_{\mu\nu}$ is the "source" and $g_{\mu\nu}$ the "field".

But unlike in electromagnetism, one cannot first specify $T^{\mu\nu}$ and then compute $g_{\mu\nu}$. The physical interpretation of $T^{\mu\nu}$ requires a metric (e.g.: for a perfect fluid, $T^{\mu\nu} = (\rho + p/c^2) u^\mu u^\nu - P g^{\mu\nu}$). One has to solve the equations for the metric $g_{\mu\nu}$ and the matter fields at the same time. Note that the Einstein equations cannot be rigorously derived. We found them by observing that

1. they automatically lead to $\nabla_\mu T^{\mu\nu} = 0$ (continuity equation for another field).
2. they lead to $\Delta\Phi = 4\pi G_N \rho$ in some limit.

These are some other promising properties! In fact, there is a theorem by Lovelock (1972) that states that $G_{\mu\nu}$ is the only $(0,2)$ tensor which

a) is formed from $g_{\mu\nu}$, $g_{\mu\nu,\rho}$ and $g_{\mu\nu,\rho\tau}$.

b) has vanishing divergence $\nabla_\mu G^{\mu\nu} = 0$.

c) allows for the Minkowski metric as a vacuum solution to $G_{\mu\nu} = 0$.

So $G_{\mu\nu} = -\kappa T_{\mu\nu}$ seems like a good choice. Alternative choices that sometimes are used include

- $-\kappa T_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu} R + g_{\mu\nu} \Lambda$ with the *cosmological constant* Λ.

  This satisfies a) and b) ($\nabla_\mu g^{\mu\nu} = 0$), but not c) (a space of constant curvature is a solution).

- $-\kappa T_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu} R + \alpha R_{\mu\sigma} R^\sigma_\mu$ or $+\beta R R_{\mu\nu}$ or other terms added. These lead to higher order derivatives which violate a), satisfy b) but not c), e.g. the "Starobinski model" which leads to inflation.

- $-\kappa T_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu} R + f(T^\mu{}_{\nu\rho})$ where the torsion $T^\mu{}_{\nu\rho}$ is nonzero because the independent variables used are $g_{\mu\nu}$ and $\Gamma^\rho_{\mu\nu}$, $\Gamma = \Gamma^{(\mathrm{LC})}(g)$.

But in fact, $G_{\mu\nu} = -\kappa T_{\mu\nu}$ has been confirmed (experimentally) so far by every test in the solar system. In particular by the following two which are not correctly predicted by Newtonian gravity:

1. The perihelion shift of Mercury. Mercury's path around the sun resembles an ellipse, but the perihelion (the closest point to the sun) and the aphelion (the farthest point from the sun) shift with each revolution by $\Delta\Phi$. Newtonian theory predicts a different value for $\Delta\Phi$ than is actually observed (by ~ $43''$/century).

2. The bending of light rays passing close to a massive object. Again, Newtonian theory predicts a different value than the one observed (by a factor of 2).

Further, more modern hints are GPS, the gravitational redshift, frame dragging, gravitational lensing etc. For every gravitational effect in the solar system, Einstein's equations are enough. On scales of more than the size of one galaxy cluster, the cosmological constant term has to be added.

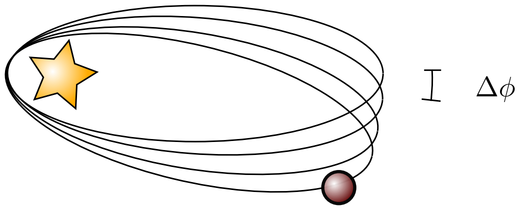

Figure 3.8: The shift $\Delta\phi$ of the Perihelion (the point of the orbit closest to the central mass) can be computed using General Relativity, or Newtonian gravity. The computed numbers differ by about $43''$/century. The value obtained by GR completely agrees with experiment.

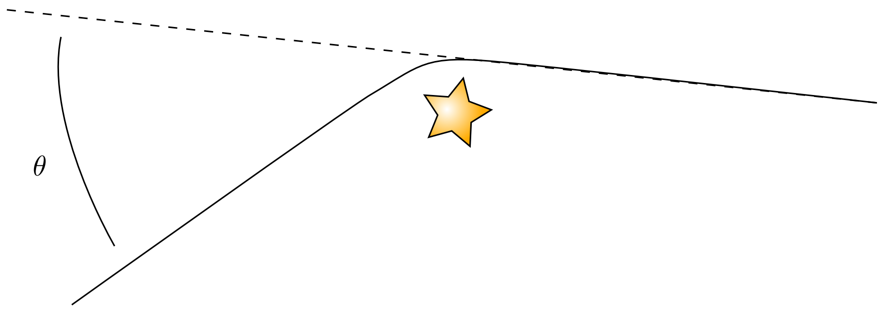

Figure 3.9: The bending of light θ at a central star can be computed using General Relativity, or Newtonian gravity. The computed numbers differ by a factor of 2. Again, the value obtained by GR (unlike that obtained by Newtonian gravity) completely agrees with experiment. This was observed by the famous experiment by Sir Arthur Eddington during a solar eclipse in 1919.

3.2.3 The Einstein equations and non-uniqueness of solutions

Consider the structure of the field equations: The fact that $g_{\mu\nu} = g_{(\mu\nu)}$ (symmetric) means there are ten independent functions ("variables per point"), so $G_{\mu\nu} = G_{(\mu\nu)}$ means that

$G_{\mu\nu} = -\kappa T_{\mu\nu}$ looks like ten equations. But, for every metric one has $\nabla_\mu G^{\mu\nu} = 0$ which means there are four non-trivial relations between the ten equations. So $G_{\mu\nu} = -\kappa T_{\mu\nu}$ are actually only $10 - 4 = 6$ equations. That means that the system is under-determined. In fact, this is true for every generally covariant theory.

Einstein's hole argument

Assume we have a solution to $G_{\mu\nu} = -\kappa T_{\mu\nu}$, e.g. for a star. This solution is given in a coordinate system $\{x^\mu\}$. Now look at another coordinate system y^μ the coordinates of which should coincide with the x^μ coordinates outside of a region R, but differ inside. Let $g_{\mu\nu}(x^\sigma)$ satisfy $G_{\mu\nu}(g) = 0$ in R. Same in y^μ coordinates:

$$\hat{g}_{\mu\nu}(y^\tau) = \frac{\partial x^{\mu'}}{\partial y^\mu} \frac{\partial x \nu'}{\partial y^\nu} g_{\mu'\nu'}(x^\sigma(y^\tau)) \tag{3.80}$$

satisfies $G_{\mu\nu}(\hat{g}) = 0$ in R. But because they are tensor equations, $G_{\mu\nu}(x) = 0$ and $G_{\mu\nu}(y) = 0$ are the same differential equation for g, one written with the x-variables, one with y's instead of x's, $\hat{g}_{\mu\nu}(x^\sigma)$ is the metric $g_{\mu\nu}$ in y-coordinates, with y's replaced by x's. This gives a new metric $\hat{g}$ in the x^μ coordinate system, $\hat{g} \neq g$. They both satisfy $G_{\mu\nu} = -\kappa T_{\mu\nu}$ in the x-coordinate system, so specifying $T_{\mu\nu}$ is not enough to uniquely determine the solutions to $g_{\mu\nu}$. The freedom lies precisely in choosing four functions $y^\mu(x^\sigma)$. Six equations $G_{(\mu\nu)} = -\kappa T_{(\mu\nu)}$ plus four choices $y^\mu(x^\sigma)$ yields exactly ten functions $g_{(\mu\nu)}(x)$.

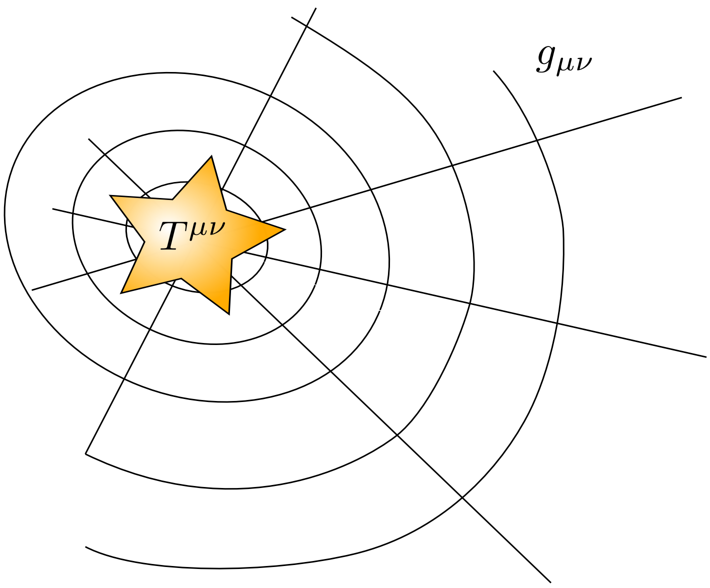

Figure 3.10: A matter distribution is the source of the gravitational field $g_{\mu\nu}$ via its energy-momentum tensor $T^{\mu\nu}$

The interpretation is that the transformation $g_{\mu\nu} \to \hat{g}_{\mu\nu}$ is a gauge transformation. Just as $\partial_\mu (dA)^{\mu\nu} = 4\pi j^\mu$ does not uniquely specify A_μ in terms of j^μ. The gauge

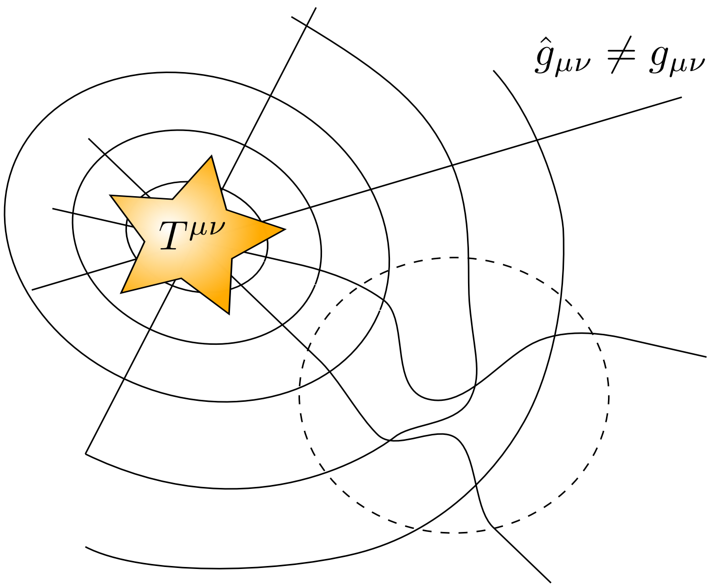

Figure 3.11: Because GR is diffeomorphism-invariant, the metric $g_{\mu\nu}$ is not uniquely determined by $T^{\mu\nu}$. For any solution $g_{\mu\nu}$, a physically equivalent solution $\hat{g}_{\mu\nu}$ can be obtained via pull-back with a diffeomorphism. Since diffeomorphisms act locally, the two metrics can, for instance, only differ inside a small region where there is vacuum (a "hole"), but not where there is matter, i.e. where $T^{\mu\nu} \neq 0$.

transformation in that case is $A \to A + d\chi$. Similarly, $G_{\mu\nu} = -\kappa T_{\mu\nu}$ does not uniquely determine the metric $g_{\mu\nu}$, given $T_{\mu\nu}$. The gauge transformation is $g \to \phi^* g$ where $\phi : M \to M$ is a diffeomorphism. So $g_{\mu\nu}$ doesn't have a complete physical meaning and $g \sim \hat{g}$ should be regarded as physically equivalent. They describe the same geometry.

In general, metrics g modulo diffeomorphisms $g \sim \phi^* g$ are geometries $[g]$ (equivalence classes). General covariance ("background independence") leads to a gauge symmetry in the equations of motion for general relativity. That can make it quite difficult to interpret solutions.

3.2.4 The Einstein-Hilbert action

For a matter field with action S_{matter}, the equation of motion can be derived by a variational principle. The same is true for the Einstein equations, they lead to an action principle. For field theories, actions are always integrals over Lagrange densities. For integrals over spacetime, $\mathcal{L}$ is a function on a manifold with Lorentzian metric g. The action is given by

$$S = \int_M d^n x \sqrt{(-1)^{n-1} \det g} \mathcal{L}. \tag{3.81}$$

It is a number independent of the chosen coordinates. Here, $\det g$ is the determinant of the $n \times n$ matrix $\{g_{\mu\nu}\}_{\mu,\nu=1}^{n}$. The metric having a Lorentzian metric means (in our convention) that it has one positive eigenvalue, all other eigenvalues are negative (an eigenvalue equal to zero would mean that the metric is degenerate). From this follows that $\det g$ is positive if n is odd and negative for n even.

First we show that the expression (3.81) is independent on the choice of coordinates.

Let us consider a change of coordinates (without loss of generality one global coordinate chart) $x^\mu \to \tilde{x}^\mu$. For simplicity, only look at orientation preserving changes of coordinates. This means that the Jacobi determinant $\det(J^\mu{}_\nu)$ of the matrix

$$J^\mu{}_\nu := \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \tag{3.82}$$

is positive. The Lagrangian is invariant under this change of coordinates,

$$\tilde{\mathcal{L}}(\tilde{x}) = \mathcal{L}(x). \tag{3.83}$$

We also need the identities

$$d^n\tilde{x} = \det\left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right) d^n x \tag{3.84}$$

$$\begin{aligned}\det \tilde{g} &= \det \tilde{g}_{\mu\nu} \\ &= \det\left(\frac{\partial x^{\mu'}}{\partial \tilde{x}^\mu}\frac{\partial x^{\nu'}}{\partial \tilde{x}^\nu} g_{\mu'\nu'}\right) \\ &= \left(\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right)\right)^2 \det g_{\mu\nu}.\end{aligned} \tag{3.85}$$

Using equations 3.83 to 3.85 we obtain

$$\begin{aligned}\tilde{S} &= \int d^n\tilde{x}\sqrt{(-1)^{n-1}\det \tilde{g}}\,\tilde{\mathcal{L}} \\ &= \int d^n x\sqrt{(-1)^{n-1}\det g}\,\mathcal{L}\underbrace{\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right)}_{\det(J^{-1})}\underbrace{\det\frac{\partial \tilde{x}^\mu}{\partial x^\nu}}_{\det(J)} \\ &= S.\end{aligned} \tag{3.86}$$

Here, we used $\det(J^{-1}) = \det(J)^{-1}$.

It follows that the action (3.81), even though expressed in coordinates, is actually a number independent on the choice of coordinate system. This is a general feature, which has to do with the fact that the expression $\sqrt{(-1)^{n-1}\det g}\,d^n x$ transforms like a density (similar to an n-form, just without regard for orientation). All actions for field theories on manifolds are formulated in this way. In fact, one needs a metric to define integration of scalar functions over manifolds in a coordinate-independent way.

We now show a specific feature of energy-momentum tensors for field theories on manifolds, the dynamics of which is given by some action S_{matter}. As we have seen, this action depends on the metric $g_{\mu\nu}$ itself.

$$S_{\text{matter}}[\phi, g_{\mu\nu}] = \int d^4x\sqrt{-\det g}\,\mathcal{L}_{\text{matter}}(\phi, g_{\mu\nu}). \tag{3.87}$$

A popular choice for how to construct these actions on general manifolds is *minimal coupling to gravity*: Take the Lagrangian from some matter theory on Minkowski space and replace $\eta_{\mu\nu} \to g_{\mu\nu}$, as well as $\partial_\mu \to \nabla_\mu$. As an example, consider the Klein-Gordon scalar field ϕ. On Minkowski space, its Lagrangian is given by

$$\mathcal{L}_{\mathrm{KG}} = \eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - m^2\phi^2. \tag{3.88}$$

The minimally coupled Klein-Gordon field then looks like

$$\mathcal{L}^{(\mathrm{mc})}_{\mathrm{KG}} = g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) - m^2\phi^2 = g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - m^2\phi^2 \tag{3.89}$$

since $\nabla_\mu = \partial_\mu$ for scalar fields. Actually, in this specific case, there is another popular choice for an action of the KG field on an arbitrary Lorentzian manifold, which is the so-called *conformally coupled* Klein-Gordon field:

$$\mathcal{L}^{(\mathrm{cc})}_{\mathrm{KG}} = g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \left(m^2 + \frac{R}{6}\right)\phi^2. \tag{3.90}$$

This behaves much nicer under *conformal transformations* $g_{\mu\nu}(x) \to \Omega(x)g_{\mu\nu}(x)$ in the case of $m = 0$.

The energy momentum tensor $T^{\mu\nu}$ of a theory can, in general, be computed from the action by a variation of the metric:

$$g_{\mu\nu} \to g_{\mu\nu} + \epsilon\delta g_{\mu\nu} =: \tilde{g}_{\mu\nu}. \tag{3.91}$$

Then $T^{\mu\nu}$ is defined to be the coefficient under the integral of the variation:

$$\begin{aligned} S_{\mathrm{matter}}[\phi, \tilde{g}_{\mu\nu}] &= S_{\mathrm{matter}}[\phi, g_{\mu\nu} + \epsilon\delta g_{\mu\nu}] \\ &= S_{\mathrm{matter}}[\phi, g_{\mu\nu}] + \frac{\epsilon}{2}\int d^4x\sqrt{-\det g}\,T^{\mu\nu}_{\mathrm{matter}}\delta g_{\mu\nu} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{3.92}$$

In other words,

$$T^{\mu\nu}_{\mathrm{matter}} = \frac{2}{\sqrt{-\det g}}\frac{\delta S}{\delta g_{\mu\nu}}. \tag{3.93}$$

A different way of writing this is

$$\begin{aligned} T^{\mu\nu}_{\mathrm{matter}} &= \frac{2}{\sqrt{-\det g}}\frac{\delta}{\delta g_{\mu\nu}}\int\sqrt{-\det g}\,\mathcal{L}d^4x \\ &= 2\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} + 2\frac{\mathcal{L}}{\sqrt{-\det g}}\frac{\partial\sqrt{-\det g}}{\partial g_{\mu\nu}}. \end{aligned} \tag{3.94}$$

For an invertible matrix valued function $M_{ij}(t)$, a useful formula is

$$\frac{d}{dt}\det(M_{ij}(t)) = \det(M_{ij}(t))\sum_{i,j}(M^{-1})^{ij}\frac{dM_{ij}}{dt}. \tag{3.95}$$

With the replacement 3.91 we obtain

$$\begin{aligned}\frac{\partial\sqrt{-\det g}}{\partial g_{\mu\nu}}\delta g_{\mu\nu} &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0}\sqrt{-\det(g_{\mu\nu}+\epsilon\delta g_{\mu\nu})} \\ &= -\frac{1}{2\sqrt{-\det g}}\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\det(g_{\mu\nu}+\epsilon\delta g_{\mu\nu}) \\ &\overset{3.95}{=} \frac{-\det g}{2\sqrt{-\det g}}g^{\mu\nu}\delta g_{\mu\nu}.\end{aligned} \tag{3.96}$$

With this we arrive at the very useful formula

$$\frac{\partial\sqrt{-\det g}}{\partial g_{\mu\nu}} = \frac{1}{2}\sqrt{-\det g}g^{\mu\nu}. \tag{3.97}$$

This allows us to write the energy-momentum tensor as

$$T^{\mu\nu}_{matter} = 2\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} + \mathcal{L}g^{\mu\nu}. \tag{3.98}$$

The action of composed systems is the sum of the actions of each individual system:

$$S_{\text{gravity + matter}} = S_{\text{gravity}} + S_{\text{matter}} \tag{3.99}$$

where S_{gravity} depends only on the metric while S_{matter} depends on both the metric and the field. We have

$$\frac{\delta}{\delta g_{\mu\nu}}S_{\text{matter}} = \frac{1}{2}\sqrt{-\det g}T^{\mu\nu}_{\text{matter}}, \tag{3.100}$$

so in order to get Einstein's equations from varying $S_{\text{gravity + matter}}$ with respect to $g_{\mu\nu}$, we need to have an action S_{gravity} such that

$$\frac{\delta}{\delta g_{\mu\nu}}S_{\text{gravity}} = \frac{1}{2\kappa}\sqrt{-\det g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) \tag{3.101}$$

because then

$$\frac{\delta}{\delta g_{\mu\nu}}S_{\text{gravity + matter}} = 0 \Leftrightarrow \text{Einstein's equations.} \tag{3.102}$$

We claim that the action

$$S_{\text{gravity}} = -\frac{1}{2\kappa}\int d^4x\sqrt{-\det g}R \tag{3.103}$$

satisfies equation 3.101. In the following, we show only the main points of the derivation.

We use the same replacement 3.91 to write

$$\begin{aligned}
-2\kappa S_{\text{gravity}} &= \int \sqrt{-\det \tilde{g}}\tilde{R} d^4x \\
&= \int \sqrt{-\det(g_{\mu\nu} + \epsilon\delta g_{\mu\nu})}\, \underbrace{R[g_{\mu\nu} + \epsilon\delta g_{\mu\nu}]}_{=:R(\epsilon)} d^4x \\
&= \int d^4x \left(\sqrt{-\det g} + \frac{6}{2}\sqrt{-\det g} g^{\mu\nu}\delta g_{\mu\nu} + \mathcal{O}(\epsilon^2)\right)\left(R + \left.\frac{d\tilde{R}}{d\epsilon}\right|_{\epsilon=0} + \mathcal{O}(\epsilon^2)\right) \\
&= -2\kappa\left(S_{\text{gravity}}[g_{\mu\nu}] + \epsilon \int d^4x \sqrt{-\det g}\left(\frac{R}{2}g^{\mu\nu} + \left.\frac{d\tilde{R}}{d\epsilon}\right|_{\epsilon=0}\right)\right) + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{3.104}$$

We calculate

$$\begin{aligned}
\left.\frac{d\tilde{R}}{d\epsilon}\right|_{\epsilon=0} &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left(\underbrace{R_{\mu\nu}[g_{\mu\nu} + \epsilon\delta g_{\mu\nu}]}_{=:\tilde{R}_{\mu\nu}} \underbrace{g^{\mu\nu}[g_{\mu\nu} + \epsilon\delta g_{\mu\nu}]}_{=g^{\mu\nu} - \epsilon g^{\mu\sigma}g^{\nu\tau}\delta g_{\sigma\tau} + \mathcal{O}(\epsilon^2)}\right) \\
&= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left(R_{\mu\nu} + \epsilon \left.\frac{d\tilde{R}_{\mu\nu}}{d\epsilon}\right|_{\epsilon=0}\right)(g^{\mu\nu} - \epsilon g^{\mu\sigma}g^{\nu\tau}\delta g_{\sigma\tau}) + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{3.105}$$

Without proof:

$$\left.\frac{d\tilde{R}_{\mu\nu}}{d\epsilon}\right|_{\epsilon=0} g^{\mu\nu} = \nabla_\mu V^\mu \tag{3.106}$$

with

$$V^\mu := \nabla^\lambda \delta g_{\mu\lambda} - g^{\sigma\tau}\nabla_\mu \delta g_{\sigma\tau} \sim \int d^4x \underbrace{\sqrt{-\det g}\nabla_\mu V^\mu}_{\partial_\mu(\sqrt{-\det g}V^\mu)}. \tag{3.107}$$

This is a total derivative term which means that on a manifold without boundary (and suitable fall-off behavior of $\delta g_{\mu\nu}$) one can neglect this term. Then we get for the action

$$S_{\text{gravity}}[g_{\mu\nu} + \epsilon\delta g_{\mu\nu}] = S_{\text{gravity}}[g_{\mu\nu}] + \left(-\frac{\epsilon}{2\kappa}\right)\int d^4x\sqrt{-\det g}\left(\frac{1}{2}Rg^{\mu\nu} - R^{\mu\nu}\right)\delta g_{\mu\nu} + \mathcal{O}(\epsilon^2). \tag{3.108}$$

Then

$$\frac{\delta S_{\text{gravity}}}{\delta g_{\mu\nu}} = \frac{1}{2\kappa}\sqrt{-\det g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right). \tag{3.109}$$

We then obtain the action for general relativity which is called the **Einstein-Hilbert action**:

$$S_{\textbf{gravity}} = -\frac{1}{2\kappa}\int d^4x\sqrt{-\det g}R. \tag{3.110}$$

There are modifications of this action which allow for the inclusion of a cosmological constant, of higher order terms as in the Starobinsky model.

4 Applications of General Relativity

The fundamentals of general relativity are

- Spacetime (the set of all events) is a manifold equipped with a non-degenerate Lorentzian metric $g_{\mu\nu}$ which describes the geometry of spacetime.

- A freely moving/falling (only influenced by gravity) (point-like) test particle has a world line satisfying the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho = 0. \tag{4.1}$$

- The matter fields and the metric satisfy Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu} \tag{4.2}$$

  and the equations of motion for matter.

It is very hard to find solutions for equations 4.2 in general, even numerically, for several reasons:

1. Equation 4.2 is a highly complicated set of non-linear, coupled partial differential equations.

2. General covariance means spacetime diffeomorphisms are a gauge transformation group. Every differently looking solution might actually describe the same physics. We don't know how many solutions to 4.2 exist, only a few examples are known.

One way of simplification is looking for solutions to 4.2 which have a large amount of symmetry.

4.1 Killing vector fields

A vector field X is called *Killing* or *Killing vector field (KVF)* for the metric g iff

$$\mathcal{L}_X g = 0 \tag{4.3}$$

where $\mathcal{L}_X$ is the Lie-derivative in direction X which is defined as

$$(\mathcal{L}_X g)(Y, Z) = X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) \tag{4.4}$$

where

$$g(Y, Z) := g_{\mu\nu} Y^\mu Z^\nu. \tag{4.5}$$

That means for the components

$$(\mathcal{L}_X g)_{\mu\nu} = X^\lambda g_{\mu\nu,\lambda} + g_{\mu\lambda}\partial_\nu X^\lambda + g_{\lambda\nu}\partial_\mu X^\lambda. \tag{4.6}$$

Assume that the vector field X is some coordinate field ∂_σ. Then

$$(\mathcal{L}_{\partial_\sigma} g)_{\mu\nu} \;=\; g_{\mu\nu,\sigma} \;=\; 0 \tag{4.7}$$

means that "g is constant in σ-direction".

Instead of looking for general solutions to 4.2, it is easier to look for those with (one or many) KVFs. This is what we will do in the following.

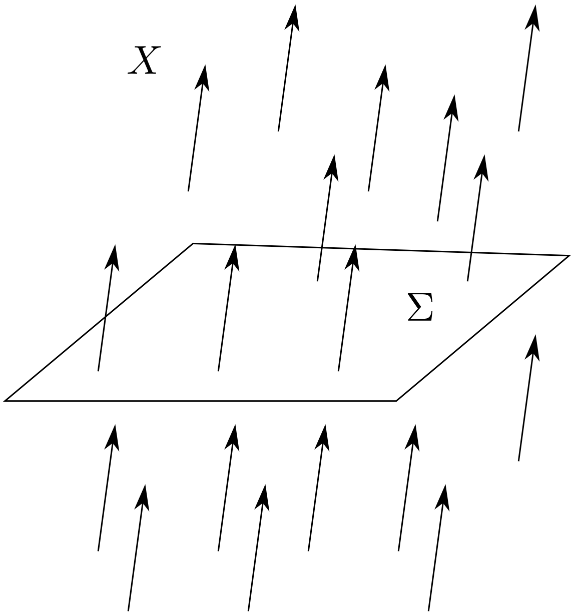

Figure 4.1: A static metric has a time-like KVF X and a space-like hypersurface Σ orthogonal to it.

We call a metric *stationary* if there exists a time-like KVF ("time-translation symmetry") that is nowhere near zero. A metric is called *static* if it is stationary and there exists a space-like hyper-surface Σ orthogonal to the KVF everywhere ("time-translation symmetry and no rotations"). An example for a stationary but not static metric involves a vector field with an internal rotation so that it is not perpendicular to a hyper-surface Σ in M.

In the static case, there is a nice and convenient set of coordinates X (in the near vicinity of Σ): Pick any coordinates (x^1, x^2, x^3) on Σ (assuming they cover all of Σ). Any point on Σ has $x^0 = 0$. For any other point p in the neighbourhood of Σ, follow the vector field X until one reaches Σ at some point q. If q has coordinates $(0, x^1, x^2, x^3)$, then p shall have coordinates (x^0, x^1, x^2, x^3) where x^0 is the curve parameter of the integral curve at p. This gives a coordinate chart in the vicinity of Σ, and if there are no closed

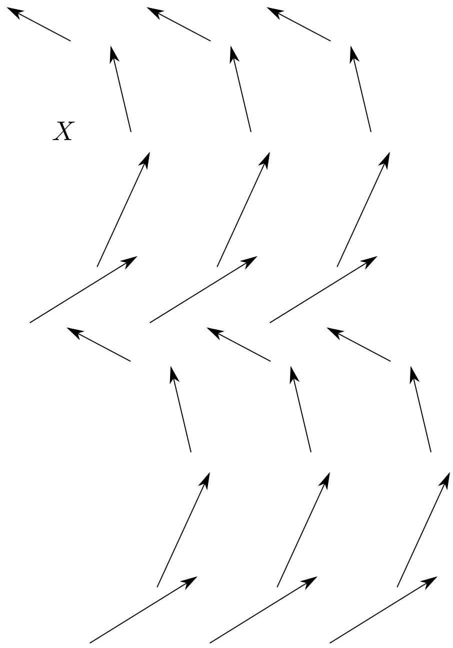

Figure 4.2: A stationary metric has a time-like KVF, but there might be no Σ, e.g. if X is spiraling.

time-like curves this gives coordinates for all of M. In these coordinates, $X = \partial_0$. The metric in these coordinates has the form

$$ds^2 = A(dx^0)^2 - B_i dx^i dx^0 - h_{ij} dx^i dx^j. \tag{4.8}$$

Since $\partial_0 = X$ is a KVF, A, B and h_{ij} do only depend on (x^1, x^2, x^3). We can write

$$A(x^1, x^2, x^3) = g_{\mu\nu} X^\mu X^\nu = |X|^2, \tag{4.9}$$

independent of x^0. Because ∂_0 is orthogonal to ∂_i on Σ, it is orthogonal to ∂_i everywhere. Because of that, $B_i = 0$ leading to

$$ds^2 = A(dx^0)^2 - h_{ji} dx^i dx^j. \tag{4.10}$$

By similar construction, a stationary metric can be cast into the form

$$ds^2 = A(x^1, x^2, x^3)(dx^0)^2 - B_i(x^1, x^2, x^3) dx^i dx^0 - h_{ij}(x^1, x^2, x^3) dx^i dx^j \tag{4.11}$$

with $B \neq 0$. A static metric is a stationary metric which is invariant under time-reversal $x^0 \to -x^0$.

4.2 The Schwarzschild metric

The **Schwarzschild metric** was published by Karl Schwarzschild (not „Schwartz-Child“!) in 1916. It was the first exact non-trivial solution to Einstein's equations

4.2 (in vacuum). It is a static metric that has spherical symmetry, meaning that there are three space-like KVFs L_1, L_2, L_3 satisfying $[L_i, L_j] = \epsilon_{ijk} L_k$, and the orbit of a point under the vector fields L_i is diffeomorphic to a sphere S^2.

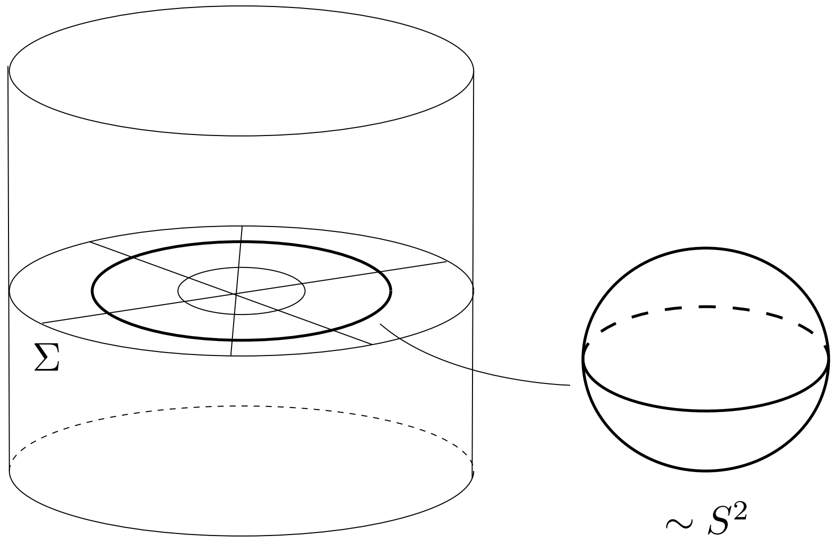

Figure 4.3: A spherically symmetric metric has three space-like KVFs L_i, $i = 1, 2, 3$, satisfying $[L_i, L_j] = \epsilon_{ijk} L_k$. At each point, the L_i are tangent to a $2d$ sphere. In other words, at each point, the L_i are linearly dependent!

All L_i are orthogonal to X (time-like) and should describe the exterior field of a spherically symmetric matter distribution. With this information, one can derive the Schwarzschild solution. Because of staticity and rotational symmetry, the space-like hyper-surface Σ foliates into spheres that never overlap, so they are all located inside of each other (think of a Russian Matryoshka doll). On each of these spheres, the induced metric necessarily is proportional to $d\theta^2 + \sin^2\theta d\varphi^2$, because that is the only rotationally invariant metric on S^2. We call this proportionality factor r^2, and take t, θ, φ as coordinates so that each of the spheres equals a set of points with constant r. The metric is of the form

$$ds^2 = A(dx^0)^2 - C dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.12}$$

Because of rotational symmetry, A, C can only depend on r, not on θ, ϕ. Careful: We have chosen r such that the area of a sphere with constant coordinate r is given by $4\pi r^2$, but r is not a radius in a geometrical sense.

Now determine $A(r), C(r)$ by using Einstein's equations in vacuum. From the metric

$$g_{\mu\nu} = \begin{pmatrix} A(r) & & & \\ & -C(r) & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix}, \tag{4.13}$$

we obtain the Christoffel symbols

$$\Gamma^0_{r0} = \Gamma^0_{0r} = \frac{A'}{2A}, \quad \Gamma^r_{00} = \frac{A'}{2C} \tag{4.14}$$

$$\Gamma^r_{rr} = \frac{C'}{2C}, \quad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \tag{4.15}$$

$$\Gamma^r_{\theta\theta} = -\frac{r}{C}, \quad \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{C} \tag{4.16}$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta \tag{4.17}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta. \tag{4.18}$$

All other Christoffel Symbols are zero. The dash denotes the derivative with respect to r. The Ricci tensor is diagonal with

$$R_{00} = -\frac{A''}{2C} + \frac{A'}{4C}\left(\frac{C'}{C} + \frac{A'}{A}\right) - \frac{A'}{rC} \tag{4.19}$$

$$R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{C'}{C} + \frac{A'}{A}\right) - \frac{C'}{rC} \tag{4.20}$$

$$R_{\theta\theta} = -1 - \frac{r}{2C}\left(\frac{C'}{C} - \frac{A'}{A}\right) + \frac{1}{C} \tag{4.21}$$

$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta. \tag{4.22}$$

Einstein's equation in a vacuum can be written as $R_{\mu\nu} = 0$, so

$$0 = \frac{R_{00}}{A} + \frac{R_{rr}}{C} = -\frac{1}{rC}\left(\frac{A'}{A} + \frac{C'}{C}\right) \tag{4.23}$$

which implies

$$\frac{A'}{A} + \frac{C'}{C} = 0. \tag{4.24}$$

By noticing that this is

$$\frac{d}{dr}(\ln(AC)) = 0 \tag{4.25}$$

we can conclude

$$AC = \text{const} = 1, \tag{4.26}$$

obtaining

$$A(r) = \frac{1}{C(r)}. \tag{4.27}$$

From

$$0 = R_{\theta\theta} = -1 + rA' + A \tag{4.28}$$

we conclude that

$$\frac{d}{dr}(rA) = 1 \tag{4.29}$$

which one can integrate to

$$rA = r + const. \tag{4.30}$$

We call this constant of integration $-r_S$, and obtain

$$A = 1 - \frac{r_S}{r} = C^{-1}. \tag{4.31}$$

With this, we can write finally write down the metric:

$$ds^2 \quad = \left(1 - \tfrac{r_S}{r}\right)(dx^0)^2 - \tfrac{1}{1-\frac{r_S}{r}} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.32}$$

Here, r_S is a free parameter called **Schwarzschild radius**.[1] The Schwarzschild metric (4.32) describes the exterior of a spherically symmetric matter distribution, and by its formula one can see that it is only defined for $0 < r \neq r_S$. In fact, we will always assume $r > r_S$.

We attempt to interpret r_S by comparison with Newton's law of gravity. The world line of an observer who keeps a constant distance to the central mass, i.e. who has constant r, is

$$x^0_{\text{obs}}(s) = \frac{1}{\sqrt{1 - r_S/r}} s \tag{4.33}$$

with $r > r_S$. Note that this world line is parametrized by proper time! At some time s the observer lets go of an object which then obeys the geodesics equation

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0. \tag{4.34}$$

We calculate

$$\frac{d^2r}{ds^2} = -\Gamma^r_{00}\left(\frac{dx^0}{ds}\right)^2 = \frac{1}{2}g^{rr}g_{00,r}\frac{1}{1 - r_S/r} = -\frac{r_S}{2r^2}. \tag{4.35}$$

This is only true in the instant of the let-go and comes from the initial condition that first derivatives vanish. For the outside (large r), we expect space-time to have almost no curvature, so r has the interpretation of a radius in Newtonian physics. In Newtonian physics the dropped object would undergo an acceleration (with $s = ct$):

$$\frac{d^2r}{ds^2} \;=\; \frac{1}{c^2}\frac{d^2r}{dt^2} \;=\; -\frac{G_N M}{r^2c^2}, \tag{4.36}$$

where G_N is Newton's constant and M is the total mass of the matter in the center. This means that

$$r_S = \frac{2G_N M}{c^2} \tag{4.37}$$

In retrospect, this justifies the sign choice for r_S in the derivation.

Putting in numerical values for the sun, the Schwarzschild radius is $r^\odot \approx 2.95$ km. From this we get $r_S = (M/M^\odot)r_S^\odot$, e.g. $r_S^{(\text{earth})} \approx 8.87$ mm. So in these cases, the Schwarzschild radius is well inside the object itself.

[1] Birkhoff's theorem states that any vacuum solution to Einstein's equations with spherical symmetry is static. So, actually, one could have derived (4.32) with fewer conditions.

4.3 Interior solution of Einstein's equations

We now consider the interior solution of Einstein's equations, i.e. for the inside of a spherically symmetric arrangement of matter.

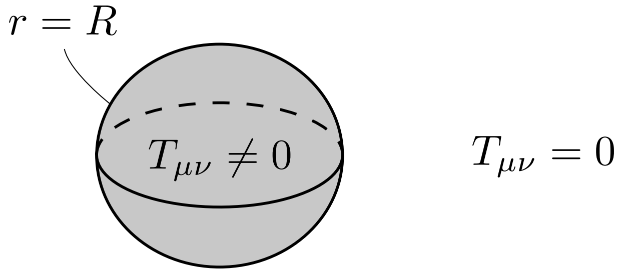

Figure 4.4: The interior solution has $T_{\mu\nu} \neq 0$, i.e. $\rho(r), p(r) \neq 0$ for $r < R$, and $= 0$ for $r \geq R$.

We show the main steps of the derivation of the interior solution. Einstein's equations with matter have the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}. \tag{4.38}$$

We write

$$R = \kappa T := \kappa T_{\mu\nu} g^{\mu\nu}, \tag{4.39}$$

so we get

$$R_{\mu\nu} = \kappa \left(\frac{1}{2} g_{\mu\nu} T - T_{\mu\nu} \right). \tag{4.40}$$

In a simplified model of a star as an ideal fluid, the energy momentum tensor is given by

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu} \tag{4.41}$$

where ρ is the density, p the pressure and u_μ the rest frame of the fluid with $g_{\mu\nu} u^\mu u^\nu = 1$. Because of staticity and spherical symmetry, ρ and p only depend on r. Repeating the analysis with the ansatz $ds^2 = A(r)(dx^0)^2 - C(r)dr^2 - r^2 d\Omega^2$ leads to

$$R_{00} = = -\frac{A''}{2C} + \frac{A'}{4C}\left(\frac{C'}{C} + \frac{A'}{A}\right) - \frac{A'}{rC} = -\frac{\kappa}{2}(\rho + 3p)A \tag{4.42}$$

$$R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{C'}{C} + \frac{A'}{A}\right) - \frac{C'}{rC} = -\frac{\kappa}{2}(\rho - p)C \tag{4.43}$$

$$R_{\theta\theta} = -1 - \frac{r}{2C}\left(\frac{C'}{C} - \frac{A'}{A}\right) + \frac{1}{C} = -\kappa\rho. \tag{4.44}$$

From this it follows that

$$\frac{R_{00}}{2A} + \frac{R_{rr}}{2C} + \frac{R_{\theta\theta}}{r^2} = -\frac{C'}{rC^2} - \frac{1}{r^2} + \frac{1}{r^2 C} = -\kappa\rho. \tag{4.45}$$

This means that

$$\frac{d}{dr}\left(\frac{r}{C(r)}\right) = 1 - \kappa\rho(r)r^2. \tag{4.46}$$

Integration from 0 to r with boundary condition $r/C|_{r=0} = 0$ yields

$$C(r) = \frac{1}{1 - \frac{2G_N M(r)}{r}} \tag{4.47}$$

where

$$M(r) = 4\pi \int_0^r dr'(r')^2\rho(r') \tag{4.48}$$

is the total mass. To get $A(r)$ is a bit more complicated. From $\nabla_\mu T^{\mu\nu} = 0$ we obtain

$$\frac{A'}{A} = -\frac{2p'}{\rho + p} \tag{4.49}$$

Equations 4.42 to 4.44 together with 4.49 lead to

$$p' = -\frac{G_N M(r)}{r^2}(\rho + p)\left(1 + \frac{4\pi r^3 p}{M(r)}\right)\left(1 - \frac{2G_N M(r)}{r}\right)^{-1}. \tag{4.50}$$

This is known as the *Tollmann-Oppenheimer-Volkoff equation* (TOV). Inserting the TOV into equation 4.49 gives an ordinary differential equation for A'/A. Integrating this from ∞ to r with $A(\infty) = 1$ yields

$$A(r) = \exp\left(-2G_N \int_r^\infty dr' \frac{1}{(r')^2}\frac{M(r') + 4\pi(r')^2 p(r')}{1 - 2G_N M(r')/r'}\right). \tag{4.51}$$

This gives $A(r), C(r)$ in terms of $p(r), \rho(r)$. To solve $\rho(r), p(r)$, one has to solve the TOV equation and some equation of state (matter-dependent relation between p and ρ, e.g. $p \sim \rho^\gamma$ for some constant γ).

Quick consistency check: Is the solution with matter outside of the star equal to the vacuum solution? For $r > R$, we have $M(r) = M(R) = M$. We then write

$$A(r) = \exp\left(-G_N \int_r^\infty dr' \frac{1}{(r')^2}\frac{M}{1 - r_S/r'}\right). \tag{4.52}$$

Substituting $x' = 1 - r_S/r$ (with $dx' = dr' r_S/(r')^2$) leads to

$$A(r) = \exp\left(\int_1^x \frac{dx'}{x'}\right) = x = 1 - \frac{r_S}{r}, \tag{4.53}$$

so for $r > R$ the solution coincides with the Schwarzschild solution for total mass M.

Note: For "normal" stars with radii $R > r_S$, the factor $1/(1 - 2G_N M(r)/r)$ always stays finite for $0 < r < R$. So for the sun, the earth and the stars in the universe the metric coefficients are finite for all $r > 0$.

For quite some time, the singularity at $r = r_S$ was regarded to be physically unimportant (e.g. like the Landau pole in QED). Stars with a density so large that their radius was smaller than r_S were thought to be unrealistic. However, if we look at the stability for a star with $\rho(r) = \rho_0 = const$ for $r < R$ and $\rho = 0$ for $r \geq R$ ("incompressible matter"), we find (without proof) for $r < R$

$$A(r) = \frac{1}{4}\left(3\sqrt{1-\frac{r_S}{R}} - \sqrt{1-\frac{r_S r^2}{R^3}}\right)^2 \tag{4.54}$$

and

$$C(r) = \left(1-\frac{r_S r^2}{R^3}\right)^{-1}. \tag{4.55}$$

We then obtain

$$p(r) = \rho_0 c^2 \frac{\sqrt{1-\frac{r_S r^2}{R^3}} - \sqrt{1-\frac{r_S}{R}}}{3\sqrt{1-\frac{r_S}{R}} - \sqrt{1-\frac{r_S r^2}{R^3}}}, \tag{4.56}$$

$p(r)$ can diverge:

$$p(r=0) = \rho_0 c^2 \frac{1-\sqrt{1-\frac{r_S}{R}}}{3\sqrt{1-\frac{r_S}{R}}-1} \xrightarrow{R\to 9/8 r_S} \infty. \tag{4.57}$$

So for a given radius R, a star can only have a maximum total mass M. Infinite pressure means that matter collapses, so the star is unstable. Note that also in Newtonian physics there can be instabilities of the matter, but the one talked about here is of a fundamental nature: It doesn't come from properties of matter, but from relativistic effects in the TOV equation.

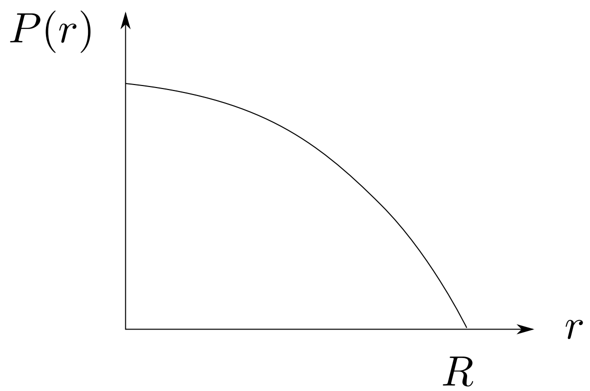

Figure 4.5: The pressure in an incompressible star depends on r and r_S, diverging at $r = 0$ for $R < \frac{9}{8} r_S$.

If even incompressible matter collapses, then every realistic matter will collapse, independent of its equation of state. There is a theoretical possibility that, when a very massive star collapses (because its fusion fuel runs out), it becomes so compressed that it cannot sustain its own weight no matter what the matter does (degenerate, Pauli's principle, ...). This leads to a total gravitational collapse behind $r = r_S$. This is what is referred to as **black hole**.

For normal celestial bodies, the Schwarzschild metric is only valid for $r > R \gg r_S$. However, there are numerous indications that there actually exist objects with $R < r_S$ in the universe where all of the mass is concentrated behind the Schwarzschild radius. This is the result of complete gravitational collapse. At the end of the life cycle of a star (greatly simplified), it explodes, leaving a remnant mass M_{rem}. If this is smaller than roughly 1.4 solar masses (**Chandrasekhar limit**), it turns into a white dwarf. If it is smaller than roughly 3 solar masses (**Tollmann-Oppenheimer-Volkhoff limit**) it becomes a neutron star. If the remnant mass is even bigger than that, a black hole forms. It is expected that every star with a remnant mass bigger than three solar masses becomes a black hole eventually. Since there are very heavy stars in the universe, it is expected that black holes exist. What is the physical significance of r_S for a black hole? The light cones become more and more narrow as $r \to r_S$. In fact, the surface $r = r_S$ is light-like. The curve $x^0(\lambda) = \lambda$, $r(\lambda) = const = r_S$ is light-like. At $r = r_S$, an observer would have to travel at the speed of light to keep a constant "distance to the center" (r constant). There are no light rays which originate at $r < r_S$ and cross the boundary $r = r_S$, hence the name black hole. The boundary $r = r_S$ is also called the *event horizon*.

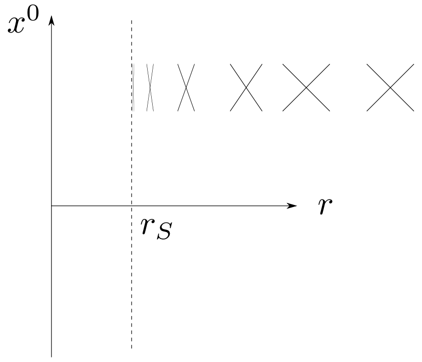

Figure 4.6: Light cones near $r = r_S$ become more and more vertical, similarly to the Rindler horizon.

Let's take a more detailed look at the geodesics: Consider an observer at constant $r = r_0$, dropping some small object. Look at the geodesic of that object. Since $\dot{\phi} = \dot{\theta} = 0$, only r, x^0 are important in what follows. The metric is

$$ds^2 = A(r)(dx^0)^2 - C(r)dr^2 \tag{4.58}$$

with

$$A(r) = \frac{1}{C(r)} = 1 - \frac{r_S}{r}. \tag{4.59}$$

The Christoffel symbols are

$$\Gamma^0_{0r} = \frac{A'}{2A} \tag{4.60}$$

$$\Gamma^r_{00} = \frac{A'}{2C} \tag{4.61}$$

$$\Gamma^r_{rr} = \frac{C'}{2C} \tag{4.62}$$

and the geodesic equation is

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0. \tag{4.63}$$

With the curve parameter s and $\mu = 0$:

$$\frac{d^2x^0}{ds^2} + \frac{A'}{A}\frac{dx^0}{ds}\frac{dr}{ds} = 0 \tag{4.64}$$

$$\Leftrightarrow \frac{\frac{d^2x^0}{ds^2}}{\frac{dx^0}{ds}} + \frac{\frac{dA}{ds}}{A} = 0 \tag{4.65}$$

$$\Leftrightarrow \frac{d}{ds}\left(\ln\frac{dx^0}{ds} + \ln A\right) = 0 \tag{4.66}$$

$$\Leftrightarrow \frac{dx^0}{ds}A = const =: F \tag{4.67}$$

determined by the initial conditions. The world line of the rocket is

$$x^\mu_{\text{rocket}}(s) = \begin{pmatrix} \frac{1}{\sqrt{A(r_0)}}s \\ r_0 \\ 0 \\ 0. \end{pmatrix} \tag{4.68}$$

That means that

$$F = \sqrt{A(r_0)} \tag{4.69}$$

if the geodesic of the object is parameterized by proper time. Also, the curve is always time-like of norm squared equal to 1:

$$\begin{aligned} 1 &= g_{00}\left(\frac{dx^0}{ds}\right)^2 + g_{rr}\left(\frac{dr}{ds}\right)^2 \\ &= A(r)\left(\frac{dx^0}{ds}\right)^2 - C(r)\left(\frac{dr}{ds}\right)^2 \\ &= \frac{F^2}{A(r)} - C\left(\frac{dr}{ds}\right)^2. \end{aligned} \tag{4.70}$$

From this, we obtain

$$\frac{dr}{ds} = -\sqrt{F^2 - \frac{1}{C(r)}} = -\sqrt{A(r_0) - A(r)}, \tag{4.71}$$

where we have chosen the negative sign of the square root because we treat the case of an object falling towards the center. Look at the ordinary differential equation with x^0 as curve parameter

$$\frac{dr}{dx^0} = \frac{\frac{dr}{ds}}{\frac{dx^0}{ds}} = \frac{A(r)}{F}\frac{dr}{ds} = -A(r)\sqrt{1-\frac{A(r)}{A(r_0)}}. \tag{4.72}$$

If $x^0 \to \infty$, then $r \to r_S$ so that $dr/dx^0 \to 0$. Now look at the proper time passing for the object along the geodesics:

$$\begin{aligned} l &= \int_{r_S}^{r_0} dr \sqrt{\left(\frac{dx^0}{dr}\right)^2 A(r) - C(r)\left(\frac{dr}{dr}\right)^2} \\ &= \int_{r_S}^{r_0} dr \sqrt{A(r)\frac{1}{\left(A(r)\sqrt{1-\frac{A(r)}{A(r_0)}}\right)^2} - C(r)} \\ &= \int_{r_S}^{r_0} dr \sqrt{\frac{1}{A(r)}\frac{1}{1-\frac{A(r)}{A(r_0)}} - \frac{1}{A(r)}} \\ &= \int_{r_S}^{r_0} dr \frac{1}{\sqrt{A(r)-A(r_0)}} \\ &= \sqrt{\frac{r_0}{r_S}} \int_{r_S}^{r_0} dr \sqrt{\frac{r}{r_0 - r}} \\ &< \infty. \end{aligned} \tag{4.73}$$

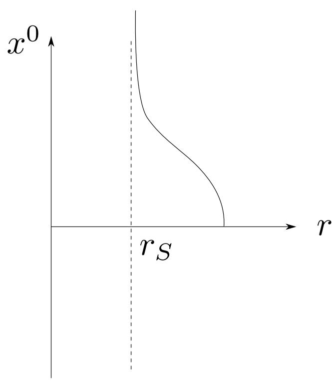

Figure 4.7: The world line of an object, dropped from a rocket which keeps a constant distance r_0 to the black hole. Again, this is similar to the Rindler horizon case.

So the object itself does not perceive that it takes forever to reach the horizon. It reaches $r = r_S$ after finite proper time. Again, this is similar to what happened to the Rindler observer. Indeed, the event horizon is in very close analogy to the Rindler horizon. Just as in that case, $r = r_S$ is a coordinate singularity. The metric coefficients diverge, because x^0, r, θ, ϕ are only good coordinates for $r > r_S$, just as Rindler coordinates were for $y^1 > -1/d$. So what happens at $r < r_S$? We need to find different coordinates which cover more than just $r > r_S$, just as Minkowski coordinates cover more than $x^1 > |x^0|$.

4.3.1 The Kruskal extension

The Kruskal extension of the Schwarzschild metric is a metric in a specific set of coordinates (the Kruskal coordinates), which extends the Schwarzschild metric. This means that there is a coordinate transformation of the domain of Schwarzschild coordinates (x^0, r, θ, ϕ) to a *subset* of Kruskal coordinates T, X, θ, ϕ. The central idea is to replace r, x^0 by different coordinates which are the parameters for ingoing and outgoing light rays. We start with

$$ds^2 = \left(1 - \frac{r_S}{r}\right)(dx^0)^2 - \frac{1}{1 - \frac{r_S}{r}} dr^2. \tag{4.74}$$

A light ray satisfies

$$\left(\frac{dx^0}{dr}\right)^2 = \left(\frac{r}{r - r_S}\right)^2, \tag{4.75}$$

which means that radial null geodesics satisfy

$$x^0 = \pm r_* + const \tag{4.76}$$

where

$$r_* = r + r_S \ln\left(\frac{r}{r_S} - 1\right) \tag{4.77}$$

with

$$\frac{dr_*}{dr} = \frac{1}{1 - \frac{r_S}{r}}. \tag{4.78}$$

This is called the *Regge-Wheeler-tortoise coordinate*. Now define null coordinates u, v as

$$u := x^0 - r_* \tag{4.79}$$
$$v := x^0 + r_*. \tag{4.80}$$

In these coordinates, the metric looks like

$$ds^2 = \left(1 - \frac{r_S}{r}\right) du dv \tag{4.81}$$

where a symmetric product between du and dv is implied such that

$$g = \begin{pmatrix} 0 & \frac{1}{2}\left(1 - \frac{r_S}{r}\right) \\ \frac{1}{2}\left(1 - \frac{r_S}{r}\right) & 0 \end{pmatrix}. \tag{4.82}$$

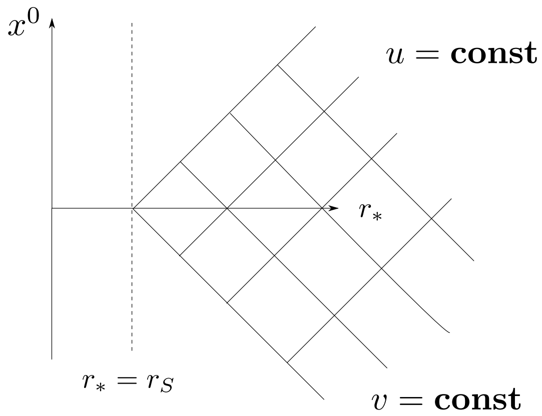

Figure 4.8: Change from (x^0, r_*) to (u, v).

Now r is a complicated function of u and v, implicitely defined by

$$\frac{v-u}{2} = r_* = r + r_S \ln\left(\frac{r}{r_S} - 1\right). \tag{4.83}$$

Taking the exponential of both sides gives

$$\exp\left(\frac{v-u}{2r_S}\right) = \exp\left(\frac{r}{r_S}\right)\left(\frac{r}{r_S} - 1\right) \tag{4.84}$$

$$\Rightarrow \frac{r_S}{r}\exp\left(-\frac{r}{r_S}\right)\exp\left(\frac{v}{2r_S}\right)\exp\left(-\frac{u}{2r_S}\right) = \left(1 - \frac{r_S}{r}\right) \tag{4.85}$$

$$\Rightarrow ds^2 = \frac{r_S}{r} e^{-\frac{r}{r_S}} e^{\frac{v}{2r_S}} e^{-\frac{u}{2r_S}} du dv. \tag{4.86}$$

Defining new coordinates

$$U := -e^{-\frac{u}{2r_S}} \tag{4.87}$$

$$V := e^{\frac{v}{2r_S}} \tag{4.88}$$

leads to

$$ds^2 = \frac{4r_S^3}{r} e^{-\frac{r}{r_S}} dU dV. \tag{4.89}$$

Yet new coordinates

$$T := \frac{U+V}{2} \tag{4.90}$$

$$X := \frac{V-U}{2} \tag{4.91}$$

give

$$ds^2 = \frac{4r_S^3}{r} e^{-\frac{r}{r_S}} (dT^2 - dX^2) \tag{4.92}$$

with

$$\left(\frac{r}{r_S} - 1\right) e^{\frac{r}{r_S}} = X^2 - T^2 \tag{4.93}$$

$$\frac{x^0}{r_S} = 2\text{arctanh}\frac{T}{X}. \tag{4.94}$$

This implicitly defines r since the equations can't be solved for r. One can extend X, T from $X > |T|$ to more values, up to $X^2 - T^2 = -1$ which corresponds to a singularity at $r = 0$. The metric can be written as

$$ds^2 = \frac{4r_S^3}{r} dT^2 - \frac{4r_S^3}{r} dX^2 - r^2 d\Omega^2. \tag{4.95}$$

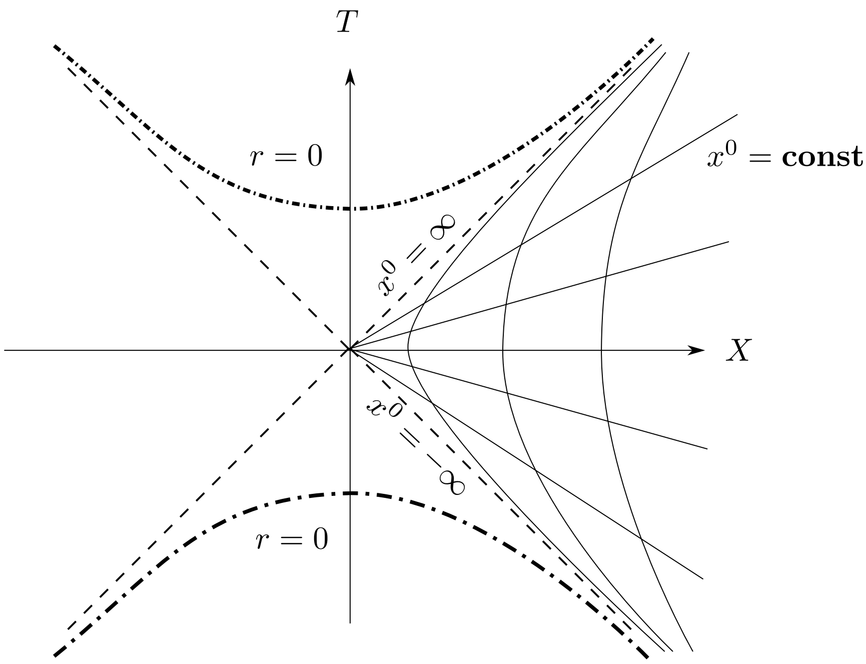

Figure 4.9: Kruskal diagram, where light rays travel along lines of slope 1, every point (T, X) is a 2-sphere (i.e. represents all points (T, X, θ, ϕ) for fixed (T, X)).

In a so-called **Kruskal diagram**, every point is a 2-sphere. Light rays always move on world lines with slope 1. Note that a Kruskal diagram is not quite what is called a Penrose diagram: For that it would have to be bounded.

The interior of region II is not static (or even stationary). There is a KVF ξ which, in region, coincides with ∂_0 ("time"): $\xi = \partial_0$ in region I. But in region II, ξ becomes space-like. Region II is the opposite of static: Every time-like curve (not only geodesics) runs into the singularity at $r = 0$ $(X^2 - T^2 = -1)$ after a finite proper time.[2] Region III is the "time-reversed" version of region II: Every time-like curve needs to have its origin

[2] The geodesics actually take the longest time – the more you struggle, the quicker you die.

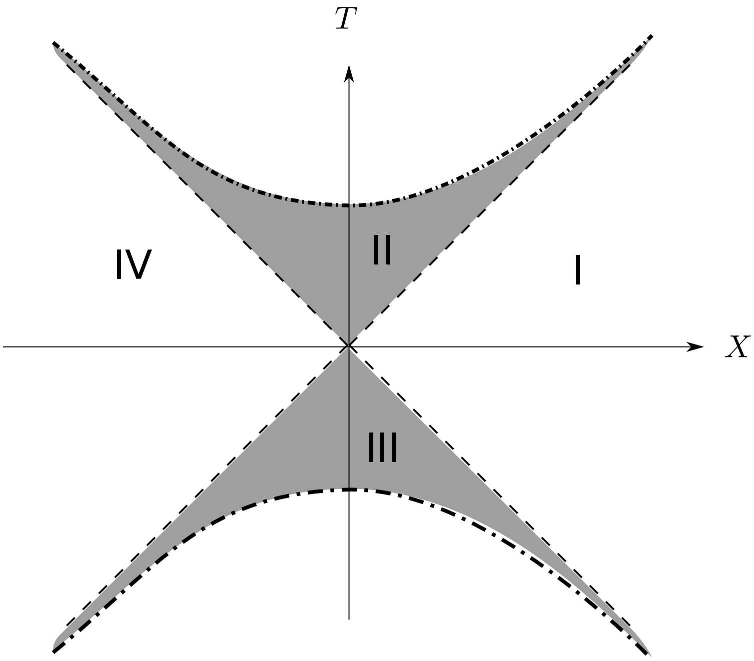

Figure 4.10: Different regions in the Kruskal diagram: region I is the outside of the black hole, which is where $r > r_S$, i.e. where Schwarzschild coordinates dare defined. Region II is inside the black hole, region III is inside the white hole, region IV is a causally disconnected universe which also touches both black and white hole.

at the singularity $r = 0$ and has to leave region III after a finite proper time. This is called a **white hole**. Region IV is a "different universe" which is causally disconnected from ours. The whole Kruskal diagram is probably not a realistic model for the black holes in our universe.

4.4 Gravitational red-shift

A very useful tool to examine the physical properties of space-time is the so-called **geometrical optics approximation**. It replaces electromagnetic waves with light-like curves.

1. A light ray is represented by a light-like geodesic $\lambda \mapsto x_L^\mu(\lambda)$.

2. The velocity vector $k^\mu = \frac{dx_L^\mu}{d\lambda}$ is the wave-vector of the light ray.

3. An observer with world line $s \mapsto x^\mu(s)$, who intersects the world line of the light ray, measures a frequency of

$$\omega = g_{\mu\nu} \frac{dx^\mu}{ds} k^\nu. \tag{4.96}$$

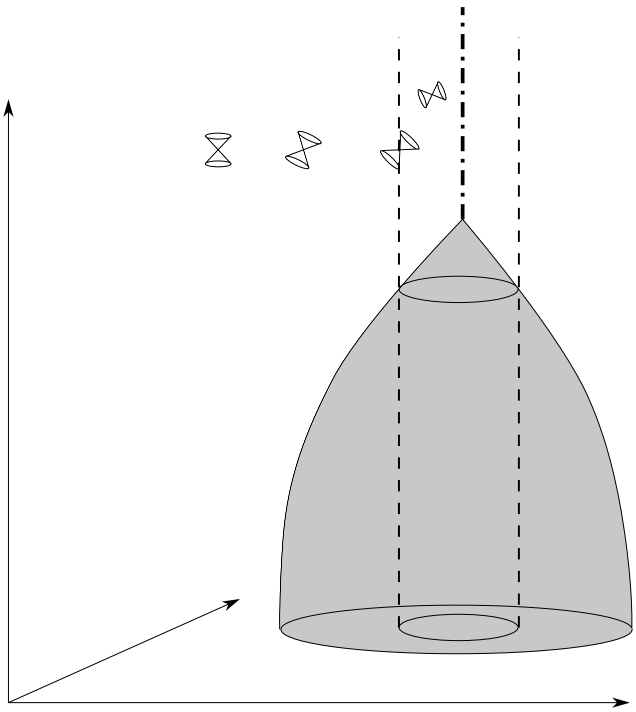

Figure 4.11: Gravitational collapse of a star. The diagram shows Kruskal coordinates "folded together" at the singularity $r = 0$ (bold dot-dashed line). This means that in this image, light cones "flip over" inside the event horizon (small circle, dashed line).

The geometrical optics approximation is valid whenever the lateral extension of the light ray is very small (in particular compared to scales on which the curvature varies).

Let us consider the Schwarzschild metric, and two observers O_1 and O_2, both keeping constant distance to the event horizon: $r_S < r_1 < r_2$. They send each other light signals. Their respective world lines (parametrized by proper time) are

$$x_I^\mu(s) = \begin{pmatrix} \frac{1}{\sqrt{1-r_S/r_I}} s \\ r_I \\ 0 \\ 0 \end{pmatrix} \tag{4.97}$$

with $I = 1, 2$. Now O_1 sends a light ray to O_2. Assume that the curve parameters at which the light ray starts and ends are λ_1 and λ_2, respectively, i.e.

$$r_L(\lambda_1) = r_1, \qquad r_L(\lambda_2) = r_2.$$

Then the velocity vectors of the light-like geodesic at these points are

$$\frac{dx_L^\mu}{d\lambda}(\lambda_1) =: k_1^\mu, \qquad \frac{dx_L^\mu}{d\lambda}(\lambda_2) =: k_2^\mu \tag{4.98}$$

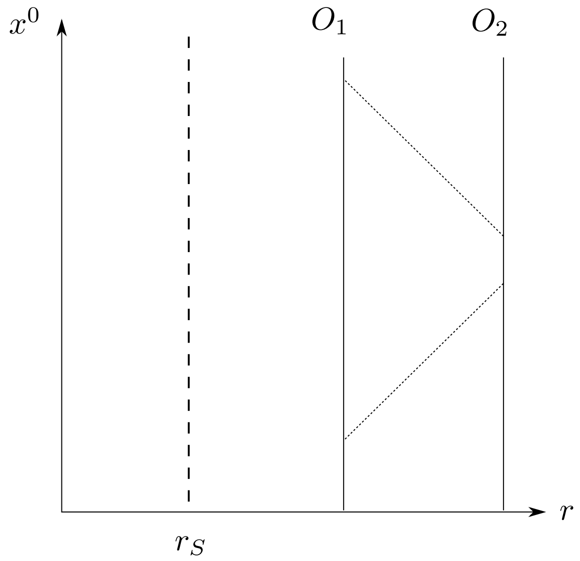

Figure 4.12: Two observers keeping a constant distance to the event horizon, sending ieach other light signals. They observe a shift in frequency $\omega_1 = \omega_2$ of the same light ray.

such that

$$\omega_1 = g_{\mu\nu}\frac{dx_1^\mu}{ds}k_1^\nu = A(r_1)k_1^0\frac{dx_1^0}{ds} \tag{4.99}$$

with

$$A(r) = 1 - \frac{r_S}{r}. \tag{4.100}$$

The other observer measures the frequency

$$\omega_2 = A(r_2)k_2^0\frac{dx_2^\mu}{d\lambda} = g_{\mu\nu}\frac{dx_2^\mu}{ds}\frac{dx_1^\nu}{d\lambda}. \tag{4.101}$$

Remember that for any geodesic in Schwarzschild spacetime (even light-like ones, see last section or the exercises)

$$\frac{dx_L^0}{d\lambda}A(r_L(\lambda)) = \text{const} =: F \tag{4.102}$$

is constant along the geodesic Then

$$k_L^0(\lambda)A(r_L(\lambda)) = \text{const}. \tag{4.103}$$

We can then write

$$\frac{\omega_1}{\frac{dx_1^0}{ds}} = \frac{\omega_2}{\frac{dx_2^0}{ds}} \quad \Leftrightarrow \quad \omega_1\sqrt{1-\frac{r_S}{r_1}} = \omega_2\sqrt{1-\frac{r_S}{r_2}} \tag{4.104}$$

$$\Leftrightarrow \frac{\omega_2}{\omega_1} = \frac{\sqrt{1-\frac{r_S}{r_1}}}{\sqrt{1-\frac{r_S}{r_2}}}. \tag{4.105}$$

Since $r_2 > r_1 > r_S$, this means that $\omega_2 < \omega_1$, so O_2 sees the light ray with a different (i.e. red-shifted) frequency compared to O_1. Slower-running frequency means that O_1's time seems to run slower from the point of view of O_2. Also, $\omega_2/\omega_1 \to 0$ as $r_1 \to r_2$, so from the point of view of an external observer, anything falling into the black hole becomes infinitely red-shifted eventually.

4.5 Motion in rotationally symmetric gravity field

Computing the motion around a rotationally symmetric matter distribution was one of the main successes of Newtonian physics, allowing to derive Kepler's laws of motion which had been found empirically before. A similar computation can be carried out in General relativity, using the Schwarzschild metric.

We consider the geodesic equation of motion

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0. \tag{4.106}$$

In what follows we will consider both massive and massless particles. In the case of massive particles, we assume that the curve parameter $\lambda = s$ coincides with proper time. In the massless case, the geodesic will be light-like, so proper time is zero, so it cannot be used as curve parameter. In that case, we assume a parameterisation from the geometrical optics approximation.

The normalisation of the world lines will be indicated by the parameter ϵ:

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \epsilon = \begin{cases} 1 & \text{falls } m > 0, \\ 0 & \text{falls } m = 0. \end{cases} \tag{4.107}$$

We work in our usual coordinates

$$\left(x^0, x^1, x^2, x^3\right) = \left(ct, r, \theta, \phi\right). \tag{4.108}$$

In what follows, the "dot" shall always indicate derivation w.r.t. λ.

First we simplify the situation somewhat: Due to rotational symmetry, we choose our curve to have an initial condition of starting in the $\theta = \pi/2$-plane, and having an initial velocity vector also lying in that plane, i.e. $\theta(0) = \frac{\pi}{2}$ and $\dot\theta(0) = 0$. Consider (4.106) for $\mu = 2$, i.e. the θ-coordinate, we get with (4.14) – (4.18):

$$\ddot\theta = -\frac{2}{r}\dot r\dot\theta + \sin\theta\cos\theta\,(\dot\phi)^2. \tag{4.109}$$

One can see that this part of the geodesic equations can be solved via

$$\theta(\lambda) = \frac{\pi}{2} = \text{constant}. \tag{4.110}$$

So we assume that indeed that $\dot{\theta} = 0$ throughout the whole curve, so the movement of the particle takes place entirely in the $\theta = \frac{\pi}{2}$-plane. With this choice, the rest of the geodesic equations ($\mu = 0, 1, 3$) can be written as:

$$\ddot{x}^0 = -\frac{A'}{A}\dot{x}^0\dot{r} \tag{4.111}$$

$$\ddot{r} = -\frac{A'}{2C}(\dot{x}^0)^2 - \frac{C'}{2C}\dot{r}^2 + \frac{r}{C}\dot{\phi}^2 \tag{4.112}$$

$$\ddot{\phi} = -\frac{2}{r}\,\dot{r}\,\dot{\phi}. \tag{4.113}$$

The last equation (4.113) can be rewritten as

$$\frac{1}{r^2}\frac{d}{d\lambda}\left(r^2\,\dot{\phi}\right) = 0. \tag{4.114}$$

In other words, the quantity

$$\ell := r^2\,\dot{\phi} = \text{const} \tag{4.115}$$

is a constant of motion. This is reminiscent of the angular momentum, and indeed, the quantity ℓ is called angular momentum of a particle in GR. Indeed, for large r this quantity tends towards the Newtonian angular momentum $\ell \approx L/c$.

The metric we consider is static, so there is a time-like KVF, ∂_0. As is shown in an exercise, the inner product between the velocity vector of a geodesic an a KVF are constant. In our case, this means that

$$F := g_{\mu\nu}\dot{x}^\mu X^\nu = \left(1 - \frac{r_S}{r(\lambda)}\right)\dot{x}^0(\lambda) = \text{const.} \tag{4.116}$$

One can show that this is equivalent to the geodesic equation for $x^0(\lambda)$ (4.111). We have already encountered this constant of motion in our treatment of the central fall.

The fourth and last geodesic equation is the one for r (4.112). We can rewrite it with the help of the constants F and ℓ:

$$\frac{d^2r}{d\lambda^2} = -\frac{F^2A'}{2A} - \frac{C'}{2C}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{Cr^3}. \tag{4.117}$$

Multiplying with $2C\frac{dr}{d\lambda}$, this yields

$$\frac{d}{d\lambda}\left[C\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2} - \frac{F^2}{A}\right] = 0. \tag{4.118}$$

Indeed, the expression the square brackets is nothing else but the normalisation of the velocity vector, i.e.

$$C\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2} - \frac{F^2}{A} = -\epsilon = \text{const.} \tag{4.119}$$

This can be written as

$$\left(\frac{dr}{d\lambda}\right)^2 = F^2 - \frac{\ell^2}{Cr^2} - \frac{\epsilon}{C} \tag{4.120}$$

Now we have used all components of the geodesic equations. To find a solution to the geodesic equations for certain given initial conditions, one proceeds as follows: First compute F and ℓ from the initial condition. Depending on whether one wants to solve for massive or massless particles, one either has to use $\epsilon = 1$ or $\epsilon = 0$. Then one has to solve the equation (4.120). This will usually be the main difficulty, since it might not be possible exactly to solve that ODE. If one can solve it, however, receiving a solution $r(\lambda)$ (i.e. numerically), then one can, by integrating (4.115) compute $\phi(\lambda)$, and by intgration of (4.116) one can get $x^0(\lambda)$. With (4.110) this gives all components of the trajectory.

In general this is complicated, but one can make some qualitative statements using (4.120). It can be rewritten as:

$$\frac{\dot{r}^2}{2} + V_{\text{eff}}(r) = E = \text{const.} \tag{4.121}$$

Here the effective potential is given by

$$V_{\text{eff}}(r) = -\frac{r_S}{2r}\epsilon + \frac{\ell^2}{2r^2} - \frac{\ell^2 r_S}{2r^3}. \tag{4.122}$$

The effective energy E is given by

$$E := \frac{F^2 - \epsilon}{2}. \tag{4.123}$$

Equation (4.121) describes a classical one-dimensional movement of a particle of mass $m = 1$ and position r in a potential (4.122), with total energy E. Careful: these are not necessarily the actual energy of the particle, the name is just chosen to highlight the similarity of (4.121) with the 1d-movement.

Solutions to these equations have several well-understood properties. The solutions are constricted to the region $E \geq V_{\text{eff}}$. If there is a minimum of V_{eff}, and E is only slightly above it, the system will make small oscillations around that minimum.

4.5.1 The massive case: $\epsilon = 1$

An equation like (4.121) also appears for the Kepler problem in Newtonian mechanics. In that case the effective potential has a very similar form. The first two terms in (4.122) also appear. One recognise the attractive Newtonian potential $\sim 1/r$, as well as the repulsive term $\sim 1/r^2$, signifying the angular momentum barrier. For large r, these terms dominate the third term, which is propertional to $1/r^3$. That third term does not appear in the Kepler problem, and is a direct consequence of relativistic physics. It is attractive, and dominates the other two, in case a particle comes too close to the center $r = 0$.

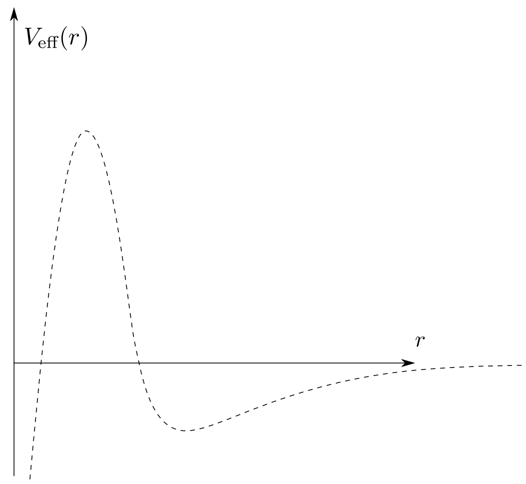

Figure 4.13: Effective potential for $\epsilon = 1$, for some fixed value of ℓ: For large r, this looks like the one from Newtonian mechanics. For small r, the angular momentum barrier can be crossed, when the particle comes too close to the black hole.

For a fixed value of ℓ, the effective potential $V_{\text{eff}}(r)$ is depicted in figure 4.13.

In Newtonian mechanics, there are essentially two cases: a bound trajectory (ellipses), and unbound trajectories (hyperboloids or paraboloids). In the case of a black hole, there are, however, three cases:

1. The bound case: $E < 0$ and $r \geq r_{\text{max}}$. These are similar to the Kepler ellipses, on which the particle orbits around the black hole indefinitely.

2. The unbound case: $0 < E < V_{\text{max}}$ and $r \geq r_{\text{max}}$. These are similar to scattering trajectories, where an object comes from infinity, gets scattered, and flies off to infinity again.

3. The singular case: $E > V_{\text{max}}$ or $r < r_{\text{max}}$. These can only appear in the relativistic case. The only ones which are physically relevant are the ones that start at infinity, and are able to cross the angular momentum barrier, coming so close to the black hole that they get sucked in.

4.5.2 The massless case: $\epsilon = 0$

The trajectory for $m = 0$, i.e. for $\epsilon = 0$, looks quite different from the one in Newtonian physics. Here we have

$$V_{\text{eff}}(r) = \frac{\ell^2}{2r^2} - \frac{\ell^2 r_S}{2r^3}. \tag{4.124}$$

The attractive term from the Newtonian potential $\sim -1/r$ is missing.

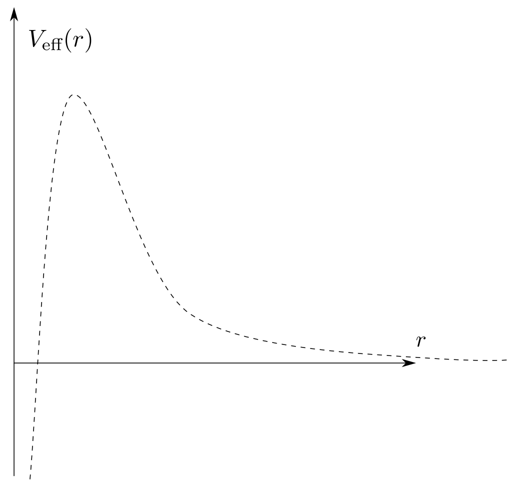

Figure 4.14: Effective potential for $m = 0$. There is a local maxímum, leading to an unstable circular orbit of light particles (the photosphere).

In figure 4.14 we have some effective potential for $\epsilon = 0$. There are no (stable) bound trajectories, since the effective potential has no local minimum. There are only those trajectories which are being scattered, or those which vanish in the black hole.

Since the potential has a maximum, there is an circular orbit around the black hole. However, it is unstable: a tiny disturbance will send it either to infinity or into the black hole.

4.5.3 Light scattering

The light scattering in the gravitational field of a star was the first experimental verification of GR. These light rays correspond to geodesics in figure 4.14 which come from infinity, get close to the star (up to $r = r_{\text{min}}$, and then fly off to infinity again. During that time, the accumulated angle, which describes the scatterinag angle of the light ray, can be computed in the limit of $r \gg r_S$.

To compute this, we are mainly interested in the connection between r and ϕ. From (4.120) we get:

$$\frac{d\phi}{dr} = \frac{d\phi}{d\lambda}\left(\frac{dr}{d\lambda}\right)^{-1} = \pm\frac{\ell}{r^2}\frac{1}{\sqrt{F^2 - \ell^2/(r^2C) - \epsilon/C}}. \tag{4.125}$$

Here the sign has to be chosen to be the correct one of $dr/d\lambda$, depending on whether the light ray is outgoing or incoming. Integrating once and using $A = C^{-1}$, one gets

$$\phi(r_2) - \phi(r_1) = \pm\int_{r_1}^{r_2} dr\,\frac{1}{r^2}\frac{\sqrt{C(r)}}{\sqrt{\frac{F^2}{A(r)\ell^2} - \frac{1}{r^2} - \frac{\epsilon}{\ell^2}}}. \tag{4.126}$$

In most cases this integral will not be analytically solvable However, for the case of $r \gg r_S$, one can compute it for $r_2 = r_{\text{min}}$ and $r_1 = \infty$.

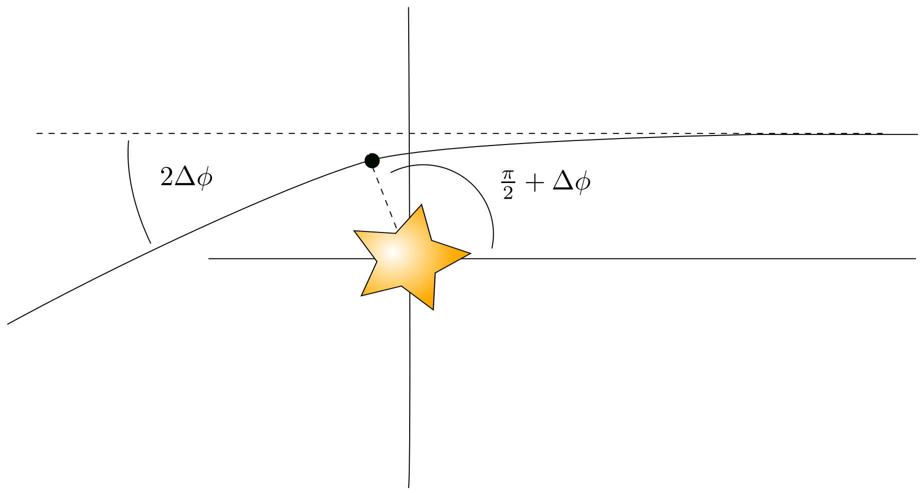

Figure 4.15: A light ray scattered in the gravity field of a star.

We consider $\epsilon = 0$, looking at the deflection of light. We set $r_1 = \infty$, $\phi(r_1) = 0$, and call the total angle of deflection $2\Delta\phi$. In figure 4.15 one can see that the angle, for which $r = r_{\text{min}}$, has to satisfy

$$\phi(r_{\text{min}}) = \frac{\pi}{2} + \Delta\phi \tag{4.127}$$

The point with $r = r_{\text{min}}$ is the one where

$$\frac{dr}{d\phi} = 0. \tag{4.128}$$

Hence the inverse of (4.125) has to vanish. With $\epsilon = 0$ we get

$$\frac{F^2}{A(r_{\text{min}})\ell^2} - \frac{1}{r_{\text{min}}^2} = 0, \tag{4.129}$$

or

$$\frac{F^2}{\ell^2} = \frac{A(r_{\text{min}})}{r_{\text{min}}^2}. \tag{4.130}$$

With this, and $\epsilon = 0$, the integrand of (4.126) becomes

$$\frac{1}{r^2}\frac{\sqrt{C(r)}}{\sqrt{\frac{F^2}{A(r)\ell^2} - \frac{1}{r^2} - \frac{\epsilon}{\ell^2}}} = \frac{1}{r}\frac{\sqrt{C(r)}}{\sqrt{\frac{r^2 A(r_{\text{min}})}{r_{\text{min}}^2 A(r)} - 1}}, \tag{4.131}$$

and the deflection angle satisfies (choosing the correct sign for incoming light rays)

$$\Delta\phi + \frac{\pi}{2} = \phi(r_{\text{min}}) - \phi(\infty) = -\int_\infty^{r_{\text{min}}} dr\,\frac{1}{r}\frac{\sqrt{C(r)}}{\sqrt{\frac{r^2 A(r_{\text{min}})}{r_{\text{min}}^2 A(r)} - 1}}. \tag{4.132}$$

Using the approximation $r_S/r_{\text{min}} \ll 1$, i.e. that the light ray does not come close to the Schwarzschild radius, we get.

$$\sqrt{C(r)} = \frac{1}{\sqrt{1-\frac{r_S}{r}}} = 1 + \frac{r_S}{2r} + \ldots \tag{4.133}$$

Also, we have that

$$\begin{aligned}
\frac{r^2}{r_{\text{min}}^2}\frac{A(r_{\text{min}})}{A(r)} - 1 &= \frac{r^2}{r_{\text{min}}^2}\frac{1-\frac{r_S}{r_{\text{min}}}}{1-\frac{r_S}{r}} - 1 \\
&= \frac{r^2}{r_{\text{min}}^2}\left(1 - \frac{r_S}{r_{\text{min}}} + \frac{r_S}{r}\right) - 1 + \ldots \\
&= \left(\frac{r^2}{r_{\text{min}}^2} - 1\right)\left(1 - \frac{r r_S}{r_{\text{min}}(r + r_{\text{min}})}\right) + \ldots
\end{aligned}$$

With this we obtain

$$\frac{\pi}{2} + \Delta\phi \approx \int_{r_{\text{min}}}^{\infty} dr\, \frac{r_{\text{min}}}{r}\frac{1 + r_S/(2r)}{\sqrt{(r^2 - r_{\text{min}}^2)\left(1 - \frac{r r_S}{r_{\text{min}}(r+r_{\text{min}})}\right)}}. \tag{4.134}$$

The first term under the integral assumes all values from 0 to ∞.The second, however, is always close to 1, which is why we can expand it with the help of $1/\sqrt{1+x} \approx 1 - x/2$ for $|x| \ll 1$. With this we get, while only considering terms of first order in r_S/r and r_S/r_{min}:

$$\Delta\phi + \frac{\pi}{2} \approx \int_{r_{\text{min}}}^{\infty} \frac{dr}{\sqrt{r^2 - r_{\text{min}}^2}}\frac{r_{\text{min}}}{r}\left(1 + \frac{r_S}{2r} + \frac{r r_S}{2r_{\text{min}}(r + r_{\text{min}})}\right). \tag{4.135}$$

This integral can be solved, and one gets:

$$\Delta\phi + \frac{\pi}{2} = \frac{\pi}{2} + \frac{r_S}{2r_{\text{min}}} + \frac{r_S}{2r_{\text{min}}} \tag{4.136}$$

or, in other words:

$$\Delta\phi = \frac{r_S}{r_{\text{min}}}. \tag{4.137}$$

We consider the case of the sun, and a light ray just barely grazing the surface. We have the values $r_S^{\odot} = 2.97\text{km}$, and $r_{\text{min}}^{\odot} = 695500\text{km}$, in other words:

$$2\Delta\phi^{\odot} = \frac{2r_S^{\odot}}{r_{\text{min}}^{\odot}} = 0.00000854 = 1.76 \text{ arc seconds.}$$

This value is different from the value predicted by Newtonian mechanics, by a factor of 2. This is significant, and it was possible to measure this value with the technical possibilities of the early 20th century. Sir Arthur Eddington used this in order to confirm GR this way in 1919, during a total solar eclipse.

Figure 4.16: On the scale of, say, our Milky Way, matter is arranged quite irregularly.

4.6 Cosmology

Cosmology attempts to describe the large scale-structure of the universe using general relativity. Observational data about the universe shows that on large scales, matter (that we can see) is distributed homogeneously any isotropically. So on very large scales, the geometry should be **isotropic** and **homogeneous**.

Homogeneous means that "every point looks the same", while isotropic means that "at a point every direction looks the same". Because of homogeneity, space-time foliates into hyper-surfaces Σ_t. For two points $p, q \in \Sigma_t$, there is a map $\phi : M \to M$, so that

$$\phi^* g = g \tag{4.138}$$
$$\phi(p) = q. \tag{4.139}$$

Such a map is called an isometry. Isotropy means there is a unit time-like vector field X, so that for every point p and s_1, s_2 orthonormal to X, $|s_1|^2 = |s_2|^2$, there is a map $\phi : M \to M$ with

$$\phi^* g = g \tag{4.140}$$

with $\phi(p) = p$, $d\phi(X(p)) = X(p)$, $d\phi(s_1) = s_2$. In formulas:

$$\phi^\mu(x^\nu) : M^\mu{}_\nu := \frac{\partial \phi^\mu}{\partial x^\nu}, \quad M^\mu{}_\nu X^\nu(p) = X^\mu(p), \quad M^\mu{}_\nu s_1^\nu = s_2^\mu. \tag{4.141}$$

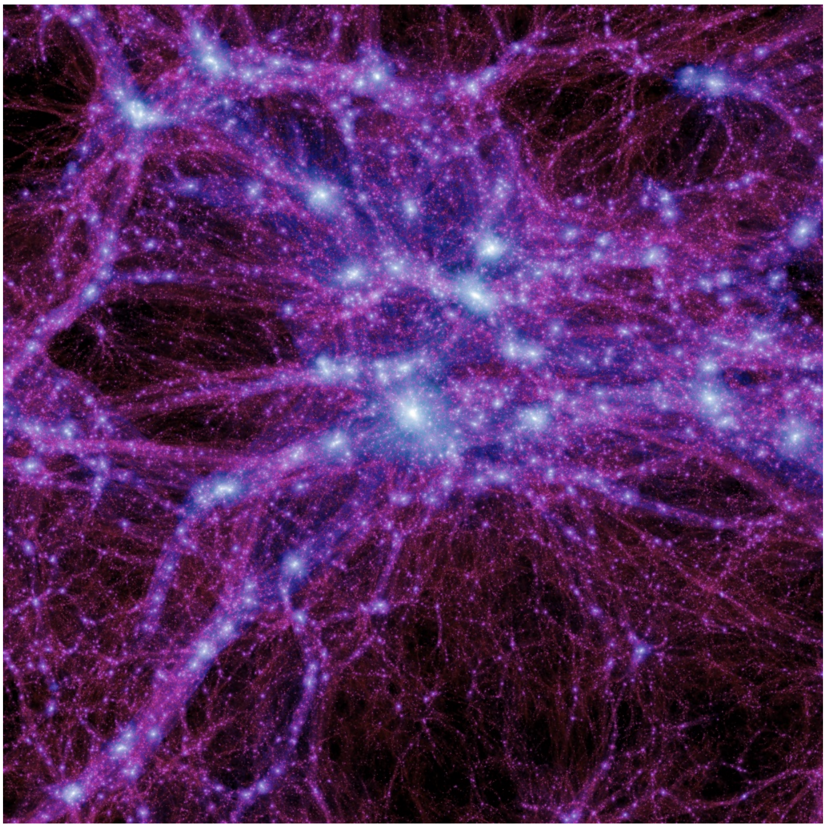

Figure 4.17: On larger scales, galaxy clusters and super clusters arrange in filaments and voids.

Homogeneity and isotropy together mean that the Σ_t are orthogonal to X. The metric needs to have the form

$$ds^2 = dt^2 - h_{ij}dx^i dx^j \tag{4.142}$$

where h is the **spatial metric** on each Σ_t. This metric h_{ij} is Riemannian, and describes homogenous and isotropic universe Σ_t. Note that, unlike in the static case, we allow h_{ij} to depend on t here. Consider the Riemann curvature tensor of h: ${}^{(3)}R^i{}_{jkl}$. Note that in general, ${}^{(3)}R^i{}_{jkl} \neq R^i{}_{jkl}$. The symmetries are ${}^{(3)}R^{ij}{}_{kl} = h^{jj'}\,{}^{(3)}R^i{}_{j'kl}$, ${}^{(3)}R^{ij}{}_{kl} = {}^{(3)}R^{[ij]}{}_{[kl]}$. These coefficients define a map $\Phi : V \to V$, where $V = (T_p\Sigma_t) \wedge (T_p\Sigma_t)$ via

$$(\Phi v)^{ij} \coloneqq {}^{(3)}R^{ij}{}_{kl}v^{kl} \tag{4.143}$$

with $v^{ij} = v^{[ij]}$ and $\dim V = 3$ the space of 3d bivectors. There is a positive definite inner product H on V:

$$H(v,w) \coloneqq h_{i[j}h_{k]l}v^{ij}w^{kl} \tag{4.144}$$

with $H(v,w) = H(w,v)$ and $H(v,v) \geq 0$. From the symmetries of ${}^{(3)}R^{ij}{}_{kl}$, we get

$$H(\Phi v, w) = H(v, \Phi w), \tag{4.145}$$

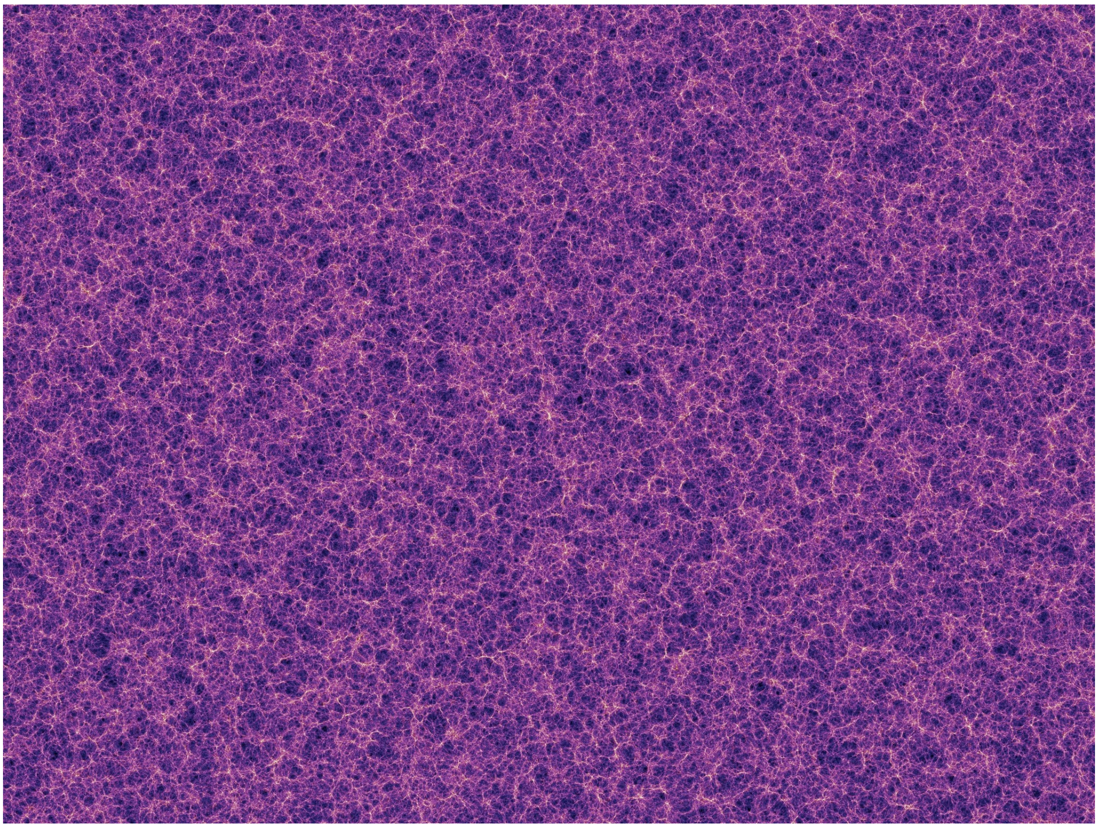

Figure 4.18: On the largest accessible scale, matter is approximately homogenous and isotropic.

meaning Φ is self-adjoint, so it has three orthogonal eigenvectors. Because of isotropy, all eigenvalues are the same (otherwise, we could chose a preferred direction in V, and therefore in $T_p\Sigma_t$). So

$$\Phi = K\mathrm{Id}_V \tag{4.146}$$

$$\Leftrightarrow {}^{(3)}R^{ij}{}_{kl} = K\delta^i{}_{[j}\delta^j{}_{l]} \tag{4.147}$$

$$\Leftrightarrow {}^{(3)}R_{ijkl} = K(h_{ik}h_{jl} - h_{il}h_{jk}) \tag{4.148}$$

with a constant K and Id_V the identity of V. Because of homogeneity, K only depends on t, so each Σ_t is a space of constant curvature because the Ricci tensor is ${}^{(3)}R_{jl} = 2Kh_{jl}$ and the Ricci scalar is ${}^{(3)}R = 6K$. We need to distinguish three cases:

- $K > 0$: $\Sigma_t \sim S^3$ of radius $\frac{1}{\sqrt{K}}$
- $K = 0$: $\Sigma_t = \mathbb{R}^3$, $h_{ij} = \delta_{ij}$
- $K < 0$: $\Sigma_t \sim H^3$, hyperbolic space of radius $\frac{1}{\sqrt{-K}}$.

We can write

$$\Sigma_t \simeq \left\{ (w,x,y,z) \in \mathbb{R}^4 \;\middle|\; \begin{array}{ll} K > 0 & w^2 + x^2 + y^2 + z^2 = \frac{1}{K^2} \\ K = 0 & w^2 + x^2 + y^2 + z^2 = 0 \\ K < 0 & w^2 - x^2 - y^2 - z^2 = \frac{1}{K^2} \end{array} \right\}, \tag{4.149}$$

at least locally: e.g. one could have $\Sigma_t \sim \mathbb{RP}^3 = S^3/\mathbb{Z}_2 \sim \mathrm{SO}(3)$ (a 3-sphere module discrete symmetry). But we ignore the not-simply-connected alternatives. We write the metric as

$$ds^2 = dt^2 - a(t)^2 \begin{cases} d\psi^2 + \sin^2\psi d\Omega^2 & K > 0 \\ dx^2 + dy^2 + dz^2 & K = 0 \\ d\psi^2 + \sinh^2\psi d\Omega^2 & K < 0 \end{cases} \tag{4.150}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi$. We define

$$k = \begin{cases} +1 \\ 0 \\ -1 \end{cases} \text{if } K = \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases} \tag{4.151}$$

and write

$$dx^2 + dy^2 + dz^2 = r^2 d\Omega^2 + dr^2 \tag{4.152}$$

and for $k = 1$ with $r := \sin\psi$

$$d\psi^2 = \frac{dr^2}{1 - r^2}, \tag{4.153}$$

for $k = -1$ and $r := \sinh\psi$

$$d\psi^2 = \frac{dr^2}{1 + r^2}. \tag{4.154}$$

The general metric can be written as

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right). \tag{4.155}$$

4.6.1 Homogeneous and isotropic matter

Consider the energy-momentum-tensor

$$T^{\mu\nu} = \rho u^\mu u^\nu + P(g^{\mu\nu} + u^\mu u^\nu). \tag{4.156}$$

In our case

$$u^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.157}$$

We now consider only the case $k = 0$ with the metric

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2). \tag{4.158}$$

The non-vanishing Christoffel symbols are

$$\Gamma^t_{xx} = \frac{1}{2}g^{tt}(-g_{xx,t}) = a\dot{a} = \Gamma^t_{yy} = \Gamma^t_{zz} \text{ (isotropy)} \tag{4.159}$$

$$\Gamma^x_{tx} = \frac{1}{2}g^{xx}(g_{xx,t}) = \frac{\dot{a}}{a} = \Gamma^x_{xt}, \text{ same for } y, z. \tag{4.160}$$

The Riemann curvature tensor is

$$R^t{}_{ttt} = 0 \tag{4.161}$$

$$R^x{}_{txt} = -\frac{\ddot{a}}{a}, \text{ same for } y, z. \tag{4.162}$$

$$R^t{}_{xtx} = \ddot{a}a \tag{4.163}$$

$$R^y{}_{xyx} = \dot{a}^2, \text{ same for all combinations of } x, y, z. \tag{4.164}$$

Then the Ricci tensor is

$$R_{tt} = -3\frac{\ddot{a}}{a} \tag{4.165}$$

$$R_{xx} = \ddot{a}a + 2\dot{a}^2 \tag{4.166}$$

and the Ricci scalar is

$$R = R_{tt} - a^{-2}(R_{xx} + R_{yy} + R_{zz}) = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \tag{4.167}$$

The energy-momentum tensor is given by

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & -a^2P & & \\ & & -a^2P & \\ & & & -a^2P \end{pmatrix}. \tag{4.168}$$

From the Einstein equations, we get

$$R_{tt} - \frac{1}{2}g_{tt}R \quad = \quad = -\kappa T_{tt} \quad \Leftrightarrow \quad -3\frac{\dot{a}^2}{a^2} = -\kappa\rho \tag{4.169}$$

$$R_{xx} - \frac{1}{2}g_{xx}R \quad = \quad -\kappa T_{xx} \quad \Leftrightarrow \quad 2\ddot{a}a + \dot{a}^2 = \kappa a^2 P \text{ (same for } x, y) \tag{4.170}$$

$$\Rightarrow \frac{3\ddot{a}}{a} \quad = \quad -\frac{\kappa}{2}(\rho + 3P), \quad \frac{3\dot{a}^2}{a^2} = \kappa\rho. \tag{4.171}$$

With $\kappa = 8\pi G_N$ and setting $c = G_N = 1$:

$$\frac{3\ddot{a}}{a} = -4\pi(\rho + 3P) \tag{4.172}$$

and

$$\frac{3\ddot{a}^2}{a^2} = 8\pi\rho. \tag{4.173}$$

For $k \neq 0$, the result is (without proof):

$$\frac{3\ddot{a}}{a} = -4\pi(\rho + 3P) \tag{4.174}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{\rho} - \frac{k}{a^2}. \tag{4.175}$$

These are called the **Friedmann-Roberston-Walker equations** where a is the so-called **scale factor**. The immediate observation is that a matter-filled universe cannot be static: We have $\ddot{a} < 0$ unless $\rho = 0, P = 0$ (and $k = 0$).

Historically, Einstein published his equations in 1915 and in 1917 introduced the **cosmological constant** Λ as a modification:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}. \tag{4.176}$$

In 1923, The FRW equations were published. With the cosmological constant term they take the form

$$\begin{aligned} \frac{3\ddot{a}}{a} &= -4\pi(\rho + 3P) + \Lambda & (4.177)\\ \frac{\dot{a}^2}{a^2} &= \frac{8\pi}{3}\rho - \frac{k}{a^2} - \frac{\Lambda}{3}. & (4.178)\end{aligned}$$

This way, Λ and ρ might have been tuned to allow for a static universe where $\ddot{a} = \dot{a} = 0$, so that $k \stackrel{!}{=} 1$. However, in 1939 Hubble observed the expansion of the universe.

In the FRW equations, $\rho(t)$ is a matter density which in this context means it contains everything that is not gravity, e.g. dust, radiation, dark matter and so on.

Hubble's law (which holds no matter what the precise solutions for a, ρ, P are) can be formulated as follows: Consider the distance of two galaxies. Their world lines correspond to the unit time-like vector field (they are the isotropic observers). Their time-dependent distance $d(t)$ is measured in the induced (Riemannian) metric on Σ_t:

$$d(t) = \int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \tag{4.179}$$

where x^μ is a curve in Σ_t. We have

$$0 = g_{\mu 0}\dot{x}^\mu \dot{x}^0 \tag{4.180}$$

$$0 = g_{00}(\dot{x}^0)^2 \tag{4.181}$$

$$\Rightarrow d(t) = a(t) \int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} = a(t) d_0. \tag{4.182}$$

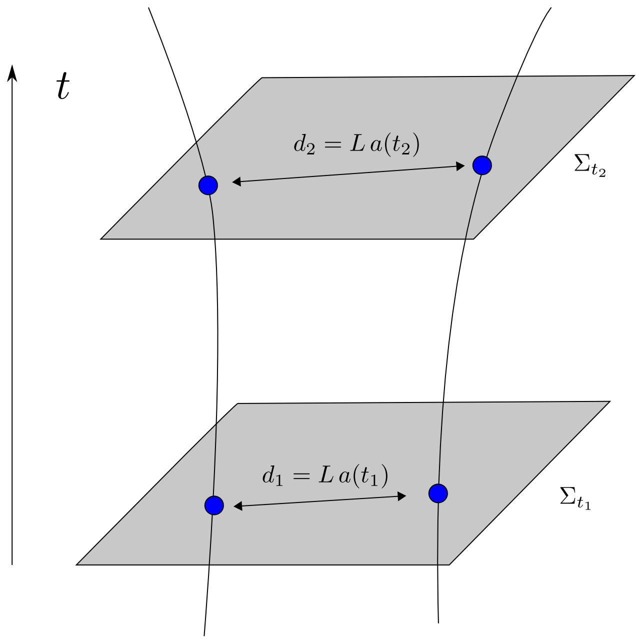

Figure 4.19: The distance of two isotropic observers can change over time.

Since d_0 does not depend on t, we get

$$\dot{d} = \dot{a} d_0 = \frac{\dot{a}}{a} d(t) =: H d \tag{4.183}$$

with the **Hubble parameter** H. So whether the universe expands ($H > 0$) or contracts ($H < 0$), objects have a "relative velocity" to one another proportional to their distance. This is the observation Hubble made: Galaxies move away from us, the further away, the faster. This is a crucial, important confirmation of general relativity. The best value for H nowadays is $H \approx 1/(14.4bn\text{ ys})$. This value can change in time. Note that $\dot{d}$ can easily become greater than the speed of light. This is no violation of relativity. In fact, each galaxy is (nearly) at rest. They do not move with respect to one another, rather the space between them expands.

4.6.2 Light propagation in FRW spacetimes

In the geometrical optics approximation, $\lambda \mapsto x^\mu(\lambda)$ is a light-like geodesic and the velocity vector is equal to the wave vector, i.e. $dx^\mu/d\lambda = k^\mu$.

Consider the "projection" of the light-like geodesic on one Σ_t: $\lambda \mapsto (x^1(\lambda), x^2(\lambda), x^3(\lambda))$.

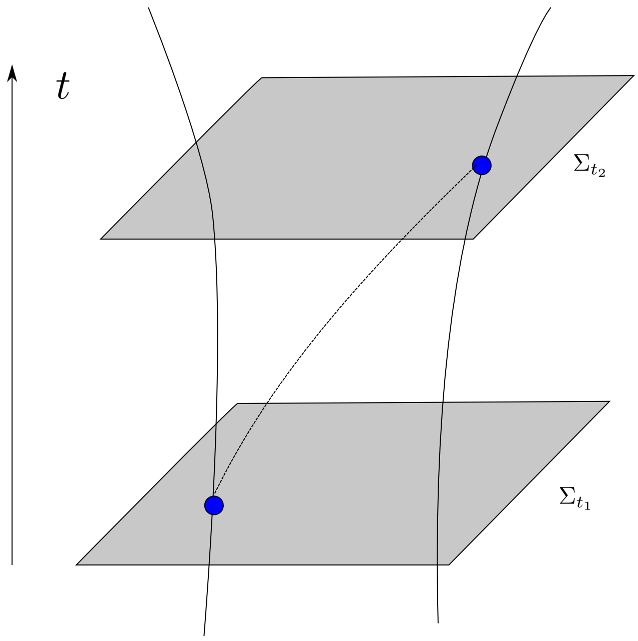

Figure 4.20: Two galaxies sending a light ray to one another.

The Christoffel symbols are

$$\Gamma^i_{00} = \frac{1}{2} g^{i\lambda}(-g_{00,\lambda}) = 0 \tag{4.184}$$

$$\Gamma^i_{0j} = \frac{1}{2} g^{i\lambda}(g_{\lambda j,0}) = \delta_{ij}\frac{\dot{a}}{a} \tag{4.185}$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\lambda}(\ldots) = \frac{1}{2} g^{im}(\ldots). \tag{4.186}$$

with equation 4.186 being the same as for the spatial metric. The geodesic equation can be written as

$$\begin{aligned} 0 &= \ddot{x}^i + \Gamma^i_{j0}\dot{x}^j\dot{x}^0 \\ &= \ddot{x}^i + \frac{\dot{a}}{a}\dot{x}^0\dot{x}^i + \Gamma^i_{jk}\dot{x}^k\dot{x}^j. \end{aligned} \tag{4.187}$$

So the "projection onto space" of a geodesic is not necessarily one due to the extra term. But it is the representation of one. Let $y^i(\sigma) \coloneqq x^i(\lambda(\sigma))$ for some $\sigma \mapsto \lambda(\sigma)$. In the following, a dash $'$ denotes the derivative with respect to σ and a dot $\dot{}$ denotes the derivative with respect to λ. Then

$$(y^i)'' = (\lambda'\dot{x}^i)' = \lambda''\dot{x}^i + (\lambda')^2\ddot{x}^i. \tag{4.188}$$

Then solve the equation

$$\lambda'' = (\lambda')^2\frac{\dot{a}}{a}\dot{x}^0 \tag{4.189}$$

which depends on λ via $t(\lambda)$. The precise solution depends on the precise solutions of $a(t)$ and $t(\lambda)$. We can then write

$$(y^i)'' + \Gamma^i_{jk}(y^j)'(y^k)' = (\lambda')^2 \underbrace{\left(\ddot{x}^i + \frac{\dot{a}}{a}\dot{x}^0\dot{x}^i + \Gamma_{jk^i\dot{x}^j\dot{x}^k}\right)}_{=0} = 0. \tag{4.190}$$

With this new parameterization, the projection of a light ray onto space traces out a geodesic on Σ_t (a great circle on S^3, a straight line on $\mathbb{R}^3$, a great arc on H^3). Each of these lines follows a Killing vector field, corresponding to a translation symmetry of Σ_t. Call this X_t, and expand it to all of spacetime. Make these X_t into a 4-dimensional Killing vector field on all of spacetime: ${}^{(4)}X(t, x^1, x^2, x^3) = X_t(x^1, x^2, x^3)$ with $g_{\mu\nu}{}^{(4)}X^{\mu\,(4)}X^\nu = -a^2(t)$. The velocity vector of the light ray is

$$k(t) = \dot{x}(t) = \alpha(t)\frac{\partial}{\partial t} + \beta(t)^{(4)}X. \tag{4.191}$$

The vector k is light-like, so

$$0 = g_{\mu\nu}k^\mu k^\nu = \alpha^2 - a^2\beta^2. \tag{4.192}$$

On the other hand, we use the fact that $\lambda \mapsto x^\mu(\lambda)$ is a geodesic, so

$$\langle k, {}^{(4)}X\rangle|_{t=t_1} = \langle k, {}^{(4)}X\rangle|_{t=t_2} \tag{4.193}$$

which means that $(-a^2\beta)$ is constant in λ (so also constant in t). That means that an observer at rest in galaxy 1 measures the frequency

$$\omega_1 = \left\langle \frac{\partial}{\partial t}, k(t_1)\right\rangle = \alpha(t_1) = a(t_1)\beta(t_1) \tag{4.194}$$

and an observer at rest in galaxy 2 measures the frequency

$$\omega_2 = \left\langle \frac{\partial}{\partial t}, k(t_2)\right\rangle = \alpha(t_2) = a(t_2)\beta(t_2). \tag{4.195}$$

From this follows that $\omega_1 a(t_1) = \omega_2 a(t_2)$ or

$$\frac{\omega_2}{\omega_1} = \frac{a(t_1)}{a(t_2)}. \tag{4.196}$$

So as the universe expands, frequencies get red-shifted since for a wave at t_1 with wave length $\lambda_1 = 1/\omega_1$, the wave length at t_2 is $\lambda_2 = \lambda_1 \cdot a(t_2)/a(t_1)$. One defines the so-called **red-shift**

$$z := \frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{\omega_1}{\omega_2} - 1 = \frac{a(t_2)}{a(t_1)} - 1 \tag{4.197}$$

or

$$1 + z = \frac{a(t_2)}{a(t_1)}. \tag{4.198}$$

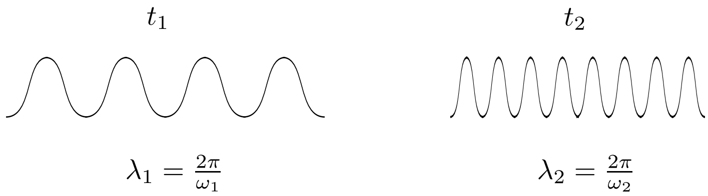

Figure 4.21: After light has been travelling through the universe, it has been reduced in frequency, due to the expansion of the universe.

For "nearby" galaxies (= during the time light rays need between them and us, the universe has not expanded significantly), one has $t_2 - t_1 \approx d(t_1)$, also

$$\frac{a(t_2) - a(t_1)}{t_2 - t_1} \approx \dot{a}(t_1) \tag{4.199}$$

which means that

$$z = \frac{a(t_2)}{a(t_1)} - 1 \approx dH. \tag{4.200}$$

So the red-shift of galaxies (near us) is proportional to the distance. This will be violated for larger distances, depending on $\ddot{a}$. This is precisely what was measured in the SNIa observations in $\sim$ 1998, and one found $\ddot{a} > 0$. Compare this with the FRW equation (4.177): $3\ddot{a}/a = -4\pi(\rho + 3P) + \Lambda$. If $\ddot{a} > 0$, then necessarily $\Lambda > 0$. After more than 70 years, Einstein's "greatest blunder" had been rehabilitated.

FRW-spacetimes

In the following, we will look at different solutions to the Einstein equations.

First, we look at the so-called **Einstein universe** with $\dot{a} = 0$. Matter is dominated by galaxies with negligible relative motion ($P = 0$). Then

$$\Lambda = 4\pi\rho \tag{4.201}$$

$$0 = \frac{8\pi}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{4.202}$$

from which follows that $k = 1 = 4\pi\rho a^2$. Then

$$a = \sqrt{\frac{1}{4\pi\rho}}\left(= \sqrt{\frac{c^2}{4\pi\rho G_N}}\right). \tag{4.203}$$

The relativistic matter density today is $\rho \approx 6$ atoms/m^3 $\approx 10^{-26}$ kg/m^3. Then a is a constant of 10 billion light years. This model was proposed by Einstein in 1917, but is actually unstable. In other words, even the tiniest disturbance of the matter density would lead to either contraction or expansion of the universe. This model has been, eventually, superseded by more realistic ones, in particular after Hubble measured the expansion of the universe.

Vacuum solutions with $\rho = 0, P = 0$ lead to $\ddot{a} = a\Lambda/3$ and $\dot{a}^2 = a^2\Lambda/3 - k$. We distinguish several cases:

a) $\Lambda = 0, k = 0 \Rightarrow a$ is an arbitrary constant. This is Minkowski space.

b) $\Lambda = 0, k = -1 \Rightarrow a(t) = t$. Here, the universe is a linearly expanding hyperboloid. These are the same solution.

c) $\Lambda > 0, k = 1 \Rightarrow a(t) = \alpha \cosh t/\alpha$ with $\alpha = \sqrt{3/\Lambda}$. This is deSitter space dS_4.

d) $\Lambda > 0, k = 0 \Rightarrow a(t) = e^{t/\alpha}$

e) $\Lambda > 0, k = -1 \Rightarrow a(t) = \alpha \sinh t/\alpha$. Again, these all describe (different parts of) the same solution. In all of these cases, ∂_t and Σ_t are not unique. This is very special for vacuum solutions, otherwise one can use the rest system of the matter for ∂_t.

f) $\Lambda < 0, k = -1 \Rightarrow$ "Anti-deSitter space" AdS_4. ($k = -1$ is the only allowed value for $\Lambda < 0$.)

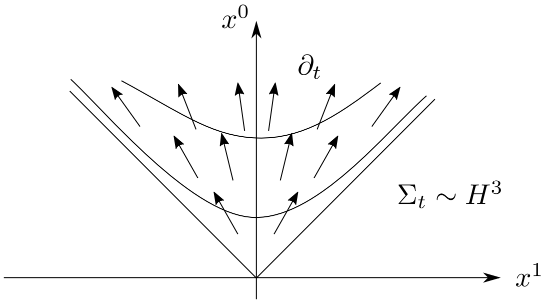

Figure 4.22: Case b) of the previous list. This covers part of Minkowski space, namely the forwards light cone. The parameter t is not x^0, which is why with this coordinate this solution has $\eta_{\mu\nu} \neq g_{\mu\nu}$.

Models with matter have

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}\frac{3\ddot{a}}{a} \qquad = -4\pi(\rho + 3P) + \Lambda. \tag{4.204}$$

We define the volume of a small spherical region

$$C := \frac{8}{3}\pi a^3 \rho \tag{4.205}$$

and write

$$\dot{C} = -8\pi P \dot{a} a^2 \tag{4.206}$$

or

$$\frac{d}{dt}\left(\frac{4\pi}{4}\rho a^3\right) + 4\pi a^2 P \dot{a} = 0. \tag{4.207}$$

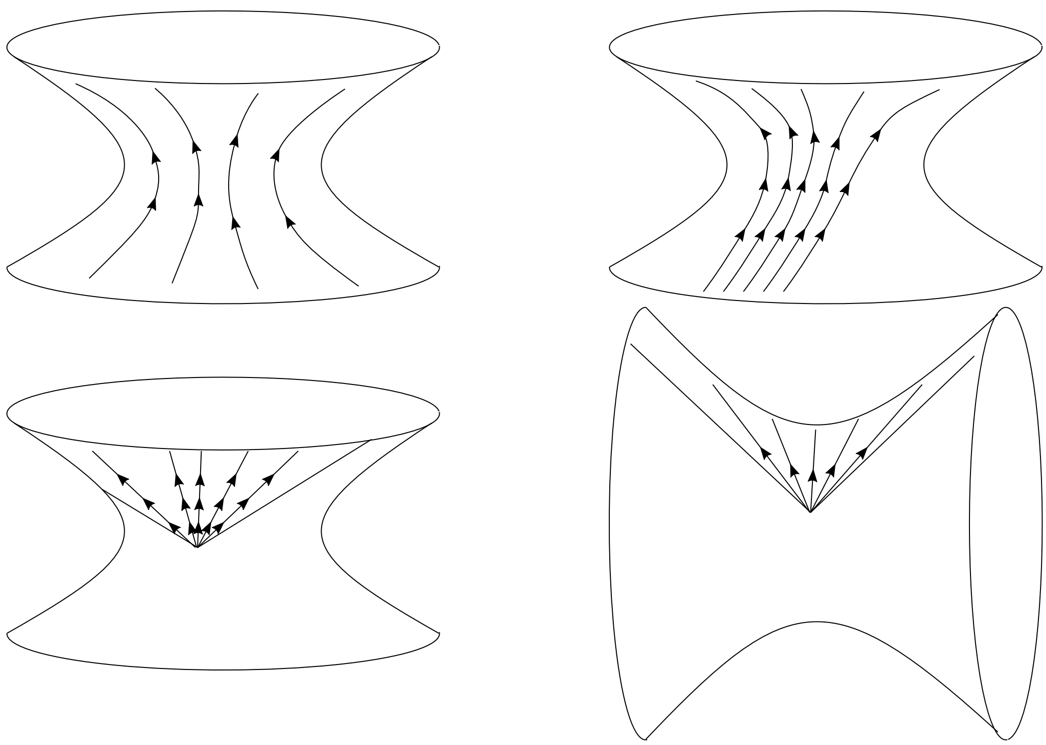

Figure 4.23: Different cases of the previous list, each with the vector field ∂_t drawn, for comparison. The first three are $c), d), e)$, different patches on deSitter space, while the last one is $f)$, some patch on Anti-deSitter space.

The total matter inside that region is $\sim \frac{4}{3}\pi a^3 \rho = C/2 =: U$. Its surface area is $\sim 4\pi a^2$, which means that

$$\dot{U} + AP\dot{a} = 0 \tag{4.208}$$

which is like $dU = -PdV$: The energy balance of a system without thermal flow. If one assumes a very simple equation of state, e.g. $P = w\rho$ for some constant w, we get

$$\frac{\dot{\rho}}{\rho} = -(1+w)\frac{\dot{(a^3)}}{a^3} \tag{4.209}$$

which integrates to

$$\rho \sim a^{-3(1+w)}. \tag{4.210}$$

For dust with $P = 0$, we have $\rho a^3 = const = C$, "matter conservation". For radiation with $P = \rho/3$, we have $\rho a^4 = const$. One can show that with mixed composition

$$\rho = \rho_{\text{dust}} + \rho_{\text{rad}} \tag{4.211}$$

$$P = \underbrace{P_{\text{dust}}}_{=0} + \underbrace{P_{\text{rad}}}_{\rho_{\text{rad}}/3}. \tag{4.212}$$

But each part separately satisfies $\rho_{\text{dust}} a^3 = const_1, \rho_{\text{rad}} a^4 = const_2$ for the same a. Nowadays, measurements suggest that $1/a_{\text{today}} = \rho_{\text{rad}}/\rho_{\text{dust}} \sim 10^{-3}$. The radiation density ρ_{rad} is mostly the cosmic microwave background (radiation from stars etc. is negligible compared to that). That means that at the time t_{rec}, when $a(\text{today})/a(\text{recombination}) \sim 10^3$, one had $\rho_{\text{rad}}(t_{\text{rec}}) \approx \rho_{\text{dust}}(t_{\text{rec}})$. If the universe is expanding, then the universe was radiation dominated before t_{rec} and matter dominated afterwards.

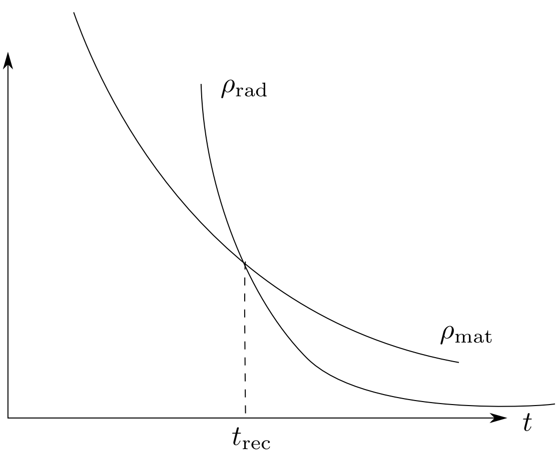

Figure 4.24: Before recombination, radiation dominated the matter content of the universe. After recombination, dust did.

Let us now consider "classical solutions" with $\Lambda = 0$. For a long time, this was supposed to be the only sensible type of models. We have

- $\Lambda = 0, k = 0$, dust $(w = 0)$, $a(t) \sim t^{2/3}$ or radiation $(w = 1/3)$, $a(t) \sim t^{1/2}$.

- $\Lambda = 0, k = -1$, dust: $t = 1/2C(\sinh\chi - \chi)$, $a = 1/2C(\cosh\chi - 1)$, $\chi > 1$, $\dot{C} = 0 = 8/3\pi\rho a^3$.

- $\Lambda = 0, k = +1$, dust: $t = 1/2C(\chi - \sinh\chi)$, $a = 1/2C(1 - \cos\chi)$, $\chi \in [0, \pi]$.

The value of k depends on the value of $\rho_0 = \rho_{\text{today}}$. We introduce the density parameter

$$\Omega_0 = \frac{8\pi\rho_0}{3H_0^2}\begin{cases} > 1 \\ = 1 \\ < 1 \end{cases} \text{if } k = \begin{cases} +1 \\ 0 \\ -1 \end{cases}. \tag{4.213}$$

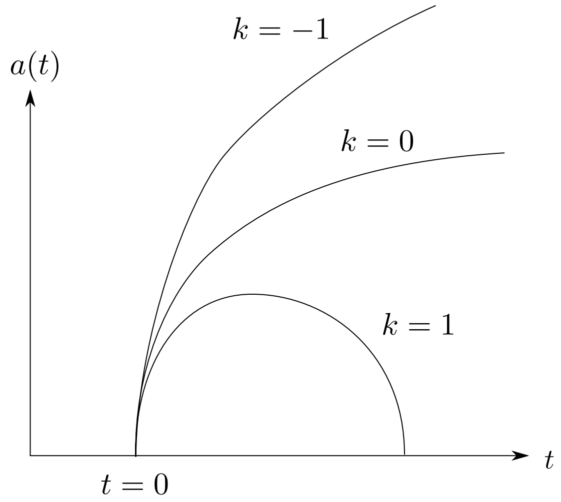

Figure 4.25: The "classical" solutions to the FRW equations, i.e. with $\Lambda = 0$. One can still find these three possibilities (collapsing universe, asymptotically, and infinitely expanding universe) in some textbooks, although they are nowadays obsolete.

For $\Lambda \neq 0$, there is a very large set of possible solutions, but $\Lambda > 0, k = 0$ seems to fit the observations best. The value of k is related to the present day density parameter Ω_0 and $\Omega_{\Lambda,0} := \Lambda/(3H_0^2)$. We have

$$\Omega_0 + \Omega_{\Lambda,0}\begin{cases} > 1 \\ = 1 \\ < 1 \end{cases} \text{if } k = \begin{cases} +1 \\ 0 \\ -1 \end{cases}. \tag{4.214}$$

Observation suggests $k = 0$. The density parameter splits into a density parameter $\Omega_{0,\text{ba}} \approx 0.05$ for baryons and one $\Omega_{0,\text{d}} \approx 0.26$ for dark matter, also $\Omega_{\Lambda,0} \approx 0.69$. This leads to a "$\Lambda$CDM" model with six free parameters containing properties of matter and fluctuations etc. This is nowadays the accepted model of our universe.

4.7 Linearised solutions and gravitational waves

While few exact solution to Einstein's equations are known, some approximate solutions can be computed. One example for this is the case of linear perturbations around Minkowski space. This is also called the linearised case.

One makes the ansatz

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4.215}$$

and assumes that $h_{\mu\nu}$, as well as all of its derivatives, are small. Of course, this violates the principle of general covariance, since components that are small in one coordinate system are not necessarily small in some other coordinate system. To this end, we also allow to only make coordinate transformations

$$x^\mu \longrightarrow \tilde{x}^\mu = x^\mu + \epsilon^\mu(x) \tag{4.216}$$

which are themselves small, i.e. the functions ϵ^μ and all of its derivatives are small. In the following, we disregard all terms quadratic in h, ϵ, and products $h\epsilon$, as well as terms containing more than one derivative.

First we note that, for any tensor which is small, i.e. contains h or ϵ, we can use either g or η to raise or lower indices, since the difference is a second order term, which we neglect.

For the inverse metric, we make the ansatz $g^{\mu\nu} = \eta^{\mu\nu} + k^{\mu\nu}$, with small coefficients $k^{\mu\nu}$. With this and (4.215) we get

$$\begin{aligned} \delta^\mu{}_\nu &= g^{\mu\rho} g_{\rho\nu} = \left(\eta^{\mu\rho} + k^{\mu\rho}\right)\left(\eta_{\rho\nu} + h_{\rho\nu}\right) \\ &= \delta^\mu{}_\nu + k^{\mu\rho}\eta_{\rho\nu} + \eta^{\mu\rho} h_{\rho\nu}. \end{aligned}$$

We define $h^{\mu\nu}$ to be the result of $h_{\mu\nu}$ after raising both indices (again, it does not matter whether we raise them with $g^{\mu\nu}$ or $\eta^{\mu\nu}$):

$$h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma}. \tag{4.217}$$

With this we get that $k^{\mu\nu} = -h^{\mu\nu}$, i.e.

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{4.218}$$

Note the sign difference compared to (4.215).

Next we note that the Christoffel symbols in the linearised case are themselves small. So products of two or more can be neglected This means that the Riemann curvature tensor (with all indices down), given by (2.151), is simply

$$R_{\mu\nu\sigma\rho} = \frac{1}{2}\left(h_{\mu\rho,\nu\sigma} - h_{\mu\sigma,\nu\rho} + h_{\nu\sigma,\mu\rho} - h_{\rho\nu,\sigma\mu}\right). \tag{4.219}$$

The Ricci tensor and Ricci scalar are then

$$R_{\nu\rho} = \eta^{\mu\sigma} R_{\mu\nu\sigma\rho} = \frac{1}{2}\left(h_{\mu\rho,\nu}{}^{\mu} - h^{\mu}{}_{\mu,\nu\rho} + h_{\nu}{}^{\mu}{}_{,\mu\rho} - \Box h_{\nu\rho}\right), \tag{4.220}$$

$$R = \eta^{\nu\rho} R_{\nu\rho} = h_{\mu\nu}{}^{,\mu\nu} - \Box(h^\mu{}_\mu). \tag{4.221}$$

Here we have used the (unfortunately, quite standard) notation of „upper commas". In particle physics (much more than in general relativity, actually), one off-handedly moves indices upstairs or downstairs with the Minkowski metric. For instance, one writes

$$h_{\mu\nu}{}^{,\rho} := \eta^{\rho\sigma} h_{\mu\nu,\sigma}, \text{ or even } h_{\mu\nu,\rho}{}^{\sigma} := \eta^{\sigma\lambda} h_{\mu\nu,\rho\lambda}. \tag{4.222}$$

This raising and lowering of indices actually works only because we are in the linearised case, which means that the covariant derivative can be replaced by the ordinary partial derivative. So all expressions above can, for this chapter, be treated as if they were tensor equations.

The $\Box$ is called the **d'Alembert** operator, or wave-operator, and is shorthand for $\Box = \eta_{\mu\nu}\partial_\mu\partial_\nu$, the Minkowski analogue of the Laplace operator.

Putting all of this together, the Einstein equations in vacuum $R_{\mu\nu} = 0$ become

$$R_{\mu\nu} = \frac{1}{2}\left(h_{\mu\rho,\nu}{}^{\rho} - h^{\rho}{}_{\rho,\nu\mu} + h_{\nu}{}^{\rho}{}_{,\rho\mu} - \Box h_{\mu\nu}\right) = 0. \tag{4.223}$$

4.7.1 Linearised gauge transformations

In chapter 3.2.3 we have talked about the fact that diffeomorphisms are acting as gauge transformations on the metric. Practically, this is realised by using what is called active change of coordinates. Start from a metric in some coordinates $g_{\mu\nu}(x)$. One can take any change of coordinates $x^\mu \mapsto \tilde{x}^\mu(x)$, compute the transformed metric coefficients $\tilde{g}_{\mu\nu}(\tilde{x})$, and replace, in its formula, all $\tilde{x}^\mu$ by x^μ. This leads to a new metric $\tilde{g}_{\mu\nu}(x)$ in the same coordinate system, which is regarded to be physically equivalent to the original one $g_{\mu\nu}(x)$.

With the linearised coordinate transformations (4.216), the inverse transformation is, to first order, given by

$$x^\mu = \tilde{x}^\mu - \epsilon^\mu(x) = \tilde{x}^\mu - \epsilon^\mu(\tilde{x}), \tag{4.224}$$

where we have used that, to first order,

$$\epsilon^\mu(x) = \epsilon^\mu(\tilde{x} - \epsilon) = \epsilon^\mu(\tilde{x}) - \epsilon^\nu(x)\partial_\nu\epsilon^\mu(x) + \cdots = \epsilon^\mu(\tilde{x}). \tag{4.225}$$

Thus, the partial derivatives are

$$\frac{\partial x^\mu}{\partial \tilde{x}^\nu} = \delta^\mu{}_\nu - \frac{\partial \epsilon^\mu}{\partial \tilde{x}^\nu}. \tag{4.226}$$

Putting these into the formula (4.215) for the metric, keeping only first order terms, and replacing $\tilde{x}$ with x everywhere then leads to

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \epsilon_{\mu,\nu} - \epsilon_{\nu,\mu}, \tag{4.227}$$

where $\epsilon_\mu := \eta_{\mu\nu}\epsilon^\nu$, and all functions are takes at the same argument x.

It is noteworthy that (4.227) is the linearised gravity analogue of the gauge transformation $a_\mu \to A_\mu + \chi_{,\mu}$ in electromagnetism. The presence of this gauge transformation means that we can impose a specific condition on our field, which are such that for every metric, there is a gauge transformed one which satisfies the condition. Such a condition is called **gauge fixing**. Examples in electromagnetism for this are, e.g. the time gauge $A^0 = 0$ or the Coulomb gauge $A^\mu{}_{,\mu} = 0$.

Consider the expression $h^\mu{}_{\nu,\mu} - \frac{1}{2}h^\mu{}_{\mu,\nu}$. Then, under a gauge transformation (4.227), one has that

$$h^\mu{}_{\nu,\mu} \longrightarrow \tilde{h}^\mu{}_{\nu,\mu} = h^\mu{}_{\nu,\mu} - \epsilon^\mu{}_{,\nu\mu} - \Box\epsilon_\nu \tag{4.228}$$

$$h^\mu{}_{\mu,\nu} \longrightarrow \tilde{h}^\mu{}_{\mu,\nu} = h^\mu{}_{\mu,\nu} - 2\epsilon^\mu{}_{,\mu\nu}. \tag{4.229}$$

This means that

$$\tilde{h}^\mu{}_{\nu,\mu} - \frac{1}{2}\tilde{h}^\mu{}_{\mu,\nu} = h^\mu{}_{\nu,\mu} - \frac{1}{2}h^\mu{}_{\mu,\nu} - \Box\epsilon_\nu. \tag{4.230}$$

In other words, by applying gauge transformations (4.227), we can change $h^\mu{}_{\nu,\mu} - \frac{1}{2}h^\mu{}_{\mu,\nu}$ by $\Box\epsilon_\nu$. Since the ϵ_ν are arbitrary (as long as they are small), we can use it to make the above term vanish. To do this, first solve the equation $\Box_\nu = h^\mu{}_{\nu,\mu} - \frac{1}{2}h^\mu{}_{\mu,\nu}$ for ϵ_μ[3], and use that ϵ_μ in the gauge transformation (4.227). The resulting $\tilde{h}_{\mu\nu}$ then satisfies $\tilde{h}^\mu{}_{\nu,\mu} - \frac{1}{2}\tilde{h}^\mu{}_{\mu,\nu} = 0$. In other words, without loss of generality, we can always find a gauge transformation to make our field satisfy this condition, so we impose

$$h^\mu{}_{\nu,\mu} - \frac{1}{2}h^\mu{}_{\mu,\nu} = 0. \tag{4.231}$$

This is also called **de Donder gauge**, and is often used in general relativity. Using it, Einstein's equations (4.223) simply become

$$-R_{\mu\nu} = \Box h_{\mu\nu} = 0. \tag{4.232}$$

So every component of the metric tensor solves the massless wave equation on Minkowski space. In other words, small perturbations in the Minkowski metric propagate with the speed of light. These perturbations are called **gravitational waves**.

4.7.2 Gravitational waves

Let us solve (4.232). To this end, we first note that the equation is easily solved[4] by

$$h_{\mu\nu}(x) = e_{\mu\nu}\exp\left(ik_\lambda x^\lambda\right) \tag{4.233}$$

for a constant 4×4-matrix $e_{\mu\nu}$ and a wave vector k^μ. Putting this into (4.232), we immediately get

$$k^\lambda k_\lambda = \eta_{\mu\nu}k^\mu k^\nu = 0. \tag{4.234}$$

[3] Which is possible, since this is just the wave equation with a source term, which can be solved e.g. by using the Green's function of the wave operator. See (...) for details.

[4] Since the equations (4.232) are linear, we can pretend that the $h_{\mu\nu}$ are allowed to be complex, and, in the end, simply take the real part of our solution.

So the wave vector k is light like, which confirm that the wave propagates at the speed of light. The matrix $e_{\mu\nu}$ is not arbitrary, however, since we have assumed the de Donder gauge condition (4.231). This condition, using (4.233) translates to

$$k_\mu e^\mu{}_\nu - k_\nu e^\mu{}_\mu = 0. \tag{4.235}$$

We assume that the wave travels in 3-direction, so

$$k^\mu = \begin{pmatrix} \omega/c \\ 0 \\ 0 \\ -k \end{pmatrix}, \qquad k = \frac{\omega}{c} \tag{4.236}$$

For the four different values of $\nu = 0, 1, 2, 3$, the equation for $e_{\mu\nu}$ are then

$$2(e_{00} + e_{30}) = e_{00} - e_{11} - e_{22} - e_{33} \tag{4.237}$$

$$2(e_{01} + e_{31}) = 0 \tag{4.238}$$

$$2(e_{02} + e_{32}) = 0 \tag{4.239}$$

$$2(e_{03} + e_{33}) = -e_{00} + e_{11} + e_{22} + e_{33}. \tag{4.240}$$

Since $h_{\mu\nu}$ is symmetric, so is $e_{\mu\nu}$. Subtracting (4.240) from (4.237) leads to

$$e_{11} = -e_{22}, \tag{4.241}$$

while the sum of both equations results in

$$e_{30} = \frac{e_{00} + e_{33}}{2}. \tag{4.242}$$

From this we can see that there are, in fact not 10 independent components in the matrix $e_{\mu\nu}$, but the do Donder gauge reduces that to six independent components, which can e.g. be taken to be e_{00}, e_{11}, e_{33}, e_{12}, e_{13}, and e_{23}.

However, we are not done at this point: Although we have used the gauge transformations (4.227) to bring the metric into a specific form, i.e. one that satisfies (4.231), there is still some residular gauge symmetry let. One can see this by considering the gauge-fixing procedure described above, and note that the equation

$$\Box\epsilon^\mu = h^\mu{}_{\nu,\mu} - \frac{1}{2}h^\mu{}_{\mu,\nu} \tag{4.243}$$

does not uniquely determine ϵ^μ. In particular, we can add another solution to $Box\epsilon^\mu = 0$, since this leaves (4.231) invariant.

Let us investigate how such a gauge transformation would change $e_{\mu\nu}$: The equation $\Box e^\mu = 0$ can be solved by

$$\epsilon_\mu(x) = \delta_\mu \exp\left(il_\lambda l^\lambda\right) \tag{4.244}$$

for some light-like l^μ. We choose $l^\mu = k^\mu$, and put this and (4.233) into (4.227), and we see that this leads to

$$e_{\mu\nu} \longrightarrow \tilde{e}_{\mu\nu} = e_{\mu\nu} + ik_\mu\delta_\nu + ik_\nu\delta_\mu. \tag{4.245}$$

Since k is of the form (4.236), we get

$$\begin{aligned}
\tilde{e}_{00} &= e_{00} + 2ik\delta_0 \\
\tilde{e}_{11} &= e_{11} \\
\tilde{e}_{33} &= e_{33} - 2ik\delta_3 \\
\tilde{e}_{12} &= e_{12} \\
\tilde{e}_{13} &= e_{13} + ik\delta_1 \\
\tilde{e}_{23} &= e_{23} - ik\delta_2
\end{aligned}$$

Since the δ_μ are arbitrary, we can chose them so that the only non-zero components are $e_{12} = e_{21}$ and $e_{11} = -e_{22}$. These are also denoted $e_\times$ and e_+, and are called the two **polarisations** of the gravitational wave. With the notation

$$h_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad h_\times = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.246}$$

one can write this (in a slightly non-covariant form) as

$$h_{\mu\nu}(x) = \left(e_+h_+ + e_\times h_\times\right)\exp\left(ik_\lambda x^\lambda\right) + c.c. \tag{4.247}$$

4.7.3 Polarisations of gravitational waves

After solving the linearised Einstein equationsm we have seen that the solution space contains two massless degrees of freedom per mode $\vec{k}$. Let us gain an intuition of the effect of these metrics (4.247). First we note that this metric leads to the Christoffel symbols

$$\Gamma^0_{00} = 0 = \Gamma^i_{00}. \tag{4.248}$$

There are other, non-vanishing Christoffel symbols, but these two are enough for us to show that the worldl ines

$$x^\mu(s) = \begin{pmatrix} s \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{4.249}$$

for constant x^1, x^2, x^3 are geodesics. This is remarkable, since it means that particles at rest stay at rest, as soon as a gravitational wave passes through them. Of course, the notion of being at rest is misleading here, since it is only in the Minkowski coordinate system that they appear as being at rest. In general, it does not make any sense to ask whether a gravitational wave moves particles or not.

In fact, one can show that, although it appears that all particles are at rest, the distance of nearby particles changes over time. To see this, we consider a number of dust particles, arranged at rest in the (x, y)-plane, on a circle around the origin of length L. Each of the particles can be identified by an angle ϕ, i.e. if it has the spatial coordinates $x^1 = L\cos\phi$, $x^2 = L\sin\phi$. In the space-time described by the metric (4.247), we consider the distance of that point to the center of the circle.

To first order, one can show that straight lines in the (x, y)-plane are indeed (reparametrisations of) geodesics. Hence, the distance of the dust particle at ϕ is equal to (minus) the length of the straight line from the center of the circle to the particle, i.e. the curve

$$x^0(\lambda) = ct, x^1(\lambda) = \lambda L\cos\phi,\ x^2(\lambda) = \lambda L\sin\phi,\ x^3(\lambda) = 0. \tag{4.250}$$

The length of the curve depends on t, and is given by

$$\ell_t = \int_0^1 d\lambda\sqrt{-(\eta_{\mu\nu} + h_{\mu\nu})\dot{x}^\mu\dot{x}^\nu}.$$

This can be easily computed, since the integrand is, in fact, independent of λ. One has, to first order,

$$\ell_t^2 = L^2\Big(1 + 2\big(e_\times \sin(2\phi) - e_+\cos(2\phi)\big)\cos(\omega t)\Big). \tag{4.251}$$

The two cases $e_\times = 0$, and $e_+ = 0$ can be considered separately. The two describe points lying on an ellipse, with small eccentricity, which is either stretching and contracting over time t, in the x^1-direction, or with angle $\pi/4$ relative to it.

So the gravitational wave does not "move" points, but compresses/stretches space between them in such a way that the distance between them changes over time.

If one solves the Einstein equations with matter, one can show that rotating bodies indeed generate gravitational waves, which radiate away. One can show that the quadrupole moment of the matter distribution is the first non-vanishing moment which contributes to this radiation. This means that there are no scalar and dipole moments of the radiation. The most important example of a stellar phenomenon generating gravitational waves consists of two massive bodies orbiting one another.

While gravitational waves have, for a long time, been only a theoretical possibility, there are several ways to directly or indirectly measure gravitational radiation.

- The most famous indirect method to confirm the existence of gravitational waves is the PSR1913+16. This is the name of a binary star system, in which a pulsar (quickly rotating neutron star) and another neutron star orbit around each other, once every 7.75 hours. Since they are quite heavy, they generate gravitational waves, which radiate energy away from the binary system, which leads to a slow

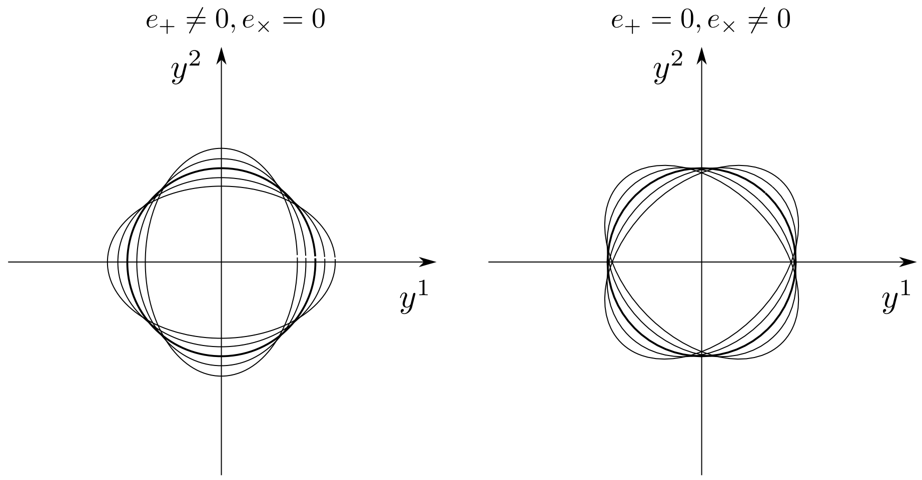

Figure 4.26: The two polarisations of a gravitational wave. Depicted are the collection of dust particles in Cartesian coordinates y^1, y^2, for different t. The circle is continuously deformed into an oscillating ellipse.

increase in their rotation period. Since that can be measured quite accurately (a pulsar sends out very regular bursts of energy), the increase in the rotation time has been measured, and confirmed the prediction fro GR to an astonishing degree.

The involved scientists Hulse and Taylor received the Nobel prize in physics 1993 for this discovery.

- More recent is, however, the direct verification of gravitational waves. These have been measured using large interferometers, with an arm lenth of several kilometres. The first confirmed detection of two colliding black holes, (rather, the gravitational wave signal emitted by them) was in September 2015, almost exactly a century after the publication of Einstein's theory of general relativity. Apart form technical difficulties, the detection relied on very accurate numerical simulations, which predicted the exact wave form of such an event. In 2017, Wise, Barish and Thorne received the Nobel Prize in Physics for their research related to this discovery.

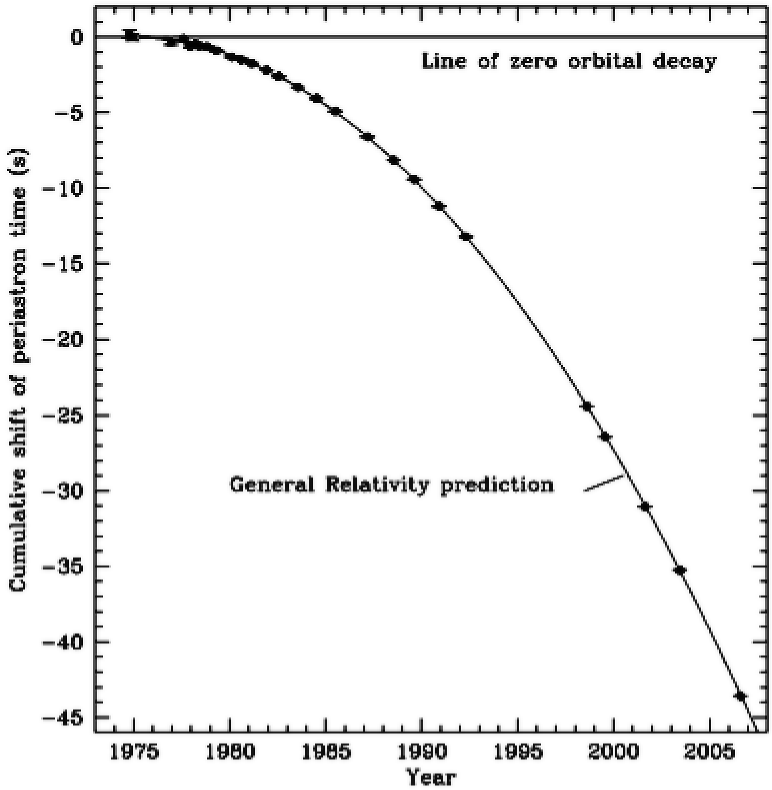

Figure 4.27: Measurement and theoretical prediction of orbital decline of the Pulsar PSR1913+16.

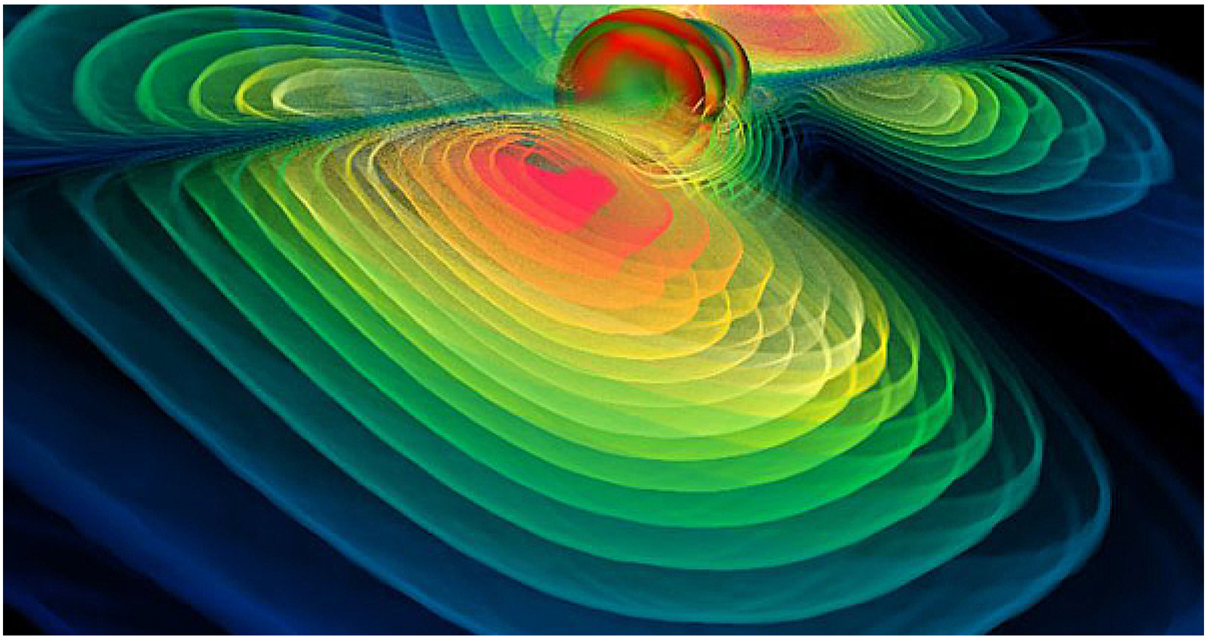

Figure 4.28: Numerical simulation of colliding black holes, and emitted gravitational radiation.

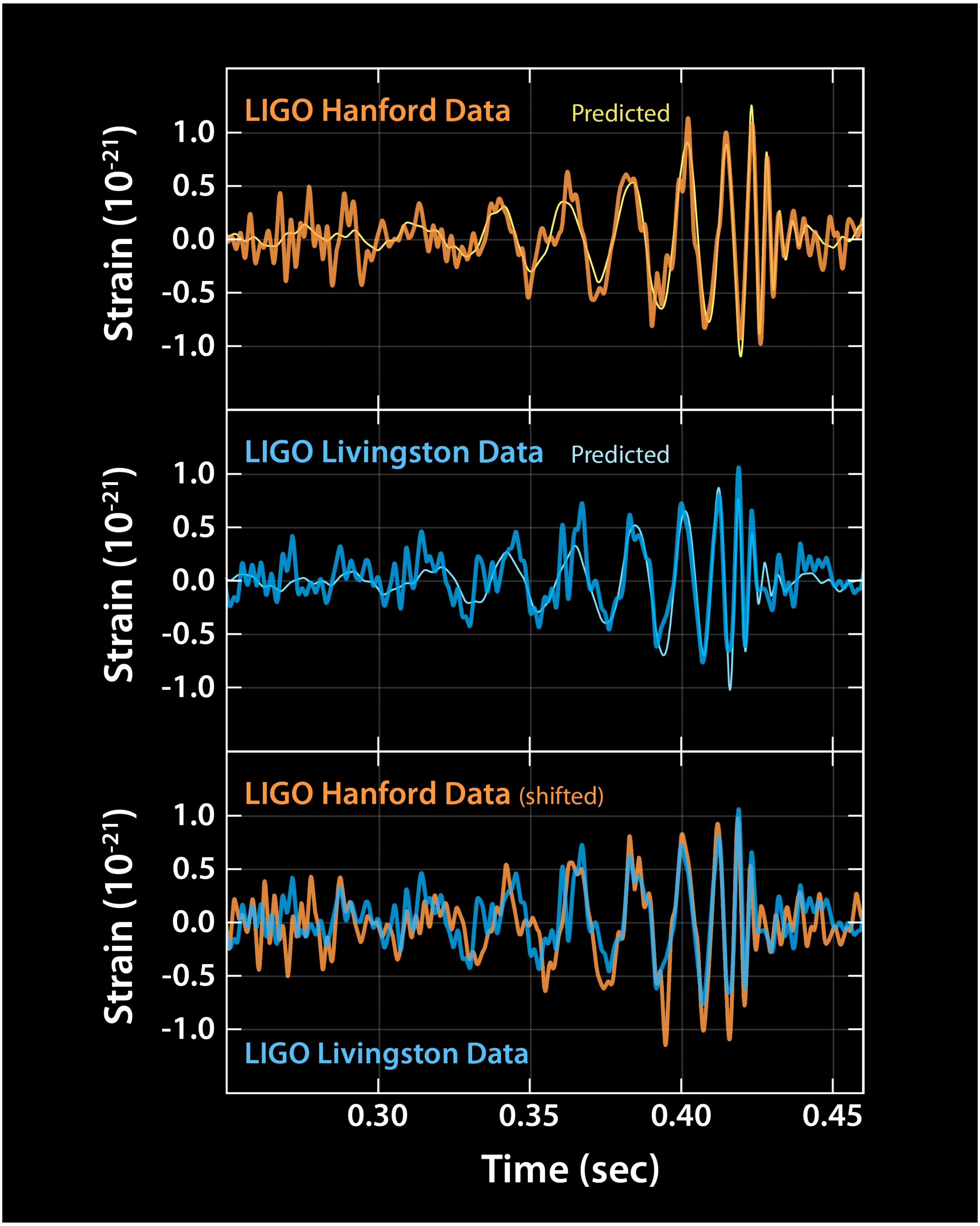

Figure 4.29: Detection of the gravitational wave signal GW150914, of two black holes colliding. The two signals were measured independently 7ms apart, in two stations roughly 3000km from one another.